# Level sets and Composition operators on the Dirichlet space

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# LEVEL SETS AND COMPOSITION OPERATORS ON THE DIRICHLET SPACE

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Abstract. We consider composition operators in the Dirichlet space of the unit disc in the plane. Various criteria on boundedness, compactness and Hilbert-Schmidt class membership are established. Some of these criteria are shown to be optimal.

## 1. Introduction

In this note we consider composition operators in the Dirichlet space of the unit disc. A comprehensive study of composition operators in function spaces and their spectral behavior could be found in [3, 10, 16]. See also [6, 7, 8, 12, 13, 17] for a treatment of some of the questions addressed in this paper.

Let $\mathbb{D}$ be the unit disc in the complex plane and let $\mathbb{T} = \partial\mathbb{D}$ be its boundary. We denote by $\mathcal{D}$ the classical Dirichlet space. This is the space of all analytic functions $f$ on $\mathbb{D}$ such that

$$\mathcal{D}(f) := \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$. We call $\mathcal{D}(f)$ the Dirichlet integral of $f$. The space $\mathcal{D}$ is endowed with the norm

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \mathcal{D}(f).$$

It is standard that a function $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)\, z^n$, holomorphic on $\mathbb{D}$, belongs to $\mathcal{D}$ if and only if

$$\sum_{n\geq 0} |\widehat{f}(n)|^2 \,(1+n) < \infty,$$

and that this series defines an equivalent norm on $\mathcal{D}$.

Since the Dirichlet space is contained in the Hardy space $\mathrm{H}^2(\mathbb{D})$, every function $f \in \mathcal{D}$ has non-tangential limits $f^*$ almost everywhere

2000 *Mathematics Subject Classification.* 47B38, 30H05, 30C85, 47A15.

*Key words and phrases.* Dirichlet space, composition operators, capacity.

1. Research partially supported by a grant from Egide Volubilis.

2. Research partially supported by grants from Egide Volubilis and ANR Dynop.

on $\mathbb{T}$. In this case, however, more can be said. Indeed, Beurling [2] showed that if $f \in \mathcal{D}$ then $f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$ exists for $\zeta \in \mathbb{T}$ outside of a set of logarithmic capacity zero.

Let $\varphi$ be a holomorphic self-map of $\mathbb{D}$. The composition operator $C_\varphi$ on $\mathcal{D}$ is defined by

$$C_\varphi(f) = f \circ \varphi, \qquad f \in \mathcal{D}.$$

We are interested herein in describing the spectral properties of the composition operator $C_\varphi$, such as compactness and Hilbert-Schmidt class membership, in terms of the size of the level set of $\varphi$. For $s \in (0,1)$, the level set $E_\varphi(s)$ of $\varphi$ is given by

$$E_\varphi(s) = \{\zeta \in \mathbb{T} \colon \, |\varphi(\zeta)| \geq s\}.$$

We give new characterizations of Hilbert-Schmidt class membership in the case of the Dirichlet space. We also establish the sharpness of these results.

## 2. A general criterion

For $\alpha > -1$, $dA_\alpha$ will denote the finite measure on $\mathbb{D}$ given by

$$dA_\alpha(z) := (1+\alpha)(1-|z|^2)^\alpha dA(z).$$

For $p \geq 1$ and $\alpha > -1$, the weighted Bergman space $\mathcal{A}^p_\alpha$ consists of the holomorphic functions $f$ on $\mathbb{D}$ for which

$$\|f\|_{p,\alpha} := \Big[\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z)\Big]^{1/p} < \infty.$$

We denote by $\mathcal{D}^p_\alpha$ the space consisting of analytic functions $f$ on $\mathbb{D}$ such that

$$\|f\|^p_{\mathcal{D}^p_\alpha} := |f(0)|^p + \|f'\|^p_{p,\alpha} < \infty.$$

Appropriate choices of the parameter $\alpha$ give, with equivalent norm, all the standard holomorphic function spaces. Indeed, The Hardy space $\mathrm{H}^2$ can be identified with $\mathcal{D}^2_1$. The classical Besov space is precisely $\mathcal{D}^p_{p-2}$, and if $p < \alpha + 1$, $\mathcal{D}^p_\alpha = \mathcal{A}^p_{\alpha-2}$. Finally, the classical Dirichlet space $\mathcal{D}$ is identical to $\mathcal{D}^2_0$.

We recall that, by the reproducing formula, one has

$$f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{2+\alpha}} \, dA_\alpha(w), \qquad z \in \mathbb{D}, \tag{1}$$

for every $f \in \mathcal{A}^p_\alpha$ (see [16]).

**Lemma 2.1.** *Let $p \geq 1$ and let $\sigma > -1$. Then, there exists a constant $C$ depending only on $p$ and $\sigma$ such that*

$$|f(z)|^p \ \leq \ C \int_{\mathbb{D}} \frac{|f(\lambda)|^p}{|1-\overline{\lambda}z|^{2+\sigma}}\, dA_\sigma(\lambda),$$

*for every $f \in \mathcal{A}^p_\sigma$ and $z \in \mathbb{D}$.*

*Proof.* By the above reproducing formula,

$$\frac{f(z)}{1-z\overline{w}} \ = \ \int_{\mathbb{D}} \frac{f(\lambda)}{1-\lambda\overline{w}}\, \frac{dA_\sigma(\lambda)}{(1-\overline{\lambda}z)^{2+\sigma}}, \qquad z, w \in \mathbb{D},$$

for every $f \in \mathcal{A}^p_\sigma$. By Hölder's inequality, with $q = p/(p-1)$,

$$\frac{|f(z)|^p}{|1-z\overline{w}|^p} \leq \int_{\mathbb{D}} \frac{|f(\lambda)|^p dA_\sigma(\lambda)}{|1-\overline{\lambda}z|^{2+\sigma}} \times \left( \int_{\mathbb{D}} \frac{dA_\sigma(\lambda)}{|1-\lambda\overline{w}|^q |1-\lambda\overline{z}|^{(2+\sigma)p}} \right)^{\frac{p}{q}}.$$

Taking $w = z$, and using the standard estimate ([16, Lemma 3.10])

$$\int_{\mathbb{D}} \frac{dA_c(\lambda)}{|1-z\overline{\lambda}|^{2+c+d}} \asymp \frac{1}{(1-|z|^2)^d}, \qquad \text{if } d > 0 \ ,\ c > -1, \tag{2}$$

we get the desired conclusion. □

For $\lambda \in \mathbb{D}$, consider the test function

$$F_{\lambda,\beta}(z) = \frac{1}{(1-\overline{\lambda}z)^{1+\beta}}, \qquad z \in \mathbb{D}.$$

If $\beta \geq 0$ is chosen such that $\delta := \delta(p,\alpha,\beta) = 2 + \beta - (2+\alpha)/p > 0$, by (2), we have

$$\|F_{\lambda,\beta}\|^p_{\mathcal{D}^p_\alpha} \asymp (1-|\lambda|^2)^{-p\delta}.$$

The following theorem unifies and generalizes the previously known results of MacCluer [3, Theorem 3.12], Tjani [12, Theorem 3.5] and Wriths-Xiao [13, Theorem 3.2] on Hardy, Besov and weighted Dirichlet spaces, respectively.

As mentioned before, the proof we provide here is short and simple.

**Theorem 2.2.** *Let $p > 1$. Suppose $\varphi \in \mathcal{D}^p_\alpha$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$. Fix $\beta \geq 0$ such that $\delta := \delta(p,\alpha,\beta) = 2 + \beta - (2+\alpha)/p > 0$. Then*

(a) *$C_\varphi$ is bounded on $\mathcal{D}^p_\alpha$* $\iff \sup_{\lambda \in \mathbb{D}} (1-|\lambda|^2)^\delta \|F_{\lambda,\beta} \circ \varphi\|_{\mathcal{D}^p_\alpha} < \infty$;

(b) *$C_\varphi$ is compact on $\mathcal{D}^p_\alpha$* $\iff \lim_{|\lambda| \to 1} (1-|\lambda|^2)^\delta \|F_{\lambda,\beta} \circ \varphi\|_{\mathcal{D}^p_\alpha} = 0.$

*Proof.* To prove (a), we observe that if $C_\varphi$ is bounded , then

$$\|F_{\lambda,\beta}\circ\varphi\|_{\mathcal{D}^p_\alpha} = O((1-|\lambda|^2)^{-\delta}).$$

For the converse we may assume, without loss of generality, that $\varphi$ fixes the origin. It follows from Lemma 2.1 that, for $f\in\mathcal{D}^p_\alpha$,

$$\begin{aligned}
&\int_{\mathbb{D}} |\varphi'(z)|^p |f'(\varphi(z))|^p\, dA_\alpha(z)\\
\leq\ & C\int_{\mathbb{D}} |\varphi'(z)|^p \Big(\int_{\mathbb{D}} \frac{|f'(\lambda)|^p}{|1-\overline{\lambda}\varphi(z)|^{(2+\beta)p}}\, dA_{2p+\beta p-2}(\lambda)\Big)\, dA_\alpha(z)\\
=\ & C\int_{\mathbb{D}} |f'(\lambda)|^p \Big((1-|\lambda|^2)^{2p+\beta p-2-\alpha}\int_{\mathbb{D}} \frac{|\varphi'(z)|^p}{|1-\overline{\lambda}\varphi(z)|^{(2+\beta)p}}\, dA_\alpha(z)\Big)\, dA_\alpha(\lambda)\\
=\ & C\int_{\mathbb{D}} |f'(\lambda)|^p (1-|\lambda|^2)^{p\delta} \|(F_{\lambda,\beta}\circ\varphi)'\|^p_{p,\alpha}\, dA_\alpha(\lambda).
\end{aligned}$$

Therefore part (a) follows.

(b) Without loss of generality we assume that $\varphi(0)=0$. Note that $C_\varphi$ is compact on $\mathcal{D}^p_\alpha$ if and only if for every bounded sequence $(f_n)_n\subset\mathcal{D}^p_\alpha$ such that $f_n\to 0$ uniformly on compact subsets of $\mathbb{D}$, we have $\|C_\varphi(f_n)\|_{\mathcal{D}^p_\alpha}\to 0$, as $n\to\infty$.

Suppose that $C_\varphi$ is compact. Since $(1-|\lambda|^2)^\delta F_{\lambda,\beta}\to 0$ uniformly on compact subsets of the unit disc, as $|\lambda|\to 1$, we see that

$$\|C_\varphi(F_{\lambda,\beta})\|_{\mathcal{D}^p_\alpha} = o((1-|\lambda|^2)^{-\delta}).$$

Conversely, assume that $\lim\limits_{|\lambda|\to 1}(1-|\lambda|^2)^\delta\|F_{\lambda,\beta}\circ\varphi\|_{\mathcal{D}^p_\alpha}=0$. Let $(f_n)_n$ be a bounded sequence of $\mathcal{D}^p_\alpha$ such that $f_n\to 0$ uniformly on compact sets. Since $f_n'\to 0$ uniformly on compact sets, it follows from the proof of part $(a)$ and the hypothesis that, for $r$ close enough to 1,

$$\begin{aligned}
\|C_\varphi(f_n)\|^p_{\mathcal{D}^p_\alpha} - |f_n(0)|^p &\leq \int_{r\mathbb{D}} |f_n'(\lambda)|^p (1-|\lambda|^2)^{p\delta}\|(F_{\lambda,\beta}\circ\varphi)\|^p_{p,\alpha} dA_\alpha(\lambda)\\
&+\int_{\mathbb{D}\setminus r\mathbb{D}} |f_n'(\lambda)|^p (1-|\lambda|^2)^{p\delta}\|(F_{\lambda,\beta}\circ\varphi)'\|^p_{p,\alpha} dA_\alpha(\lambda)\to 0,\quad n\to\infty
\end{aligned}$$

which finishes the proof. □

The following result is an immediate consequence of Theorem 2.2.

**Corollary 2.3.** *Let $\varphi:\mathbb{D}\to\mathbb{D}$ such that $\varphi\in\mathcal{D}$.*

*(a) If $\sup\limits_{n\geq 1}\mathcal{D}(\varphi^n)<\infty$, then $C_\varphi$ is bounded;*

*(b) If $\lim\limits_{n\to\infty}\mathcal{D}(\varphi^n)=0$, then $C_\varphi$ is compact.*

*Proof.* We consider the test function $F_{\lambda,0}$ with $\beta = \alpha = 0$ and $p = 2$. Both (a) and (b) follow from the following inequality:

$$\begin{aligned}
\mathcal{D}(F_{\lambda,0} \circ \varphi) &\leq 2\,(1 - |\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\lambda|^2 \varphi(z)|^2)^4}\, dA(z) \\
&\leq c\,(1 - |\lambda|^2)^2 \sum_{n \geq 0} (n+1)^3\, |\lambda|^{2n} \int_{\mathbb{D}} |\varphi'(z)|^2 |\,\varphi^n(z)|^2\, dA(z) \\
&= c\,(1 - |\lambda|^2)^2 \sum_{n \geq 0} (1+n)\, |\lambda|^{2n}\, \mathcal{D}(\varphi^{n+1}) \\
&\leq c\, \limsup_{n \to \infty} \mathcal{D}(\varphi^{n+1}).
\end{aligned}$$

□

**Remarks 2.4.**

1. The compactness criterion for $C_\varphi$ in the Bloch space is equivalent to $\|\varphi^n\|_{\mathcal{B}} \to 0$ as was shown in [15] (see also [11, 12]). In the case of the Hardy space $\mathrm{H}^2$, however, we know that if $C_\varphi$ is compact on $\mathrm{H}^2$ then $\|\varphi^n\|_{\mathrm{H}^2} \to 0$ but the converse does not hold [3]. Note that as before in the proof of Corollary 2.3 ($\beta = 0, \alpha = 1$ and $p = 2$) if $\|\varphi^n\|_{\mathrm{H}^2} = o(1/\sqrt{n})$, then $C_\varphi$ is compact on $\mathrm{H}^2$.

2. The characterization of compact composition operators on the Dirichlet space in terms of Carleson measures can be found in [3, 12, 17]. A positive Borel measure $\mu$ given on $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} |f(z)|^2\, d\mu(z) \leq \|f\|_{2,0}^2, \qquad f \in \mathcal{A}_0^2,$$

is called a Carleson measure for $\mathcal{A}_0^2$, i.e., the identity map $i_0 : \mathcal{A}_0^2 \to \mathrm{L}^2(\mu)$ is a bounded operator. Such a measure has the following equivalent properties (see [16, Theorem 7.4]). A positive Borel measure $\mu$ is a Carleson measure for $\mathcal{A}_0^2$ if and only if

$$\sup_{\lambda \in \mathbb{D}} (1 - |\lambda|^2)^2 \int_{\mathbb{D}} \frac{d\mu(z)}{|1 - \overline{\lambda} z|^4} < \infty,$$

or, equivalently,

$$\sup_{I \subset \mathbb{T}} \mu(S(I))/|I|^2 < \infty,$$

for any subarc $I \subset \mathbb{T}$ with arclengh $|I|$, and $S(I)$ is the Carleson box.

The measure $\mu$ is called vanishing (or compact) Carleson measure for $\mathcal{A}_0^2$ if the identity map $i_\alpha : \mathcal{A}_0^2 \to \mathrm{L}^2(\mu)$ is a compact operator. This happens if and only if

$$\lim_{|\lambda|\to 1}(1-|\lambda|^2)^2\int_{\mathbb{D}}\frac{d\mu(z)}{|1-\overline{\lambda}z|^4}=0 \iff \lim_{|I|\to 0}\ \mu(S(I))/|I|^2=0.$$

Let $\varphi:\mathbb{D}\to\mathbb{D}$ be analytic and denote by $n_\varphi(z)$ the multiplicity of $\varphi$ at $z$. By the change of variable formula [10],

$$\|F_{\lambda,0}\|_{2,0}^p=(1-|\lambda|^2)^2\int_{\mathbb{D}}\frac{n_\varphi(z)\,dA(z)}{|1-\overline{\lambda}z|^4}.$$

Therefore, as a consequence of Theorem 2.2, $C_\varphi$ is bounded in $\mathcal{D}$ if and only if $n_\varphi(z)dA(z)$ is a Carleson measure for $\mathcal{A}_0^2$ and $C_\varphi$ is compact in $\mathcal{D}$ if and only if $n_\varphi(z)dA(z)$ is a vanishing Carleson measure for $\mathcal{A}_0^2$. More explicitly, we have

$$\begin{cases} C_\varphi \quad \text{is bounded in } \mathcal{D} & \iff \sup_{I\subset\mathbb{T}}\dfrac{1}{|I|^2}\displaystyle\int_{S(I)} n_\varphi(z)\,dA(z)<\infty; \\ C_\varphi \quad \text{is compact in } \mathcal{D} & \iff \lim_{|I|\to 0}\dfrac{1}{|I|^2}\displaystyle\int_{S(I)} n_\varphi(z)\,dA(z)=0. \end{cases}$$

## 3. Hilbert-Schmidt membership

In the case of the Hardy space $\mathrm{H}^2$, one can completely describe the membership of $C_\varphi$ in the Hilbert-Schmidt class in terms of the size of the level sets of the inducing map $\varphi$. Indeed, $C_\varphi$ is Hilbert-Schmidt in $\mathrm{H}^2$ if and only if

$$\sum_{n\geq 0}\|\varphi^n\|_{\mathrm{H}^2}^2=\int_{\mathbb{T}}\frac{|d\zeta|}{1-|\varphi(\zeta)|^2}<\infty.$$

Given an arbitrary measurable function $f$ on $\mathbb{T}$, consider the associated distribution function $m_f$ defined by

$$m_f(\lambda)=|\{\zeta\in\mathbb{T}:\,|f(\zeta)|>\lambda|\},\qquad \lambda>0.$$

It then follows that $C_\varphi$ is in the Hilbert-Schmidt class of $\mathrm{H}^2$ if and only if

$$\int_{\mathbb{T}}\frac{|d\zeta|}{1-|\varphi(\zeta)|^2}=\int_1^\infty m_{(1-|\varphi|^2)^{-1}}(\lambda)\,d\lambda\asymp\int_0^1\frac{|E_\varphi(s)|}{(1-s)^2}\,ds<\infty.$$

It was shown by Gallardo–González [8, Theorem] that there is a mapping $\varphi$ taking $\mathbb{D}$ to itself such that $C_\varphi$ is compact in $\mathrm{H}^2$, and that the level set $E_\varphi(1)$ has Hausdorff measure equal to one. Recall that the Hausdorff dimension of $E$

$$d(E)=\inf\{\alpha:\,\Lambda_\alpha(E)=0\}$$

where $\Lambda_\alpha(E)$ is the $\alpha$–Hausdorff measure of $E$ given by

$$\Lambda_\alpha(E) = \lim_{\epsilon\to 0}\inf\Big\{\sum_i |\Delta_i|^\alpha : E\subset\bigcup_i \Delta_i\ ,\ |\Delta_i|<\epsilon\Big\}.$$

Given $E\subset\mathbb{T}$ and $t>0$, let us write $E_t=\{\zeta : d(\zeta,E)\le t\}$ where $d$ denotes the arclength distance and $|E_t|$ denotes the Lebesgue measure of $E$.

Let $E$ be a closed subset of $\mathbb{T}$ with $|E_t| = O((\log(e/t))^{-3})$ and $E$ has Hausdorff dimension one. (such examples can be given by generalized Cantor sets [2]). Let $\omega(t) = (\log(e/t))^{-2}$, and consider the outer function given by

$$|f_{\omega,E}(\zeta)| = e^{-w(d(\zeta,E))}, \qquad \text{a.e on } \mathbb{T}.$$

Since $\omega$ satisfies the Dini condition

$$\int_0 \frac{\omega(t)}{t}dt<\infty,$$

it follows that $f_{\omega,E}\in A(\mathbb{D}) := \mathrm{Hol}(\mathbb{D})\cap C(\overline{\mathbb{D}})$, disc algebra (see [9] p.105–106) and so $E_{f_{\omega,E}}(1)=E$. On the other hand

$$\int_{\mathbb{T}}\frac{|d\zeta|}{1-|f_{\omega,E}(\zeta)|^2}\asymp\int_{\mathbb{T}}\frac{|d\zeta|}{\omega(d(\zeta,E))}\asymp\int_0|E_t|\frac{\omega'(t)}{\omega(t)^2}dt,$$

(see [4, Proposition A.1 ] for the last equality). Since the last integral converges, $C_\varphi$ is a Hilbert-Schmidt operator in $\mathrm{H}^2$.

We have the following more precise result.

**Theorem 3.1.** *Let $E$ be a closed subset of $\mathbb{T}$ with Lebesgue measure zero. There exists a mapping $\varphi:\mathbb{D}\to\mathbb{D}$, $\varphi\in A(\mathbb{D})$ such that $C_\varphi$ is a Hilbert-Schmidt operator on $\mathrm{H}^2$ and that $E_\varphi(1)=E$.*

*Proof.* The proof is based a well known construction of peak functions in the disc algebras. Let $\mathbb{T}\setminus E=\cup_{n\ge1}(e^{ia_n},e^{ib_n})$. For $t\in(a_n,b_n)$, we define

$$g(e^{it}) = \tau_n\frac{(b_n-a_n)^{1/2}}{((b_n-a_n)^2-(2t-(b_n+a_n))^2)^{1/4}},$$

where $(\tau_n)_n\subset(0,\infty)$ will be chosen later, and $g(e^{it}) := +\infty$ if $e^{it}\in E$. Note that

$$\int_0^{2\pi} g(e^{it})^2\,dt = \pi\sum_{n=1}^\infty \tau_n^2\,(b_n-a_n).$$

Since $\sum_{n=1}^{\infty}(b_n - a_n) = 2\pi$, there exists a sequence $(\tau_n)_n$ such that

$$\lim_{n\to+\infty} \tau_n = +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \tau_n^2\,(b_n - a_n) < \infty.$$

Let $U$ denote the harmonic extension of $g$ on the unit disc given by

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|e^{it} - re^{i\theta}|^2}\, g(e^{it})\, dt = \sum_{n\in\mathbb{Z}} \widehat{g}(n)\, r^{|n|}\, e^{in\theta}.$$

Since $\tau_n \to \infty$, one can easily verify that $\lim_{t\to\theta} g(e^{it}) = +\infty$, for $e^{i\theta} \in E$. Hence, $\lim_{r\to 1^-} U(re^{i\theta}) = +\infty$, for $e^{i\theta} \in E$.
Let $V$ be the harmonic conjugate of $U$, with $V(0) = 0$. It is given by

$$V(re^{i\theta}) = \sum_{n\neq 0} \frac{n}{|n|}\, \widehat{g}(n)\, r^{|n|}\, e^{in\theta}.$$

Now, since $g$ is a $C^1$ function on $\mathbb{T}\setminus E$, we see that the holomorphic function $f = U + iV$ is continuous on $\overline{\mathbb{D}}\setminus E$. Knowing that $\lim_{r\to 1^-} U(re^{it}) = +\infty$, for $e^{it} \in E$, we get that $\varphi = \dfrac{f}{f+1} \in A(\mathbb{D})$, disc algebra, and $E_\varphi(1) = E$. Finally

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{1-|\varphi(e^{it})|^2} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{(U(e^{it})+1)^2 + V^2(e^{it})}{(U(e^{it})+1)^2 - U^2(e^{it})} dt \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} (U(e^{it})+1)^2 + V^2(e^{it}) dt \\ &\leq 1 + 2\sum_{n\in\mathbb{Z}} |\,\widehat{g}(n)|^2, \end{aligned}$$

which shows that $C_\varphi$ is Hilbert-Schmidt because $g \in L^2(\mathbb{T})$. □

Let $E$ be a closed subset of the unit circle $\mathbb{T}$. Fix a non-negative function $w \in C^1(0,\pi]$ such that

$$\int_{\mathbb{T}} w(d(\zeta, E))\, |d\zeta| < \infty,$$

where d denotes the arclength distance. Now, let $f_{w,E}$ be the outer function given by

$$|f^*_{w,E}(\zeta)| = e^{-w(d(\zeta,E))}, \qquad \text{a.e. on } \mathbb{T}. \tag{3}$$

The following lemma gives an estimate for the Dirichlet integral of $f_{w,E}$ in terms of $w$ and the distance function on $E$. The proof is based on Carleson's formula, and can be achieved by slightly modifying the arguments used in [5, Theorem 4.1].

**Lemma 3.2.** *Assume that the function $\omega$ is nondecreasing and $\omega(t^\gamma)$ is concave for all $\gamma > 2$. Then*

$$\mathcal{D}(f_{w,E}) \asymp \int_{\mathbb{T}} \omega'(d(\zeta, E))^2 \, e^{-2w(d(\zeta,E))} \, d(\zeta, E) \, |d\zeta|.$$

Since the sequence $\{z^n/\sqrt{n+1}\}_{n=0}^\infty$ is an orthonormal basis of $\mathcal{D}$, the operator $C_\varphi$ is Hilbert-Schmidt on the Dirichlet space if and only if

$$\frac{1}{\pi} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \, dA(z) = \sum_{n \geq 1} \frac{\mathcal{D}(\varphi^n)}{n} \; < \; \infty.$$

**Theorem 3.3.** *Assume that the function $\omega$ is nondecreasing and $\omega(t^\gamma)$ is concave for some $\gamma > 2$. Then $C_{f_{w,E}}$ is in the Hilbert-Schmidt class in $\mathcal{D}$ if and only if*

$$\int_{\mathbb{T}} \frac{\omega'(d(\zeta, E))^2}{w(d(\zeta, E))^2} \, d(\zeta, E) \, |d\zeta| \; < \; \infty.$$

*Proof.* We first note that $f^n_{w,E} = f_{nw,E}$. Therefore, by Lemma 3.2, we have

$$\begin{aligned}
\int_{\mathbb{D}} \frac{|f'_{w,E}(z)|^2}{(1-|f_{w,E}(z)|^2)^2} \, dA(z) &= \sum_{n=1}^\infty \frac{\mathcal{D}(f_{nw,E})}{n} \\
&\asymp \int_{\mathbb{T}} \omega'(d(\zeta, E))^2 \, d(\zeta, E) \sum_{n=1}^\infty n e^{-2nw(d(\zeta,E))} \, |d\zeta| \\
&\asymp \int_{\mathbb{T}} \frac{\omega'(d(\zeta, E))^2}{[1-e^{-2w(d(\zeta,E))}]^2} \, d(\zeta, E) \, |d\zeta|.
\end{aligned}$$

Since $1 - e^{-2w(d(\zeta,E))} \asymp w(d(\zeta, E))$, the result flows. $\square$

Given a (Borel) probability measure $\mu$ on $\mathbb{T}$, we define its $\alpha$-energy, $0 \leq \alpha < 1$, by

$$I_\alpha(\mu) = \sum_{n=1}^\infty \frac{|\widehat{\mu}(n)|^2}{n^{1-\alpha}}.$$

For a closed set $E \subset \mathbb{T}$, its $\alpha$-capacity $\mathrm{cap}_\alpha(E)$ is defined by

$$\mathrm{cap}_\alpha(E) := 1/\inf\{I_\alpha(\mu) : \mu \text{ is a probability measure on } E\}.$$

If $\alpha = 0$, we simply note $\mathrm{cap}(E)$ and this means the logarithmic capacity of $E$.

The weak–type inequality for capacity [2] states that, for $f \in \mathcal{D}$ and $t \geq 4\|f\|_{\mathcal{D}}^2$,

$$\operatorname{cap}(\{\zeta : |f(\zeta)| \geq t\}) \leq \frac{16\|f\|_{\mathcal{D}}^2}{t^2}.$$

As a result of this inequality, we see that if $\liminf \|\varphi^n\|_{\mathcal{D}} = 0$, then $\operatorname{cap}(E_\varphi(1)) = 0$. Indeed, since $E_\varphi(1) = E_{\varphi^n}(1)$, the weak capacity inequality implies that

$$\operatorname{cap}(E_\varphi(1)) = \operatorname{cap}(E_{\varphi^n}(1)) \leq 16\|\varphi^n\|_{\mathcal{D}}^2.$$

Now let $n \to \infty$. Hence, in particular, if the operator $C_\varphi$ is in the Hilbert-Schmidt class in $\mathcal{D}$, then $\operatorname{cap}(E_\varphi(1)) = 0$. This result was first obtained by Gallardo–González [6, 7] using a completely different method. Theorems 3.4 and 3.6 give quantitative versions of this result.

**Theorem 3.4.** *If $C_\varphi$ is a Hilbert-Schmidt operator in $\mathcal{D}$, then*

$$\int_0^1 \frac{\operatorname{cap}(E_\varphi(s))}{1-s} \log \frac{1}{1-s}\, ds < \infty. \tag{4}$$

*Proof.* Fix $\lambda \in \mathbb{T}$ and let

$$\varphi_\lambda(\zeta) = \log \operatorname{Re} \frac{1+\lambda\varphi(\zeta)}{1-\lambda\varphi(\zeta)}, \qquad \zeta \in \mathbb{T}.$$

Since

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)^2|)^2}\, dA(z) < \infty,$$

it follows that $\varphi_\lambda \in \mathcal{D}(\mathbb{T})$, see [6], where

$$\mathcal{D}(\mathbb{T}) := \{f \in L^2(\mathbb{T}) : \|f\|_{\mathcal{D}(\mathbb{T})}^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2(1+|n|) < \infty\}.$$

Setting $\Delta_\lambda := \{\zeta \in \mathbb{T} : |1-\lambda\varphi(\zeta)| \geq 1\}$, we see that

$$|\varphi_\lambda(\zeta)| \asymp \log \frac{1}{1-|\varphi(\zeta)|^2}, \qquad \forall \zeta \in \Delta_\lambda.$$

Applying the strong capacity inequality [14, Theorem 2.2] to $\varphi_\lambda$, we get

$$\begin{aligned}
\infty > \|\varphi_\lambda\|^2_{\mathcal{D}(\mathbb{T})} &\geq c\int^\infty \operatorname{cap}\{\zeta\in\mathbb{T}: |\varphi_\lambda(\zeta)|>s\}\,ds^2\\
&= c\int^\infty \operatorname{cap}\Big\{\zeta\in\mathbb{T}: \Big|\log\frac{1-|\varphi(\zeta)|^2}{|1-\lambda\varphi(\zeta)|^2}\Big|>s\Big\}\,ds^2\\
&\geq c\int^\infty \operatorname{cap}\Big\{\zeta\in\mathbb{T}\cap\Delta_\lambda: \Big|\log\frac{1-|\varphi(\zeta)|^2}{|1-\lambda\varphi(\zeta)|^2}\Big|>s\Big\}\,ds^2\\
&\geq c\int^\infty \operatorname{cap}\Big\{\zeta\in\mathbb{T}\cap\Delta_\lambda: \log\frac{1}{1-|\varphi(\zeta)|^2}>4s\Big\}\,ds^2\\
&\geq c_1\int^1 \operatorname{cap}\Big\{\zeta\in\mathbb{T}\cap\Delta_\lambda: |\varphi(\zeta)|>u\Big\}\,d\big(\log\frac{1}{1-u}\big)^2.
\end{aligned}$$

Since $\mathbb{T}=\Delta_1\cup\Delta_{-1}$, the subadditivity of the capacity implies that

$$\begin{aligned}
&\infty > \|\varphi_1\|^2_{\mathcal{D}(\mathbb{T})}+\|\varphi_{-1}\|^2_{\mathcal{D}(\mathbb{T})}\geq\\
&\qquad\qquad c_2\int^1 \operatorname{cap}\Big\{\zeta\in\mathbb{T}: |\varphi(\zeta)|>u\Big\}\,d\big(\log\frac{1}{1-u}\big)^2,
\end{aligned}$$

and hence the theorem follows. □

**Remarks 3.5.**

Since $\{z^n/(1+n)^{\frac{1-\alpha}{2}}\}_{n=0}^\infty$ is an orthonormal basis in $\mathcal{D}_\alpha$, $\alpha\in(0,1)$, $C_\varphi$ is a Hilbert-Schmidt operator in $\mathcal{D}_\alpha$ if and only if

$$\sum_{n=1}^\infty\frac{\mathcal{D}_\alpha(\varphi^n)}{n^{1-\alpha}}\asymp\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2+\alpha}}\,dA_\alpha(z)<\infty.$$

Therefore, for fixed $\lambda\in\mathbb{T}$, the function

$$\varphi_\lambda(\zeta)=\Big(\mathrm{Re}\frac{1+\lambda\varphi(\zeta)}{1-\lambda\varphi(\zeta)}\Big)^{-\alpha/2},\qquad(\zeta\in\mathbb{T}),$$

belongs to the weighted harmonic Dirichlet space

$$\mathcal{D}_\alpha(\mathbb{T}):=\{f\in L^2(\mathbb{T}):\ \|f\|^2_{\mathcal{D}_\alpha(\mathbb{T})}=\sum_{n\in\mathbb{Z}}|\widehat{f}(n)|^2(1+|n|)^{1-\alpha}<\infty\},$$

(see [7]) . Applying again the strong capacity inequality [14, Theorem 2.2] for $\mathcal{D}_\alpha$ to $\varphi_\lambda$, we get as before

$$\int_0^1\frac{\operatorname{cap}_\alpha(E_\varphi(s))}{(1-s)^{1+\alpha}}\,ds<\infty.$$

The following theorem is the analogue of Proposition 3.1 for the Dirichlet space. It shows that condition (4) is optimal.

**Theorem 3.6.** *Let* $h : [1,+\infty[\to [1,+\infty[$ *be a function such that* $\lim_{x\to\infty} h(x) = +\infty$. *Let* $E$ *be a closed subset of* $\mathbb{T}$ *such that* $\operatorname{cap}(E) = 0$. *Then there is* $\varphi \in A(\mathbb{D}) \cap \mathcal{D}$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ *such that :*

*(1)* $E_\varphi(1) = E$;
*(2)* $C_\varphi$ *is in the Hilbert–Schmidt class in* $\mathcal{D}$;
*(3)* $\displaystyle\int^1 \frac{\operatorname{cap}(E_\varphi(s))}{1-s} \log\frac{1}{1-s} h\Big(\frac{1}{1-s}\Big)\, ds = +\infty.$

*Proof.* Let $k(x) = h(e^x)$, there exists a continuous decreasing function $\psi$ such that

$$\int^{+\infty} \psi(x)\, dx^2 < \infty \quad \text{and} \quad \int^{+\infty} \psi(x)\, k(x)\, dx^2 = \infty.$$

Set $\eta(t) = \psi^{-1}(\operatorname{cap}(E_t))$. We have

$$\begin{aligned}\int_0 \operatorname{cap}(E_t)\, |d\eta^2(t)| &\asymp \int_0 \psi(\eta(t))\, |d\eta^2(t)| \\ &\asymp \int^{+\infty} \psi(x)\, dx^2 < \infty,\end{aligned}$$

and,

$$\begin{aligned}\int_0 \operatorname{cap}(E_t)\, h(e^{\eta(t)}))\, |d\eta^2(t)| &\asymp \int_0 \psi(\eta(t))\, k(\eta(t))\, |d\eta^2(t)| \\ &\asymp \int^{+\infty} \psi(x)\, k(x)\, dx^2 = \infty.\end{aligned}$$

Since

$$\int_0 \operatorname{cap}(E_t)\, |d\eta^2(t)| < \infty,$$

by [4, Theorem 5], there exists a function $f \in \mathcal{D}$ such that

$$\operatorname{Re} f(\zeta) \geq \eta(d(\zeta, E)) \quad \text{and} \quad |\operatorname{Im} f(\zeta)| < \pi/4, \qquad \text{q.e. on } \mathbb{T}.$$

By harmonicity,

$$|\operatorname{Im} f(z)| < \pi/4\,, \qquad |z| < 1,$$

Now take

$$\varphi = \exp(-e^{-f}).$$

By a simple modification in the construction of $f$ as in [1], we can suppose that $\varphi \in A(\mathbb{D})$. Hence $E_\varphi(1) = E$ and

$$\begin{aligned}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} &\asymp \int_{\mathbb{D}} \frac{|f'(z)|^2\, e^{-2\,\mathrm{Re} f(z)}\, e^{-2\,e^{-\mathrm{Re} f(z)}\cos(\mathrm{Im} f(z))}}{e^{-2\,\mathrm{Re} f(z)}\cos^2(\mathrm{Im} f(z))}\, dA(z) \\ &\leq \int_{\mathbb{D}} |f'(z)|^2 \exp(-\sqrt{2}\, e^{-\mathrm{Re} f(z)})\, dA(z) \\ &\leq c\int_{\mathbb{D}} |f'(z)|^2\, dA(z) < \infty.\end{aligned}$$

Hence $C_\varphi$ is in the Hilbert–Schmidt class. Finally, since

$$E_\varphi(s) \supseteq \{\zeta \in \mathbb{T} : \eta(d(\zeta, E)) \geq \log(1/1-s)\},$$

we get

$$\int_0 \mathrm{cap}(E_\varphi(s)) h(1/1-s)\, d\big(\log(1/1-s)\big)^2 \geq \int_0 \mathrm{cap}(E_t) h(e^{\eta(t)})\, |d\eta^2(t)| = +\infty.$$

□

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