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Sergio Albeverio, Arnaud Debussche, Lihu Xu

# EXPONENTIAL MIXING OF THE 3D STOCHASTIC NAVIER-STOKES EQUATIONS DRIVEN BY MILDLY DEGENERATE NOISES

SERGIO ALBEVERIO, ARNAUD DEBUSSCHE, AND LIHU XU

Abstract. We prove the strong Feller property and exponential mixing for 3D stochastic Navier-Stokes equation driven by mildly degenerate noises (i.e. all but finitely many Fourier modes are forced) via Kolmogorov equation approach.

## 1. Introduction

The ergodicity of SPDEs driven by *degenerate* noises have been intensively studied in recent years (see for instance [1], [2], [7], [6], [8], [12], [13], [16], [15], [14], [17], [22]). For the 2D stochastic Navier-Stokes equations (SNS), there are several results on ergodicity, among which the most remarkable one is by Hairer and Mattingly ([12]). They proved that the 2D stochastic dynamics has a unique invariant measure as long as the noise forces at least two linearly independent Fourier modes. As for the 3D SNS, most of ergodicity results are about the dynamics driven by non-degenerate noises (see [4], [10], [20], [21], [22]). Partial results can be found for degenerate noises in [21] and [24]. In the case of an essentially elliptic degenerate noise, *i.e.* when *all but a finite number* of Fourier modes are driven, [23] obtained the ergodicity by combining Markov selection and Malliavin calculus. For the case of very degenerate noises (as in [12] in the 2D case), ergodicity is still open.

In this paper, we shall still study the 3D SNS driven by essentially elliptic noises as above, but our approach is essentially different from that in [23]. Rather than Markov selection and cutoff technique, we prove the strong Feller property by studying some Kolmogorov equations with a large negative potential, which was developed in [4]. Comparing with the method in [4] and [5], we cannot apply the Bismut-Elworthy-Li formula ([9]) due to the degeneracy of the noise. To fix this problem, we follow the ideas in [8] and split the dynamics into high and low frequency parts, applying the formula to the dynamics at high modes and Malliavin calculus to those at low ones. Due to the degeneracy of the noise again, when applying Duhamel formula as in [4] and [5], we shall encounter an obstruction of non integrability in time (see (5.1)). Two techniques are developed in Proposition 5.1

2000 *Mathematics Subject Classification.* Primary 76D05; Secondary 60H15, 35Q30, 60H30, 76M35.

*Key words and phrases.* stochastic Navier-Stokes equation (SNS), Kolmogorov equation, Galerkin approximation, strong Feller, exponential mixing, mildly degenerate noise, Malliavin calculus.

and 5.2 to conquer this problem, and the underlying idea is to trade off the spatial regularity for the time integrability. Thus, we are able to generalize the arguments of [4] and [5] and construct markov solutions and ergodicity. Then, using the coupling method of [19], in which the noise has to be non-degenerate, we prove the exponential mixing. Note that our construction of the coupling is simpler. Finally, we remark that the large coefficient $K$ in front of the potential (see (2.11)), besides suppressing the nonlinearity $B(u,u)$ as in [4] and [5], also conquers the crossing derivative flows (see (3.10) and (3.11)).

Let us discuss the further application of the Kolmogorov equation method in [4], [5] and this paper. For another essentially elliptic setting where sufficiently large (but still finite) modes are forced ([12], section 4.5), due to the large negative potential, it is easy to show the *asymptotic strong Feller* ([12]) for the semigroup $S_t^m$ (see (2.13)). There is a hope to transfer this asymptotic strong Feller to the semigroup $P_t^m$ (see (2.14)) using the technique in Proposition 5.2. If $P_t^m$ satisfies asymptotic strong Feller, then we can also prove the ergodicity. This is the further aim of our future research in 3D SNS.

The paper is organized as follows. Section 2 gives a detailed description of the problem, the assumptions on the noise and the main results (Theorems 2.4 and 2.5). Section 3 proves the crucial estimate in Theorem 3.1, while Section 4 applies Malliavin calculus to prove the important Lemma 3.6. Section 5 gives a sketch proof for the main theorems, and the last section contains the estimate of Malliavin matrices and the proof of some technical lemmas.

**Acknowledgements**: We would like to thank Prof. Martin Hairer for pointing out a serious error in Malliavin calculus of the original version. We also would like to thank Dr. Marco Romito for the stimulating discussions and some helpful suggestions on correcting several errors.

## 2. Preliminary and main results

2.1. **Notations and assumptions.** Let $\mathbb{T}^3=(\mathbb{R}/2\pi\mathbb{Z})^3$ be the three-dimensional torus, let

$$\mathcal{H}=\{x\in C^\infty(\mathbb{T}^3,\mathbb{R}^3):\int_{\mathbb{T}^3}x(\xi)d\xi=0,\ div\,x(\xi)=0, x \text{ is periodic}\},$$

and $H$ be the closure of $\mathcal{H}$ in $L^2(\mathbb{T}^3,\mathbb{R}^3)$. We define:

$$\mathcal{P}:L^2(\mathbb{T}^3,\mathbb{R}^3)\to H$$

be the orthogonal projection operator. We shall study the equation

$$\begin{cases} dX+[\nu AX+B(X,X)]dt=QdW_t, \\ X(0)=x, \end{cases} \tag{2.1}$$

where

- $A=-\mathcal{P}\Delta \quad D(A)$ is the closure of $\mathcal{H}$ in $H^2(\mathbb{T}^3,\mathbb{R}^3)$.
- The nonlinear term $B$ is defined by

$$B(u,v)=\mathcal{P}[(u\cdot\nabla)v],\ B(u)=B(u,u)\quad \forall\ u,v\in H^1(\mathbb{T}^3,\mathbb{R}^3)\cap H.$$

- $W_t$ is the cylindrical Brownian motion on $H$ and $Q$ is the covariance matrix to be defined later.
- We shall assume the value $\nu = 1$ later on, indeed its exact value plays no essential role.

Define $\mathbb{Z}^3_+ = \{k \in \mathbb{Z}^3; k_1 > 0\} \cup \{k \in \mathbb{Z}^3; k_1 = 0, k_2 > 0\} \cup \{k \in \mathbb{Z}^3; k_1 = 0, k_2 = 0, k_3 > 0\}$, $\mathbb{Z}^3_- = -\mathbb{Z}^3_+$ and $\mathbb{Z}^3_* = \mathbb{Z}^3_+ \cup \mathbb{Z}^3_-$, for any $n > 0$, denote

$$Z_\ell(n) = [-n, n]^3 \setminus (0,0,0), \quad Z_h(n) = \mathbb{Z}^3_* \setminus Z_\ell(n).$$

Let $k^\perp = \{\eta \in \mathbb{R}^3; k \cdot \eta = 0\}$, define the projection $\mathcal{P}_k : \mathbb{R}^3 \to k^\perp$ by

$$\mathcal{P}_k \eta = \eta - \frac{k \cdot \eta}{|k|^2} k \qquad \eta \in \mathbb{R}^3. \tag{2.2}$$

Let $e_k(\xi) = cos k\xi$ if $k \in \mathbb{Z}^3_+$, $e_k(\xi) = sin k\xi$ if $k \in \mathbb{Z}^3_-$ and let $\{e_{k,1}, e_{k,2}\}$ be an orthonormal basis of $k^\perp$, denote

$$e^1_k(\xi) = e_k(\xi) e_{k,1}, \quad e^2_k(\xi) = e_k(\xi) e_{k,2} \qquad \forall\, k \in \mathbb{Z}^3_*;$$

$\{e^i_k; k \in \mathbb{Z}^3_*, i = 1,2\}$ is a Fourier basis of $H$ (up to the constant $\sqrt{2}/(2\pi)^{3/2}$). With this Fourier basis, we can write the cylindrical Brownian motion $W$ on $H$ by

$$W_t = \sum_{k \in \mathbb{Z}^3_*} w_k(t) e_k = \sum_{k \in \mathbb{Z}^3_*} \sum_{i=1}^{2} w^i_k(t) e^i_k$$

where each $w_k(t) = (w^1_k(t), w^2_k(t))^T$ is a 2-d standard Brownian motion. Moreover,

$$B(u,v) = \sum_{k \in \mathbb{Z}^3_*} B_k(u,v) e_k$$

where $B_k(u,v)$ is the Fourier coefficient of $B(u,v)$ at the mode $k$. Define

$$\tilde{B}(u,v) = B(u,v) + B(v,u), \qquad \tilde{B}_k(u,v) = B_k(u,v) + B_k(v,u).$$

We shall calculate $\tilde{B}_k(a_j e_j, a_l e_l)$ with $a_j \in j^\perp, a_l \in l^\perp$ in Appendix 6.1.

Furthermore, given any $n > 0$, let $\pi_n : H \longrightarrow H$ be the projection from $H$ to the subspace $\pi_n H := \{x \in H : x = \sum_{k \in Z_\ell(n)} x_k e_k\}$.

**Assumption 2.1** (Assumptions for $Q$)**.** *We assume that $Q : H \longrightarrow H$ is a linear bounded operator such that*

(A1) *(Diagonality) There are a sequence of linear maps $\{q_k\}_{k \in \mathbb{Z}^3_*}$ with $q_k : k^\perp \longrightarrow k^\perp$ such that*

$$Q(y e_k) = (q_k y) e_k \qquad y \in k^\perp.$$

(A2) *(Finitely Degeneracy) There exists some nonempty sublattice $Z_\ell(n_0)$ of $\mathbb{Z}^3_*$ such that*

$$q_k = 0 \qquad k \in Z_\ell(n_0).$$

(A3) $(Id - \pi_{n_0}) A^r Q$ *is* bounded invertible *on $(Id - \pi_{n_0})H$ with $1 < r < 3/2$ and moreover $Tr[A^{1+\sigma} QQ^*] < \infty$ for some $\sigma > 0$.*

*Remark* 2.2. Under the Fourier basis of $H$, $Q$ has the following representation

$$Q = \sum_{k \in Z_{\mathfrak{h}}(n_0)} \sum_{i,j=1}^{2} q_k^{ij} e_k^i \otimes e_k^j \tag{2.3}$$

where $x \otimes y : H \longrightarrow H$ is defined by $(x \otimes y)z = \langle y, z \rangle x$ and $(q_k^{ij})$ is a matrix representation of $q_k$ under some orthonormal basis $(e_{k,1}, e_{k,2})$ of $k^{\perp}$. By (A3), $rank(q_k) = 2$ for all $k \in Z_{\mathfrak{h}}(n_0)$. Take $Q = (Id - \pi_{n_0})A^{-r}$ with some $5/4 < r < 3/2$, it clearly satisfies (A1)-(A3).

With the above notations and assumptions, equation (2.1) can be represented under the Fourier basis by

$$\begin{cases} dX_k + [|k|^2 X_k + B_k(X)]dt = q_k dw_k(t), & k \in Z_{\mathfrak{h}}(n_0) \\ dX_k + [|k|^2 X_k + B_k(X)]dt = 0, & k \in Z_{\ell}(n_0) \\ X_k(0) = x_k, & k \in \mathbb{Z}_*^3 \end{cases} \tag{2.4}$$

where $x_k, X_k, B_k(X) \in k^{\perp}$.

We further need the following notations:

- $\mathcal{B}_b(B)$ denotes the Borel measurable bounded function space on the given Banach space $B$. $|\cdot|_B$ denotes the norm of a given Banach space $B$
- $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the norm and the inner product of $H$ respectively.
- Given any $\phi \in C(D(A), \mathbb{R})$, we denote

$$D_h \phi(x) := \lim_{\epsilon \to 0} \frac{\phi(x + \varepsilon h) - \phi(x)}{\varepsilon}, \tag{2.5}$$

  provided the above limit exists, it is natural to define $D\phi(x) : D(A) \to \mathbb{R}$ by $D\phi(x)h = D_h\phi(x)$ for all $h \in D(A)$. Clearly, $D\phi(x) \in D(A^{-1})$. We call $D\phi$ the first order derivative of $\phi$, similarly, one can define the second order derivative $D^2\phi$ and so on. Denote $C_b^k(D(A), \mathbb{R})$ the set of functions from $D(A)$ to $\mathbb{R}$ with bounded 0-th, ..., $k$-th order derivatives.
- Let $B$ be some Banach space and $k \in \mathbb{Z}_+$, define $C_k(D(A), B)$ as the function space from $D(A)$ to $B$ with the norm

$$||\phi||_k := \sup_{x \in D(A)} \frac{|\phi(x)|_B}{1 + |Ax|^k} \quad \phi \in C_k(D(A), B).$$

- For any $\gamma > 0$ and $0 \le \beta \le 1$, define the Hölder's norm $||\cdot||_{2,\beta}$ by

$$||\phi||_{2,\beta} = \sup_{x,y \in D(A)} \frac{|\phi(x) - \phi(y)|}{|A^{\gamma}(x-y)|^{\beta}(1 + |Ax|^2 + |Ay|^2)},$$

  and the function space $C_{2,\gamma}^{\beta}(D(A), \mathbb{R})$ by

$$C_{2,\gamma}^{\beta}(D(A), \mathbb{R}) = \{\phi \in C_2(D(A), \mathbb{R}); ||\phi||_{C_{2,\gamma}^{\beta}} = ||\phi||_2 + ||\phi||_{2,\beta} < \infty\}. \tag{2.6}$$

- For any signed measure $\mu$ on a measurable space $(E, \mathcal{F})$, its total variation is defined by

$$||\mu||_{var} = \sup_{A \in \mathcal{F}} |\mu(A)|.$$

2.2. **Main results.** The following definition of Markov family follows that in [5].

**Definition 2.3.** Let $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)_{x\in D(A)}$ be a family of probability spaces and let $(X(\cdot,x))_{x\in D(A)}$ be a family of stochastic processes on $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)_{x\in D(A)}$. Let $(\mathcal{F}_x^t)_{t\geq 0}$ be the filtration generated by $X(\cdot,x)$ and let $\mathcal{P}_x$ be the law of $X(\cdot,x)$ under $\mathbb{P}_x$. The family of $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x, X(\cdot,x))_{x\in D(A)}$ is a Markov family if the following condition hold:

(1) For any $x \in D(A)$, $t \geq 0$, we have

$$\mathbb{P}_x(X(t,x) \in D(A)) = 1.$$

(2) The map $x \to \mathcal{P}_x$ is measurable. For any $x \in D(A)$, $t_0, \cdots, t_n \geq 0$ , $A_0, \cdots, A_n \subset D(A)$ Borel measurable, we have

$$\mathbb{P}_x(X(t+\cdot) \in \mathcal{A}|\mathcal{F}_x^t) = \mathcal{P}_{X(t,x)}(\mathcal{A})$$

where $\mathcal{A} = \{(y(t_0), \cdots, y(t_n)); y(t_0) \in A_0, \cdots, y(t_n) \in A_n\}$.

The Markov transition semigroup $(P_t)_{t\geq 0}$ associated to the family is then defined by

$$P_t\phi(x) = \mathbb{E}_x[\phi(X(t,x))], \quad x \in D(A) \quad t \geq 0.$$

for all $\phi \in \mathcal{B}_b(D(A), \mathbb{R})$.

The main theorems of this paper are as the following. The proofs are given in Section 5.

**Theorem 2.4.** *There exists a Markov family of martingale solution* $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x, X(\cdot,x))_{x\in D(A)}$ *of the equation* (2.1)*. Furthermore, the transition semigroup* $(P_t)_{t\geq 0}$ *is stochastically continuous.*

**Theorem 2.5.** *The transition semigroup* $(P_t)_{t\geq 0}$ *in the previous theorem is strong Feller and irreducible. Moreover, it admits a unique invariant measure* $\nu$ *supported on* $D(A)$ *such that, for any probability measure* $\mu$ *supported on* $H$*, we have*

$$||P_t^*\mu - \nu||_{var} \leq Ce^{-ct}\left(1 + \int_H |x|^2\mu(dx)\right) \tag{2.7}$$

*where* $C, c > 0$ *are the constants depending on* $Q$*.*

2.3. **Kolmogorov equations for Galerkin approximation.** Let us consider the Galerkin approximations of equation (2.4):

$$\begin{cases} dX_m = -[AX_m + B_m(X_m)]dt + Q_m dW_t \\ X_m(0) = x_m \end{cases} \tag{2.8}$$

where $x_m \in \pi_m D(A)$, $B_m(x) = \pi_m B(\pi_m x)$ and $Q_m = \pi_m Q$. The Kolmogorov equation for (2.8) is

$$\begin{cases} \partial_t u_m = \frac{1}{2}Tr[Q_m Q_m^* D^2 u_m] - \langle Ax + B_m(x), Du_m\rangle \\ u_m(0) = \phi \end{cases} \tag{2.9}$$

where $\phi$ is some suitable test function and

$$\mathcal{L}_m := \frac{1}{2}Tr[Q_m Q_m^* D^2] - \langle Ax + B_m(x), D\rangle$$

is the Kolmogorov operator associated to (2.8). It is well known that (2.9) is uniquely solved by

$$u_m(t,x) = \mathbb{E}[\phi(X_m(t,x))], \qquad x \in \pi_m D(A). \tag{2.10}$$

Now we introduce an auxiliary Kolmogorov equation with a negative potential $-K|Ax|^2$ as

$$\begin{cases}\partial_t v_m = \frac{1}{2}Tr[Q_m Q_m^* D^2 v_m] - \langle Ax + B_m(x), Dv_m\rangle - K|Ax|^2 v_m, \\ v_m(0) = \phi,\end{cases} \tag{2.11}$$

which is solved by the following Feynman-Kac formula

$$v_m(t,x) = \mathbb{E}\left[\phi(X_m(t,x))\exp\{-K\int_0^t |AX_m(s,x)|^2 ds\}\right]. \tag{2.12}$$

Denote

$$S_t^m\phi(x) = v_m(t,x), \tag{2.13}$$

$$P_t^m\phi(x) = u_m(t,x), \tag{2.14}$$

for any $\phi \in \mathcal{B}(\pi_m D(A))$, it is clear that $S_t^m$ and $P_t^m$ are both contraction semigroups on $\mathcal{B}(\pi_m D(A))$. By Duhamel's formula, we have

$$u_m(t) = S_t^m\phi + K\int_0^t S_{t-s}^m[|Ax|^2 u_m(s)]ds. \tag{2.15}$$

For further use, we set

$$\mathcal{E}_{m,K}(t) = \exp\{-K\int_0^t |AX_m(s)|^2 ds\}, \tag{2.16}$$

which plays a very important role in section 3. The constant $K > 1$ in (2.16) is a large but *fixed* number. Note that in the following, we work on a fixed time interval $[0,T]$, the constant $K$ may depend on $T$. We often use the trivial fact $\mathcal{E}_{m,K_1+K_2}(t) = \mathcal{E}_{m,K_1}(t)\mathcal{E}_{m,K_2}(t)$ and

$$\int_0^t |AX_m(s)|^2 \mathcal{E}_{m,K}(s)ds = \frac{1}{K}(1-\mathcal{E}_{m,K}(t)) \le \frac{1}{K}. \tag{2.17}$$

## 3. Gradient estimate for the semigroups $S_t^m$

In this section, the main result is as follows, and it is similar to Lemma 3.4 in [5] (or Lemma 4.8 in [4]).

**Theorem 3.1.** *Given any $T > 0$ and $k \in \mathbb{Z}_+$, there exists some $p > 1$ such that for any $\max\{\frac{1}{2}, r - \frac{1}{2}\} < \gamma \le 1$ with $\gamma \ne 3/4$ and $r$ defined in Assumption 2.1, we have*

$$||A^{-\gamma}DS_t^m\phi||_k \le Ct^{-\alpha}||\phi||_k \qquad 0 < t \le T \tag{3.1}$$

*for all $\phi \in C_b^1(D(A),\mathbb{R})$, where $C = C(k,\gamma,r,T,K) > 0$ and $\alpha = p + \frac{1}{2} + r - \gamma$.*

*Remark* 3.2. The condition '$\gamma \ne 3/4$' is due to the estimate (6.11) about the nonlinearity $B(u,v)$. Note that $p$ is larger than 1 so that estimate (3.1) does not imply time integrability of the gradient of $S_t^m$.

In [4] and [5], estimate (3.1) is proved by applying the identity

(3.2)

$$D_h S_t^m\phi(x) = \frac{1}{t}\mathbb{E}[\mathcal{E}_{m,K}(t)\phi(X^m(t,x))\int_0^t \langle Q^{-1}D_h X^m(s,x), dW_s\rangle]$$
$$+ 2K\mathbb{E}\left[\mathcal{E}_{m,K}(t)\phi(X^m(t,x))\int_0^t (1-\frac{s}{t})\langle AX^m(s,x), AD_h X^m(s,x)\rangle ds\right],$$

and bounding the two terms on the r.h.s. of (3.2). Since $Q$ in Assumption 2.1 is *degenerate*, formula (3.2) is not available in our case. Alternatively, we apply the idea in [8] to fix this problem, i.e. we split $X_m(t)$ into the *low* and *high* frequency parts, and apply Malliavin calculus and Bismut-Elworthy-Li formula to treat these terms.

Let $n \in \mathbb{N}$ be a *fixed* number throughout this paper which satisfies $n > n_0$ and is determined below, see the proof of Proposition 4.6. Recal that $n_0$ is the constant given in Assumption 2.1. We split the Hilbert space $H$ into the low and high frequency parts by

$$\pi^{\ell} H = \pi_n H, \qquad \pi^{h} H = (Id - \pi_n)H. \tag{3.3}$$

(We remark that the technique of splitting frequency space into two pieces is similar to the well known Littlewood-Paley projection in Fourier analysis.) Then, the Galerkin approximation (2.8) with $m > n$ can be divided into two parts as follows:

$$\begin{aligned} dX_m^{\ell} + [AX_m^{\ell} + B_m^{\ell}(X_m)]dt &= Q_m^{\ell} dW_t^{\ell}, \\ dX_m^{h} + [AX_m^{h} + B_m^{h}(X_m)]dt &= Q_m^{h} dW_t^{h}, \end{aligned} \tag{3.4}$$

where $X_m^{\ell} = \pi^{\ell} X_m$, $X_m^{h} = \pi^{h} X_m$ and the other terms are defined in the same way. In particular,

$$Q_m^{\ell} = \sum_{k \in Z_{\ell}(n) \setminus Z_{\ell}(n_0)} \sum_{i,j=1}^{2} q_k^{ij} e_k^i \otimes e_k^j \,, \quad Q_m^{h} = \sum_{k \in Z_{\ell}(m) \setminus Z_{\ell}(n)} \sum_{i,j=1}^{2} q_k^{ij} e_k^i \otimes e_k^j, \tag{3.5}$$

with $x \otimes y : H \longrightarrow H$ defined by $(x \otimes y)z = \langle y, z \rangle x$

With such separation for the dynamics, it is natural to split the Frechet derivatives on $H$ into the low and high frequency parts. More precisely, for any stochastic process $\Phi(t,x)$ on $H$ with $\Phi(0,x) = x$, the Frechet derivative $D_h \Phi(t,x)$ is defined by

$$D_h \Phi(t,x) := \lim_{\epsilon \to 0} \frac{\Phi(t, x + \epsilon h) - \Phi(t,x)}{\epsilon} \qquad h \in H,$$

provided the limit exists. The map $D\Phi(t,x) : H \longrightarrow H$ is naturally defined by $D\Phi(t,x)h = D_h \Phi(t,x)$ for all $h \in H$. Similarly, one can easily define $D^{\ell} \Phi(t,x)$, $D^{h} \Phi(t,x)$, $D^{\ell} \Phi^{h}(t,x)$, $D^{h} \Phi^{\ell}(t,x)$ and so on, for instance, $D^{h} \Phi^{\ell}(t,x) : \pi^{h} H \to \pi^{\ell} H$ is defined by

$$D^{h} \Phi^{\ell}(t,x)h = D_h \Phi^{\ell}(t,x) \quad \forall \ h \in \pi^{h} H$$

with $D_h \Phi^{\ell}(t,x) = \lim_{\epsilon \to 0} [\Phi^{\ell}(t, x + \epsilon h) - \Phi^{\ell}(t,x)]/\epsilon$.

Recall that for any $\phi \in C_b^1(D(A), \mathbb{R})$ one can define $D\phi$ by (2.5), in a similar way as above, $D^{\ell} \phi(x)$ and $D^{h} \phi(x))$ can be defined (e.g. $D^{\ell} \phi(x)h = \lim_{\varepsilon \to 0} [\phi(x + \varepsilon h) - \phi(x)]/\varepsilon \quad h \in D(A)^{\ell}$).

**Lemma 3.3.** *Denote $Z(t) = \int_0^t e^{-A(t-s)} Q dW_s$, for any $T > 0$ and $\varepsilon < \sigma/2$ with the $\sigma$ as in Assumption 2.1, one has*

$$\mathbb{E}\left[\sup_{0 \le t \le T} |A^{1+\varepsilon} Z(t)|^{2k}\right] \le C(\alpha) T^{2k(\sigma - 2\varepsilon - 2\alpha)} \tag{3.6}$$

*where $0 < \alpha < \sigma/2 - \varepsilon$ and $k \in \mathbb{Z}_+$. Moreover, as $K > 0$ is sufficiently large, for any $T > 0$ and any $k \geq 2$, we have*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \mathcal{E}_{m,K}(t)|AX_m(t)|^k\right] \leq C(k,T)(1+|Ax|^k). \tag{3.7}$$

*Proof.* The proof of (3.6) is standard (see Proposition 3.1 of [5]). Writing $X_m(t) = Y_m(t) + Z_m(t)$, and differentiating $|AY_m(t)|^2$ (or seeing (3.1) in Lemma 3.1 of [4]), we have

$$\mathcal{E}_{m,K}(t)|AY_m(t)|^2 \leq |Ax|^2 + \sup_{0 \leq t \leq T} |AZ_m(t)|^2.$$

as $K > 0$ is sufficiently large. Hence,

$$\mathcal{E}_{m,K}(t)|AX_m(t)|^2 \leq \mathcal{E}_{m,K}(t)|AY_m(t)|^2 + \mathcal{E}_{m,K}(t)|AZ_m(t)|^2 \leq |Ax|^2 + 2\sup_{0 \leq t \leq T} |AZ(t)|^2.$$

Hence, by (3.6) and the above inequality, we immediately have (3.7). □

**Lemma 3.4.** *$S_t^m : C_k(D(A), \mathbb{R}) \to C_k(D(A), \mathbb{R})$ is bounded with the estimate*

$$||S_t^m \phi||_k \leq C||\phi||_k. \tag{3.8}$$

*where $C = C(k) > 0$.*

*Proof.* For any $x \in D(A)$, by (2.11), (2.13) and (3.7), one clearly has

$$|S_t^m \phi(x)| \leq ||\phi||_k \mathbb{E}[(1 + |AX^m(t)|^k)\mathcal{E}_{m,K}(t)] \leq C(1+|Ax|^k)||\phi||_k.$$

□

The main ingredients of the proof of Theorem 3.1 are the following two lemmas, and they will be proven in Appendix 6.3 and Section 4.2 respectively.

**Lemma 3.5.** *Let $x \in D(A)$ and let $X_m(t)$ be the solution to (2.8). Then, for any $\max\{\frac{1}{2}, r - \frac{1}{2}\} < \gamma \leq 1$ with $\gamma \neq 3/4$, $h \in \pi_m H$ and $v \in L^2_{loc}(0,\infty;H)$, as $K$ is sufficiently large, we have almost surely*

$$|A^\gamma D_h X_m(t)|^2 \mathcal{E}_{m,K}(t) + \int_0^t |A^{1/2+\gamma} D_h X_m(s)|^2 \mathcal{E}_{m,K}(s) ds \leq |A^\gamma h|^2 \tag{3.9}$$

$$|A^\gamma D_{h^h} X_m^\ell(t)|^2 \mathcal{E}_{m,K}(t) \leq \frac{C}{K}|A^\gamma h|^2 \tag{3.10}$$

$$|A^\gamma D_{h^\ell} X_m^h(t)|^2 \mathcal{E}_{m,K}(t) \leq \frac{C}{K}|A^\gamma h|^2 \tag{3.11}$$

$$\int_0^t |A^r D_h X_m(s)|^2 \mathcal{E}_{m,K}(s) ds \leq C t^{1-2(r-\gamma)}|A^\gamma h|^2 \tag{3.12}$$

$$\mathbb{E}[\mathcal{E}_{m,K}(t)\int_0^t \langle v(s), dW(s)\rangle] \leq \mathbb{E}[\int_0^t \mathcal{E}^2_{m,K}(s)|v(s)|^2 ds] \tag{3.13}$$

*where all the $C = C(\gamma) > 0$ above are independent of $m$ and $K$.*

**Lemma 3.6.** *Given any $\phi \in C_b^1(D(A))$ and $h \in \pi^\ell H$, there exists some $p > 1$ (possibly very large) such that for any $k \in \mathbb{Z}_+$, we have some constant $C = C(p,k) > 0$ such that*

$$|\mathbb{E}[D^\ell \phi(X_m(t)) D_h X_m^\ell(t,x)\mathcal{E}_{m,K}(t)]| \leq C t^{-p} e^{Ct}||\phi||_k(1+|Ax|^k)|h| \tag{3.14}$$

*Proof of Theorem 3.1.* For the notational simplicity, we shall drop the index in the quantities if no confusion arises. For $S_{t-s}\phi(X(s))$, applying Itô formula to $X(s)$ and the equation (2.11) to $S_{t-s}$, (differentiating on $s$), we have

$$\begin{aligned} d\left[S_{t-s}\phi(X(s))\mathcal{E}_K(s)\right] = &\ \mathcal{L}_m S_{t-s}\phi(X(s))\mathcal{E}_K(s)ds + DS_{t-s}\phi(X(s))\mathcal{E}_K(s)QdW_s \\ & - \mathcal{L}_m S_{t-s}\phi(X(s))\mathcal{E}_K(s)ds + K|AX(s)|^2 S_{t-s}\phi(X(s))\mathcal{E}_K(s)ds \\ & - S_{t-s}\phi(X(s))K|AX(s)|^2\mathcal{E}_K(s)ds \\ = &\ DS_{t-s}\phi(X(s))\mathcal{E}_K(s)QdW_s \end{aligned}$$

where $\mathcal{L}_m$ is the Kolmogorov operator defined in (2.9), thus

$$\phi(X(t))\mathcal{E}_K(t) = S_t\phi(x) + \int_0^t DS_{t-s}\phi(X(s))\mathcal{E}_K(s)QdW_s \tag{3.15}$$

Given any $h \in \pi_m H$, by (A3) of Assumption 2.1 and (3.15), we have $y_t^{\mathfrak{h}} := (Q^{\mathfrak{h}})^{-1}D_{h^{\mathfrak{h}}}X^{\mathfrak{h}}(t)$ so that

$$\begin{aligned} &\mathbb{E}[\phi(X(t))\mathcal{E}_K(t)\int_0^{t/2}\langle y_s^{\mathfrak{h}}, dW_s^{\mathfrak{h}}\rangle] \\ &= \mathbb{E}[\int_0^t DS_{t-s}\phi(X(s))\mathcal{E}_K(s)Q^{\mathfrak{h}}dW_s^{\mathfrak{h}}\int_0^{t/2}\langle (Q^{\mathfrak{h}})^{-1}D_{h^{\mathfrak{h}}}X^{\mathfrak{h}}(s), dW_s^{\mathfrak{h}}\rangle] \\ &= \int_0^{t/2}\mathbb{E}[D^{\mathfrak{h}}S_{t-s}\phi(X(s))D_{h^{\mathfrak{h}}}X^{\mathfrak{h}}(s)\mathcal{E}_K(s)]ds, \end{aligned} \tag{3.16}$$

hence,

$$\begin{aligned} \int_0^{t/2}\mathbb{E}[D_{h^{\mathfrak{h}}}S_{t-s}\phi(X(s))\mathcal{E}_K(s)]ds = &\ \mathbb{E}[\phi(X(t))\mathcal{E}_K(t)\int_0^{t/2}\langle (Q^{\mathfrak{h}})^{-1}D_{h^{\mathfrak{h}}}X^{\mathfrak{h}}(s), dW_s^{\mathfrak{h}}\rangle] \\ & + \int_0^{t/2}\mathbb{E}[D^{\mathfrak{l}}S_{t-s}\phi(X(s))D_{h^{\mathfrak{h}}}X^{\mathfrak{l}}(s)\mathcal{E}_K(s)]ds. \end{aligned} \tag{3.17}$$

By the fact $S_t\phi(x) = \mathbb{E}[S_{t-s}\phi(X(s))\mathcal{E}_K(s)]$, (3.16) and (3.17), we have

$$\begin{aligned} D_{h^{\mathfrak{h}}}S_t\phi(x) &= \frac{2}{t}\int_0^{t/2}D_{h^{\mathfrak{h}}}\mathbb{E}[S_{t-s}\phi(X(s))\mathcal{E}_K(s)]ds \\ &= \frac{2}{t}\mathbb{E}[\phi(X(t))\mathcal{E}_K(t)\int_0^{t/2}\langle (Q^{\mathfrak{h}})^{-1}D_{h^{\mathfrak{h}}}X^{\mathfrak{h}}(s), dW_s^{\mathfrak{h}}\rangle] \\ &\quad + \frac{2}{t}\int_0^{t/2}\mathbb{E}[D^{\mathfrak{l}}S_{t-s}\phi(X(s))D_{h^{\mathfrak{h}}}X^{\mathfrak{l}}(s)\mathcal{E}_K(s)]ds \\ &\quad - \frac{4K}{t}\int_0^{t/2}\mathbb{E}[S_{t-s}\phi(X(s))\mathcal{E}_K(s)\int_0^s\langle AX(r), AD_{h^{\mathfrak{h}}}X(r)\rangle dr]ds \\ &= \frac{2}{t}I_1 + \frac{2}{t}I_2 - \frac{4K}{t}I_3. \end{aligned}$$

We now fix $T, \gamma, k, r$ and let $C$ be constants depending on $T, \gamma, k$ and $r$ (whose values can vary from line to line), then $I_1, I_2$ and $I_3$ above can be estimated as follows:

$$\begin{aligned}|I_1| &\le ||\phi||_k \mathbb{E}\left[\mathcal{E}_{K/2}(t)(1+|AX(t)|^k)\mathcal{E}_{K/2}(t)\int_0^{t/2}\langle (Q^{\mathfrak{h}})^{-1}D_{h^{\mathfrak{h}}}X^{\mathfrak{h}}(s), dW_s^{\mathfrak{h}}\rangle\right] \\ &\le ||\phi||_k \mathbb{E}\left(\sup_{0\le s\le T}(1+|AX(s)|^k)^2\mathcal{E}_K(s)\right)^{\frac{1}{2}}\mathbb{E}\left(|\mathcal{E}_{\frac{K}{2}}(t)\int_0^{t/2}\langle (Q^{\mathfrak{h}})^{-1}D_{h^{\mathfrak{h}}}X^{\mathfrak{h}}(s), dW_s^{\mathfrak{h}}\rangle|^2\right)^{\frac{1}{2}} \\ &\le C||\phi||_k(1+|Ax|^k)\mathbb{E}\left(\int_0^{t/2}|A^r D_{h^{\mathfrak{h}}}X^{\mathfrak{h}}(s)|^2\mathcal{E}_K(s)ds\right)^{1/2} \\ &\le Ct^{1/2-(r-\gamma)}||\phi||_k(1+|Ax|^k)|A^\gamma h|\end{aligned}$$

where the last two inequalities are by (3.7), (3.13) and (3.12) in order. By (3.10) and (3.7),

$$\begin{aligned}|I_2| &\le \frac{C}{K}\int_0^{t/2}||A^{-\gamma}D^{\ell}S_{t-s}\phi||_k\mathbb{E}\left[(1+|AX(s)|^k)\mathcal{E}_{K/2}(s)\right]ds|A^\gamma h| \\ &\le \frac{C}{K}\int_0^{t/2}||A^{-\gamma}DS_{t-s}\phi||_k ds(1+|Ax|^k)|A^\gamma h|.\end{aligned}$$

By Markov property of $X(t)$ and (3.7), we have

$$\begin{aligned}|I_3| &= \int_0^{t/2}\mathbb{E}\left\{\mathbb{E}[\phi(X(t))e^{-K\int_s^t|AX(r)|^2dr}|\mathcal{F}_s]\mathcal{E}_K(s)\int_0^s\langle AX(r), AD_{h^{\mathfrak{h}}}X(r)\rangle dr\right\}ds \\ &\le C||\phi||_k\int_0^t\mathbb{E}[(1+|AX(s)|^k)\mathcal{E}_{K/2}(s)\int_0^s\mathcal{E}_{K/2}(r)|AX(r)|\cdot|AD_{h^{\mathfrak{h}}}X(r)|dr]ds,\end{aligned}$$

moreover, and by Hölder inequality, Poincare inequality $|A^{\gamma+\frac{1}{2}}x|\ge|Ax|$, (2.17) and (3.9),

$$\begin{aligned}&\int_0^s\mathcal{E}_{K/2}(r)|AX(r)|\cdot|AD_{h^{\mathfrak{h}}}X(r)|dr \\ &\le(\int_0^s\mathcal{E}_{K/2}(r)|AX(r)|^2dr)^{\frac{1}{2}}(\int_0^s|AD_{h^{\mathfrak{h}}}X(r)|^2\mathcal{E}_{K/2}(r)dr)^{\frac{1}{2}} \\ &\le[\int_0^s\mathcal{E}_{K/2}(r)|A^{\frac{1}{2}+\gamma}D_{h^{\mathfrak{h}}}X(r)|^2dr]^{\frac{1}{2}}\le|A^\gamma h|;\end{aligned}$$

hence, by (3.7) and the above,

$$|I_3|\le Ct||\phi||_k(1+|Ax|^k)|A^\gamma h|.$$

Collecting the estimates for $I_1$, $I_2$ and $I_3$, we have

(3.18)
$$|D_{h^{\mathfrak{h}}}S_t\phi(x)|\le C\left\{(t^{-\frac{1}{2}-(r-\gamma)}+K)||\phi||_k+\frac{1}{Kt}\int_0^{t/2}||A^{-\gamma}DS_{t-s}\phi||_k ds\right\}(1+|Ax|^k)|A^\gamma h|$$

For the low frequency part, according to Lemma 3.6, we have

(3.19)
$$
\begin{aligned}
|D_{h^\ell} S_t\phi(x)| &= |D_{h^\ell} S_{t/2}(S_{t/2}\phi)(x)| \\
&\le |\mathbb{E}[D^h S_{t/2}\phi(X(t/2))D_{h^\ell} X^h(t/2)\mathcal{E}_K(t/2)]| \\
&\quad + |\mathbb{E}[D^\ell S_{t/2}\phi(X(t/2))D_{h^\ell} X^\ell(t/2)\mathcal{E}_K(t/2)]| \\
&\quad + \mathbb{E}[|S_{t/2}\phi(X(t/2))|\mathcal{E}_K(t/2)K\int_0^{t/2}|AX(s)||AD_{h^\ell}X(s)|ds] \\
&\le C\left\{\frac{1}{K}||A^{-\gamma}DS_{t/2}\phi||_k + t^{-p}e^{Ct}||\phi||_k + K||\phi||_k\right\}(1+|Ax|^k)|A^\gamma h|
\end{aligned}
$$

where the last inequality is due to (3.11), (3.7) and (3.14), and to the following estimate (which is obtained by (3.8) and the same argument as in estimating $I_3$):

$$
\mathbb{E}[S_{t/2}\phi(X(t/2))\mathcal{E}_K(t/2)\int_0^{t/2}|AX(s)||AD_{h^\ell}X(s)|ds] \le C||\phi||_k(1+|Ax|^k)|A^\gamma h|
$$

Denote $\alpha = p + \frac{1}{2} + r - \gamma$ and

$$
\phi_T = \sup_{0\le t\le T} t^\alpha ||A^{-\gamma}DS_t\phi||_k,
$$

by (3.7), (3.9) and the similar argument as estimating $I_3$, we have

$$
\begin{aligned}
|D_h S_t\phi(x)| &= |D_h\mathbb{E}[\phi(X(t))\mathcal{E}_K(t)]| \\
&\le \mathbb{E}\left[|A^{-\gamma}D\phi(X(t))|\mathcal{E}_{\frac{K}{2}}(t)|A^\gamma D_hX(t)|\mathcal{E}_{\frac{K}{2}}(t)\right] \\
&\quad + 2K\mathbb{E}\left[|\phi(X(t))|\mathcal{E}_{\frac{K}{2}}(t)\mathcal{E}_{\frac{K}{2}}(t)\int_0^t|AX(s)||AD_hX(s)|ds\right] \\
&\le C(T,K,\gamma,k)(||A^{-\gamma}D\phi||_k + ||\phi||_k)(1+|Ax|^k)|A^\gamma h|,
\end{aligned}
$$

which implies $||A^{-\gamma}DS_t\phi||_k \le C(T,K,\gamma,k)(||A^{-\gamma}D\phi||_k + ||\phi||_k)$, thus $\phi_T < \infty$.

Combine (3.18) and (3.19), we have for every $t \in [0,T]$

$$
\begin{aligned}
& t^\alpha||A^{-\gamma}DS_t\phi||_k \\
&\le Ct^p||\phi||_k + \frac{C}{K}t^{\alpha-1}\int_0^{t/2}(t-s)^{-\alpha}(t-s)^\alpha||A^{-\gamma}DS_{t-s}\phi||_k ds \\
&\quad + CKt^\alpha||\phi||_k + \frac{C}{K}t^\alpha||A^{-\gamma}DS_{t/2}\phi||_k + Ct^{\alpha-p}e^{Ct}||\phi||_k \\
&\le Ct^p||\phi||_k + \phi_T\frac{C}{K}t^{\alpha-1}\int_0^{t/2}(t-s)^{-\alpha}ds \\
&\quad + CKt^\alpha||\phi||_k + \phi_T\frac{C}{K} + Ct^{\alpha-p}e^{Ct}||\phi||_k \\
&\le \phi_T\frac{C}{K} + KCe^{CT}||\phi||_k,
\end{aligned}
$$

this easily implies

$$
\phi_T \le \phi_T\frac{C}{K} + KCe^{CT}||\phi||_k.
$$

As $K>0$ is sufficiently large, we have for all $t\in[0,T]$

$$t^{\alpha}||A^{-\gamma}DS_t\phi||_k \le \frac{K}{1-C/K}Ce^{CT}||\phi||_k,$$

from which we conclude the proof. □

## 4. Malliavin Calculus

4.1. **Some preliminary for Malliavin calculus.** Given $v\in L^2_{loc}(\mathbb{R}^+,\pi_m H)$, the Malliavin derivative of $X_m(t)$ in direction $v$, denoted as $\mathcal{D}_v X_m(t)$, is defined by

$$\mathcal{D}_v X_m(t)=\lim_{\epsilon\to 0}\frac{X_m(t,W+\epsilon V)-X_m(t,W)}{\epsilon}$$

where $V(t)=\int_0^t v(s)ds$, provided the above limit exists; $v$ can be random and is adapted with respect to the filtration generated by $W$.

Recall $\pi^{\ell} H=\pi_n H$ and $Z_{\ell}(n)=[-n,n]^3\setminus(0,0,0)$ with $n_0<n<m$ to be determined in Proposition 4.6. The Malliavin derivatives on the low and high frequency parts of $X_m(t)$, denoted by $\mathcal{D}_v X^{\ell}_m(t)$ and $\mathcal{D}_v X^{\hbar}_m(t)$, can be defined in a similar way as above. Moreover, $\mathcal{D}_v X^{\ell}_m(t)$ and $\mathcal{D}_v X^{\hbar}_m(t)$ satisfy the following two SPDEs respectively ([12]):

$$\partial_t \mathcal{D}_v X^{\ell}_m + A\mathcal{D}_v X^{\ell}_m + \tilde{B}^{\ell}_m(\mathcal{D}_v X^{\ell}_m, X_m) + \tilde{B}^{\ell}_m(\mathcal{D}_v X^{\hbar}_m, X_m) = Q^{\ell}_m v^{\ell} \tag{4.1}$$

with $\mathcal{D}_v X^{\ell}_m(0)=0$, and

$$\partial_t \mathcal{D}_v X^{\hbar}_m + A\mathcal{D}_v X^{\hbar}_m + \tilde{B}^{\hbar}_m(\mathcal{D}_v X^{\ell}_m, X_m) + \tilde{B}^{\hbar}_m(\mathcal{D}_v X^{\hbar}_m, X_m) = Q^{\hbar}_m v^{\hbar} \tag{4.2}$$

with $\mathcal{D}_v X^{\hbar}_m(0)=0$, where $\tilde{B}(u,v)=B(u,v)+B(v,u)$. Moreover, we define a flow between $s$ and $t$ by $J^m_{s,t}$ $(s\le t)$, where $J^m_{s,t}\in\mathcal{L}(\pi^{\ell}H,\pi^{\ell}H)$ satisfies the following equation: $\forall\ h\in\pi^{\ell}H$

$$\partial_t J^m_{s,t}h + AJ^m_{s,t}h + \tilde{B}^{\ell}_m(J^m_{s,t}h, X_m(t)) = 0 \tag{4.3}$$

with $J^m_{s,s}=Id\in\mathcal{L}(\pi^{\ell}H,\pi^{\ell}H)$. It is easy to see that the inverse $(J^m_{s,t})^{-1}$ exists and satisfies

$$\partial_t (J^m_{s,t})^{-1}h - (J^m_{s,t})^{-1}[Ah + \tilde{B}^{\ell}_m(h, X_m(t))] = 0. \tag{4.4}$$

Simply writing $J^m_t=J^m_{0,t}$, clearly, $J^m_{s,t}=J^m_t(J^m_s)^{-1}$.

We shall follow the ideas in section 6.1 of [8] to develop a Malliavin calculus for $X_m$, one of the key points for this approach is to find an adapted process $v\in L_{loc}(\mathbb{R}^+;\pi_m H)$ such that

$$Q^{\hbar}_m v^{\hbar}(t) = \tilde{B}^{\hbar}_m(\mathcal{D}_v X^{\ell}_m(t), X_m(t)), \tag{4.5}$$

which, combining with (4.2), implies $\mathcal{D}_v X^{\hbar}_m(t)=0$ for all $t>0$. More precisely,

**Proposition 4.1.** *There exists some $v\in L^2_{loc}(\mathbb{R}^+;\pi_m H)$ satisfying (4.5), and*

$$\mathcal{D}_v X^{\ell}_m(t) = J^m_t\int_0^t (J^m_s)^{-1}Q^{\ell}_m v^{\ell}(s)ds, \qquad \mathcal{D}_v X^{\hbar}_m(t)=0.$$

*Proof.* When $\mathcal{D}_v X_m^{\hbar}(t) = 0$ for all $t \geq 0$, equation (4.1) is simplified to

$$\partial_t \mathcal{D}_v X_m^{\ell} + [A\mathcal{D}_v X_m^{\ell} + \tilde{B}_m^{\ell}(\mathcal{D}_v X_m^{\ell}, X_m)] = Q_m^{\ell} v^{\ell}$$

with $\mathcal{D}_v X_m^{\ell}(0) = 0$, which is solved by

$$\mathcal{D}_v X_m^{\ell}(t) = \int_0^t J_{s,t}^m Q_m^{\ell} v^{\ell}(s) ds = J_t^m \int_0^t (J_s^m)^{-1} Q_m^{\ell} v^{\ell}(s) ds. \tag{4.6}$$

Due to (A3) of Assumption 2.1, there exists some $v \in L^2_{loc}(\mathbb{R}^+, \pi_m H)$ so that $v^{\hbar}$ satisfies (4.5), therefore, (4.2) is a homogeneous linear equation and has a unique solution $\mathcal{D}_v X_m^{\hbar}(t) = 0, \quad \forall\, t > 0$. □

With the previous lemma, we see that the Malliavin derivative is essentially restricted in *low* frequency part. Take

$$N := 2[(2n+1)^3 - 1]$$

vectors $v_1, \dots, v_N \in L^2_{loc}(\mathbb{R}^+; \pi_m H)$ with each satisfying Proposition 4.1 ($N$ is the dimension of $\pi^l H$). Denote by

$$v = [v_1, \dots, v_N], \tag{4.7}$$

we have

$$\mathcal{D}_v X_m^{\hbar} = 0, \quad \mathcal{D}_v X_m^{\ell}(t) = J_t^m \int_0^t (J_s^m)^{-1} Q_m^{\ell} v^{\ell}(s) ds, \tag{4.8}$$

where $Q_m^{\ell}$ is defined in (3.5). In particular, $\mathcal{D}_v X_m^{\ell}(t)$ is an $N \times N$ matrix. Choose

$$v^{\ell}(s) = [(J_s^m)^{-1} Q_m^{\ell}]^*$$

and set

$$\mathcal{M}_t^m = \int_0^t [(J_s^m)^{-1} Q_m^{\ell}][(J_s^m)^{-1} Q_m^{\ell}]^* ds, \tag{4.9}$$

$\mathcal{M}_t^m$ is called *Malliavin matrix*, and is clearly a symmetric operator in $\mathcal{L}(\pi^{\ell} H, \pi^{\ell} H)$. $\forall\, \eta \in \pi^{\ell} H$, we have by Parseval's identity

$$\begin{aligned}
\langle \mathcal{M}_t \eta, \eta \rangle &= \int_0^t \langle [(J_s^m)^{-1} Q_m^{\ell}]^* \eta, [(J_s^m)^{-1} Q_m^{\ell}]^* \eta \rangle ds \\
&= \sum_{k \in Z_{\ell}(n)} \sum_{i=1}^2 \int_0^t |\langle (J_s^m)^{-1} Q_m^{\ell} e_k^i, \eta \rangle|^2 ds \\
&= \sum_{k \in Z_{\ell}(n) \setminus Z_{\ell}(n_0)} \sum_{i=1}^2 \int_0^t |\langle (J_s^m)^{-1} q_k^i e_k, \eta \rangle|^2 ds
\end{aligned} \tag{4.10}$$

where $q_k^i$ is the i-th column vector of the $2 \times 2$ matrix $q_k$ (recall (2.3)).

The following lemma is crucial for proving Lemma 3.6 and is proven in Appendix 6.3.

**Lemma 4.2.** *1. For any $h \in \pi^{\ell} H$, we have*

$$|J_t^m h|^2 \mathcal{E}_{m,K}(t) \leq |h|^2, \tag{4.11}$$

$$|D_h X^{\ell}(t)|^2 \mathcal{E}_{m,K}(t) \leq |h|^2, \tag{4.12}$$

$$|(J_t^m)^{-1}h|^2\mathcal{E}_{m,K}(t) \le Ce^{Ct}|h|^2 \tag{4.13}$$

$$\left|\mathcal{E}_{m,K}(t)(J_t^m)^{-1} - Id\right|_{\mathcal{L}(H)} \le t^{1/2}Ce^{Ct} \tag{4.14}$$

$$\mathbb{E}\left(\int_0^t |[(J_s^m)^{-1}Q_m^\ell]^* h|^2\mathcal{E}_{m,K}(s)ds\right) \le te^{Ct}tr[Q_m^\ell(Q_m^\ell)^*]|h|. \tag{4.15}$$

*where the above $C = C(n) > 0$ can vary from line to line and the $n$ is the size of $\pi^\ell H$ defined in (3.3).*

*2. Suppose that $v_1, v_2$ satisfy Proposition 4.1 and $h \in \pi^\ell H$, we have*

$$|A\mathcal{D}_{v_1}X_m^\ell(t)|^2\mathcal{E}_{m,K}(t) \le C\int_0^t e^{1/2(t-s)}|v_1^\ell(s)|^2\mathcal{E}_{m,K}(s)ds \tag{4.16}$$

$$|\mathcal{D}_{v_1}D_hX_m^\ell(t)|^2\mathcal{E}_{m,K}(t) \le Ce^{Ct}|h|^2\left(\int_0^t |v_1^\ell(s)|^2\mathcal{E}_{m,\frac{K}{2}}(s)ds\right) \tag{4.17}$$

(4.18)
$$|\mathcal{D}^2_{v_1v_2}X_m^\ell(t)|^2\mathcal{E}_{m,K}(t) \le Ce^{Ct}\left(\int_0^t |v_1^\ell(s)|^2\mathcal{E}_{m,\frac{K}{2}}(s)ds\right)\left(\int_0^t |v_2^\ell(s)|^2\mathcal{E}_{m,\frac{K}{2}}(s)ds\right)$$

*where the above $C = C(n) > 0$ can vary from line to line and the $n$ is the size of $\pi^\ell H$ defined in (3.3).*

4.2. **Hörmander's systems and proof of Lemma 3.6.** Recall the Galerkin equation satisfied by $X_m^\ell$

$$dX_m^\ell + [A_m^\ell X_m^\ell + B_m^\ell(X)]dt = \sum_{k\in Z_\ell(n)\setminus Z_\ell(n_0)}\sum_{i=1}^2 q_k^i dw_k^i(t)e_k \tag{4.19}$$

where $A^\ell$ is the Stokes operator restricted on $\pi^\ell H$ and $q_k^i$ is the $i$-th column vector in the $2\times 2$ matrix $q_k$ (under the orthonormal basis $(e_{k,1}, e_{k,2})$ of $k^\perp$). Given any two Banach spaces $B_1$ and $B_2$, denote $P(B_1, B_2)$ the collections of functions from $B_1$ to $B_2$ with polynomial growth. We introduce the Lie bracket on $\pi^\ell H$ as follows: $\forall\ K_1 \in P(\pi_m H, \pi^\ell H),\ K_2 \in P(\pi_m H, \pi^\ell H)$, define $[K_1, K_2]$ by

$$[K_1, K_2](x) = DK_1(x)K_2(x) - DK_2(x)K_1(x) \quad \forall\ x \in \pi_m H.$$

The brackets $[K_1, K_2]$ will appear when *differentiating* $J_t^{-1}K_1(X(t))$ in the proof of Lemma 4.7.

**Definition 4.3.** The Hömander's system $\mathbf{K}$ for equation (4.19) is defined as follows: given any $y \in \pi_m H$, define

$$\mathbf{K_0}(y) = \{q_k^i e_k; k \in Z_\ell(n) \setminus Z_\ell(n_0), i = 1, 2\}$$
$$\mathbf{K_1}(y) = \{[A_m^\ell + B_m^\ell(\cdot,\cdot), q_k^i e_k](y); k \in Z_\ell(n) \setminus Z_\ell(n_0), i = 1, 2\}$$
$$\mathbf{K_2}(y) = \{[q_k^i e_k, K](y); K \in \mathbf{K_1}(y), k \in Z_\ell(n) \setminus Z_\ell(n_0), i = 1, 2\}$$

and $\mathbf{K}(y) = span\{\mathbf{K_0}(y) \cup \mathbf{K_1}(y) \cup \mathbf{K_2}(y)\}$, where each $q_k^i$ is the column vector defined in (2.3).

**Definition 4.4.** The system $\mathbf{K}$ satisfies the *restricted Hörmander condition* if there exist some $\delta > 0$ such that for all $y \in \pi_m H$

$$\sup_{K\in\mathbf{K}} |\langle K(y), \ell\rangle| \geq \delta|\ell|, \qquad \ell \in \pi^{\mathfrak{l}} H. \tag{4.20}$$

The following lemma gives some inscription for the elements in $\mathbf{K_2}$ (see (4.21)) and plays a key role for the proof of Proposition 4.6. It is proven in Section 6.1.

**Lemma 4.5.** *For each* $k \in Z_{\mathfrak{l}}(n_0)$*, define* mixing set $Y_k$ *by*

$$Y_k = \left\{ \tilde{B}_{m,k}\left(q_j\ell_j e_j, q_l\ell_l e_l\right) : j, l \in Z_{\mathfrak{h}}(n_0); \ell_j \in j^\perp, \ell_l \in l^\perp \right\},$$

*where* $\tilde{B}_{m,k}(x, y)$ *is the Fourier coefficient of* $\tilde{B}_m(x, y)$ *at the mode* $k$*. For all* $k \in Z_{\mathfrak{l}}(n_0)$*, span*$\{Y_k\} = k^\perp$*.*

**Proposition 4.6.** *It is possible to choose* $n$ *such that* $\mathbf{K}$ *in Definition 4.3 satisfies the restricted Hörmander condition.*

*Proof.* It suffices to show that for each $k \in Z_{\mathfrak{l}}(n_0)$, the Lie brackets in Definition 4.3 can produce at least two linearly independent vectors of $Y_k$ in Lemma 4.5. (We note that [21] proved a similar proposition).

As $k \in Z_{\mathfrak{l}}(n_0) \cap \mathbb{Z}^3_+$, by Lemma 4.5, $Y_k$ has at least two linearly independent vectors $h^1_k \nparallel h^2_k$. Without loss of generality, assume $h^1_k = \tilde{B}_k(q^1_{j_k}e_{j_k}, q^1_{l_k}e_{l_k})$ and $h^2_k = \tilde{B}_k(q^2_{j_k}e_{j_k}, q^2_{l_k}e_{l_k})$ with $j_k, -l_k \in Z_{\mathfrak{h}}(n_0)$ and $j_k + l_k = k$. We can easily have (*simply writing* $j = j_k, l = l_k$)

$$[q^i_j e_j, [A^{\mathfrak{l}} y + B^{\mathfrak{l}}(y, y), q^i_l e_l]] = -\tilde{B}^{\mathfrak{l}}(q^i_l e_l, q^i_j e_j), \tag{4.21}$$

and by (6.1)-(6.3),

$$[q^i_j e_j, [A^{\mathfrak{l}} y + B^{\mathfrak{l}}(y, y), q^i_k e_k]] = -\frac{1}{2}\tilde{B}_{j-l}(q^i_j e_j, q^i_l e_l) - \frac{1}{2}\tilde{B}_k(q^i_j e_j, q^i_l e_l).$$

Clearly, $j - l \in Z_{\mathfrak{h}}(n_0)$, by (A2) of Assumption 2.1, $\tilde{B}_{j-l}(q^1_j e_j, q^1_l e_l)$ and $\tilde{B}_{j-l}(q^2_j e_j, q^2_l e_l)$ must both be equal to a linear combination of $q^i_{j-l}e_{j-l}$ $(i = 1, 2)$. Combining this observation with the previous assumption $\tilde{B}_k(q^1_j, q^1_l) \nparallel \tilde{B}_k(q^2_j, q^2_l)$, one immediately has that $[q^i_j e_j, [A^{\mathfrak{l}} y + B^{\mathfrak{l}}(y, y), q^i_l e_l]]$ $(i = 1, 2)$ and $q^i_{j-l}e_{j-l}$ $(i = 1, 2)$ span $k^\perp$.

Similarly, we have the same conclusion for $k \in Z_{\mathfrak{l}}(n_0) \cap \mathbb{Z}^3_-$. Choose the $n$ in (3.3) sufficiently large so that $j_k, l_k, j_k + l_k, j_k - l_k \in Z_{\mathfrak{l}}(n)$ for all $k \in Z_{\mathfrak{l}}(n_0)$. $\square$

With Proposition 4.6, we can show the following key lemma (see the proof in Section 6.2).

**Lemma 4.7.** *Suppose that* $X_m(t, x)$ *is the solution to equation* (2.8) *with initial data* $x \in \pi_m H$*. Then* $\mathcal{M}^m_t$ *is invertible almost surely. Denote* $\lambda_{min}(t)$ *the minimal eigenvalue of* $\mathcal{M}^m_t$*, then there exists some constant* $q > 0$ *such that for all* $p > 0$ *we have a constant* $C = C(p) > 0$ *satisfying*

$$\mathbb{P}\left\{\frac{1}{\lambda_{min}(t)} \geq \frac{1}{\varepsilon^q}\right\} \leq \frac{C\varepsilon^p}{t^p}. \tag{4.22}$$

*Proof of Lemma 3.6.* We shall simply write $X(t) = X_m(t), J_t = J_t^m$, $\mathcal{M}_t = \mathcal{M}_t^m$, $Q^\ell = Q_m^\ell$ and $\mathcal{E}_K(t) = \mathcal{E}_{m,K}(t)$ for notational simplicity. Under an orthonormal basis of $\pi^\ell H$, the operators $J_t$, $\mathcal{M}_t$, $\mathcal{D}_v X^\ell(t)$ with $v$ defined in (4.7), and $D^\ell X^\ell(t)$ can all be represented by $N \times N$ matrices, where $N$ is the dimension of $\pi^\ell H$. Noticing $\mathcal{D}_v X^\ell(t) = J_t \mathcal{M}_t$ (see (4.8)), the following $\phi_{il}$ is well defined:

$$\phi_{il}(X(t)) = \phi(X(t)) \sum_{j=1}^{N} [(\mathcal{D}_v X^\ell(t))^{-1}]_{ij} [D^\ell X^\ell(t)]_{jl} \mathcal{E}_K(t) \quad i, l = 1, \dots, N,$$

where $v$ is defined in (4.7) with $v^\ell(t) = (J_t^{-1} Q^\ell)^*$. For any $h \in \pi^\ell H$, by our special choice of $v$, we have

$$\begin{aligned} \mathcal{D}_{vh}\phi_{il}(X(t)) = D^\ell \phi(X(t))[\mathcal{D}_v X^\ell(t) h] \sum_{j=1}^{N} [(\mathcal{D}_v X^\ell(t))^{-1}]_{ij} [D^\ell X^\ell(t)]_{jl} \mathcal{E}_K(t) \\ + \phi(X(t)) \sum_{j=1}^{N} \mathcal{D}_{vh} \left\{ [(\mathcal{D}_v X^\ell(t))^{-1}]_{ij} [D^\ell X^\ell(t)]_{jl} \right\} \mathcal{E}_K(t) \\ - 2K \phi_{il}(X(t)) \int_0^t \langle AX(s), A\mathcal{D}_{vh} X(s) \rangle ds \end{aligned} \tag{4.23}$$

Note that $\pi^\ell H$ is isomorphic to $\mathbb{R}^N$ under the orthonormal basis. Take the standard orthonormal basis $\{h_i; i = 1, \dots, N\}$ of $\mathbb{R}^N$, which is a representation of the orthonormal basis of $\pi^\ell H$. Set $h = h_i$ in (4.23) and sum over $i$, we obtain

$$\begin{aligned} &\mathbb{E}\left(D^\ell \phi(X(t)) D_{h_l}^\ell X^\ell(t) \mathcal{E}_K(t)\right) \\ &= \mathbb{E}\left(\sum_{i=1}^{N} \mathcal{D}_{vh_i} \phi_{il}(X(t))\right) - \mathbb{E}\left(\sum_{i,j=1}^{N} \phi(X(t)) \mathcal{D}_{vh_i} \left\{ [(\mathcal{D}_v X^\ell(t))^{-1}]_{ij} [D^\ell X^\ell(t)]_{jl} \right\} \mathcal{E}_K(t)\right) \\ &\quad + 2K\mathbb{E}\left(\sum_{i=1}^{N} \phi_{il}(X(t)) \int_0^t \langle AX(s), A\mathcal{D}_{vh_i} X(s) \rangle ds\right) \end{aligned} \tag{4.24}$$

Let us first bound the first term on the r.h.s. of (4.24) as follows: By Bismut formula (simply write $v_i = vh_i$), (3.7) and the identity $\mathcal{D}_v X^\ell(t) = J_t \mathcal{M}_t$, one has

$$\begin{aligned} &|\mathbb{E}\left(\sum_{i=1}^{N} \mathcal{D}_{v_i} \phi_{il}(X(t))\right)| = |\mathbb{E}\left(\sum_{i,j=1}^{N} \phi(X(t)) [\mathcal{M}_t^{-1} J_t^{-1}]_{ij} [D^\ell X^\ell(t)]_{jl} \mathcal{E}_K(t) \int_0^t \langle v_i^\ell, dW_s \rangle\right)| \\ &\le C||\phi||_k (1 + |Ax|^k) \sum_{i,j=1}^{N} \mathbb{E}\left(\frac{\mathcal{E}_{K/2}(t)}{\lambda_{min}} |J_t^{-1} h_j| |D_{h_l}^\ell X^\ell(t)| \left|\int_0^t \langle v_i^\ell, dW_s \rangle\right|\right), \end{aligned} \tag{4.25}$$

moreover, by Hölder's inequality, Burkholder-Davis-Gundy's inequality, (4.22), (4.13), (4.12) and (4.15) in order,

(4.26)

$$\begin{aligned}
&\mathbb{E}\left(\frac{\mathcal{E}_{K/2}(t)}{\lambda_{min}}|J_t^{-1}h_j||D^\ell_{h_l}X^\ell(t)||\int_0^t \langle v_i^\ell, dW_s\rangle|\right) \\
&\leq \left[\mathbb{E}\left(\frac{1}{\lambda^6_{min}}\right)\right]^{\frac{1}{6}}\left[\mathbb{E}\left(|J_t^{-1}h_j|^6\mathcal{E}_K(t)\right)\right]^{\frac{1}{6}}\left[\mathbb{E}\left(|D^\ell_{h_l}X^\ell(t)|^6\mathcal{E}_K(t)\right)\right]^{\frac{1}{6}}\left[\mathbb{E}\left(\int_0^t \mathcal{E}_{\frac{K}{3}}(s)|(J_s^{-1}Q^\ell)^*h_i|^2ds\right)\right]^{\frac{1}{2}} \\
&\leq \frac{Ce^{Ct}}{t^p}
\end{aligned}$$

where $p > 6q+1$ and $C = C(p)$. Combining (4.25) and (4.26), we have

$$|\mathbb{E}\left(\sum_{i=1}^N \mathcal{D}_{v_i}\phi_{il}(X(t))\right)| \leq \frac{Ce^{Ct}}{t^p}||\phi||_k(1+|Ax|^k) \tag{4.27}$$

where $C = C(p,k) > 0$. By a similar method but a little more complicate calculations (using Lemma 4.7 and the estimates in Lemma 4.2), we have the same bounds for the other two terms on the r.h.s. of (4.24). Hence,

$$|\mathbb{E}\left[D^\ell\phi(X(t))D^\ell_{h_l}X^\ell(t)\mathcal{E}_K(t)\right]| \leq t^{-p}Ce^{Ct}||\phi||_k(1+|Ax|^k)$$

for all $t > 0$. Since the above argument is in the framework of $\pi^\ell D(A)$ with the orthonormal base $\{h_l; 1 \leq l \leq N\}$, we have

$$|\mathbb{E}\left[D^\ell\phi(X(t))D_hX^\ell(t)\mathcal{E}_K(t)\right]| \leq t^{-p}Ce^{Ct}||\phi||_k(1+|Ax|^k)|h| \qquad h \in \pi^\ell H.$$

□

## 5. Proof of the main theorems

5.1. **Gradient estimates of $u_m(t)$.** To prove the strong Feller property of the semigroup $P_t^m$ (recall $P_t^m\phi = u_m(t)$) and the later limiting semigroup $P_t$, a typical method is to show that $P_t^m$ has a gradient estimate similar to (3.1). In [5], one has the same estimate as (3.1) *but* with $\alpha = \frac{1}{2}+r-\gamma$ therein, thanks to the property $0 < \frac{1}{2}+r-\gamma < 1$, one can easily show

$$||A^{-\gamma}Du_m(t)||_2 \leq C(t^{-\frac{1}{2}-r+\gamma}+1)||\phi||_0,$$

this is exactly the second inequality in Proposition 3.5 of [5].

In our case, by the same method as in [5] (i.e. applying (3.1) to bound the r.h.s. of (2.15)), we *formally* have

$$||A^{-\gamma}Du_m(t)||_2 \leq Ct^{-\alpha}||\phi||_0 + KC\int_0^t (t-s)^{-\alpha}ds||\phi||_0, \tag{5.1}$$

however, the integral on the r.h.s. of (5.1) blows up due to $\alpha > 1$ in (3.1) .

We have two ways to overcome this problem of non integrability in (5.1). One is by an interpolation argument (see Proposition 5.1), the other is by some more delicate analysis (see Proposition 5.2). The underlying ideas of the two methods are the same, i.e. trading off the regularity of the space for the integrability of the time.

**Proposition 5.1.** *Given $T>0$, for any $0<t\le T$, $\max\{\frac{1}{2}, r-\frac{1}{2}\}<\gamma\le 1$ with $\gamma\ne\frac{3}{4}$ and $0<\beta<1$, if $\phi\in C_b^1(D(A),\mathbb{R})$, then $S_t^m\phi$ and $u_t^m$ are both functions in $C_{2,\gamma}^{\beta/\alpha}(D(A),\mathbb{R})$, which is the Hölder space defined by (2.6). Moreover,*

$$||S_t^m\phi||_{C_{2,\gamma}^{\beta/\alpha}}\le Ct^{-\beta}||\phi||_2, \tag{5.2}$$

$$||u_m(t)||_{C_{2,\gamma}^{\beta/\alpha}}\le C(t^{-\beta}+1)||\phi||_0, \tag{5.3}$$

*where $\alpha=p+\frac{1}{2}+r-\gamma$ is defined in (3.1) and $C=C(T,\alpha,\beta,\gamma)>0$.*

*Proof.* By (3.1), one has $S_t^m: C_2(D(A),\mathbb{R})\to C_{2,\gamma}^1(D(A),\mathbb{R})$ with

$$||S_t^m\phi||_{C_{2,\gamma}^1}\le Ct^{-\alpha}||\phi||_2.$$

By a simple calculation with the the above estimate and (3.8), we have

$$||S_t^m\phi||_{C_{2,\gamma}^{\beta/\alpha}}\le C||S_t^m\phi||_{C_{2,\gamma}^1}^{\beta/\alpha}||S_t^m\phi||_2^{1-\beta/\alpha}\le Ct^{-\beta}||\phi||_2,$$

for any $0\le\beta\le\alpha$. Take any $0<\beta<1$, applying the above estimate on the Duhamel formula (2.15) and the clear fact $||u_m(t)||_0\le||\phi||_0$, we immediately have (5.3). □

**Proposition 5.2.** *Given any $T>0$, there exists some $C=C(T,\alpha,\gamma)>0$ such that*

$$||A^{-\gamma}Du_m(t)||_{2+2\alpha}\le Ct^{-\alpha}||\phi||_0 \tag{5.4}$$

*where $\max\{r-\frac{1}{2},\frac{1}{2}\}<\gamma\le 1$ with $\gamma\ne\frac{3}{4}$.*

*Proof.* The idea of the proof is to split the integral $\int_0^t|D_hS_{t-s}^m(|Ax|^2u_m(s))|ds$ into two pieces, '$\int_0^{\beta t}\cdots$' and '$\int_{\beta t}^t\cdots$' with some special $\beta\in(0,1)$, applying (3.1) to the first piece and the probability presentation of $S_{t-s}^m$ to the other. Roughly speaking, '$\int_0^{\beta t}\cdots$' takes away the singularity of $(t-s)^{-\alpha}$ at $s=t$, while '$\int_{\beta t}^t\cdots$' conquers the extra polynomial growth of $|Ax|^2$ in $S_{t-s}^m[|Ax|^2u_m(s)]$. However, we have to pay a price of an extra polynomial growth of $|Ax|^{2\alpha}$ for $Du^m(t)$.

For the notational simplicity, we shall drop the index $m$ of the quantities if no confusions arise. Denote

$$\beta=1-\frac{1}{K^2(1+|Ax|^2)},$$

by (3.1) with $k=2$, we have

$$\begin{aligned}|A^{-\gamma}Du(t,x)|&\le Ct^{-\alpha}||\phi||_2(1+|Ax|^2)+KC\int_0^{\beta t}(t-s)^{-\alpha}ds||\phi||_0(1+|Ax|^2)\\&\quad+K\int_{\beta t}^t|A^{-\gamma}DS_{t-s}(|Ax|^2u(s))|ds\\&\le Ct^{-\alpha}||\phi||_0(1+|Ax|^2)+K^{2\alpha+1}Ct^{-\alpha+1}(1+|Ax|^{2+2\alpha})||\phi||_0\\&\quad+K\int_{\beta t}^t|A^{-\gamma}DS_{t-s}(|Ax|^2u(s))|ds,\end{aligned}$$

thus

$$
\begin{aligned}
t^{\alpha}|A^{-\gamma}D_h u(t,x)| \le C||\phi||_0(1+|Ax|^2) + K^{2\alpha+1}Ct||\phi||_0(1+|Ax|^{2+2\alpha}) \\
+ Kt^{\alpha}\int_{\beta t}^{t}|A^{-\gamma}DS_{t-s}(|Ax|^2 u(s))|ds.
\end{aligned} \tag{5.5}
$$

Define

$$u_{\phi,T} = \sup_{0\le s\le T} s^{\alpha}||A^{-\gamma}Du(s)||_{2+2\alpha},$$

let us estimate the integral on the r.h.s. of (5.5) in the following way: it is easy to see that

$$
\begin{aligned}
&\int_{\beta t}^{t}|D_h S_{t-s}(|Ax|^2 u(s)]|ds \\
&= \int_{\beta t}^{t}|\mathbb{E}(D_h|AX(t-s)|^2 u(s,X(t-s))\mathcal{E}_K(t-s)]|ds \\
&+ \int_{\beta t}^{t}|\mathbb{E}(|AX(t-s)|^2 u(s,X(t-s))D_h\mathcal{E}_K(t-s)]|ds \\
&+ \int_{\beta t}^{t}|\mathbb{E}(|AX(t-s)|^2 D_h u(s,X(t-s))\mathcal{E}_K(t-s)]|ds.
\end{aligned} \tag{5.6}
$$

By the same argument as estimating $I_3$ in the proof of Theorem 3.1 and the easy fact $||u(t)||_0 \le ||\phi||_0$ for all $t\ge 0$, the first two integrals on the r.h.s. of (5.6) can both be bounded by

$$C(1+|Ax|^2)||\phi||_0|A^{\gamma}h|.$$

The last integral can be estimated as follows: By (3.7), (3.9) and the definition of $u_{T,\phi}$, one has

$$
\begin{aligned}
&\int_{\beta t}^{t}|\mathbb{E}(|AX(t-s)|^2 D_h u(s,X(t-s))\mathcal{E}_K(t-s)]|ds \\
&\le \int_{\beta t}^{t}\mathbb{E}[(1+|AX(t-s)|^{4+2\alpha})\mathcal{E}_{\frac{K}{2}}(t-s)||A^{-\gamma}Du(s)||_{2+2\alpha}\mathcal{E}_{\frac{K}{2}}(t-s)|A^{\gamma}D_h X(t-s)|]ds \\
&\le C(1+|Ax|^{4+2\alpha})|A^{\gamma}h|\int_{\beta t}^{t}||A^{-\gamma}Du(s)||_{2+2\alpha}ds \\
&\le C(1+|Ax|^{4+2\alpha})|A^{\gamma}h|\left(\int_{\beta t}^{t}s^{-\alpha}ds\right)u_{T,\phi} \\
&\le \frac{Ct^{-\alpha+1}}{K^2}u_{T,\phi}(1+|Ax|^{2+2\alpha})|A^{\gamma}h|.
\end{aligned}
$$

Collecting the above three estimates, we have

$$\int_{\beta t}^{t}|A^{-\gamma}DS_{t-s}(|Ax|^2 u(s))|ds \le C(1+|Ax|^2)||\phi||_0 + \frac{Ct^{-\alpha+1}}{K^2}u_{T,\phi}(1+|Ax|^{2+2\alpha}).$$

Plugging this estimate into (5.5) and dividing the both sides of the inequality by $(1+|Ax|^{2+2\alpha})$, one has

$$u_{T,\phi} \le C||\phi||_0 + CK^{2\alpha+1}T||\phi||_0 + CKT^{\alpha}||\phi||_0 + \frac{CT}{K}u_{T,\phi}.$$

As $K > 0$ is sufficiently large,

$$u_{T,\phi} \leq \frac{C(1 + K^{2\alpha+1}T + KT^{\alpha})}{1 - CT/K} ||\phi||_0,$$

from this inequality, we immediately have (5.4). $\square$

5.2. **Proof of Theorem 2.4.** One can pass to the Galerkin approximation limit of $u_m(t)$ by the same procedures as in [5]. For the completeness, we sketch out the main steps as following.

The following proposition is nearly the same as Proposition 3.6 in [5], only with a small modification in which (5.3) plays an essential role.

**Proposition 5.3.** *Let $\phi \in C_b^1(D(A), \mathbb{R})$ and $T > 0$. For any $0 < \beta < 1/2$, $t_1 \geq t_2 > 0$, $m \in \mathbb{N}$ and $x \in D(A)$, we have some $C(T, \beta) > 0$ such that*

$$|u_m(t_1, x) - u_m(t_2, x)| \leq C||\phi||_{C^1_{2,1}}(1 + |Ax|^6)(|t_2 - t_1|^{\beta} + |A(e^{-At_2} - e^{-At_1})x|). \tag{5.7}$$

Define $K_R = \{x \in D(A); |Ax| \leq R\}$, which is compact in $D(A^{\gamma})$ for any $\gamma < 1$, we have the following lemma (which is Lemma 4.1 in [5]) by applying Proposition 5.3.

**Lemma 5.4.** *Assume $\phi \in C_b^1(D(A), \mathbb{R})$, then there exists a subsequence $(u_{m_k})_{k \in \mathbb{N}}$ of $(u_m)$ and a function $u$ on $[0, T] \times D(A)$, such that*

1. *$u \in C_b((0, T] \times D(A))$ and for all $\delta > 0$ and $R > 0$*
$$\lim_{k \to \infty} u_{m_k}(t, x) = u(t, x) \ \text{ uniformly on } [\delta, \mathrm{T}] \times \mathrm{K_R}.$$
2. *For any $x \in D(A)$, $u(\cdot, x)$ is continuous on $[0, T]$.*
3. *For any $\max\{\frac{1}{2}, r - \frac{1}{2}\} < \gamma \leq 1$ with $\gamma \neq \frac{3}{4}$, $\delta > 0$, $R > 0$ and $\beta < min\{1/2, \sigma/2\}$, there exists some $C = C(\gamma, \beta, \delta, R, T, \phi)$ such that for any $x, y \in K_R$, $t \geq s \geq \delta$,*
$$|u(t, x) - u(s, y)| \leq C(|A^{\gamma}(x - y)| + |t - s|^{\beta}).$$
4. *For any $t \in [0, T]$, $u(t, \cdot) \in C_b(D(A), \mathbb{R})$.*
5. *$u(0) = \phi$.*

**Lemma 5.5.** *For any $\delta \in (1/2, 1 + \sigma]$, there exists some constant $C(\delta) > 0$ such that for any $x \in H$, $m \in \mathbb{N}$, and $t \in [0, T]$, we have*

1. $\mathbb{E}[|X_m(t, x)|^2] + \mathbb{E}\int_0^t |A^{1/2} X_m(s, x)|^2 ds \leq |x|^2 + tr(QQ^*)t$.
2. $\mathbb{E}\int_0^T \frac{|A^{\frac{1+\delta}{2}} X_m(s,x)|^2}{(1+|A^{\frac{\delta}{2}} X_m(s)|^2)^{\gamma_\delta}} ds \leq C(\delta)$, *with $\gamma_\delta = \frac{2}{2\delta - 1}$ if $\delta \leq 1$ and $\gamma_\delta = \frac{2\delta+1}{2\delta-1}$ if $\delta > 1$.*

By (1) of Lemma 5.5, we can prove that the laws $\mathcal{L}(X_m(\cdot, x))$ is tight in $L^2([0, T], D(A^{s/2}))$ for $s < 1$ and in $C([0, T], D(A^{-\alpha}))$ for $\alpha > 0$. By Skohorod's embedding Theorem, one can construct a probability space $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$ with a random variable $X(\cdot, x)$ valued in $L^2([0, T], D(A^{s/2})) \cap C([0, T], D(A^{-\alpha}))$ such that for any $x \in D(A)$ there exists some subsequence $\{X_{m_k}\}$ satisfying

$$X_{m_k}(\cdot, x) \to X(\cdot, x) \qquad d\mathbb{P}_x \ a.s. \tag{5.8}$$

in $L^2([0,T],D(A^{s/2}))\cap C([0,T],D(A^{-\alpha}))$. Moreover, by (3) of Lemma 5.5, for $x\in D(A)$ we have (see (7.7) in [4])

$$X_{m_k}(t,x)\to X(t,x)\quad \text{in } D(A)\ \ dt\times d\mathbb{P}_x\ a.s.\ [0,T]\times\Omega_x. \tag{5.9}$$

Note that the subsequence $\{u_{m_k}\}$ in Lemma 5.4 depends on $\phi$, by the separable property of $C(D(A),\mathbb{R})$, we can find a subsequence $\{m_k\}$ of $\{m\}$, independent of $\phi$, such that $\{u_{m_k}\}_k$ converges. That is, we have the following lemma, which is Lemma 7.5 of [4].

**Lemma 5.6.** *There exists a subsequence $\{m_k\}$ of $\{m\}$ so that for any $\phi\in C_b^1(D(A),\mathbb{R})$, one has a function $u^\phi\in C_b([0,T]\times D(A))$ satisfying*

$$\lim_{k\to\infty}u^\phi_{m_k}(t,x)=u^\phi(t,x)\quad \textit{for all } (t,x)\in(0,T]\times D(A) \tag{5.10}$$

*and*

$$u^\phi_{m_k}(t,x)\to u^\phi(t,x)\quad \text{uniformly in } [\delta,\mathrm{T}]\times\mathrm{K_R} \text{ for any } \delta>0,\mathrm{R}>0.$$

*where $u^\phi_{m_k}(t,x)=\mathbb{E}[\phi(X_{m_k}(t,x))]$.*

Take the subsequence $\{m_k\}$ in Lemma 5.6 and define

$$P_t\phi(x)=u^\phi(t,x). \tag{5.11}$$

for all $(t,x)\in[0,T]\times D(A)$, where $u^\phi$ is defined by (5.10). By Riesz Representation Theorem for functionals ([11], page 223) and the easy fact $P_t\mathbf{1}=1$, (5.11) determines a unique probability measure $P_t^*\delta_x$ supported on $D(A)$. By (5.8), for any $x\in D(A)$, we have some subsequence $\{m_k^x\}$ of $\{m_k\}$ such that $X_{m_k^x}(\cdot,x)\to X(\cdot,x)$ in $L^2([0,T],D(A^{s/2}))\cap C([0,T],D(A^{-\alpha}))$ a.s. $d\mathbb{P}_x$, hence

$$P_t\phi(x)=\mathbb{E}_x[\phi(X(t,x))]$$

for all $\phi\in C_b^1(D(A),\mathbb{R})\cap C_b(D(A^{-\alpha}),\mathbb{R})$. Since the measure $P_t^*\delta_x$ is supported on $D(A)$, $\mathbb{P}_x(X(t,x)\in D(A))=1$, which is (1) of Definition 2.3. By a classic approximation ($\mathcal{B}_b(D(A),\mathbb{R})$ can be approximated by $C(D(A),\mathbb{R})$), we have

$$P_t\phi(x)=\mathbb{E}_x[\phi(X(t,x))] \text{ is well defined for all } \phi\in\mathcal{B}_{\mathrm{b}}(\mathrm{D(A)},\mathbb{R}). \tag{5.12}$$

With the above observation, we can easily prove Theorem 2.4 as follows:

*Proof of Theorem 2.4.* Since $X_{m_k}(\cdot,x)\to X(\cdot,x)$ a.s. $\mathbb{P}_x$ in $C([0,T],D(A^{-\alpha}))$ and the map $x\to\mathcal{P}_x^{m_k}$ is measurable ($\mathcal{P}_x^{m_k}$ is the law of $X_{m_k}(\cdot,x)$), the map $x\to\mathcal{P}_x$ is also measurable. The following lemma is exactly Lemma 4.5 in [5] and expressed as

**Lemma 5.7.** *Let $X(\cdot,x)$ be the limit process of a subsequence $\{X_{m_k}\}_k$. Then, for any $M,N\in\mathbb{N}$, $t_1,\cdots,t_n\geq 0$ and $(f_k)_{k=0}^M$ with each $f_k\in C_c^\infty(\pi_N H,\mathbb{R})$, we have*

$$\mathbb{E}_x[f_0(X(0,x))f_1(X(t_1,x))\cdots f_M(X(t_1+\cdots+t_M,x))]=f_0(x)P_{t_1}[f_1P_{t_2}(f_2P_{t_3}f_3\cdots)](x) \tag{5.13}$$

*where each $f_k(x)=f_k(\pi_N x)$ and $P_t$ is defined by (5.12).*

One can easy extend (5.13) from $C_c^\infty(\pi_N H,\mathbb{R})$ to $\mathcal{B}_b(D(A),\mathbb{R})$, which easily implies the Markov property of the family $(\Omega_x,\mathcal{F}_x,\mathbb{P}_x,X(\cdot,x))_{x\in D(A)}$. ∎

5.3. **Proof of Theorem 2.5.** To prove the ergodicity, we first prove that (2.1) has at least one invariant measure, and then show the uniqueness by Doob's Theorem. With the ergodic measure, we follow the coupling method in [19] to prove the exponential mixing property (2.7).

**Lemma 5.8.** *Each approximate stochastic dynamics $X_m(t)$ has a unique invariant measure $\nu_m$.*

*Proof.* By Proposition 5.1 or Proposition 5.2, we easily obtain that $P_t^m$ is strong Feller. The existence of the invariant measures for $X^m(t)$ is standard (see [3]), and it is easy to prove that 0 is the support of each invariant measure (see Lemma 3.1, [7]). Therefore, by Corollary 3.17 of [12], we conclude the proof. $\square$

The following lemma is the same as Lemma 7.6 in [4] (or Lemma 5.1 in [5]).

**Lemma 5.9.** *There exists some constant $C>0$ so that*

$$\int_H [|Ax|^2+|A^{1/2}x|^{2/3}+|A^{1+\sigma/2}x|^{(1+\sigma)/(10+8\sigma)}]\nu_{m_k}(dx)<C \tag{5.14}$$

*where $\sigma>0$ is the same as in Assumption 2.1.*

With the above lemma, it is easy to see that $\{\nu_{m_k}\}$ is tight on $D(A)$, and therefore there exists a limit measure $\nu$ which satisfies $\nu(D(A))=1$. Taking any $\phi\in C_b^1(D(A),\mathbb{R})$, we can check via the Galerkin approximation (or see the detail in pp. 938 of [4]) that

$$\int_H P_t\phi(x)\nu(dx)=\int_H\phi(x)\nu(dx) \tag{5.15}$$

for any $t>0$. Hence $\nu$ is an invariant measure of $P_t$.

**Proposition 5.10.** *The system $X(t)$ is irreducible on $D(A)$. More precisely, for any $x,y\in D(A)$, we have*

$$P_t[1_{B_\delta(y)}](x)>0. \tag{5.16}$$

*for arbitrary $\delta>0$, where $B_\delta(y)=\{z\in D(A);|Az-Ay|\le\delta\}$.*

*Proof.* We first prove that the following control problem is solvable: Given any $T>0$, $x,y\in D(A)$ and $\varepsilon>0$, there exist $\rho_0=\rho_0(|Ax|,|Ay|,T)$, $u$ and $w$ such that

- $w\in L^2([0,T];H)$ and $u\in C([0,T];D(A))$,
- $u(0)=x$ and $|Au(T)-Ay|\le\varepsilon$,
- $\sup_{t\in[0,T]}|Au(t)|\le\rho_0$,

and $u$, $w$ solve the following problem,

$$\partial_t u+Au+B(u,u)=Qw, \tag{5.17}$$

where $Q$ is defined in Assumption 2.1.

This control problem is exactly Lemma 5.2 of [23] with $\alpha=1/4$ therein, but we give the sketch of the proof for completeness. First, it is easy to find some $z\in D(A^{5/2})$ with $|Ay-Az|\le\epsilon/2$, therefore it suffices to prove there exists some control $w$ so that

$$|Au(T)-Az|\le\epsilon/2. \tag{5.18}$$

Second, decompose $u = u^{h} + u^{\ell}$ where $u^{h} = (I - \pi_{n_0})u$ and $u^{\ell} = \pi_{n_0}u$ and $n_0$ is the number in Assumption 2.1, then equation (5.17) can be written as

$$\partial_t u^{\ell} + Au^{\ell} + B^{\ell}(u,u) = 0, \tag{5.19}$$

$$\partial_t u^{h} + Au^{h} + B^{h}(u,u) = Q^{h}w. \tag{5.20}$$

We prove (5.18) in the following four steps:

(1) *Regularization of the initial data*: Let $w \equiv 0$ on $[0, T_1]$, by some classical arguments about the regularity of Navier-Stokes equation, one has $u(T_1) \in D(A^{5/2})$, where $T_1 > 0$ depends on $|Ax|$.

(2) *High modes lead to zero*: Choose a smooth function $\psi$ on $[T_1, T_2]$ such that $0 \le \psi \le 1$, $\psi(T_1) = 1$ and $\psi(T_2) = 0$, and set $u^{h}(t) = \psi(t)u^{h}(T_1)$ for $t \in [T_1, T_2]$. Plugging this $u^{h}$ into (5.20), we obtain

$$w(t) = \psi'(t)(Q^{h})^{-1}u^{h}(T_1) + \psi(t)(Q^{h})^{-1}Au^{h}(T_1) + (Q^{h})^{-1}B^{h}(u(t), u(t)).$$

(3) *Low modes close to $z^{\ell}$*: Let $u^{\ell}(t)$ be the linear interpolation between $u^{\ell}(T_2)$ and $z^{\ell}$ for $t \in [T_2, T_3]$. Write $u(t) = \sum u_k(t)e_k$, then (5.19) in Fourier coordinates is given by

$$\dot{u}_k + |k|^2 u_k + B_k(u,u) = 0, \qquad k \in Z_{\ell}(N_0),$$

where $B_k(u,u) = B_k(u^{\ell}, u^{\ell}) + B_k(u^{\ell}, u^{h}) + B_k(u^{h}, u^{\ell}) + B_k(u^{h}, u^{h})$. We can choose a suitable simple $u^{h}$ to $B_k(u^{\ell}, u^{h}) = B_k(u^{h}, u^{\ell}) = 0$ and make the above equation explicitly solvable.

(4) *High modes close to $z^{h}$*: In the interval $[T_3, T]$ we choose $u^{h}$ as the linear interpolation between $u^{h}(T_3)$ and $z^{h}$. By continuity, as $T - T_3$ is sufficiently small (thanks to that $T_3 \in (T_2, T)$ can be arbitrary), $u^{\ell}(T)$ is still close to $z^{\ell}$.

From the above four steps, we can see that $\sup_{T_1 \le t \le T} |A^{5/2}u(t)| < \infty$. Moreover, since $w \equiv 0$ on $[0, T_1]$ and $\sup_{0 \le t \le T_1} |Au(t)|^2 < \infty$, by differentiating $|Au(t)|^2$ and applying (6.12), we have the energy inequality

$$|Au(T_1)|^2 + \int_0^{T_1} |A^{3/2}u(s)|^2 ds \le C \int_0^{T_1} |Au(s)|^4 ds + |Ax|^2 < \infty \tag{5.21}$$

Hence $u \in L^2([0,T], D(A^{3/2}))$. With this observation and the controllability, we can apply Lemma 7.7 in [4] to obtain the conclusion (Note that our control $w$ is different from the $\bar{w}$ in [4], this is the key point that we can apply the argument there with $Q$ not invertible.)

Alternatively, with the solvability of the above control problem, we can apply the argument in the proof of Proposition 5.1 in [23] to show irreducibility. $\square$

From Proposition 5.1 or Proposition 5.2, $P_t$ is strong Feller. By the irreducibility, there exists a unique invariant measure $\nu$ for $X(t)$ by Doob's Theorem.

Finally, let us prove the exponential mixing property (2.7). To show this, it suffices to prove that

$$||(P_t^m)^*\mu - \nu_m||_{var} \le Ce^{-ct}\left(1 + \int_H |x|^2 \mu(dx)\right) \tag{5.22}$$

where $c, C > 0$ are independent of $m$, and $\nu_m$ is the unique measure of the approximate dynamics (see Lemma 5.8). We follow exactly the coupling method in [19] to prove (5.22), let us sketch out the key point as follows.

For two independent cylindrical Wiener processes $W$ and $\tilde{W}$, denote $X_m$ and $\tilde{X}_m$ the solutions of the equation (2.8) driven by $W$ and $\tilde{W}$ respectively. For any fixed $0 < T \le 1$, given any two $x_1, x_2 \in D(A)$, we construct the coupling of the probabilities $(P_T^m)^*\delta_{x_1}$ and $(P_T^m)^*\delta_{x_2}$ as follows

$$(V_1, V_2) = \begin{cases} (X_m(T, x_0), X_m(T, x_0)) & \text{if } x_1 = x_2 = x_0, \\ (Z_1(x_1, x_2), Z_2(x_1, x_2)) & \text{if } x_1, x_2 \in B_{D(A)}(0, \delta) \ with \ x_1 \ne x_2, \\ (X_m(T, x_1), \tilde{X}_m(T, x_2)) & \text{otherwise,} \end{cases}$$

where $(Z_1(x_1, x_2), Z_2(x_1, x_2))$ is the maximal coupling of $(P_T^m)^*\delta_{x_1}$ and $(P_T^m)^*\delta_{x_2}$ (see Lemma 1.14 in [19]) and $B_{D(A)}(0, \delta) = \{x \in D(A); |Ax| \le \delta\}$. It is clear that $(V_1, V_2)$ is a coupling of $(P_T^m)^*\delta_{x_1}$ and $(P_T^m)^*\delta_{x_2}$. We construct $(X^1, X^2)$ on $T\mathbb{N}$ by induction: set $X^i(0) = x^i$ $(i = 1, 2)$ and define

$$X^i((n+1)T) = V_i(X^1(nT), X^2(nT)) \quad i = 1, 2.$$

The key point for using this coupling to show the exponential mixing is the following lemma, which plays the same role as Lemma 2.1 in [19], but we prove it in a little simpler way.

**Lemma 5.11.** *There exist some $0 < T, \delta < 1$ such that for any $m \in \mathbb{N}$, one has a maximal coupling $(Z_1(x_1, x_2), Z_2(x_1, x_2))$ of $(P_T^m)^*\delta_{x_1}$ and $(P_T^m)^*\delta_{x_2}$ which satisfies*

$$\mathbb{P}(Z_1(x_1, x_2) = Z_2(x_1, x_2)) \ge 3/4 \tag{5.23}$$

*if $|Ax_1| \vee |Ax_2| \le \delta$ with $\delta > 0$ sufficiently small.*

*Proof.* Since $(Z_1(x_1, x_2), Z_2(x_1, x_2))$ is maximal coupling of $(P_T^m)^*\delta_{x_1}$ and $(P_T^m)^*\delta_{x_2}$ (see Lemma 1.14 in [19]), one has

$$||(P_T^m)^*\delta_{x_1} - (P_T^m)^*\delta_{x_2}||_{var} = \mathbb{P}\{Z_1(x_1, x_2) \ne Z_2(x_1, x_2)\}.$$

It is well known that

$$\begin{aligned} ||(P_T^m)^*\delta_{x_1} - (P_T^m)^*\delta_{x_2}||_{var} &= \sup_{||g||_\infty = 1} |\mathbb{E}[g(X_m(T, x_1))] - \mathbb{E}[g(X_m(T, x_2)]| \\ &= \sup_{||g||_\infty = 1} |P_T^m g(x_1) - P_T^m g(x_2)|, \end{aligned}$$

where $||\cdot||_\infty$ is the supremum norm. By Proposition 5.2, (noticing $||g||_0 = ||g||_\infty$=1 with $||\cdot||_0$ defined in section 2), one has

$$\begin{aligned} |P_T^m g(x_1) - P_T^m g(x_2)| &\le \int_0^1 |A^{-1} D P_T^m g(\lambda x_1 + (1-\lambda) x_2)||Ax_1 - Ax_2| d\lambda \\ &\le C T^{-\alpha}(1 + |Ax_1| + |Ax_2|)^{2+2\alpha} |A(x_1 - x_2)| \le 1/4 \end{aligned}$$

if $\delta = T^\beta$ with $\beta > 0$ sufficiently large. Hence $\mathbb{P}(Z_1 = Z_2) = 1 - \mathbb{P}(Z_1 \ne Z_2) \ge \frac{3}{4}$. □

With this lemma, one can prove the exponential mixing (2.7) by exactly the same procedure as in [19].

## 6. Appendix

### 6.1. Some calculus for $\tilde{B}_k$ and Proof of Lemma 4.5.

*Some calculus for $\tilde{B}_k$.* By $B(u,v)=\mathcal{P}[(u\cdot\nabla)v]$, we have

$$B(a_j cos j\xi, a_l sin l\xi)=\frac{1}{2}(l\cdot a_j)\mathcal{P}[a_l cos(j+l)\xi]+\frac{1}{2}(l\cdot a_j)\mathcal{P}[a_l cos(j-l)\xi],$$
$$B(a_j sin j\xi, a_l cos l\xi)=\frac{1}{2}(l\cdot a_j)\mathcal{P}[a_l cos(j+l)\xi]-\frac{1}{2}(l\cdot a_j)\mathcal{P}[a_l cos(j-l)\xi],$$

where $\mathcal{P}$ is the projection from $L^2(\mathbb{T}^3,\mathbb{R}^3)$ to $H$. If $j,-l\in\mathbb{Z}^3_+$ with $j+l\in\mathbb{Z}^3_+$, $\forall\ a_j\in j^\perp, a_l\in l^\perp$, we have from the above two expressions

$$\tilde{B}_{j-l}(a_je_j,a_le_l)=\frac{1}{2}[(l\cdot a_j)\mathcal{P}_{j-l}a_l-(j\cdot a_l)\mathcal{P}_{j-l}a_l], \tag{6.1}$$
$$\tilde{B}_{j+l}(a_je_j,a_le_l)=\frac{1}{2}[(l\cdot a_j)\mathcal{P}_{j+l}a_l+(j\cdot a_l)\mathcal{P}_{j+l}a_j], \tag{6.2}$$
$$\tilde{B}_{k}(a_je_j,a_le_l)=0 \quad \text{if} \quad k\neq j+l,j-l. \tag{6.3}$$

where the projection $\mathcal{P}_k:\mathbb{R}^3\longrightarrow k^\perp$ is defined by (2.2). For the case of $j,l\in\mathbb{Z}^3_+$ with $j-l\in\ Z^3_-$, we can calculate $\tilde{B}_{-j-l}(a_je_j,a_le_l)$, $\tilde{B}_{j-l}(a_je_j,a_le_l)$ and so on by the same method. □

*Proof of Lemma 4.5.* As $k\in Z_\ell(n_0)\cap\mathbb{Z}^3_+$, for any $j,l\in\mathbb{Z}^3_*$ such that

$$j\in Z_\hbar(n_0)\cap\mathbb{Z}^3_+,\quad l\in Z_\hbar(n_0)\cap\mathbb{Z}^3_-,\quad j\nparallel l,\quad |j|\neq|l|,\quad j+l=k; \tag{6.4}$$

taking an *orthogonal* basis $\{k,h_1,h_2\}$ of $\mathbb{R}^3$ where $\{h_1,h_2\}$ is an *orthogonal* basis of $k^\perp$ with $h_1$ defined by

$$h_1=l\quad\text{if}\quad k\cdot l=0,\qquad h_1=j-\frac{j\cdot k}{k\cdot l}l\quad\text{otherwise.}$$

Let $p_j\in j^\perp, p_l\in l^\perp$ be represented by $p_j=ak+b_1h_1+b_2h_2$ and $p_l=\alpha k+\beta_1h_1+\beta_2h_2$. Clearly, $j,l\perp h_2$, by some basic calculation, we have

$$(j\cdot p_l)\mathcal{P}_kp_j+(l\cdot p_j)\mathcal{P}_kp_l=\begin{cases}-\left[\frac{(k\cdot l)(|j|^2-|l|^2)}{|j|^2|l|^2-(j\cdot l)^2}a\alpha\right]h_1+(\alpha b_2+\beta_2 a)h_2 & \text{if } h_1=j-\frac{j\cdot k}{k\cdot l}l\\ -\left[\frac{j\cdot k}{j\cdot l}a\alpha\right]h_1+(\alpha b_2+\beta_2 a)h_2 & \text{if } h_1=l\end{cases}$$

Since $b_2,\beta_2,a,\alpha\in\mathbb{R}$ can be arbitrarily chosen, one clearly has

$$\{j\cdot p_l)\mathcal{P}_kp_j+(l\cdot p_j)\mathcal{P}_kp_l:\quad p_j\in j^\perp,\ p_l\in l^\perp\}=\ k^\perp. \tag{6.5}$$

By (A3) of Assumption 2.1, we have $rank(q_j), rank(q_l)=2$, therefore, by (6.2) and (6.5),

$$\left\{\tilde{B}_k\left(q_j\ell_je_j,q_l\ell_le_l\right):\ell_j\in j^\perp,\ell_l\in l^\perp\right\}=k^\perp.$$

Hence, $span\{Y_k\}=k^\perp$. For $k\in Z_\ell(n_0)\cap Z^3_-$, we have the same conclusion by the same argument as above. □

6.2. **Proof of Lemma 4.7.** The key points for the proof are Proposition 4.6 and the following Norris' Lemma, which is exactly Lemma 4.1 in [18].

**Lemma 6.1. (Norris' Lemma)** *Let $a, y \in \mathbb{R}$. Let $\beta_t, \gamma_t = (\gamma_t^1, \dots \gamma_t^m)$ and $u_t = (u_t^1, \dots, u_t^m)$ be adapted processes. Let*

$$a_t = a + \int_0^t \beta_s ds + \int_0^t \gamma_s^i dw_s^i, \quad Y_t = y + \int_0^t a_s ds + \int_0^t u_s^i dw_s^i,$$

*where $(w_t^1, \dots, w_t^m)$ are i.i.d. standard Brownian motions. Suppose that $T < t_0$ is a bounded stopping time such that for some constant $C < \infty$:*

$$|\beta_t|, |\gamma_t|, |a_t|, |u_t| \le C \quad \textit{for all } t \le T.$$

*Then for any $r > 8$ and $\nu > \frac{r-8}{9}$*

$$P\{\int_0^T Y_t^2 dt < \epsilon^r, \int_0^T (|a_t|^2 + |u_t|^2) dt \ge \epsilon\} < C(t_0, q, \nu) e^{-\frac{1}{\epsilon^\nu}}.$$

*Proof of Lemma 4.7.* We shall drop the index $m$ of the quantities if no confusions arise. The idea of the proof is from Theorem 4.2 of [18], it suffices to show the inequality in the lemma, which is equivalent to

$$P\left(\inf_{\eta \in \mathcal{S}^\ell} \langle \mathcal{M}_t \eta, \eta \rangle \le \varepsilon^q\right) \le \frac{C(p)\varepsilon^p}{t^p} \quad (\forall\ p > 0) \tag{6.6}$$

where $\mathcal{S}^\ell = \{\eta \in \pi^\ell H; |\eta| = 1\}$. From (4.10), (6.6) is equivalent to

$$P\left(\inf_{\eta \in \mathcal{S}^\ell} \sum_{k \in Z_\ell(n) \setminus Z_\ell(n_0)} \sum_{i=1}^2 \int_0^t |\langle J_s^{-1}(q_k^i e_k), \eta\rangle|^2 ds \le \varepsilon^q\right) \le \frac{C(p)\varepsilon^p}{t^p},$$

(recall $q_k^i$ is the i-th column vector of the matrix $q_k$, see Assumption 2.1), which is implied by

$$D_\theta \sup_j \sup_{\eta \in \mathcal{D}_j} P\left(\int_0^t \sum_{k \in Z_\ell(n) \setminus Z_\ell(n_0)} \sum_{i=1}^2 |\langle J_s^{-1}(q_k^i e_k), \eta\rangle|^2 ds \le \varepsilon^q\right) \le \frac{C(p)\varepsilon^p}{t^p} \tag{6.7}$$

where $\{\mathcal{D}_j\}_j$ is a *finite* $\theta$-radius disk cover of $\mathcal{S}^\ell$ (due to the compactness of $\mathcal{S}^\ell$) and $D_\theta = \#\{\mathcal{D}_j\}$. Define a stopping time $\tau$ by

$$\tau = \inf\{s > 0; \left|\mathcal{E}_K(s) J_s^{-1} - Id\right|_{\mathcal{L}(H)} > c\}. \tag{6.8}$$

where $c > 0$ is a sufficiently small but fixed number. It is easy to see that (6.7) holds as long as for any $\eta \in \mathcal{S}^\ell$, we have some neighborhood $\mathcal{N}(\eta)$ of $\eta$ and some $k \in Z_\ell(n) \setminus Z_\ell(n_0)$, $i \in \{1, 2\}$ so that

$$\sup_{\eta' \in \mathcal{N}(\eta)} P\left(\int_0^{t \wedge \tau} |\langle J_s^{-1}(q_k^i e_k), \eta'\rangle|^2 ds \le \varepsilon^q\right) \le \frac{C(p)\varepsilon^p}{t^p} \quad (\forall\ p > 0). \tag{6.9}$$

The above argument is according to [18] (see Claim 1 of the proof of Theorem 4.2), one may see the greater details there.

Let us prove (6.9). According to the restricted Hörmander condition and Definition 4.3, for any $\eta \in \mathcal{S}^\ell$, there exists a $K \in \mathbf{K}$ satisfying for all $y \in \pi_m H$

$$|\langle K(y), \eta\rangle|^2 \ge \delta |\eta|^2$$

where $\delta > 0$ is a constant independent of $y$. Without loss of generality, assume that $K \in \mathbf{K}_2$, so there exists some $q_k^i e_k$ and $q_l^j e_l$ such that

$$K_0 := q_k^i e_k,\ \ K_1 := [A_m y + B_m^\ell(y,y), q_k^i e_k],\ \ K = K_2 := [q_l^j e_l, K_1].$$

Take

$$Y(t) = \langle J_t^{-1} K_1(X(t)), \eta\rangle, \quad u^i(t) = 0, \quad a(t) = \langle J_t^{-1} K_2(X(t)), \eta\rangle,$$

applying Norris lemma with $t_0 = 1$ therein, we have

$$P\left(\int_0^{t\wedge\tau} |\langle J_s^{-1} K_1(X(s)), \eta\rangle|^2 ds \le \varepsilon^r, \int_0^{t\wedge\tau} |\langle J_s^{-1} K_2(X(s)), \eta\rangle|^2 ds \ge \varepsilon\right) \le C(p,\nu) e^{-\frac{1}{\varepsilon^\nu}}$$

On the other hand, by (4.14), (6.8) and Chebyshev's inequality, it is easy to have

$$\begin{aligned} &P\left(\int_0^{t\wedge\tau} |\langle J_s^{-1} K_2(X(s)), \eta\rangle|^2 ds \le \varepsilon\right) \\ &= P\left(\int_0^{t\wedge\tau} \frac{1}{\mathcal{E}_K(s)^2} |\langle \mathcal{E}_K(s) J_s^{-1} K_2(X(s)), \eta\rangle|^2 ds \le \varepsilon\right) \\ &\le P\left(\int_0^{t\wedge\tau} |\langle \mathcal{E}_K(s) J_s^{-1} K_2(X(s)), \eta\rangle|^2 ds \le \varepsilon\right) \le P\left(\tau \wedge t \le \frac{2\varepsilon}{\delta}\right) \le \frac{C(p)\varepsilon^p}{t^p}. \end{aligned}$$

Hence,

$$P\left(\int_0^{t\wedge\tau} |\langle J_s^{-1} K_1(X(s)), \eta\rangle|^2 ds \le \varepsilon^r\right) \le \frac{C(p)\varepsilon^p}{t^p}.$$

By a similar but simpler arguments, (recalling $K_0 = q_k^i e_k$), we have

$$P\left(\int_0^{t\wedge\tau} |\langle J_s^{-1}(q_k^i e_k), \eta\rangle|^2 ds \le \varepsilon^{r^2}\right) \le \frac{C(p)\varepsilon^p}{t^p} \tag{6.10}$$

for all $p > 0$.

Hence, for any $\eta \in \mathcal{S}^\ell$, we have some $q_k^i e_k$ satisfying (6.10). Take the neighborhood $\mathcal{N}(\eta)$ small enough and $q = r^2$, by the continuity, we have (6.9) immediately. □

6.3. **Proof of some technical lemmas.** In this subsection, we need a key estimate as follows (see Lemma D.2 in [10]): For any $\gamma > 1/4$ with $\gamma \neq 3/4$, we have

$$|A^{\gamma-1/2} B(u,u)| \le C(\gamma) |A^\gamma u|^2 \ \text{ for any } u \in D(A^\gamma). \tag{6.11}$$

By (6.11) and Young's inequality, we have

$$\begin{aligned} |\langle A^\gamma u, A^\gamma B(u,v)\rangle| &\le |A^{\gamma+1/2} u| |A^{\gamma-1/2} B(u,v)| \\ &\le |A^{\gamma+1/2} u|^2 + C(\gamma) |A^\gamma u|^2 |A^\gamma v|^2 \end{aligned} \tag{6.12}$$

*Proof of Lemma 3.5.* We shall drop the index $m$ of quantities if no confusions arise. By Itô formula, we have

$$\begin{aligned} &d\left[|A^\gamma D_h X(t)|^2 \mathcal{E}_K(t)\right] + 2|A^{1/2+\gamma} D_h X(t)|^2 \mathcal{E}_K(t) \\ &+ 2\langle A^\gamma D_h X(t), A^\gamma \tilde{B}\left(D_h X(t), X(t)\right)\rangle \mathcal{E}_K(t) dt \\ &+ K |A^\gamma D_h X(t)|^2 |AX(t)|^2 \mathcal{E}_K(t) dt = 0. \end{aligned} \tag{6.13}$$

Recall that $\tilde{B}(D_h X(t), X(t)) = B(D_h X(t), X(t)) + B(X(t), D_h X(t))$. Thus, one has by (6.12)

$$
\begin{aligned}
&|A^\gamma D_h X(t)|^2 \mathcal{E}_K(t) + \int_0^t |A^{\gamma+1/2} D_h X(s)|^2 \mathcal{E}_K(s) ds \\
&\le |A^\gamma h|^2 + \int_0^t C |A^\gamma D_h X(s)|^2 |A^\gamma X(s)|^2 \mathcal{E}_K(s) ds \\
&\quad - K \int_0^t |A^\gamma D_h X(s)|^2 |A X(s)|^2 \mathcal{E}_K(s) ds
\end{aligned}
\tag{6.14}
$$

By Poincare inequality, we have $|Ax| \ge |A^\gamma x|$, and therefore as $K \ge C$,

$$
|A^\gamma D_h X(t)|^2 \mathcal{E}_K(t) + \int_0^t |A^{\gamma+1/2} D_h X(s)|^2 \mathcal{E}_K(s) ds \le |A^\gamma h|^2.
$$

As to (3.10) and (3.11), we only prove (3.10), similarly for the other. By an estimate similar to (6.14), (3.9) and (2.17) (noticing $D_{h^s} X^\ell(0) = 0$), we have

$$
\begin{aligned}
&|A^\gamma D_{h^s} X^\ell(t)|^2 \mathcal{E}_K(t) + \int_0^t |A^{\gamma+1/2} D_{h^s} X^\ell(s)|^2 \mathcal{E}_K(s) ds \le \\
&\le \int_0^t [C |A^\gamma D_{h^s} X(s)|^2 |A^\gamma X(s)|^2 - K |A^\gamma D_{h^s} X^\ell(s)|^2 |AX(s)|^2] \mathcal{E}_K(s) ds \\
&\le C \int_0^t \left[ |A^\gamma D_{h^s} X(s)|^2 \mathcal{E}_{\frac{K}{2}}(s) \right] \left[ |AX(s)|^2 \mathcal{E}_{\frac{K}{2}}(s) \right] ds \\
&\le C |A^\gamma h|^2 \int_0^t |AX(s)|^2 \mathcal{E}_{\frac{K}{2}}(s) ds \le \frac{2C}{K} |A^\gamma h|^2.
\end{aligned}
\tag{6.15}
$$

As to (3.12), by the classical interpolation inequality

$$
|A^r D_h X(s)|^2 \le |A^\gamma D_h X(s)|^{2(1-2(r-\gamma))} |A^{1/2+\gamma} D_h X(s)|^{4(r-\gamma)},
$$

and by Hölder's inequality and (3.9), we have

$$
\begin{aligned}
\int_0^t |A^r D_h X(s)|^2 \mathcal{E}_K(s) ds \le & \left[\int_0^t |A^{\gamma+1/2} D_h X(s)|^2 \mathcal{E}_{\frac{K}{2}}(s) ds\right]^{2(r-\gamma)} \left[\int_0^t |A^\gamma D_h X(s)|^2 \mathcal{E}_{\frac{K}{2}}(s) ds\right]^{1-2(r-\gamma)} \\
\le & C t^{1-2(r-\gamma)} |A^\gamma h|^2.
\end{aligned}
$$

(3.13) immediately follows from applying Itô formula to $|\mathcal{E}_K(t) \int_0^t \langle v, dW_s \rangle|^2$. □

*Proof of Lemma 4.2.* By (4.3) and the evolution equation governing $D_h X^\ell$, using the same method as proving (3.9), we immediately have (4.11) and (4.12). Recall that $J_t$ and $J_t^{-1}$ are both the dynamics in $\pi^\ell H$, thus the operator $J_t$ is bounded invertible. Let $C$ be some constant only depends on $n$ (see (3.3)), whose values can vary from line to line. By the fact $|A|_{\mathcal{L}(\pi^\ell H)} \le C$ and (4.4), for any $h \in \pi^\ell H$, we have by differentiating $|J_t^{-1}h|^2\mathcal{E}_K(t)$

(6.16)

$$\begin{aligned}
&|J_t^{-1}h|^2\mathcal{E}_K(t)+K\int_0^t |J_s^{-1}h|^2|AX(s)|^2\mathcal{E}_K(s)ds\\
&\le |h|^2+2\int_0^t |J_s^{-1}h||J_s^{-1}Ah|\mathcal{E}_K(s)ds\\
&\quad +C\int_0^t |J_s^{-1}h||J_s^{-1}A^{-\frac{1}{2}}|_{\mathcal{L}(\pi^\ell H)}\cdot|A^{1/2}B^\ell(h,X(s))|\mathcal{E}_K(s)ds\\
&\le |h|^2+C\int_0^t |J_s^{-1}|^2_{\mathcal{L}(H)}|h|^2\mathcal{E}_K(s)ds+C\int_0^t |J_s^{-1}|^2_{\mathcal{L}(H)}|AX(s)||h|^2\mathcal{E}_K(s)ds,
\end{aligned}$$

where the last inequality is by (6.11). Hence,

$$|J_t^{-1}|^2\mathcal{E}_K(t)+K\int_0^t |J_s^{-1}|^2_{\mathcal{L}(H)}|AX(s)|^2\mathcal{E}_K(s)ds\le 1+C\int_0^t |J_s^{-1}|^2_{\mathcal{L}(H)}(1+|AX(s)|^2)\mathcal{E}_K(s)ds,$$

as $K$ is sufficiently large, we have $|J_t^{-1}|^2_{\mathcal{L}(H)}\mathcal{E}_K(t)\le 1+C\int_0^t|J_s^{-1}|^2_{\mathcal{L}(H)}\mathcal{E}_K(s)ds$, which immediately implies (4.13).

To prove (4.14), by (4.4), we have

(6.17)

$$\begin{aligned}
\big|\mathcal{E}_K(t)J_t^{-1}h-h\big|&\le \int_0^t |J_s^{-1}Ah|\mathcal{E}_K(s)ds+\int_0^t |J_s^{-1}A^{-1/2}|_{\mathcal{L}(\pi^\ell H)}|A^{1/2}B^\ell(h,X(s))|\mathcal{E}_K(s)ds\\
&\le C\int_0^t |J_s^{-1}|_{\mathcal{L}(H)}|h|\mathcal{E}_K(s)ds+C\int_0^t |J_s^{-1}|_{\mathcal{L}(H)}|h||AX(s)|\mathcal{E}_K(s)ds,
\end{aligned}$$

thus, by (4.13) and (2.17),

$$\begin{aligned}
\big|\mathcal{E}_K(t)J_t^{-1}-Id\big|_{\mathcal{L}(H)}&\le C\int_0^t \mathcal{E}_K(s)|J_s^{-1}|_{\mathcal{L}(H)}ds+C\int_0^t \mathcal{E}_K(s)|J_s^{-1}|_{\mathcal{L}(H)}|AX(s)|ds\\
&\le Ct^{\frac{1}{2}}\left[\int_0^t \mathcal{E}_K(s)|J_s^{-1}|^2_{\mathcal{L}(H)}ds\right]^{\frac{1}{2}}+Ct^{\frac{1}{2}}\left[\int_0^t \mathcal{E}_K(s)|J_s^{-1}|^2_{\mathcal{L}(H)}\ \mathcal{E}_K(s)|AX(s)|^2ds\right]^{\frac{1}{2}}\\
&\le t^{1/2}Ce^{Ct}
\end{aligned}$$

where the last inequality is due to (4.13). As for (4.15), by Parseval's identity and (4.13),

(6.18)

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \mathcal{E}_K^2(s)|(J_s^{-1}Q^\ell)^*h|^2ds\right]=\sum_{k\in Z_\ell(n)}\sum_{i=1}^2\mathbb{E}\left[\int_0^t \mathcal{E}_{2K}(s)|\langle J_s^{-1}(q_k^ie_k),h\rangle|^2ds\right]\\
&\le \sum_{k\in Z_\ell(n)}\sum_{i=1}^2\mathbb{E}\left[\int_0^t \mathcal{E}_{2K}(s)|J_s^{-1}(q_k^ie_k)|^2ds\right]|h|^2\le tCe^{Ct}\sum_{k\in Z_\ell(n)\setminus Z_\ell(n_0)}\sum_{i=1}^2|q_k^ie_k|^2|h|^2.
\end{aligned}$$

By an estimate similar to (6.14), we have

$$\begin{aligned}
&|A\mathcal{D}_v X^\ell(t)|^2\mathcal{E}_K(t)+\int_0^t |A^{3/2}\mathcal{D}_v X^\ell(t)|^2\mathcal{E}_K(s)ds\\
&\le \frac{1}{2}\int_0^t |A\mathcal{D}_v X^\ell(s)|^2\mathcal{E}_K(s)ds+\frac{1}{2}\int_0^t |AQ^\ell v^\ell(s)|^2\mathcal{E}_K(s)ds\\
&\le \frac{1}{2}\int_0^t |A\mathcal{D}_v X^\ell(s)|^2\mathcal{E}_K(s)ds+C\int_0^t |v^\ell(s)|^2\mathcal{E}_K(s)ds
\end{aligned}$$

which implies (4.16) by Gronwall's inequality.

As to (4.17), write down the differential equation for $\mathcal{D}_v D_h X^\ell(t)$, and apply Itô formula, we have

$$\begin{aligned}
&|\mathcal{D}_v D_h X^\ell(t)|^2\mathcal{E}_K(t)+2\int_0^t |A^{1/2}\mathcal{D}_v D_h X^\ell(t)|^2\mathcal{E}_K(s)ds\\
\le &\int_0^t |\mathcal{D}_v D_h X^\ell(s)|\left(|\tilde{B}^\ell(\mathcal{D}_v D_h X^\ell(s),X(s))|+|\tilde{B}^\ell(D_h X^\ell(t),\mathcal{D}_v X(s))|\right)\mathcal{E}_K(s)ds\\
&-K\int_0^t |\mathcal{D}_v D_h X^\ell(s)|^2\mathcal{E}_K(s)ds.
\end{aligned}$$

By (6.11) and Young's inequality,

$$\begin{aligned}
&|\mathcal{D}_v D_h X^\ell(t)|^2\mathcal{E}_K(t)+\int_0^t |A^{1/2}\mathcal{D}_v D_h X^\ell(t)|^2\mathcal{E}_K(s)ds\\
\le &\int_0^t |\mathcal{D}_v D_h X^\ell(s)|^2\left(|A^{\frac{1}{2}}X(s)|^2+1\right)\mathcal{E}_K(s)ds+\int_0^t |A^{\frac{1}{2}}\mathcal{D}_v X^\ell(s)|^2|A^{\frac{1}{2}}D_h X^\ell(s)|^2\mathcal{E}_K(s)ds\\
&-K\int_0^t |\mathcal{D}_v D_h X^\ell(s)|^2|AX(s)|^2\mathcal{E}_K(s)ds,
\end{aligned}$$

as $K$ is sufficiently large, by Poincare inequality, $|A|_{\mathcal{L}(\pi^\ell H)}\le C$, (4.16) and (3.9), we have

$$\begin{aligned}
&|\mathcal{D}_v D_h X^\ell(t)|^2\mathcal{E}_K(t)+\int_0^t |A^{1/2}\mathcal{D}_v D_h X^\ell(t)|^2\mathcal{E}_K(s)ds\\
&\le \int_0^t |\mathcal{D}_v D_h X^\ell(s)|^2\mathcal{E}_K(s)ds+C\int_0^t |\mathcal{D}_v X^\ell(s)|^2\mathcal{E}_{K/2}(s)|D_h X^\ell(s)|^2\mathcal{E}_{K/2}(s)ds\\
&\le \int_0^t |\mathcal{D}_v D_h X^\ell(s)|^2\mathcal{E}_K(s)ds+C|h|^2\int_0^t e^{t-s}|v^\ell(s)|^2\mathcal{E}_{K/2}(s)ds.
\end{aligned}$$

By Gronwall's inequality, we obtain (4.17) immediately. Similarly, (4.18) can be obtained by

$$\begin{aligned}
&|\mathcal{D}^2_{v_1v_2}X^\ell(t)|^2\mathcal{E}_K(t)\\
&\le C\int_0^t [|A\mathcal{D}_{v_1}X^\ell(s)|^2\mathcal{E}_{K/2}(s)]\ [|AD_{v_2}X^\ell(s)|^2\mathcal{E}_{K/2}(s)]ds\\
&\le tCe^{Ct}\left[\int_0^t |v_1^\ell(s)|^2\mathcal{E}_{K/2}(s)ds\right]\left[\int_0^t |v_2^\ell(s)|^2\mathcal{E}_{K/2}(s)ds\right].
\end{aligned}$$

where the last inequality is due to (4.16). □

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Department of Applied Mathematics, Bonn University, Bonn, Germany

ENS Cachan Bretagne and IRMAR, Campus de Ker Lann, 35170 BRUZ, France
*E-mail address*: arnaud.debussche@bretagne.ens-cachan.fr

PO Box 513, EURANDOM, 5600 MB Eindhoven. The Netherlands
*E-mail address*: xu@eurandom.tue.nl