Towards a global view of dynamical systems, for the C1-topology.

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Preliminary version

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Dedicated to Jacob Palis, whose enlighting conjectures motivated this paper

Abstract

This paper suggests a program for getting a global view of the dynamics of diffeomorphisms, from the point of view of the C^1-topology. More precisely, given any compact manifold M, one splits $Diff^1(M)$ in disjoint C^1-open regions whose union is C^1-dense, and conjectures state that these open set, and their complement, are characterized by the presence of

- either a robust local phenomenon
- or a global structure forbiding this local phenomenon.

Other conjectures states that some of these regions are empty. This set of conjectures draws a global view of the dynamics, putting in evidence the coherence of the numerous recent results on C^1-generic dynamics.

1 Introduction

The diffeomorphisms whose dynamics is the simplest look like the gradient flow of a Morse function: all the orbits flow down to hyperbolic periodic orbits; the whole manifold splits in the basins of the periodic sinks; these basins are bounded by the stable manifolds of the saddle periodic orbits. The formalization by Smale of this simple behavior, called *Morse Smale* diffeomorphisms, defines an open subset of the space of all diffeomorphisms.

However, a simple robust phenomenon, discovered by Poincaré [Po] at the end of the 19th century, prevents this open set to be dense: Poincaré noticed that the invariant (stable and unstable) manifolds of a periodic saddle can meet transversely; such a transverse intersection point, called a *transverse homoclinic intersection*, forces a chaotic behavior of the dynamics. This phenomenon, studied in particular by Birkhoff, has been modelized by Smale's hyperbolic theory (see [Sm]), leading to a nice combinatorial description of the chaotic dynamics generated by homoclinic intersections. Then Smale defined a class of diffeomorphism whose global dynamical behavior looks similar to the Morse-Smale ones, just substituting the hyperbolic periodic orbits by hyperbolic basic sets. This class, called *Axiom A + no cycle* or *+strong transversality*[1], shortly named here *hyperbolic systems*, characterizes the stable (possibly chaotic) dynamical systems. Hyperbolic systems are now considered as well understood from the topological and ergodic points of view (see [Sh1],[PT, Chapter 0], [KH, Part 4] and [Bo]).

The hope that hyperbolic dynamics could describe *most of* dynamical systems (i.e. this open class could be dense in the space of all diffeomorphisms), was broken in 1968 when Abraham and Smale exhibit examples of C^1-robustly non-hyperbolic systems in dimension 4 (see [AS]). At the same time Newhouse [N, N_1] exhibited a C^2-open set of non-hyperbolic surface diffeomorphisms.

Then dynamicists concentrated most of the energy for understanding "non-hyperbolicity".

A first wave of results made impressive advances on critical one-dimensional behavior, non-uniform hyperbolicity, and homoclinic tangencies. In particular, the dynamics of almost all one-dimensional maps has now a nice, satisfactory description. Most of these results require some regularity, strictly more than C^1. Despite these results, a global view of dynamical systems in dimensions ≥ 2 for the C^r-topologies, $r > 1$, seems beyond reach in a close future .

[1] The Axiom A + strong transversality diffeomorphisms are the diffeomorphisms which are *structurally stable* on the whole manifold; the Axiom A + no cycle diffeomorphisms are structurally stable in restriction to the non-wandering set.

At the same time, other kind of results was developed by Mañé and Liao for ending the hyperbolic theory, proving the stability conjecture (stability implies hyperbolicity) for diffeomorphisms (see for instance [Ma_2, Li]). This kind of tools, typical for the C^1-topology, started with Pugh's C^1-closing lemma and with Franks' derivative perturbation lemma. Some extra difficulties for proving the stability conjecture for flows lead Hayashi [Ha] to improve strongly the closing lemma, allowing to connect orbits by C^1-small perturbations. This has been the starting point of a new wave of results, explorating the space of dynamical systems from the point of view of the C^1-topology. This exploration leads to two kinds of results:

- some results present *robust* or *locally generic* phenomena, that is some specific dynamical behavior satisfied either on a C^1-open subset of diffeomorphisms, or on a residual subset of it. For instance [Sh, Ma_1, BD_1, BV] present open sets of robustly transitive diffeomorphism, and [BLY] present an open set where C^1-generic diffeomorphisms have no attractors nor repellers;

- other results announce that every diffeomorphisms satisfying (or avoiding) some robust phenomenon admits some global structure. For instance, [Ma, DPU, BDP] prove that every robustly transistive diffeomorphim admits a dominated splitting, and [Cr_2] proves the existence of a partially hyperbolic structure, with at most two 1-dimensional central bundles for dynamical systems *far from homoclinic bifurcations*.

The great variety of these robust phenomena may appear as disorganized, and some times leading to opposite directions. It is not easy to find a coherence in all the possible robust behaviors.

Palis [Pa] proposed several conjectures giving goals for the study; for instance:

- (weak density conjecture): density, in the open set of diffeomorphisms far from the Morse-Smale, of those presenting a transverse homoclinic intersections.

- density, in the open set of diffeomorphisms far from the hyperbolic dynamics, of those presenting a homoclinic bifurcations (homoclinic tangencies and heterodimensional cycles) (see Conjecture 3).

- density of diffeomorphisms having finitely many attractors, whose basins cover a dense open set of the manifold (see Conjecture 18).

Stated in any C^r-topologies, some of these conjectures have been (at least partially) succesfull in the C^1-topology: in this C^1 setting, the weak density conjecture as been proved in [PS, BGW, Cr_3] and [Cr_2, CP] proved an essential part of Conjecture 3. These conjectures start an organization of the C^1-space of diffeomorphisms or vector-fields, and lead to a wish of a global panorama. However, Palis conjectures and the known results are not yet enough for drawing the global picture. The aim of this paper is to complete the panorama by proposing conjectures. We will use 3 dichotomies, splitting the space $Diff^1(M)$ in 8 disjoint regions (C^1-open subsets) whose union is C^1-dense:

- being far from tangencies / being robustly approachable by diffeomorphisms displaying homoclinic tangencies;

- being far from heterodimensional cycles / being robustly approachable by diffeomorphisms displaying heterodimensional cycles;

- being *tame*, i.e., having robustly finitely many *chain recurrence classes* whose number is locally constant / being *wild* i.e. far from tame diffeomorphisms.

We know that:

- the region *tame diffeomorphisms far from tangencies and from heterodimensional cycles* is the region of hyperbolic systems.

- the region *tame diffeomorphisms far from heterodimensional cycles but approachable by tangencies* is empty;

Conjectures by Palis and myself state that 3 other regions are empty (the regions *wild diffeomorphisms far from homoclinic tangencies* or/and *far from heterodimensional cycles*). Examples prove that the 4 remaining regions are non-empty. This leads to a panorama with 4 non-trivial regions having an increasing complexity of the dynamical behavior:

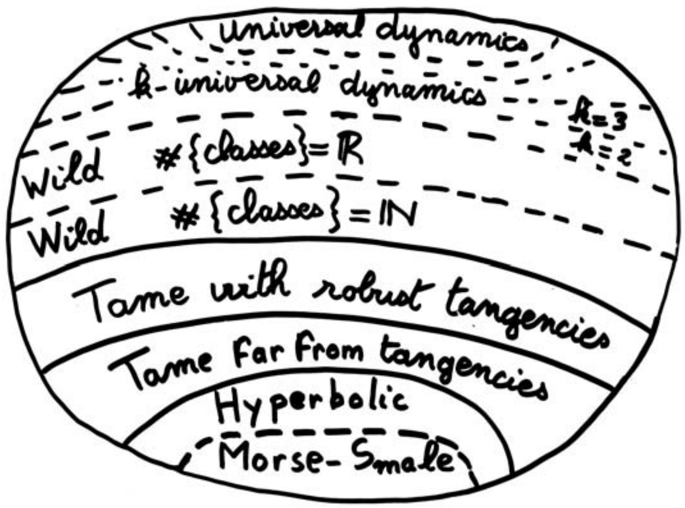

Figure 1: Conjectural map of $Diff^1(M)$

- the hyperbolic systems;
- the tame systems far from tangencies;
- the tame systems with robust tangencies;
- the wild systems.

I will try to present the dynamical behavior in each of this regions. The region corresponding to the wild systems is certainly the less understood: I will try to organize it by proposing a characterization and by splitting this region in natural sub-regions, see Picture 1.

Such a global panorama seems to be hopeless for the C^r-topology, $r > 1$. It is tempting to ask if the global picture of dynamical systems for the C^1-topology can be useful for the C^r-dynamics (allowing to combine the technics of both fields). One important difficulty is that C^r-generic dynamical systems are nowhere C^{r+1}. For instance, the global dynamics of C^0-generic homeomorphisms is mostly understood[2], see [AHK], but their dynamics is very far from what we expect from C^1-generic diffeomorphisms. For this reason, the panorama I am proposing is based on robust properties (and not on generic properties): the space $Diff^1(M)$ is splitted in C^1-open regions, whose union is C^1 dense. C^1-open regions are C^r-open regions for any $r \geq 1$, so that the regions we define certainly make sense in any topology. However, the C^1-density does not imply the C^r-density, and some C^r-robust phenomena are not seen by this C^1 classification. Thus, certainly this panorama makes sense in any C^r-topology, $r \geq 1$: it is just incomplete, forgetting phenomena which require an extra regularity. For instance, the famous C^2-robust tangencies for surface diffeomorphisms, proved by Newhouse, is announced to be non-existing in the C^1-topology.

In this paper I am proposing many conjectures. I do not strongly believe that each of them is true: in my opinion, they are important problems and any answer would have a strong influence on our global view of C^1-dynamical systems. I always proposed the answer leading to the simplest panorama. A negative answer to some of these conjectures will just add some new region, ensuring that some surprising behavior appear in some open set of diffeomorphisms.

[2] C^0-generic homeomorphisms have an uncountable family of quasi attractors whose union of the basins cover a residual subset of the manifold; furthermore, in dimension ≥ 2, generic orbits tends to quasi attractors which are adding machines; however the basin of each quasi attractor is nowhere dense.

Thanks

This paper comes from the discussions I had with Sylvain Crovisier, Lorenzo Díaz and Enrique Pujals. Certainly each of them could have wrote this paper, with a slightly different point of view. Thank you for all our exchanges, and for your comments on a previous version of this paper.

The questions and comments, after my talks, by Marguerite Flexor, motivate me for looking for the coherence between the results, and how the new results fit with the previous ones. Lan Wen convinced me that such a paper could be useful for the community. Thanks a lot.

The International Workshop on Global Dynamics Beyond Uniform Hyperbolicity in Beijing, August 2009, gave me the opportunity to present a preliminary version of this program.

2 Some definitions

This paper can certainly not be read without some background on the topological qualitative study of dynamical systems. In this section, we just recall some classical definitions which are of constant use in this paper.

We consider a diffeomorphism f of a closed manifold M.

2.1 Topological dynamics

Conley's theory [Co] defines the *chain recurrent set* $\mathcal{R}(f)$: a point x is *chain recurrent* if there are periodic ε-pseudo-orbits through x, for every $\varepsilon > 0$.

The chain recurrent set admits a natural partition in compact invariant sets called *chain recurrence classes*: two points $x, y \in \mathcal{R}(f)$ belong to the same class if one can go from x to y and from y to x by ε-pseudo orbits, for every $\varepsilon > 0$. More generally, a set K is *chain transitive* if one can go from any point of K to any point of K by ε-pseudo orbits contained in K, for $\varepsilon > 0$.

A chain recurrence class C is a *(topological) attractor* if it admits an open neighborhood U such that the closure $\overline{f(U)}$ is contained in U and $C = \bigcap_{n\in\mathbb{N}} f^n(U)$.

According to Hurley,[Hu], a chain recurrence class C is a *quasi attractor* if it admits a basis U_n of open neighborhoods such that $\overline{f(U_n)} \subset U_n$.

An *attracting region* is some set U such that $\overline{f(U)}$ is contained in the interior of U. A *repelling region* is an attracting region of f^{-1}. A *filtrating region* is the intersection $U \cap V$ where U and V are attracting and repelling regions, respectively. Conley's theory shows that every chain recurrence class admits a basis of neighborhoods which are filtrating regions.

2.2 Intersections of invariant manifolds.

If x is a hyperbolic periodic point, the *homoclinic class* $H(x, f)$ is the closure of the transverse intersections points of the invariant (stable and unstable) manifolds of the orbit of x. It is a transitive invariant compact set canonically associated to x, and contained in the chain recurrence class of x. Two hyperbolic periodic points x and y are *homoclinically related* if the stable manifold of each of them cuts transversally the unstable manifold of the orbit of the other point. This defines an equivalence relation on the set of hyperbolic perriodic points of f, and the homoclinic class $H(x, f)$ is the closure of the class of x for this relation.

A *homoclinic tangency* is a non-transverse intersection point of the stable and the unstable manifolds of a hyperbolic periodic orbit.

The *stable index* of a hyperbolic periodic point x is the dimension of its stable manifold. A *heterodimensional cycle* consists in two hyperbolic periodic points x, y with different stable indices, an intersection point p of the stable manifold of the orbit of x with the unstable manifold of the orbit of y, and an intersection point q of the stable manifold of the orbit of y with the unstable manifold of the orbit of x.

2.3 Dominated splittings

Let K be an invariant compact set. A *dominated splitting* on K is a splitting

$$T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_k(x)$$

of the tangent space over the points $x \in K$ such that

- the dimensions $dim(E_i(x))$ do not depend on $x \in K$;
- the subbundles E_i are Df-invariant: $E_i(f(x)) = Df(E_i(x))$
- there is $n > 0$ such that, for every $x \in K$, for every $1 \leq i < j \leq k$ and every unit vector $u \in E_i(x)$, $v \in E_j(x)$ one has $\|Df^n(u)\| < 2\|Df^n(v)\|$, where $\| \dots \|$ denotes an Riemannian metrics on M.

In that case, we denote

$$T_K M = E_1 \oplus_< E_2 \oplus_< \cdots \oplus_< E_k.$$

A subbundle E_i of a dominated splitting is called *uniformely contracting* (resp. *expanding*) if there is $n > 0$ such that the norm of the restriction $Df^n|_{E_i}(x)$ (resp.$Df^{-n}|_{E_i}(x)$) is less than $\frac{1}{2}$ for every $x \in K$. A subbundle E_i is called *hyperbolic* if it is either uniformely contracting or uniformely expanding. The non-hyperbolic subbundles of a dominated splitting are called *central bundles*.

A dominated splitting is *partially hyperbolic* if at least one of the subbundles E_1 or E_k is hyperbolic. A dominated splitting is *hyperbolic* if it consists in two hyperbolic bundles, one being contracting and the other being expanding.

A *hyperbolic set* is either a compact invariant set admiting a hyperbolic splitting, or a hyperbolic periodic source or sink. A *hyperbolic basic set* is a transitive invariant compact set which is hyperbolic, and which is the maximal invariant set in some neighborhood.

3 C^1-perturbation lemmas

The main tools for understanding a dense subset of the space of dynamical systems are perturbation lemmas: given a system, one tries to perturb it in order to create a specific dynamical phenomenon. The progresses for a global view of dynamical systems for the C^1-topology comes from C^1-perturbation lemmas, which are either wrong or unknown for the C^r-topologies, for $r > 1$.

3.1 Why the C^1-topology?

What is the specificity of the C^1-topology?: it is invariant by rescaling. If you imagine some ε-perturbation creating some phenomenon, and you put this perturbation in a very small ball or radius δ, it remains an ε-perturbation for the C^1-topology. In other words, let us denote by H_δ the homothety of ratio δ; now, given an ε-C^1-perturbation g of the identity map, supported on the unit disc, then $H_\delta \circ g \circ H_\delta^{-1}$ is an ε-C^1-perturbation of the identity map supported on the ball of radius δ. In contrast, the C^r-size of this perturbation will be of the order $\frac{\varepsilon}{\delta^{r-1}}$, for $r > 1$. That is, for $r > 1$, the C^r-topologies are more and more rigid when the scale tends to 0.

The Lipschitz topology shares this invariance by rescaling with the C^1-topology. As we will use generic properties, we need to consider Baire spaces: hence the Lipschitz topology is natural on the set of bi-Lipschitz homeomorphisms, when the C^1-topology is natural on C^1-diffeomorphisms. Now, C^1-diffeomorphisms are more and more simple when one consider smaller balls: they look like their differential map which is a bounded linear map, with bounded inverse. In contrast, Lipschitz homeomorphisms can present the same complexity on small scale than in large scale.

Hence, C^1-perturbations at microscopic scale look like perturbations of linear maps. This principle is the key point of most of the C^1-perturbation lemmas, starting with Pugh's closing lemma.

3.2 Approaching the global dynamics by periodic orbits

Lemmas of C^1-perturbations of the orbits, as Pugh closing lemma [Pu], Mañé's ergodic closing lemma [Ma] and Hayashi's connecting lemma [Ha] and their generalizations [Ar_1, BC], allowed us to show that the dynamics of C^1-generic diffeomorphisms (or flows) is very well approached by the periodic orbits:

- the chain recurrent set is the closure of the set of periodic orbits ([BC]). More generally, every *chain transitive set*[3] is the Hausdorff limit of periodic orbits ([Cr]).

[3]An invariant compact set K is *chain transitive* if one can go from any $x \in K$ to any $y \in K$ by pseudo orbits in K with arbitrarily small jumps

- a chain recurrence class containing a periodic orbit is the homoclinic class of the periodic orbit [BC];
- every ergodic measure is the Hausdorff and weak limit of periodic measures ([Ma]).

As C^1-generically, the global dynamics is very well approached by periodic orbits, it looks natural to hope that the chaotic behavior would be reflected on the set of periodic orbits. Answering a Conjecture by Palis, [PS] (in dimension 2), [BGW] (in dimension 3), and [Cr_3] in any dimensions proved:

Theorem 3.1. *Consider the disjoint open subsets $\mathcal{MS}(M)$ and $\mathcal{H}omocline(M)$ of $Diff^1(M)$ defined by:*

- *$f \in \mathcal{MS}$ if f is Morse Smale, and*
- *$f \in \mathcal{H}omocline(M)$ if f has a transverse homoclinic intersection associated to a hyperbolic periodic orbit.*

Then the union $\mathcal{MS}(M) \cup \mathcal{H}omocline(M)$ is dense in $Diff^1(M)$.

This result is typical of our philosophy: it splits $Diff^1(M)$ in two disjoint open subsets whose union is dense, one of them corresponding to a global structure (Morse Smale) the other to a robust local phenomenon (homoclinic intersections). It provides a characterization of chaotic dynamics, algorithmically checkable: being Morse Smale, and having a transverse homoclinic intersection, are both checkable.

Consider now dynamical systems far from hyperbolic dynamics: $f \in Diff^1(M) \setminus \overline{\{\text{Axiom A + no cycle}\}}$. As, C^1-generically, the global dynamic is very well approached by periodic orbits, this lack of hyperbolicity is reflected by a lack of hyperbolicity on the periodic orbits (this are important ideas due to Mañé and Liao in the 70-80ies).

Next conjecture (first formulated in dimension 2 in [ABCD]) is typical from our approach of the global dynamics by considering the periodic orbits: it states that the robust non hyperbolicity is due to the robust non-hyperbolicity of the class of a hyperbolic periodic orbit.

Conjecture 1. *There is a dense open subset in $Diff^1(M) \setminus \overline{\{\textit{Axiom A + no cycle}\}}$ of diffeomorphisms having a hyperbolic periodic point p_f whose chain recurrence class is robustly non hyperbolic: the chain recurrence class $C(p_g)$ is not hyperbolic for every g in a C^1-neighborhood of f.*

This conjecture remains open in any dimension ≥ 2. In dimension 2, this conjecture remains one of the main difficulties for proving Smale's conjecture (the density of Axiom A diffeomorphisms on surfaces, see Conjecture 5)[4]. More precisely, [ABCD] splits Smale's conjecture in 3 conjectures on C^1-generic surface diffeomorphisms, one of them being Conjecture 1.

Remark 3.1. *If this conjecture is false, then there is an open set $\mathcal{U}$ of $Diff^1(M)$ such that every C^1-generic diffeomorphism $f \in \mathcal{U}$ is* wild hyperbolic *in the following sense:*

- *every homoclinic class is a hyperbolic basic set (in particular is isolated)*
- *there are infinitely many homoclinic classes, accumulating on aperiodic classes*

This would contradict the philosophy which is behind most of the approaches in non-hyperbolic dynamical systems: looking at the local phenomena (in general related to 1 or finitely many periodic orbits) generating rich global behaviors.

We denote by $\mathcal{W}_{Hyp}(M)$ the maximal open set of $Diff^1(M)$ in which the generic diffeomorphisms are wild hyperbolic, that is: $\mathcal{W}_{Hyp}$ is the interior of the closure of the set of wild hyperbolic diffeomorphisms. Conjecture 1 states that

$$\mathcal{W}_{Hyp}(M) = \emptyset.$$

Assuming Conjecture 1, one needs to focus the study on non-hyperbolic homoclinic classes. Next conjecture states that everything happening in a homoclinic class of a generic diffeomorphism can be seen on periodic orbits in the class

[4]This conjecture is known as *Smale's conjecture*, even if we did not find any reference where Smale stated it

Conjecture 2. *There is a residual subset $\mathcal{R} \subset Diff^1(M)$ such that, for every $f \in \mathcal{R}$ for every periodic point p and for every ergodic measure μ supported on the homoclinic class $H(p)$, there is a sequence $\gamma_n \subset H(p)$ of periodic orbits such that the Dirac measure μ_n along γ_n tends weakly to μ.*

A simple argument in [ABC] shows that C^1-generically, the closure of the periodic measures supported in a homoclinic class is convex. Hence, Conjecture 2 implies that every measure supported in $H(p)$ is limit of periodic measures in $H(p)$.

This conjecture could appear as a simple corollary of Mañé's ergodic closing lemma, if one forgets that the periodic orbits γ_n are asked to belong to the homoclinic class $H(p)$. The ergodic closing lemma provides periodic orbits approaching the ergodic measure; the unique difficulty is to keep these periodic orbits in the class. A positive answer to this conjecture would provide a positive answer to many of the next conjectures stated in this program. In fact, many of our previous works would have been much simpler if we knew a positive answer to this conjecture.

3.3 Lack of hyperbolicity and bifurcations

Mañé proved [Ma] that diffeomorphisms, whose periodic orbits are robustly hyperbolic, satisfy the Axiom A and the no-cycle condition. In other words, the robust lack of hyperbolicity leads to weakly hyperbolic periodic orbits. Now, C^1-perturbation lemmas of the local dynamics in a neighborhood of periodic orbits, through Franks lemma [F], relate the lack of hyperbolicity and dominated splittings with the bifurcations associated with periodic orbits.

More precisely, there are two ways for loosing the uniform hyperbolicity on the set of periodic orbits:

- either one looses the uniform exponential contraction/expansion at the period: there are periodic orbits having Lyapunov exponents arbitrarily close to 0.
- or one looses the uniform domination of the stable/unstable splitting along the orbits: they are arbitrarily large time intervals where the expansion in the unstable direction is not twice the expansion in the stable direction.

These two phenomena lead to two different kinds of bifurcations:

- in the first case, up to a small perturbation, one direction changes from contracting to expanding or the contrary ([Ma]): in other words, one may perform a saddle node or a flip bifurcation. If this phenomenon happens persistently in some open region of $Diff^1(M)$ then one has the *coexistence of different indices*, and it is natural to expect that this leads to *hetero-dimensional cycles*;
- in the second case, up to a small perturbation, the stable and unstable directions make a very small angle: this leads to *homoclinic tangencies* ([PS, W_2, Go]).

This suggested the following conjecture, formulated by J. Palis in any C^r-topology, $r \geq 1$, but with many progresses mostly in the C^1-topology:

Conjecture 3 (Palis density conjecture)**.** *There is a dense open subset $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ of $Diff^1(M)$ such that $f \in \mathcal{O}_1$ statisfies the Axiom A without cycle, and there is a dense subset $\mathcal{D} \subset \mathcal{O}_2$ such that $f \in \mathcal{D}$ admits a heterodimensional cycle or a homoclinic tangency.*

This conjecture has been proved for surface diffeomorphisms by Pujals and Sambarino in [PS]. In higher dimensions, let me give two important recent results in direction of Palis density conjecture (see also [W_3] for a previous local step for Crovisier's global result):

Theorem 3.2 (Crovisier [Cr_2])**.** *Let $\mathcal{C}$ be the closure of the set of diffeomorphisms having a heterodimensional cycle or a homoclinic tangency. There is an open an dense subset $\mathcal{O}_{PH} \subset Diff^1(M) \setminus \mathcal{C}$ such that for every $f \in \mathcal{O}_{PH}$, every non-hyperbolic chain recurrence class admits a partially hyperbolic splitting $E^s \oplus_< E^c \oplus_< E^u$ or $E^s \oplus_< E^c_1 \oplus_< E^c_2 \oplus_< E^u$, where every non hyperbolic subbundle is 1-dimensional.*

This result is an important step for the proof, by Crovisier and Pujals, of the C^1-density conjecture for the attractors and repellers:

Theorem 3.3. *[CP] There is a residual subset $\mathcal{O}_{\mathcal{A}} \subset Diff^1(M) \setminus \mathcal{C}$ such that every $f \in \mathcal{O}_{\mathcal{A}}$ has finitely many hyperbolic attractors whose basins cover a dense open subset of M.*

(It remains an open question to know if, in Theorem 3.3, one can replace *residual subset* by *dense open subset*; in other words, we don't know if some small (hyperbolic) attractors can appear (by C^1-small perturbations) nearby the non-attracting classes).

However, Kupka-Smale theorem (see [Ku, Sm$_2$]) implies that, for C^r generic diffeomorphisms, the periodic orbits are hyperbolic and the stable and unstable manifolds are all transverse; hence f has no heterodimensional cycles nor homoclinic tangencies. So, far from hyperbolic systems, small perturbations can destroy all the cycles or the tangencies but, according to Palis conjecture, new perturbations can build new cycles or tangencies.

In my mind, this means that the heterodimensional cycles and the homoclinic tangencies are not responsible for the robust non-hyperbolicity, but are consequences of the robust lack of hyperbolicity. For characterizing the non-hyperbolicity, one would like to find *C^1-robust phenomena generating local density of homoclinic tangencies and/or heterodimensional cycles*.

4 Robust cycles and tangencies

4.1 Definitions of robust cycles and robust tangencies

From Abraham-Smale [AS] and Simon [Si] examples, one knows the existence of C^1-robust cycles relating hyperbolic basic sets of different indices, in dimension ≥ 3:

Definition 4.1. *Let $\mathcal{U}$ be a C^1-open set of diffeomorphisms f having hyperbolic basic sets K_f and L_f, varying continuously with $f \in \mathcal{U}$, such that the indices (dimension of the stable bundle) are different, and such that $W^s(K_f) \cap W^u(L_f) \neq \emptyset$ and $W^u(K_f) \cap W^s(L_f) \neq \emptyset$, for every $f \in \mathcal{U}$.*

Then we say that f has a C^1-robust cycle *associated to K_f and L_f.*

If $f \in \mathcal{U}$ has a robust cycle associated to K_f and L_f and if $p_f \in K_f$ and $q_f \in L_f$ are hyperbolic periodic points (of different indices), then C^∞-densely in $\mathcal{U}$, f performs an heterodimensional cycle associated to p_f and q_f. Assume for instance that $dimE^s(q) < dim(E^s(p))$ so that $dimE^s(p) + dimE^u(q) > dimM$. Then for an open and dense subset of $\mathcal{U}$, $W^s(p)$ cuts transversely $W^u(q)$ at some point. Now, small perturbations allow $W^u(p_f)$ to cross quasi-transversely the stable manifold of every point in in L_f; in particular, densely in $\mathcal{U}$, $W^u(p_f)$ will cut quasi-transversely $W^s(q_f)$, performing a heterodimensional cycle relating p_f and q_f. In other words, *the robust cycle associated to K_f and L_f induces the local density in $\mathcal{U}$ of heterodimensional cycles associated to the periodic points p_f and q_f.*

One defines robust tangencies in the same way :

Definition 4.2. *Let $\mathcal{U}$ be a C^1-open set of diffeomorphisms f having a hyperbolic basic set K_f varying continuously with $f \in \mathcal{U}$, such that $W^s(K_f) \cap W^u(K_f)$ contains a non-transverse intersection point for every $f \in \mathcal{U}$. Then we say that f has a C^1-*robust tangency *associated to K_f.*

Once again, robust tangencies associated to a hyperbolic basic set K_f , $f \in \mathcal{U}$, lead to a dense subset of $\mathcal{U}$ with a homoclinic tangency associated to p_f, where p_f is any periodic point in K_f.

4.2 Existence of robust cycles and robust tangencies

In 68, [AS] built the first example of a C^1-open set of non-Axiom A diffeomorphisms, on a 4-manifold. Then in 72,[Si] built an example in dimension 3. These examples consisted in building a robust heterodimensional cycle or a robust tangency.

More precisely, consider a hyperbolic attractor Λ (for instance a Plykin attractor) on a disk D; consider now the disk D embedded in a higher dimensional manifold, as an invariant, normally expanding submanifold; the Plykin attractor Λ is now a saddle hyperbolic set of the higher dimensional diffeomorphisms. The disk D is the stable manifold of Λ and is foliated by the 1-dimensional stable manifolds of the points in Λ; assume now that the unstable manifold of Λ cuts transversally the disk D: the intersection with the stable foliation in D may be robustly non-transverse (see Picture 2). In other words there is a robust tangency associated to Λ, as pointed out by by Newhouse and more recently by Asaoka [As].

An analogous construction (using an extra periodic point P of stable index equal to 2) leads to a robust cycle (see Picture 3).

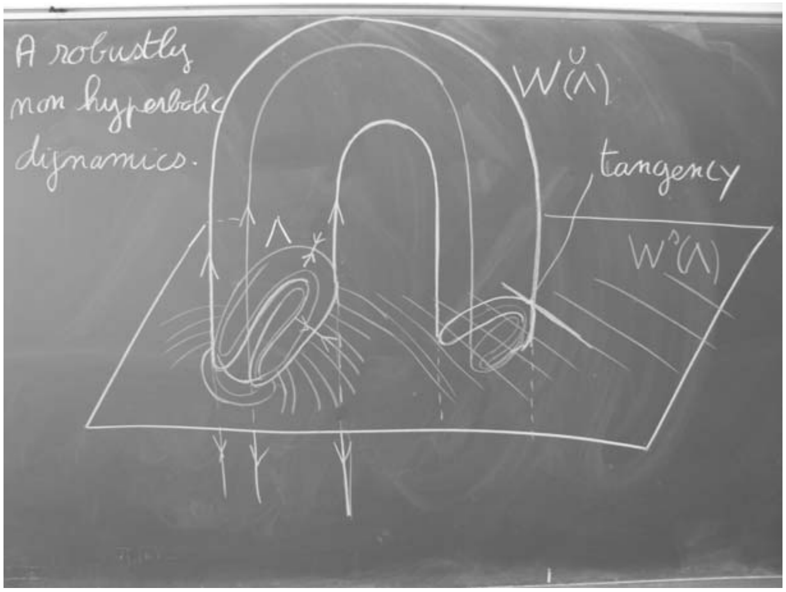

Figure 2: A robustly non-hyperbolic diffeomorphism with robust tangency

4.3 Stabilizing cycle and tangencies?

Conjecture 1 states that generic non-hyperbolic systems should contain robustly non-hyperbolic homoclinic classes. Next conjecture propose a characterization of non-hyperbolic homoclinic classes by the existence of a robust local phenomenon:

Conjecture 4 ([ABCD])**.** *For every C^1-generic diffeomorphism f, if $H(p)$ is a non-hyperbolic homoclinic class then it contains a robust cycle or a robust tangency.*

Remark 4.3. *Let us forget for a moment the C^1-topology: in 1974, [N_3] built a C^2-open set of surface diffeomorphisms having a C^2-robust tangency associated to a hyperbolic basic set Λ, assuming that Λ is* thick*: the product of its stable thickness by its unstable thickness is larger than* 1*. Furthermore, he proved that every homoclinic tangency associated to a periodic point p generates, by performing the bifurcation, a thick hyperbolic set related to p and having a homoclinic tangency: so every tangency can be turn C^2-robust (see [N_3])! Newhouse result holds for C^r-topology, $r > 1$, and there are generalizations in special cases in higher dimension (see [PV]) still for the C^r-topology, $r > 1$.*

We would like to generalize Newhouse result for the C^1-topology: Palis conjecture would give an explanation of the non-hyperbolic dynamics if it was possible to turn robust every heterodimensional cycle and homoclinic tangency.

4.4 From cycle to robust cycles

The stabilization process is essentially done for heterodimensional cycles, using some kind of thick hyperbolic sets, called *blenders*, which have been introduced in [BD_1].

Theorem 4.1. *[BD_4] If f is a diffeomorphism admitting a heterodimensional cycle associated to periodic points p,q with $ind(p) - ind(q) = 1$, then there is g arbitrarily C^1-close to f having a robust cycle.*

In most of the cases, one may ensure that the robust cycle is associated to the continuation of p and q (see [BDK]) but there is precisely one configuration where we could build counterexample (see [BD_5]).

Corollary 4.4. *For C^1-generic diffeomorphisms every hyperbolic periodic point p whose homoclinic class contains a periodic point q with $ind(p) \neq ind(q)$ belongs to a robust cycle.*

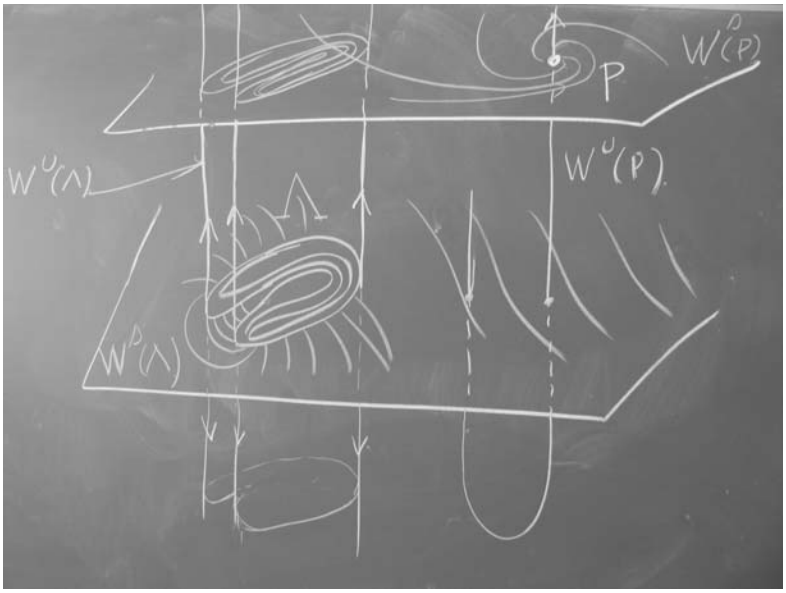

Figure 3: A robustly non-hyperbolic diffeomorphism with a robust cycle

4.5 No robust tangency in dimension 2

Is it possible to turn robust a homoclinic tangency? Answering to a question of mine, Moreira [Mo] recently proved

Theorem 4.2. *given two* 1*-dimensional dynamically defined Cantor-sets* K_f *and* K_g*, generated by expansive Markovian maps* f *and* g*, there are* C^1 *small perturbations* $\tilde{f}, \tilde{g}$ *of* f *and* g *such that* $K_{\tilde{f}}$ *and* $K_{\tilde{g}}$ *are disjoint.*

A simple argument shows that this result implies

Corollary 4.5. *Let* S *be a closed surface. In* $Diff^1(S)$ *there are no* C^1*-robust tangency associated to a hyperbolic basic set.*

More precisely: given an open subset $\mathcal{O}$ of $Diff^1(M)$ where there is a hyperbolic basic set K_f varying continuously with f. Let $W^s_\ell(K_f)$ and $W^u_\ell(K_f)$ denote the local stable and unstable manifolds of length ℓ of K_f, that is, the union of the segments of length 2ℓ of stable or unstable manifolds centered at the points of K_f. Then Moreira's results implies:

Theorem 4.3. *There is an open and dense subset of* $\mathcal{O}$ *where* $W^s_\ell(K_f)$ *is transverse to* $W^u_\ell(K_f)$*.*

Remark 4.6. *I think it is possible to prove that every transitive hyperbolic set of a surface diffeomorphism is contained in a hyperbolic basic set*[5]*. Hence, Moreira's result would imply the non existence of robust tangency associated to hyperbolic transitive sets of surface diffeomorphisms.*

This is an important step in direction of Smale's conjecture

Conjecture 5 (Smale)**.** *The Axiom A + no cycle diffeomorphisms are dense in* $Diff^1(S)$*.*

According to [ABCD] it remains 2 difficulties for proving Smale conjecture.

- one is Conjecture 1: diffeomorphisms having a C^1-robustly non-hyperbolic homoclinic class are dense far from Axiom A + no cycle
- the second is Conjecture 4: non-hyperbolic homoclinic class of C^1-generic diffeomorphisms contains robust cycle or tangencies.

[5]Todd Fisher recently announced a proof of this statement

4.6 Robust tangencies in higher dimension: from tangencies with cycles to robust tangencies

[BD_4] shows that one can turn robust any homoclinic tangency which occurs on a period point of a robust heterodimensional cycle. More precisely, in [BD_4] we define a special type of blenders (i.e. of thick hyperbolic set) called *blender horseshoe*, and we consider submanifolds in special position with respect to this blender horseshoe, called *folds*. Then we show:

Theorem 4.4. *The local stable manifold of a horseshoe blender presents (C^1-robust) tangency with any fold.*

One may think that blender horseshoes are a very special dynamical objects but, in fact, it is very common: we prove that, for C^1-generic diffeomorphisms, every homoclinic class containing periodic points with different indices contains a blender horseshoe. This allows us to prove that robust tangency are a very common phenomenon on non-hyperbolic homoclinic classes:

Theorem 4.5. *There is a residual subset $\mathcal{R} \subset Diff^1(M)$ such that if $f \in \mathcal{R}$ and p is a periodic point of f such that:*

- *the homoclinic class $H(p)$ contains a periodic point of different index as p, and*
- *the stable/unstable splitting over the periodic point homoclinically related with p is not dominated,*

then p belongs to a hyperbolic basic set having a robust tangency.

4.7 Far from cycles: no tangencies?

Robust (or persistent) tangencies associated to a periodic point p leads to accumulations of periodic orbits of a different index in a neighborhood of the homoclinic class of p: every homoclinic tangency associated to p generates periodic orbits having a complex eigenvalues corresponding to the weakest stable and unstable eigenvalues of p. Hence it is natural to espect that robust tangencies lead to heterodimensional cycles and to robust cycles.

Conjecture 6. *Let $\mathcal{U}$ being a C^1-open set of diffeomorphisms f having a hyperbolic basic set K_f varying continuously with f and presenting a robust tangency. Then there is a C^1-dense open subset $\mathcal{U}_1$ of $\mathcal{U}$ such that for f in $\mathcal{U}_1$ there is a hyperbolic basic set L_f of different index as K_f and such that (K_f, L_f) present a robust cycle.*

In dimension 2, there are no robust cycles, so this conjecture means that there are no robust tangency, which is Theorem 4.3. In higher dimension (with Crovisier, Diaz and Gourmelon [BCDG]) we have very partial results in this direction, as the following:

Theorem 4.6. *Given P a hyperbolic periodic saddle with index $2 \le i \le dimM - 2$. Assume that there is no nominated splitting on $H(P)$ neither of index $i-1$ nor of index i nor of index $i+1$.*

Then there are arbitrarily small perturbations of f creating heterodimensional cycle between P_g and a point R_g of index $i-1$ or $i+1$.

(The hypothesis *$H(p)$ has no dominated splitting of index i* is equivalent to *one can create a homoclinic tangency associated to p, by small C^1-perturbation* according to [W_2, Go]).

Conjecture 6 would be an important step in the following conjecture, which generalizes Palis density conjecture:

Conjecture 7 ([BD_4])**.** *The union of the disjoint C^1-open sets of diffeomorphisms $\mathcal{H} \cup \mathcal{RC}$, where $\mathcal{H}$ is the set of Axiom A + no cycle diffeomorphisms and $\mathcal{RC}$ is the set of diffeomorphism presenting a robust cycle, is dense in $Diff^1(M)$.*

Remark 4.7. *This conjecture provides a characterization of the non hyperbolicity which would be checkable by computers: being Axiom A + no-cycle, and having a robust cycle, are both algorithmically checkable.*

As a preliminary step for proving Conjecture 6, let me propose a conjecture generalizing Theorem 4.3:

Conjecture 8. *Let K be an index $dim(M)-1$ hyperbolic basic set of a diffeomorphism $f\colon M \to M$. Assume furthermore that K is sectionally dissipative: for every $x \in K$, and every 2-plane $P \subset T_xM$ the determinant of the restriction of D_xf to P is less than 1:*

$$|\det(D_xf)|_P| < 1.$$

Then there is no robust tangency associated to K.

The proof of that conjecture could consists, as in Moreira's theorem, to separate a dynamical Cantor set in $\mathbb{R}^{dimM-1}$ from the product by $\mathbb{R}^{dimM-2}$ of a 1-dimensional dynamical Cantor set.

Remark 4.8. *Theorem 4.4 implies that there are codimension* 1 *hyperbolic basic sets with robust tangency; hence, the hypothesis* sectionally dissipative *is essential in Conjecture 8. For instance, in [BCDG] we prove that, for C^1-generic diffeomorphisms, every homoclinic class without dominated splittings either contains a robust cycle and a robust tangency, or there is $n \neq 0$ such that f^n is sectionnaly dissipative on the class.*

It is easy to see that Conjecture 8 is wrong if K is not a Cantor set, and Theorem 4.4 also implies that it is wrong if K is a blender horseshoe. So the first step for Conjecture 8 consists in proving:

Problem 1. *Let K be a sectionally dissipative index $dim(M)-1$ hyperbolic basic set of a diffeomorphism $f\colon M \to M$. Prove that K is a Cantor set and is not a blender.*

Sylvain Crovisier gave me a simple argument which seems to solve this problem, proving that the Hausdorff dimension of a sectionally dissipative Cantor set is strictly less than one. That is a good starting point for this generalization Gugu's result.

5 Tame and wild dynamics

5.1 Definitions

A chain recurrence class C of f is called *robustly isolated* if there are a neighborhood U of C and a C^1-neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ has a unique chain recurrence class contained in U.

By an argument of genericity (using Pugh closing lemma), the fact that periodic orbits can be turned hyperbolic, and Conley theory, one can show show (see for instance [Ab, BC]):

Theorem 5.1. *There is a residual subset $\mathcal{R}$ of $Diff^1(M)$ such that for $f \in \mathcal{R}$, every isolated chain recurrence class is robustly isolated (and is a homoclinic class).*

This leads to the natural notion:

Definition 5.1. *A diffeomorphism f is* tame *if every chain recurrence class is robustly isolated.*

One denotes by $\mathcal{T}(M)$ the set of tame diffeomorphisms. It is a C^1 open set containing Axiom A + no cycle. A tame diffeomorphism has finitely many chain recurrence classes, and this number is locally constant.

A diffeomorphism f is *wild* if it is far from tame diffeomorphisms, i.e. if f belongs to the interior of the complement of $\mathcal{T}(M)$. One denotes

$$\mathcal{W}(M) = Diff^1(M) \setminus \overline{\mathcal{T}(M)}$$

the set of *wild diffeomorphisms.* Then, C^1-generic wild diffeomorphisms have infinitely many chain recurrence classes and infinitely many homoclinic classes.

5.2 Existence of tame diffeomorphisms

The first examples of non-hyperbolic tame diffeomorphisms was the *robustly transitive* diffeomorphisms built by Shub in [Sh] on T^4 and by Mañé [Ma_1] on T^3. Then [BD_1] built examples on many manifolds, for example those admitting a transitive Anosov flow, or the product $T^n \times N$ of an torus T^n by an arbitrary compact manifold N. All this examples are partially hyperbolic, with non-trivial strong stable

and strong unstable bundles. For Shub and Mañé examples, the central bundle is 1-dimensional. For [BD_1] examples, the central bundle may have an arbitrary dimension.

In [Ca], Carvalho built the first example of a robustly transitive attractor which is not the whole manifold, and which has no strong stable direction. As it was not the aim of her construction she did not emphasize this fact, which has been noticed later. The complete proof of the robust transitivity of Carvalho's attractor can be found in [BV] which also build the first examples of robustly transitive diffeomorphisms without any hyperbolic invariant subbundles.

A short presentation of the proof of most of these examples may be found in [BDV, Chapter 7.1].

5.3 Tame diffeomorphisms and dominated splittings

The relation between tame dynamics and dominated splitting started with [Ma] where Mañé proves that robustly transitive surface diffeomorphisms are Anosov diffeomorphisms. Then [DPU] proved that robustly hyperbolic diffeomorphism in dimension 3 are partially hyperbolic with at least 1 non trivial hyperbolic stable or unstable bundle; I built some of the non-published examples for correcting a first version of [DPU] which announced two non trivial bundles. [Ma, DPU] and this example may be considered as the starting point of the theory relating tame dynamics and dominated splitting.

This link has been explicited in [BDP] in any dimensions, proving the existence of a dominated splitting, where the extremal bundles are uniformly volume contracting/expanding (we called *volume hyperbolic* this structure).

Remark 5.2. *Even if nobody built all possible examples, it is clear that , for any sequences of integers $n_i > 0, i \in \{0, \cdots, k\}, k \geq 1$, there are robustly transitive diffeomorphisms whose finest dominated $E_0 \oplus_< \cdots \oplus_< E_k$ statisfies*

- *$dim(E_i) = n_i$,*
- *the bundles E_i are not hyperbolic for $i \notin \{0, k\}$,*
- *each of E_0 and E_k is not hyperbolic if its dimension is not* 1.

The different possible types of dominated splitting on a robustly isolated class C is related with the tangencies you can perform in C, and C^1-generically, is related with the robust tangencies contained in C. The following corollary of Theorem 4.5 is a good illustration of this principle:

Corollary 5.3. *There is an open an dense subset $\mathcal{O} \subset \mathcal{T}(M)$ such that, for every $f \in \mathcal{O}$ and every chain recurrence class C of f one has*

- *either there is a partially hyperbolic splitting on C*

$$TM|_C = E^s \oplus_< E_1 \oplus_< \cdots \oplus_< E_k \oplus_< E^u$$

  where E^s is uniformly contracting, E^u is uniformly expanding and $dim(E_i) = 1$.

- *or C contains a hyperbolic basic set having a robust tangency; furthermore C contains a robust heterodimensional cycle.*

Many of the conjectures are known on the set of tame diffeomorphisms. The reason is that the chain recurrence classes are robustly isolated, by definition, so that anything one can produce by perturbation remains in the same class. For example Conjecture 2 holds on $\mathcal{T}(M)$. Another example: there is an open and dense subset of $\mathcal{T}(M)$ for which every non-hyperbolic class contains a robust cycle : in particular, Conjecture 7 and Conjecture 3 hold on $\mathcal{T}(M)$.

5.4 Existence of wild diffeomorphisms

[BD_2] built the first C^1-open set of wild diffeomorphisms, proving that $\mathcal{W}(M)$ is not empty for every compact manifold M with $dim(M) \geq 3$. Then [BD_3] used a similar construction for exhibiting an open set of wild diffeomorphisms where C^1-generic diffeomorphisms present an uncountable family of *aperiodic classes*, that is, chain recurrence classes without periodic orbit.

Both constructions consists in using robust cycles relating periodic points of different indices having complex central eigenvalues: the complex eigenvalues prevent the existence of dominated splitting, so

that one gets homoclinic classes which are robustly without dominated splitting, and these homoclinic classes generates new classes by small perturbations.

The C^1-generic diffeomorphisms in the open set built in [BD$_3$] display an extra property called *universal dynamics*: they present infinitely many disjoint periodic disks on which the first return maps, up to a rescaling, define a dense subset of the set of embedding of the unit disk into its interior. As a consequence, they display simultaneously and infinitely many times any robust phenomenon you can build in the disk, and also any generic phenomenon, in an arbitrary countable list of generic phenomena. This property will be important in Section 7 where I try to organize the different types of wild dynamical behaviors.

5.5 Wild dynamics and wild homoclinic classes

My feeling is that wild dynamics are produced by a homoclinic class which generates new homoclinic classes nearby by perturbations. That is, once again, the wild behavior can be seen on the set of periodic orbits; or better said, *the wild behavior is generated by a robust local phenomenon related to periodic orbits*. This may be expressed by the following conjecture:

Conjecture 9. *There is a dense open subset $\mathcal{O}$ of $\mathcal{W}(M)$ of diffeomorphism f having a hyperbolic periodic point p_f varying continuously with f, and such that for C^1-generic $f \in \mathcal{O}$ the homoclinic class $H(p_f, f)$ is not isolated.*

This leads to the notion of *wild homoclinic class*: One says that the homoclinic class $H(p_f, f)$ is a *wild homoclinic class* if it is *locally generically non isolated*, that is, if for C^1-generic g close to f the class $H(p_g, g)$ is not isolated.

As, for C^1-generic diffeomorphisms, isolated classes are robustly isolated and the number of homoclinic classes is countable, one proves easily:

Lemma 5.4. *There is a residual subset $\mathcal{R} \subset Diff^1(M)$ such that, for $f \in \mathcal{R}$, every homoclinic class $H(p_f, f)$ which is not isolated is a wild homoclinic class.*

So Conjecture 9 may be restated as:
Conjecture 9. *There is a residual subset $\mathcal{R} \subset Diff^1(M)$ such that every $f \in \mathcal{R} \cap \mathcal{W}(M)$ has a wild homoclinic class.*

Remark 5.5. *If this conjecture is wrong, then there is a non-empty open subset $\mathcal{U} \subset Diff^1(M)$ such that, for every C^1-generic diffeomorphisms $f \in \mathcal{U}$ one has:*

- *every homoclinic class is a robustly isolated class*
- *there are sequences of homoclinic classes accumulating on aperiodic classes.*

I will denote by $\mathcal{W}_{Tame}$ the interior of the closure of the set of diffeomorphisms whose homoclinic classes are robustly isolated but which have infinitely many homoclinic classes, and I will call *wild tame diffeomorphisms* the diffeomorphisms $f \in \mathcal{W}_{Tame}$. Conjecture 9 states that

$$\mathcal{W}_{Tame} = \emptyset.$$

5.6 Wild homoclinic classes and robust tangencies

All the known examples of wild dynamical systems are based on critical behavior (homoclinic tangency or lack of dominated splitting). All my attempts for building wild non-critical dynamics failed, motivating the following conjecture:

Conjecture 10. *If $\mathcal{U}$ is an open set where p_f is a periodic point varying continuously with $f \in \mathcal{U}$ and $H(p_f, f)$ is a wild homoclinic class (that is, if $H(p_f, f)$ is not isolated for C^1-generic $f \in \mathcal{U}$), then there is a dense open subset of $\mathcal{U}$ where $H(p_f, f)$ contains a robust tangency.*

An easier step for proving this conjecture is the next conjecture (first expressed at UMALCA Cancun (2004))

Conjecture 11. *1. (weak version) There is a residual subset $\mathcal{R}$ of $Diff^1(M)$ such that, for every $f \in \mathcal{R}$, every chain recurrence class admitting a partially hyperbolic splitting*

$$E^{ss} \oplus_< E^c \oplus_< E^{uu},$$

where $dim E^c = 1$, is isolated.

2. (strong version) There is a residual subset $\mathcal{R}$ of $Diff^1(M)$ such that, for every $f \in \mathcal{R}$, every chain recurrence class admitting a dominated splitting

$$E^{ss} \oplus_< E^c_1 \oplus_< \cdots \oplus_< E^c_k \oplus_< E^{uu},$$

where $dim E^c_i = 1$, is isolated.

This conjecture expresses that, if a dominated splitting forbids homoclinic tangencies, then the local dynamics is tame.

However, Conjecture 10 is far to provide a characterization of wild dynamics: there are examples of tame dymanics which present robust homoclinic tangencies. On the other hand, there are robust local phenomena (with robust tangencies) which are known to imply a wild dynamics; but we don't know if every wild diffeomorphism presents one of these mechanisms.

Problem 2. *Find a characterization of tame diffeomorphisms, and of wild diffeomorphisms.*

It would be interesting to state a conjecture, characterizing tame and wild diffeomorphisms by using either a local phenomenon or a global structure. At this time, I am not able to propose such a conjecture.

6 Splitting $Diff^1(M)$ in 8 open regions

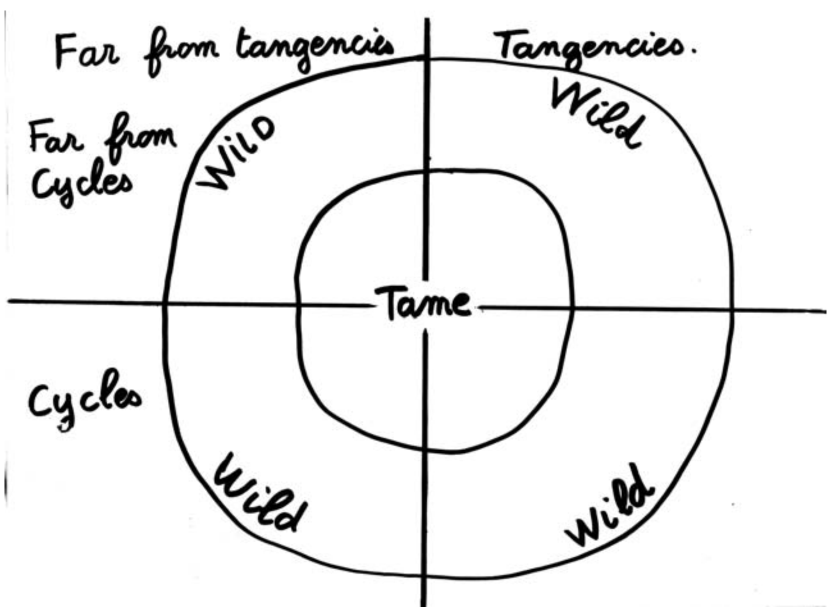

Figure 4: Splitting $Diff^1(M)$ in 8 open regions

6.1 3 dichotomies

We consider 3 criteria; each of them splits $Diff^1(M)$ in a pair of disjoint open subsets whose union is dense (see Figure 4):

- being robustly approximated by heterodimensional cycles, or being far from heterodimensional cycles. More precisely, we consider the closure $\mathcal{C}$ of the set of diffeomorphisms having a heterodimensional cycle. Then the interior of $\mathcal{C}$ and the complement of $\mathcal{C}$ are two disjoint open sets, whose union is dense
- being robustly approximated by homoclinic tangency, or being far from homoclinic tangency. This defines two disjoint open sets, whose union is dense.
- being wild or tame.

These 3 pairs of disjoint open subsets define by intersections 8 disjoint open regions whose union is dense in $Diff^1(M)$.

6.2 Known results and conjectures on the 8 regions

1. Tame diffeomorphisms far from homoclinic tangency and heterodimensional cycle are Axiom A + no cycle.
2. $$\{\text{Tame diffeomorphisms with tangency but far from cycles}\} = \emptyset$$
3. There are examples of tame diffeomorphisms far from tangencies but with robust cycles.
4. There are examples of tame diffeomorphisms with robust cycles and robust tangencies.
5. Palis density conjecture means that
   $$\{\text{wild diffeomorphisms far from tangencies and cycles}\} = \emptyset.$$
   [CP] proved an essential part of this conjecture.
6. Conjecture 7 is already known on tame diffeomorphisms. Hence, this conjecture means that
   $$\{\text{wild diffeomorphisms far from cycles}\} = \emptyset.$$
   There are very few results on diffeomorphisms far from cycles. Theorems 4.3 and 4.6 are partial results in the direction *far from cycles* $\Rightarrow$ *far from cycles and tangencies*.
7. Conjectures 9 and 10 mean that
   $$\{\text{wild diffeomorphisms far from tangencies}\} = \emptyset.$$
   The strong relation linking dominated splittings and dynamics far from tangencies explains that there are more results on the region *far from tangencies*; see for instance [Y], who proved that quasi attractors of diffeomorphisms far from tangencies are homoclinic classes, and [BGLY] who deduces that these quasi-attractors are mutually isolated, hence that they attracts generic points in a neighborhood. Another example: recently, S. Crovisier, M. Sambarino and D. Yang proved another conjecture of mine (stated at UMALCA 2004), announcing that, far from tangencies, every chain recurrence class admits a partially hyperbolic dominated splitting
   $$E^{ss} \oplus_< E_1^c \oplus_< \cdots \oplus_< E_k^c \oplus_< E^{uu}$$
   where E^{ss} and E^{uu} are non trivial (if the class is not a hyperbolic periodic sink or source) and hyperbolic and $dim E_i^c = 1$.
8. There are examples of wild diffeomorphisms, using wild homoclinic classes having robust cycles and robust tangencies.

Summarizing, one splits $Diff^1(M)$ in 4 non-empty open regions whose union would be dense, if Conjectures 3, 7, 9 and 6 are verified (see Figure 5):

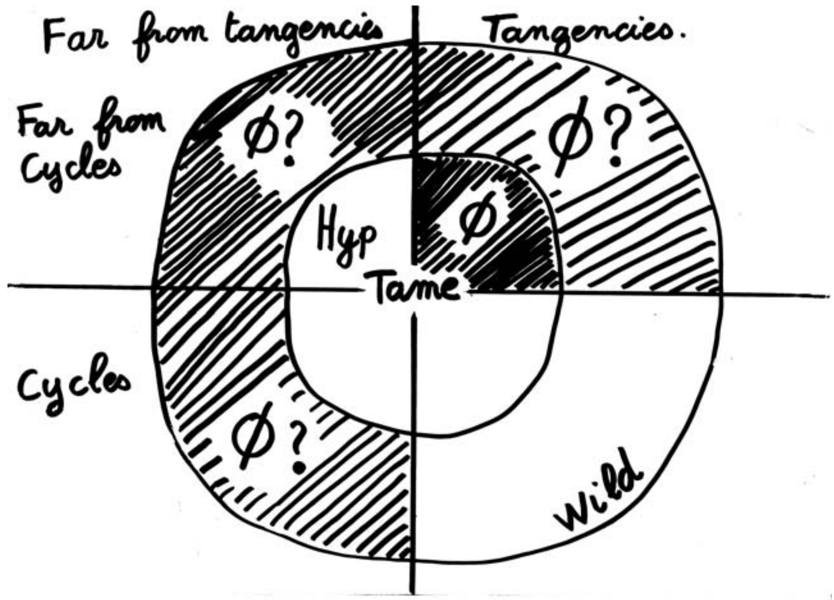

Figure 5: Four non-empty regions

6.2.1 The Axiom A + no Cycles

They admits now a very complete description

6.2.2 The non-hyperbolic tame diffeomorphisms far from tangencies

For every diffeomorphism in a dense open subset of this region, every class admits a partially hyperbolic dominated splitting

$$E^{ss} \oplus_< E_1^c \oplus_< \cdots \oplus_< E_k^c \oplus_< E^{uu}$$

where E^{ss} and E^{uu} are non trivial (if the class is not a hyperbolic periodic sink or source) and hyperbolic and $dimE_i^c = 1$.

Notice that this property is the reciprocal statement of Conjecture 11 (strong version): hence Conjecture 11 provides a characterization of tame diffeomorphisms far from tangency.

It remains many open questions on the topological dynamics of tame diffeomorphisms, even far from tangencies. For instance,(until a first version of this paper) we din't know if robustly isolated class are (at least generically) robustly transitive:

Problem 3. *Is it true that there is a dense open subset of the set of tame diffeomorphisms far from tangency, such that every chain recurrence class is robustly transitive? and is robustly a homoclinic class?*

One of the interest of writing such a survey it that you try to solve each problem before proposing it to the community: after a first version of this paper, we (with Crovisier, Gourmelon and Potrie) found an example of a non-empty C^1-open set $\mathcal{U}$ of diffeomorphisms f having a hyperbolic periodic point p_f (depending continuously on f) whose chain recurrence classe $C(p_f, f)$ admits a partial hyperbolic splitting with 1-dimensional central bundle, and such that:

- for every $f \in \mathcal{U}$, the class $C(p_f, f)$ is isolated
- for f in a dense subset in $\mathcal{U}$, the chain recurrence class $C(p_f, f)$ is not transitive (and so is different from the homoclinic class $H(p_f, f)$).

So Problem 3 admits a negative answer. However, Problem 3 remains open for attracting or repelling classes, or for classes which are the whole manifold.

An intermediary step would be a geometrical and dynamical characterization of robustly chain recurrent diffeomorphisms, using blenders, suggested by E. Pujals in the early 2000. Let me explain somewhat this characterization for a robustly chain recurrent partially hyperbolic diffeomorphism f, with a 1-dimensional central bundle. For such C^1-generic f, the class C contains a blender K for f and L for f^{-1}; each of them has a *characteristic region*. Pujals criterium is that *every strong unstable (reps. stable) manifold cuts the characteristic region of K (resp.of L).* Let call $\mathcal{P}_{blenders}$ this property. This property is open in the set of partially hyperbolic diffeomorphisms.

The conjecture is:

Conjecture 12. *(suggested by Pujals) Let $\mathcal{CT}(M)$ denote the set of* robustly chain transitive diffeomorphisms *on M, that is, the interior of the set of diffeomorphisms for which M is a chain recurrence class. Let $\mathcal{PH}(M)$ denote the open set of diffeomorphisms admitting a global partially hyperbolic structure with 1-dimensional central bundle.*

Given a closed 3-manifold M, there is a C^1-open and dense subset of $\mathcal{CT}(M) \cap \mathcal{PH}(M)$ of diffeomorphisms statifying property $\mathcal{P}_{blenders}$.

In that partially hyperbolic setting with 1-dimensional central bundle, arguments of Dìaz and Rocha should prove that the partially hyperbolic robustly chain transitive diffeomorphisms are indeed robustly transitive, giving a (partial) positive answer to Problem 3 for the robustly chain transitive diffeomorphisms. Abdenur and Crovisier recently announced progresses in the direction of Conjecture 12.

We know now that the $\mathcal{P}_{blenders}$ criterium does not holds on the isolated classes which are not the whole manifold. I guess that a better understanding of the example we built will lead to an adaptation of property $\mathcal{P}_{blenders}$ for proposing a characterization tame diffeomorphisms, far from tangencies.

This class of diffeomorphisms seems to be ready for deeper studies, as the existence and finiteness of SRB measures or of measures maximizing the entropie (...), in the C^r setting[6] ....

6.2.3 The non-hyperbolic tame diffeomorphisms with robust tangencies.

The tame diffeomorphisms admitting homoclinic tangencies is less understood. Conley theory ([Co]) provides a structure of the global dynamics similar to the structure of hyperbolic dynamics given by Smale's spectral decomposition theorem: there are finitely many classes separated by a filtration. There are finitely many (robust) attractors whose basins cover (at least C^1-generically) a dense open subset of the manifold.

Conjecture 13. *There is an open and dense subset of $\mathcal{T}(M)$ of diffeomorphisms for which the basins of the attracting classes cover a dense open subset of M.*

(This conjecture is not even proved for tame diffeomorphisms far from tangencies).

The study of the dynamics in a class C depends strongly on the type of the finest dominated splittings $E_1 \oplus_< \cdots \oplus_< E_\ell$ carried by C:

- The *stable indices* (i.e.dimension of the stable manifold) of periodic orbits in C form an interval of integers whose lower bound is given the first non-hyperbolic bundle E_i, and by the determinant of the differential on the subspaces contained in E_i[7].
- The robust cycles inside C just depend on the interval of indices.
- There are robust tangencies inside C associated to any periodic orbit whose index α is not a sum $\sum_{j=0}^{i} dimE_j$.
- There are periodic orbits in C having complex eigenvalues associated to any α which is not a sum $\sum_{j=0}^{i} dimE_j$.
- There is a deeper influence of the finest dominated splitting on the dynamics. Let say that the restriction of f to C has a *k-universal dynamics* if there are normally hyperbolic invariant k-disks D_n such that the first return maps in the D_i induces, after rescaling, a C^1-dense subset of the

[6]There are many results in these directions, for instance by Abdenur, Buzzi, Díaz, Fisher, Pacifico, Pujals, Sambarino, Vieitez (...)

[7]The unique obstruction for creating in C a periodic point of index $\sum_{j=1}^{i-1} dimE_j + r, 0 < r \leq dimE_i$ is that the determinant of Df on r-planes in E_i is uniformly expanding (this is essentially due to [BDP] and explicited in [BoBo]).

diffeomorphisms of the unit disk $\mathbb{D}^k$ (see Section 7.2 which is devoted to that notion). Then C has the k-universal dynamics if one bundle E_i with dimension $dim(E_i) \geq k$ is such that the volume in E_i is neither uniformly expanded no contracted. There are less restrictive conditions, so that having a 2-universal dynamics is very common for classes whose finest dominated splitting admits central bundles with dimension > 1. This illustrates the complexity of the topological dynamics inside these region of tame diffeomorphisms with tangencies.

- I guess that one may adapt property $\mathcal{P}_{blenders}$ for characterizing the tame property in the case where the smallest an the largest stable indices correspond to dominated splittings; in other word, when the class C is far from tangencies associated to periodic orbits of the extremal indices: this property provides strong stable an strong unstable foliations which are necessary for the formulation of property $\mathcal{P}_{blenders}$.

  However, there are tame diffeomorphisms without strong stable and strong unstable direction; in that setting, I have no candidate for a characterization of the tameness.

In my opinion, one knows too few examples. Despite the lack of examples, it is not easy to find the common argument in the proofs of the tameness of the known examples. Some of them uses some blender argument, other strongly uses the proximity to Anosov maps out of very small balls. A deeper understanding of examples, like Carvalho's attractor or Bonatti-Viana and Bonatti-Díaz robustly transitive diffeomorphisms could be very helpful.

Building new examples would be very important for proposing a characterization: I think that the time 1 map of the Lorenz attractor admits small perturbations producing a C^1-robustly transitive attractor for diffeomorphisms, different from the previous examples (some years ago a thought I have a proof of this fact, but I never wrote it).

Nevertheless, in some case, the ergodic study has been successfully performed (see [Ca, ABV, BV, HHTU]).

6.2.4 The wild diffeomorphisms.

It is certainly the less understood region. We just know some examples. In next section I try to split $\mathcal{W}(M)$ in disjoint open regions in order to organize the space of wild dynamics.

7 Wild dynamics

7.1 Examples from [BD$_3$]: universal and k-universal dynamics

In [BD$_3$] we build a robust local mechanism, which is a homoclinic class, contained in a ball and isotopic to the identity, which generates, by C^1-small perturbations, periodic orbits whose derivative is the identity. Hence a new small perturbation generates small periodic discs on which the return map is the identity. Then a new perturbation may produce the same local mechanisms inside these discs, and so on. These has two consequences:

- as the identity map generates by perturbation any orientation preserving diffeomorphisms on the disk, generic diffeomorphisms in that open set have a *universal dynamics*: renormalizing the dynamics in the (disjoint) periodic discs, one gets a dense subset of $Diff_0^1(\mathbb{D}^n)$.

- our phenomenon is a wild homoclinic class $H(p)$ which generates by perturbation the same kind of wild homoclinic classes outside $H(p)$: this leads to an infinite tree of classes whose ends correspond to an uncountable set of classes. As the homoclinic classes are countably many, these locally generic diffeomorphisms have *aperiodic classes*

One can generalize this process, in dimensions lower than $dimM$. If a homoclinic class produces, by perturbation, isolated periodic orbits having $k \geq 3$ eigenvalues arbitrarily close to 1, then by a new small perturbation one gets:

- a *filtrating region* U (i.e. the intersection of an attracting region with a repelling region) such that

- the maximal invariant set in U is contained in a normally hyperbolic k-disk D;

- the restriction of the dynamics to D is universal.

The filtrating neighborhood U isolates the dynamics in D from the remaining of the dynamics: the chain recurrence classes in restriction to D are classes of the global dynamics contained in D. So, once again, this property leads, locally generically, to uncountably many aperiodic classes.

7.2 Formal definition of k-universal dynamics

Let $\widetilde{Diff^1}(\mathbb{D}^k)$ denote the set of diffeomorphisms $\varphi\colon \mathbb{D}^k \to Int(\mathbb{D}^k)$, mapping the unit disk $\mathbb{D}^k$ of $\mathbb{R}^k$ on a compact set contained in the interior of $\mathbb{D}^k$. We denote by $\widetilde{Diff^1}_0(\mathbb{D}^k)$ the subset of those which are isotopic to the homothety $x \mapsto \frac{1}{2}x$.

Definition 7.1. *1. We say that a diffeomorphism f has the k-universal dynamics if for every open set $\mathcal{O} \in \widetilde{Diff^1}_0(\mathbb{D}^k)$, there is $\pi \in \mathbb{N}$ and a subdisc $D \subset M$ of dimension k and period π such that*

- *the restriction $f^\pi|_D$ is smoothly conjugated to a diffeomorphism $\varphi \in \mathcal{O}$*
- *D is normally hyperbolic*

2. *We say that f* has the free k-universal dynamics *or* is freely k-universal *if furthermore there is a filtrating neighborhood $U \subset M$ of D such that the maximal invariant set of f in U is contained in the f-orbit of the maximal invariant set of f^π in D.*

In Definition 7.1 if $k = dimM$ then the k-universal property is the universal property and is always freely universal. If $k < dimM$ the k-universal property can be hidden in a homoclinic class: there are robustly transitive diffeomorphisms having the k-universal property, for $k \le dimM - 2$. In contrasts, the k-freely universal dynamics implies that the global dynamics splits in infinitely many filtrating regions.

Proposition 7.2. *1. For every $k \in \mathbb{N}^*$, the (free or not) $k+1$-universal dynamics implies the (free or not) k-universal dynamics.*

2. *The (free or not) k-universal dynamics is a G_δ-property*

3. *if a C^1-generic diffeomorphism f has the free k-universal property, with $k \ge 3$, then it admits a pair (U, D) (where D is the normally hyperbolic k-disc and U the filtrating set U in the definition) such that the restriction $f^\pi|_D$ is k-universal.*

4. *if a C^1-generic diffeomorphism f has the free k-universal property, with $k \ge 3$, then it has uncountably many chain recurrence classes and aperiodic classes (contained in the orbit of a normally hyperbolic k-disc D in a filtrating set).*

Proof : Using the fact that $Diff^1(\mathbb{D}^k)$ admits a countable basis of the topology, that the disk in the definition are normally hyperbolic hence persist by perturbation, that filtrating sets persist by perturbation, one get the G_δ property.

Item 3 comes from the fact that there is a robust local mechanism in dimension $k \ge 3$, implying the k-universal property. Any C^1-generic diffeomorphism having the free k-universal property admits a pair (U, D) where the restriction of f to D presents this local mechanisms: hence, the restriction of f to D is k-universal. For $k = 2$, we don't know if such a mechanism exists: for this reason, the 2-universal property cannot be localized on one specific disk.

The item 3) implies the item 4), as in [BD$_3$]. □

We don't know if there are C^1-generic surface diffeomorphisms φ with the (free) 2-universal dynamics: this would contradict Smale conjecture. For this reason item 3 and 4 requires that $k \ge 3$.

Remark 7.3. *For any $k \ge 3$, [BD$_3$] provides a local robust mechanism which is a homoclinic class which generates, C^1-locally generically, a k-universal dynamics. As this mechanisms is robust, any diffeomorphism f displaying a free k-universal dynamics has the following properties:*

- *f admits a filtrating set U*
- *the maximal invariant set in U is contained in an invariant k-dimensional normally hyperbolic disk D,*
- *the disk D contains a homoclinic class which is the robust local mechanism described in [BD$_3$].*

This provides a checkable characterization of the (free or not) k-universal dynamics.

As a consequence of Remark 7.3 one gets:

Corollary 7.4. *If f has the free k-universal dynamics for $k \geq 3$, then there a C^1-neighborhood $\mathcal{U}_f$ of f in which generic diffeomorphisms have the free k-universal dynamics.*

One denotes by $\mathcal{W}_{k-univ}$ the maximal open set in which the free k-universal dynamics is generic, but not the free $k+1$-universal dynamics. In other words, let $\mathcal{U}_k$ denote the set of diffeomorphisms with the free k-universal dynamics. Then

$$\mathcal{W}_{k-univ} = Int(\overline{\mathcal{U}_k}) \setminus \overline{\mathcal{U}_{k+1}}.$$

7.3 Newhouse phenomenon

Newhouse built C^2-open sets of diffeomorphisms where a homoclinic class presents a robust tangency. The enfolding of this tangencies generates attracting disks, on which the diffeomorphisms present a robust tangency. As in the case of universal dynamics, this leads to uncountably many aperiodic classes (see [BDV, Chapter 3.2.1]. However, if the Jacobian is uniformly less than one, these C^2-robust tangencies cannot lead to 2-universal dynamics.

Even if Newhouse phenomenon is out of our subject (as it uses the C^2-topology) it clearly indicates that wild dynamics cannot be reduced to universal or free k-universal dynamics. This leads to the notion of self-replicating classes explained in next section.

7.4 Self replicating properties

The examples in [BD_3] and Newhouse phenomenon lead to the notion of *self-replicating* or shortly *viral* property. Let me explain this notion first on the chain recurrence class $C(p)$ where p is a hyperbolic periodic point. I will explain at the end of the section how one can generalize this notion to other classes of invariant compact sets.

Unformally, it consists in a robust property $\mathcal{P}$ such that every class $C(p)$ with property $\mathcal{P}$ admits perturbations creating new classes $C(q)$ with property $\mathcal{P}$ outside $C(p)$ but arbitrarily close to $C(p)$. However, this informal definition look like a meta-mathematic property. For this reason I am proposing a very formal definition below.

Let $\mathcal{K}(M)$ be the space of compact subset of M endowed with the Hausdorff topology.

A *local property* $\mathcal{P}$ of a diffeomorphism $f\colon M \to M$ on an invariant set K is a subset of triple (M, f, K) which is invariant by local C^1-conjugacy in a neighborhood of K: if (M, f, K) and (N, g, L) are two triples, U and V are neighborhoods of K and L and if $\varphi\colon U \to V$ is a diffeomorphisms inducing a local conjugacy between f to g, then (M, f, K) satisfies $\mathcal{P}$ if and only if (N, g, L) statisfies $\mathcal{P}$.

A local property $\mathcal{P}$ defined on the set of chain recurrence classes of hyperbolic periodic orbits is *robust* if given any $(M, f, C(p))$ satisfying $\mathcal{P}$ there is a neighborhood $\mathcal{U}_f$ of f such that $(M, g, C(P_g, g))$ satisfies $\mathcal{P}$ for every $g \in \mathcal{U}_f$, where P_g is the unique continuation of P for g.

A local property $\mathcal{P}$ defined on the set of chain recurrence classes of hyperbolic periodic points is *self replicating* or *viral* if:

- $\mathcal{P}$ is a robust property
- given any triple $(M, f, C(p))$ satisfying $\mathcal{P}$, given any neighborhood U of $C(p)$ and any C^1-neighborhood $\mathcal{U}$ of f, there is $g \in \mathcal{U}$, there is a filtrating set $V \subset U$ of g disjoint from $C(p_g, g)$, and there is a hyperbolic periodic point $q \in V$ of g such that $(M, g, C(q, g))$ statisfies $\mathcal{P}$.

One can easily adapt this definition to other families of invariant compact sets: the unique requirement is that each of the invariant compact sets in the family has a well defined continuation for the nearby diffeomorphisms.

Example 1. *The following families have well defined continuations:*

- *the homoclinic classes of hyperbolic periodic orbit,*
- *the maximal invariant sets in filtrating neighborhoods*
- *the hyperbolic sets.*

- *the closure of the unstable manifolds of hyperbolic sets.*

In contrast, we don't know if there are aperiodic classes having continuations.

By a practical abuse of language, I will speak on *a $\mathcal{P}$-viral homoclinic classes* instead of *a homoclinic class displaying a viral property $\mathcal{P}$ defined on homoclinic classes.*

Next lemma is a direct consequence of Remark 7.3 and illustrates the notion of self replicating property:

Lemma 7.5. *The property of having a filtrating set in which generic diffeomorphisms close to f have the free k-universal property, $k \geq 3$ is a self replicating property.*

Conjecture 14. *In dimension ≥ 3, the property of robust non existence of any dominated splitting on homoclinic classes is a viral property.*

In an informal discussion at Beijing 2009 workshop we thought we can prove Conjecture 14 in the case where the class C is assumed to contain points with different indices.

There are homoclinic classes which are robustly without dominated splittings but which are 3-*sectionally dissipative*: the jacobian in restriction to any 3-planes are < 1. Thus Conjecture 14 provides viral classes far from k-universal dynamics, for any $k \geq 3$.

Problem 4. *Find a finite list of robust local mechanisms associated to homoclinic classes and characterizing (at least generically) the viral property.*

Problem 5. *For universal dynamics, there is a natural graduation by the dimension of universality. Is there some similar notion for viral classes?*

As mentioned above, 3-sectionnaly dissipative classes robustly without dominated splitting leads to viral classes which are not 3 universal. One may ask if this provides the unique mechanisms for viral, not k-universal classes:

Problem 6. *Examples of viral classes which are not freely k-universal, for any $k \geq 3$, can be built by considering uniform expansion (or contraction) of the jacobian in restriction to some family of planes, in relation to the dominated splittings. It would be very interesting to know if these jacobian conditions are necessary conditions. I think that Conjecture 2 would allow to solve this problem.*

7.5 Viral classes and aperiodic classes

If a C^1-generic diffeomorphism has a self replicating property $\mathcal{P}$ defined on homoclinic classes, then it has uncountably many chain recurrent classes. As a consequence, it has uncountably many aperiodic classes.

All the known aperiodic classes are fragile: you can remove it by a small perturbation. Usually, fragile phenomena are not seen by generic diffeomorphisms, if they are countably many. In the known examples, the fact that generic diffeomorphisms display aperiodic classes comes from a tree of nested filtrating regions, each infinite branch leading to an aperiodic class. This suggests the following conjecture:

Conjecture 15. *If a C^1-generic diffeomorphism has an aperiodic class, then it has uncountably many classes.*

Our philosophy, that all generic phenomenon are produced by robust local phenomena associated to periodic orbits, suggests now:

Conjecture 16. *If a C^1-generic diffeomorphisms has uncountably many classes, it has a $\mathcal{P}$-viral homoclinic classes.*

7.6 Countable number of classes

At this time, there are no examples of C^1-open set of diffeomorphisms in which on can prove that C^1-generic diffeomorphisms have countably many chain recurrence classes.

Let us say that a homoclinic class $H(p)$ is *kicking out* if it admits a filtrating neighborhood U in which C^1-generic diffeomorphism g close to f have infinitely many classes in U, all the classes are isolated but the continuation $H(P_g)$. In other words, $H(P_g)$ kick homoclinic classes out $H(P_g)$ but only isolated classes.

At this time, there are no examples which are known to be kicking out classes. However, next conjecture looks very reasonable[8]:

Conjecture 17. *If $f \in \mathcal{W}(M)$ is a C^1-generic diffeomorphisms having countably many chain recurrence classes, then it has a kicking out homoclinic class.*

When I first tried to formalize these notion, for my mini-course at Beijing (see [B]), I mentioned a possibility for building kicking out classes: some partially hyperbolic class in dimension 3 with a strong unstable direction, which would kick out only classes contained in periodic, normally expanding 2-discs, and I mentioned that I had some candidate(a modification of the example in [BLY] forbidding the free 2-universal behavior). If such a phenomena was proved, Smale's conjecture (Conjecture 5) would imply that, generically, each of these periodic 2-disks contains at most finitely many classes, so that the set of classes is countable. Since then, I hird that such a phenomena was obtained by R. Potrie in a construction similar to Carvalho's example. As a conclusion, it seems that a positive answer of Smale's conjecture would imply the existence of non-empty open sets where generic diffeomorphisms have a countable family of chain recurrent classes.

It could be interesting to know if there are other types of kicking out classes which are kicking out classes not contained in periodic 2-disks.

7.7 Splitting the set of wild dynamical systems in regions?

The $dimM$-universal region $\mathcal{W}_{dimM-univ}$ is a clearly identified region, for $dimM \geq 3$ [9]. Diffeomorphisms in that region are the most complicated dynamics : they display any persistent phenomena appearing in balls. In some sense this stops the study.

The regions $\mathcal{W}_{k-univ}$, for $k \geq 3$ are well identified too. They have an uncountable family of classes. I guess one should split further these regions in term of the normal hyperbolicity of the k-disk carrying the k-universal dynamics. For instance, if the disks are normally contracting, this will provide infinitely many different types of attractors.

Our understanding of $\mathcal{W}_{2-univ}$ will strongly depend on Smale's conjecture.

The free 1-universality is equivalent to the locally generic coexistence of infinitely many periodic orbits with trivial homoclinic class. It is a very weak property and certainly is not enough for characterizing the dynamics.

Then one considers the open set $\mathcal{W}_{viral}$ of diffeomorphisms having a viral homoclinic class. This class contains $\mathcal{W}_{k-univ}$ for $k \geq 3$ (and maybe $\mathcal{W}_{2-univ}$ if Smale's conjecture is wrong).

This open set is contained in $\mathcal{W}_{\mathbb{R}}$, interior of the closure of the diffeomorphisms having an uncountable family of classes. Once again, $\mathcal{W}_{\mathbb{R}}$ is contained in $\mathcal{W}_{aperiodic}$ (interior of the closure of the diffeomorphisms having an aperiodic class. Conjectures 15 and 16 state

$$\mathcal{W}_{viral} = \mathcal{W}_{\mathbb{R}} = \mathcal{W}_{aperiodic}.$$

Now we consider $\mathcal{W}_{\mathbb{N}}$, the maximal open set where generic diffeomorphisms have an infinite, countable family of classes. Conjecture 15 states that all of this classes are homoclinic classes, and at least one of them would be a kicking out classes.

Some last comments:

- if Conjecture 1 is wrong we have to add $\mathcal{W}_{hyp}$.
- in the same way, if Conjecture 9 is wrong, we have to add $\mathcal{W}_{tame}$ which contains $\mathcal{W}_{hyp}$.

This ends the panorama I wanted to propose (see Figure 7.7).

[8] The idea is that countable compact sets always admits isolated points. The set of classes is not a priori compact for the Hausdorff topology however Conley theory ensures that one may separate the classes by a Lyapunov map; consider the image of the chain reccurent set by such a Lyapunov map: it is a compact subset of $\mathbb{R}$ whose points are in one to one correspondence with the classes. This shows that diffeomorphism with countable number of classes admits isolated classes; then there are classes accumulated only by this isolated classes. The isolated classes of generic diffeomorphisms are homoclinic classes; hence, the difficulty of this conjecture consists in proving that the limit of these isolated homoclinic classes is also a homoclinic class.

[9] For $dimM \geq 3$, $f \in \mathcal{W}_{dimM-univ}$ is equivalent to f has a homoclinic class containing periodic orbits of different indices linked by robust cycles, having complex eigenvalues of any ranks, and one of them having a jacobian large than 1 and another having a jacobian less than 1.

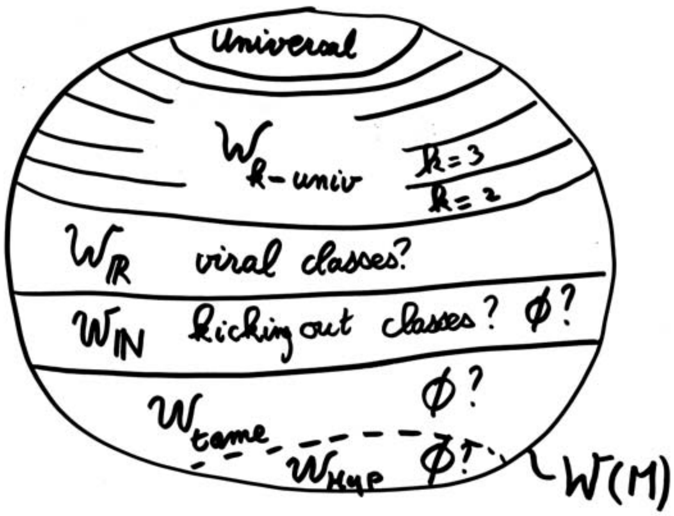

Figure 6: Conjectural panorama of the wild region $\mathcal{W}(M)$

8 Furhter questions

8.1 Density of finitely many classes?

As noticed before, wild diffeomorphisms are not defined by themself: we just know that C^1-generic wild diffeomorphisms have infinitely many classes. This leads to a natural question

Question 1. *Does it exist a non-empty open subset $\mathcal{O} \subset \mathcal{W}(M)$ where every $f \in \mathcal{O}$ has infinitely many chain recurrence classes?*

A negative answer to this question would be a good indication for a Conjecture by Jacob Palis

Conjecture 18. *There is a dense subset of $Diff^r(M)$ of diffeomorphisms having finitely many attractors. Furthermore the union of the basins is dense in M.*

In case of a positive answer to Question 1 , we could stronger the question:

Question 2. *Does it exist a non-empty open subset $\mathcal{O} \subset \mathcal{W}(M)$ where every $f \in \mathcal{O}$ has infinitely many quasi-attractors?*

8.2 Attractors

The panorama I proposed above was motivated by the wish of a complete description of the dynamics. However, it looks also reasonable to focus the study on attracting and repelling sets, neglecting the classes whose stable set is nowhere dense. This new problematic could change the panorama. For instance

- Conjecture 3 is proved by Crovisier and Pujals for attractors and repellers: generic diffeomorphisms far from tangencies and cycles admit finitely many hyperbolic attractors whose basins covers a dense open subset of the manifold.
- Tame diffeomorphisms admit finitely many robust attractors whose basins cover (C^1-generically) a dense open subset of the manifold.

  Problem 7. *Is there a dense open subset of $\mathcal{T}(M)$ where the basins of the attractors cover a dense open subset of the manifold?*

In this new problematic, the wildness would correspond to the generic existence of infinitely many attractors.

However, this problematic comes from the faith in the generic existence of attractors. This is not at all clear: in [BLY] we proved the existence of non-empty open sets where C^1-generic diffeomorphisms have no attractors nor repellers. According to Hurley, a *quasi-attractor* is a chain recurrence class admitting a basis of neighborhoods by attracting regions. [BC] proved that the ω-limit set of generic points of C^1-generic diffeomorphisms are quasi attractors.

Let call *essential attractor* every quasi-attractor whose basin contains a residual subset of a neighborhood (if the conjecture below is wrong, one could weaken the definition, replacing the essential attractors by the weaker notion of *generic attractor* just asking that the basin contains a residual subset in a non trivial open set).

Conjecture 19. *There is a residual subset $\mathcal{R}$ of $Diff^1(M)$ such that for every $f \in \mathcal{R}$ one has:*

1. *every essential attractor of f is a homoclinic class $H(p)$;*
2. *the union of the basins of the essential attractors is residual in M;*
3. *if $H(p_f)$ is a essential attractor for f, then $H(p_g)$ is a essential attractor for every C^1-generic g close to f;*
4. *the basin varies semi-continuously: given an attracting region V in the closure of the basin of f, then V it is contained in the closure of the basin of $H(p_g, g)$ for C^1-generic g close to f.*

As a first step in this direction, Hurley's conjecture in [Hu] has been proved in [BC] using [MP]:

Theorem 8.1. *For every compact manifold M, for every C^1-generic diffeomorphisms f and every generic point $x \in M$, the ω-limit set $\omega(x, f)$ is a* quasi-attractor *(i.e. a chain recurrence class admitting a basis of attracting neighborhoods).*

Remember that, for C^0-generic homeomorphisms, the basin of every quasi-attractors is nowhere dense, although the ω-limit set of every generic points is a quasi-attractor. This example shows us the distance between Theorem 8.1 and Conjecture 19.

Let us finish this section by recalling the encouraging result in[BGLY] (based on [Y]): for C^1-generic diffeomorphisms far from tangencies, the quasi-attractors are essential attractors; in fact, Conjecture 19 holds, far from tangencies.

8.3 Vector fields

The dynamics of vector fields in dimension n is very similar with the dynamics of diffeomorphisms in dimension $n-1$. Their studies followed parallel development. However there are some essential differences, the main difference being certainly the existence of singular point of vector fields.

I would say that the whole panorama I drowned for diffeomorphisms holds for the non-singular chain recurrence classes of vector fields. Thus one just need to complete this panorama by explaining the specific situation of *singular classes* (i.e. the chain recurrence classes containing the singular points of the vector field).

In dimension 3 the typical example of a robust singular class is Lorenz attractor. A large sequence of papers by Morales Pacifico and Pujals defines the notion of singular hyperbolic sets and proves that robustly isolated singular classes of 3-dimensional flows are singular hyperbolic (see for instance [MPP, MP_2]).

Conjecture 20. *Let M be a closed manifold. For every C^1-generic vector field on M, for every singular point p the singular chain recurrence class of p*

- *either reduced to $\{p\}$*
- *or coincides with the homoclinic class of a periodic orbit.*

As far as I know, this conjecture is not even proved in dimension 3. In dimension 3, as far as I know, it is not known if a singular class of a C^1-generic vector field is always isolated.

In higher dimension it is easy to build non-isolated singular class of generic vector fields (see for instance [BLY] which builds a singular essential attractor which is not an isolated class). [BPV, BKR, BLY_2]

build examples of robust singular attractors with arbitrary unstable dimensions, persistent tangencies or containing singularities of different indices, respectively. These examples are the first step for understanding the kind of structure (like dominated splittings) which would be necessary to get isolated singular classes for generic vector fields in arbitrary dimensions. In the other direction, [D, V] prove that robustly transitive vector fields are always non-singular.

It would be certainly very useful to explore all the possibilities by building new examples, in order to define the good notion of dominated splittings for singular classes which would allow us to propose a characterization of robustly isolated singular classes, and of wild singular classes.

9 Bibliography

References

[Ab] F. Abdenur, *Generic robustness of a spectral decompositions,* Ann. Sci. École Norm. Sup. **36**, 213–224, (2003).

[ABCD] Abdenur, Flavio; Bonatti, Christian; Crovisier, Sylvain; Díaz, Lorenzo J. *Generic diffeomorphisms on compact surfaces.* Fund. Math. 187 (2005), no. 2, 127–159.

[ABC] Abdenur, Flavio; Bonatti, Christian; Crovisier, Sylvain; *Nonuniform hyperbolicity for C1-generic diffeomorphisms* to appear at Israel J. of Math.

[AS] R. Abraham and S. Smale,*Nongenericity of Ω-stability,*Global Analysis I, Proc. Symp. Pure Math A.M.S., **14**, 5-8,(1968).

[Ar_1] M.-C. Arnaud, *Création de connexions en topologie C^1,* Ergod. Th. & Dynam. Syst., **21**, 339–381, (2001).

[ABV] J. Alves Ch. Bonatti, M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding,* Invent. Math., **140**(2), 351-398, (2000).

[AHK] E. Akin, M. Hurley, J. Kennedy, *Dynamics of topologically generic homeomorphisms.* Mem. Amer. Math. Soc. 164 (2003), no. 783, viii+130 pp.

[As] Asaoka, Masayuki *Hyperbolic sets exhibiting C^1-persistent homoclinic tangency for higher dimensions.* Proc. Amer. Math. Soc. 136 (2008), no. 2, 677–686

[BKR] R. Bamon, J. Kiwi, J. Rivera *Wild Lorenz like attractors* preprint 2006.

[BoBo] J. Bochi, Ch. Bonatti *Perturbation of the Lyapunov spectrum of periodic orbits* Work in progress.

[B] Ch. Bonatti *Robust tangencies* Note of the mini-course, avaliable at: http://www.math.pku.edu.cn/teachers/gansb/conference09/BeijingAout2009.pdf

[BC] Ch. Bonatti et S. Crovisier, *Recurrence and genericity,* C. R. Math. Acad. Sci. Paris **336**, 839–844 ,(2003).

[BCDG] Ch. Bonatti, S. Crovisier, L.J. Díaz, N. Gourmelon *Homoclinic tangencies and heterodimensional cycles* Work in progress.

[BD_1] Ch. Bonatti and L.J. Díaz, *Persistent nonhyperbolic transitive diffeomorphims,* Ann. Math., **143**, 357–396, (1996).

[BD_2] Ch. Bonatti, L.J. Díaz, *Connexions hétéroclines et généricité d'une infinité de puits ou de sources,* Ann. Sci. École Norm. Sup., **32**, 135–150, (1999).

[BD_3] Ch. Bonatti, L.J. Díaz, *On maximal transitive sets of generic diffeomorphisms,* Publ. Math. Inst. Hautes Études Sci. **96**, 171–197, (2003).

[BD_4] Ch. Bonatti, L.J. Díaz, *Aboundance of robust tangencies* preprint.

[BD_5] Ch. Bonatti, L.J. Díaz, *A cycle which cannot be stabelized* Work in progress.

[BDK] Ch. Bonatti, L.J. Díaz and S. Kiriki *Robust heterodimensional cycles for hyperbolic continuations*

[BDP] Ch. Bonatti, L.J. Díaz et E.R. Pujals, *A $\mathcal{C}^1$-generic dichotomy for diffeomorphisms: weak forms of hyperbolicicity or infinitely many sinks or sources,* Ann. Math. **158**, 355–418, (2003).

[BDV] Ch. Bonatti, L.J. Díaz et M. Viana, *Dynamics Beyond Uniform Hyperbolicity*

[BGW] Ch. Bonatti, Sh. Gan, L. Wen *On the existence of non-trivial homoclinic classes* Erg. Th. Dyn. Sys.27, 1473-1508, (2007).

[BGLY] Ch. Bonatti, Sh. Gan, M. Li, D. Yang *Lyapunov stable classes and essential attractors* (Work in progress).

[BLY$_2$] Ch. Bonatti, M. Li, D. Yang *Robustly chain transitive singular attractors with singularities of different indices* preprint 2008

[BLY] Ch. Bonatti, M. Li, D. Yang *On the existence of attractors* preprint 2009(ArXiv 0904.4393)

[BV] Ch. Bonatti, M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting,* Israel J. of Math. 115 157–193 (2000).

[BPV] Ch. Bonatti, A. Pumariño, M. Viana *Lorenz attractors with arbitrary expanding dimension,* C. R. Acad. Sci. Paris, 325 (1997), 883888.

[Bo] R. Bowen, *Equilibrium states and the ergodic theory of Axiom A diffeomorphisms,* Lect. Notes in Mathematics, **470**, Springer-Verlag (1975).

[Ca] M. Carvalho, *Sinaï Ruelle Bowen measures for N-dimensional derived from Anosov diffeomorphisms* Ergodic Th. & Dynam. Sys. **13**, 21–44, (1993)

[Co] C. Conley, *Isolated invariant sets and Morse index,* CBMS Regional Conference Series in Mathematics, **38**, AMS Providence, R.I., (1978).

[Cr] S. Crovisier, *Periodic orbits and chain recurrent sets of C^1-diffeomorphisms,* Pub Math.IHES 104,87-141, (2006).

[Cr$_2$] S. Crovisier,*Partial hyperbolicity far from homoclinic bifurcations* preprint 2008 (ArXiv 0809.4965).

[Cr$_3$] S. Crovisier,*Birth of homoclinic bifurcations: a model for the central dynamics of partially hyperbolic systems* , to appear at Annals of Math

[Cr$_4$] S. Crovisier *Perturbations de la dynamique de difféomorphismes en topologie C^1* Preprint (ArXiv 0912.2896).

[CP] S. Crovisier, E. Pujals *Essential hyperbolicity versus homoclinic bifurcations* (work in progress)

[DPU] L.J. Díaz, E.R. Pujals, and R. Ures, *Partial hyperbolicity and robust transitivity,* Acta Mathematica, **183**, 1-43, (1999).

[D] C. I. Doering, *Persistently transitive vector fields on three dimensional manifolds,* In Proc. on Dynamical Systems and Bifurcation Theory, Pitman Res. Notes Math. Ser. 160, 5989, 1987.

[F] J. Franks, *Necessary conditions for stability of diffeomorphisms,* Trans. Amer. Math. Soc., **158**, 301–308, (1971).

[GWY] G. Gan, L. Wen, D. Yang *Minimal non-hyperbolicity and index completeness* preprint.

[Go] N. Gourmelon *Generation of homoclinic tangencies by C^1-perturbations.* Discrete Contin. Dyn. Syst. 26 (2010), no. 1, 1–42.

[Ha] S. Hayashi, *Connecting invariant manifolds and the solution of the C^1-stability and Ω-stability conjectures for flows,* Ann. of Math., **145**, 81–137, (1997) and Ann. of Math., **150**, 353–356, (1999).

[HHTU] Rodriguez Hertz, F.; Rodriguez Hertz, M. A.; Tahzibi, A. and Ures, R. *A criterion for ergodicity of non-uniformly hyperbolic diffeomorphisms.* Electron. Res. Announc. Math. Sci. 14 (2007), 74–81

[Hu] M. Hurley, *Attractors: persistence, and density of their basins,* Trans. Amer. Math. Soc., **269**, 271–247, (1982).

[KH] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems.* With a supplementary chapter by Katok and Leonardo Mendoza. Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995. xviii+802 pp.

[Ku] Y. Kupka *Contribution la thorie des champs gnriques.* Contributions to Differential Equations 2 1963 457–484.

[Li] Liao, Shantao *Qualitative theory of differentiable dynamical systems.* Science Press, Beijing; distributed by American Mathematical Society, Providence, RI, (1996). x+383 pp.

[Ma_1] R. Mañé, *Contributions to the stability conjecture,* Topology, **17**, 386-396, (1978).

[Ma] R. Mañé, *An ergodic closing lemma,* Ann. of Math., **116**, 503-540, (1982).

[Ma_2] R. Mañé, *A proof of the C^1 stability conjecture.* Inst. Hautes tudes Sci. Publ. Math. No. 66 (1988), 161–210.

[MP] C. Morales et M. J. Pacífico, *Lyapunov stability of ω-limit sets,* Discrete Contin. Dyn. Syst., **8**, 671–674, (2002).

[MP_2] C. Morales, M.J. Pacfico *A spectral decomposition for singular-hyperbolic sets.* Pacific J. Math. 229 (2007), no. 1, 223–232.

[MPP] C. Morales, M.J. Pacifico, E. Pujals *Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers.* Ann. of Math. (2) 160 (2004), no. 2, 375–432.

[Mo] C.G. Moreira *There are no C1-stable intersections of regular Cantor set* Perprint.

[N] S. Newhouse, *Nondensity of axiom* A(a) *on* S^2. (1970) Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) pp. 191–202 Amer. Math. Soc., Providence, R.I.

[N_1] S.Newhouse, *Diffeomorphisms with infinitely many sinks,* Topology, **13**, 9–18, (1974).

[N_3] S. Newhouse, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms,* Publ. Math. Inst. Hautes Études Sci., **50**, 101–151, (1979).

[Pa] J. Palis, *A global view of dynamics and a conjecture on the denseness and finitude of attractors,* Astérisque, **261**, 335-347, (2000).

[PT] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations,* Cambridge Studies in Advanced Mathematics, **35**, (1993).

[PV] J. Palis and M. Viana, *High dimension diffeomorphisms displaying infinitely many sinks,* Ann. of Math., **140**, 207-250, (1994).

[Po] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste,* (1899), réédité par Les grands classiques Gauthier-Villars, librairie Blanchard, Paris, (1987).

[Pu] C. Pugh, *The closing lemma,* Amer. J. Math., **89**, 956–1009, (1967).

[PS] Pujals, Enrique R.; Sambarino, Martn *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms.* Ann. of Math. (2) 151 (2000), no. 3, 961–1023.

[Sh] M. Shub, *Topological transitive diffeomorphism on T^4*, Lect. Notes in Math., **206**, 39 (1971).

[Sh1] M. Shub, *Global stability of dynamical systems.* With the collaboration of Albert Fathi and Rmi Langevin. Translated from the French by Joseph Christy. Springer-Verlag, New York, 1987. xii+150 pp.

[Si] R.C. Simon, *A 3-dimensional Abraham-Smale example,* Proc. A.M.S., **34**(2), 629-630, (1972).

[Sm] S. Smale, *Differentiable dynamical systems,* Bull. A.M.S., **73**, 747–817, (1967).

[Sm$_2$] S. Smale, *Stable manifolds for differential equations and diffeomorphisms.* Ann. Scuola Norm. Sup. Pisa (3) 17 1963 97–116.

[V] T. Vivier, *Flots robustement transitifs sur les varietes compactes,* C. R. Acad. Sci. Paris, 337 (2003), 791796.

[W$_2$] L. Wen, *Homoclinic tangencies and dominated splittings,* Nonlinearity, **15**, 1445–1469, (2002).

[W$_3$] L. Wen *Generic diffeomorphisms away from homoclinic tangencies and heterodimensional cycles.* Bull. Braz. Math. Soc. (N.S.) 35 (2004), no. 3, 419–452.

[Y] J. Yang, *Lyapunov stable chain recurrent class*, preprint, 2007.

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