Refined asymptotics for the infinite heat equation with homogeneous Dirichlet boundary conditions

Philippe Laurençot* & Christian Stinner†

March 11, 2010

Abstract

The nonnegative viscosity solutions to the infinite heat equation with homogeneous Dirichlet boundary conditions are shown to converge as \( t \to \infty \) to a uniquely determined limit after a suitable time rescaling. The proof relies on the half-relaxed limits technique as well as interior positivity estimates and boundary estimates. The expansion of the support is also studied.

Key words: infinite heat equation, infinity-Laplacian, friendly giant, viscosity solution, half-relaxed limits

MSC 2010: 35B40, 35K65, 35K55, 35D40

1 Introduction

Since the pioneering work by Aronsson [4], the infinity-Laplacian \( \Delta_\infty \) defined by

\[
\Delta_\infty u := (D^2 u \nabla u, \nabla u) = \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}
\]

has been the subject of several studies, in particular due to its relationship to the theory of absolutely minimizing Lipschitz extensions [4, 5, 10]. More recently, a parabolic equation involving the infinity-Laplacian (the infinite heat equation)

\[
\partial_t u = \Delta_\infty u, \quad (t, x) \in (0, \infty) \times \Omega,
\]

has been considered in [1, 2, 13]. When \( \Omega \subset \mathbb{R}^N \) is a bounded domain and \( \partial \Omega \) is supplemented with nonhomogeneous Dirichlet boundary conditions, the large time behaviour of solutions to (1.1) is investigated in [1] and convergence as \( t \to \infty \) to the unique steady state is shown. Furthermore, for homogeneous Dirichlet boundary conditions

\[
u = 0, \quad (t, x) \in (0, \infty) \times \partial \Omega,
\]

*Institut de Mathématiques de Toulouse, CNRS UMR 5219, Université de Toulouse, F–31062 Toulouse cedex 9, France. E-mail: laurencot@math.univ-toulouse.fr
†Fakultät für Mathematik, Universität Duisburg-Essen, D–45117 Essen, Germany. E-mail: christian.stinner@uni-due.de
and nonnegative initial condition

\[ u(0, x) = u_0(x), \quad x \in \bar{\Omega}, \tag{1.3} \]

satisfying

\[ u_0 \in C_0(\bar{\Omega}) := \{ f \in C(\bar{\Omega}) : f = 0 \text{ on } \partial \Omega \}, \quad u_0 \geq 0, \quad u_0 \not\equiv 0, \tag{1.4} \]
a precise temporal decay rate is given for the \( L^\infty \)-norm of \( u \), namely

\[ C_1^{-1}(t + 1)^{-1/2} \leq \| u(t, \cdot) \|_{L^\infty(\Omega)} \leq C_1(t + 1)^{-1/2} \quad \text{for all } t > 0 \tag{1.5} \]

with some \( C_1 \geq 1 \) depending on \( u_0 \) and \( \Omega \), the unique steady state of (1.1)-(1.3) being zero in that case.

The purpose of this note is to improve (1.5) by identifying the limit of \( t^{1/2}u(t, \cdot) \) as \( t \to \infty \) (see Theorem 1.2 below). We also provide additional information on the propagation of the positivity set of \( \bar{u}_0 \) as time goes by.

Before stating our main result we first recall that the infinity-Laplacian is a quasilinear and degenerate elliptic operator which is not in divergence form and a suitable framework to study the well-posedness of the infinite heat equation is the theory of viscosity solutions (see e.g. [1]). Within this framework the well-posedness of (1.1)-(1.3) has been established in [2] when \( \Omega \) fulfills the uniform exterior sphere condition:

For all \( x_0 \in \partial \Omega \) there exists \( y_0 \in \mathbb{R}^N \) such that \( |x_0 - y_0| = R \) and \( \{ x \in \mathbb{R}^N : |x - y_0| < R \} \cap \Omega = \emptyset \) for some positive constant \( R \) independent of \( x_0 \).

(1.6)

Introducing

\[ F(s, p, X) := s - \langle Xp, p \rangle \quad \text{for } s \in \mathbb{R}, p \in \mathbb{R}^N, X \in \mathcal{S}(\mathbb{R}), \tag{1.7} \]

where \( \mathcal{S}(\mathbb{R}) \) denotes the set of all symmetric \( N \times N \) matrices, the definition of viscosity solutions to (1.1)-(1.3) reads [1, 2].

**Definition 1** Let \( Q := (0, \infty) \times \Omega \subset \mathbb{R}^{N+1} \) and let \( USC(Q) \) and \( LSC(Q) \) denote the set of upper semicontinuous and lower semicontinuous functions from \( Q \) into \( \mathbb{R} \), respectively. A function \( u \in USC(Q) \) is a viscosity subsolution to (1.1)-(1.3) in \( Q \) if

(a) \( F(s, p, X) \leq 0 \) is satisfied for all \( (s, p, X) \in \mathcal{P}^{2,+}(0, t_0, x_0) \) and all \( (t_0, x_0) \in Q \), where

\[
\mathcal{P}^{2,+}u(t_0, x_0) := \left\{ (s, p, X) \in \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(\mathbb{R}) : u(t, x) \leq u(t_0, x_0) + s(t - t_0) + \langle p, x - x_0 \rangle + \frac{1}{2} \langle X(x - x_0), x - x_0 \rangle + o(|t - t_0| + |x - x_0|^2) \right. \\
as (t, x) \to (t_0, x_0) \left. \right\},
\]

(b) \( u \leq 0 \) on \( (0, \infty) \times \partial \Omega \).

(c) \( u(0, x) \leq u_0(x) \) for \( x \in \bar{\Omega} \).

Similarly, \( u \in LSC(Q) \) is a viscosity supersolution to (1.1)-(1.3) in \( Q \) if \( F(s, p, X) \geq 0 \) for all \( (s, p, X) \in \mathcal{P}^{2,-}u(t_0, x_0) := -\mathcal{P}^{2,+}(-u)(t_0, x_0) \) and \( (t_0, x_0) \in Q \), \( u \geq 0 \) on \( (0, \infty) \times \partial \Omega \) and \( u(0, x) \geq u_0(x) \) for \( x \in \bar{\Omega} \).

Finally, \( u \in C(\bar{Q}) \) is a viscosity solution to (1.1)-(1.3) if it is a viscosity subsolution and a viscosity supersolution to (1.1)-(1.3).
With this definition, the well-posedness of (1.1)-(1.3) is shown in [2, Theorems 2.3 and 2.5] and the asymptotic behaviour of nonnegative solutions is obtained in [1, Theorem 5]. We gather these results in the next theorem.

**Theorem 1.1** ([1, 2]) Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain such that (1.6) is satisfied and assume (1.4). Then there is a unique nonnegative viscosity solution \( u \) to (1.1)-(1.3). Moreover, \( u(t, \cdot) \) converges to zero as \( t \to \infty \) in the sense that there exists a constant \( C_1 \geq 1 \) such that
\[
C_1^{-1}(t + 1)^{-1/2} \leq \|u(t, \cdot)\|_{L^\infty(\Omega)} \leq C_1(t + 1)^{-1/2} \quad \text{for all } t > 0.
\] (1.8)

Our improvement of (1.8) then reads:

**Theorem 1.2** Suppose \( \Omega \subset \mathbb{R}^N \) is a bounded domain fulfilling (1.4) and assume (1.4). If \( u \) denotes the viscosity solution to (1.1)-(1.3), then
\[
\lim_{t \to \infty} \|t^{1/2}u(t, \cdot) - f_\infty\|_{L^\infty(\Omega)} = 0,
\] (1.9)
where \( f_\infty \) is the unique positive viscosity solution to
\[
-\Delta f_\infty - \frac{f_\infty}{2} = 0 \quad \text{in } \Omega, \quad f_\infty > 0 \quad \text{in } \Omega, \quad f_\infty = 0 \quad \text{on } \partial \Omega.
\] (1.10)

Theorem 1.2 not only gives the convergence of \( t^{1/2}u(t, \cdot) \) as \( t \to \infty \), but also provides the existence and uniqueness of the positive solution \( f_\infty \) to (1.10) in \( C_0(\Omega) \). An interesting consequence of (1.10) is that the function \( (t, x) \mapsto t^{-1/2}f_\infty(x) \) is a separate variables solution to (1.1)-(1.3) with an initial data being identically infinite in \( \Omega \). Similar solutions are already known to exist for other parabolic equations such as the porous medium equation \( \partial_t u = \Delta u^m \), \( m > 1 \), or the \( p \)-Laplacian equation \( \partial_t u = \text{div}(|\nabla u|^{p-2}\nabla u) \), \( p > 2 \), (see [3, 12, 14, 20, 21] for instance). They play an important role in the description of the large time dynamics [3, 16, 21] and also provide universal bounds (and are thus called friendly giants). The function \( (t, x) \mapsto t^{-1/2}f_\infty(x) \) is a friendly giant for the infinite heat equation (1.1)-(1.3) and we have the following universal bound.

**Corollary 1.3** Suppose \( \Omega \subset \mathbb{R}^N \) is a bounded domain fulfilling (1.4) and assume (1.4). If \( u \) denotes the viscosity solution to (1.1)-(1.3), then
\[
u(t, x) \leq t^{-1/2}f_\infty(x) \quad \text{for } (t, x) \in (0, \infty) \times \Omega,
\] (1.11)
the function \( f_\infty \) being defined in Theorem 1.2.

The proof of Theorem 1.2 and Corollary 1.3 involves several steps: According to (1.8) the evolution of \( u(t, \cdot) \) takes place on a time scale of order \( t^{-1/2} \) and we first introduce a rescaled version \( v \) of \( u \) defined by \( u(t, x) = t^{-1/2}v(t, x) \). The outcome of Theorem 1.2 is then the convergence of \( v(s, \cdot) \) to the time-independent function \( f_\infty \) as \( s \to \infty \). To establish such a convergence, we use the half-relaxed limits technique introduced in [3] which is well-suited here as we have rather scarce information on \( v(s, \cdot) \) as \( s \to \infty \). This requires however a strong comparison principle for the limit problem (1.10) which will be established in Section 2 under an additional positivity assumption,
and furthermore implies the uniqueness of $f_\infty$. That the half-relaxed limits indeed enjoy this positivity property has to be proved as a preliminary step and follows from the observation that $v(s, \cdot)$ is non-decreasing with time and eventually becomes positive in $\Omega$ (see Section 3.1). At this point, boundary estimates are also needed to ensure that the half-relaxed limits vanish on $\partial \Omega$ and are shown by constructing suitable barrier functions. Thanks to these results, we deduce that the half-relaxed limits coincide, which implies that $v(s, \cdot)$ converges as $s \to \infty$ and the existence of a positive solution $f_\infty$ to (1.10) as well (see Section 3.2). We emphasize here that the existence of a positive solution to (1.10) is a consequence of the dynamical properties of $v$ and was seemingly not known previously. Finally, Corollary 1.3 is a consequence of Theorem 1.2 and the time monotonicity of $v$ (see Section 3.2).

Additionally, in Section 4 we investigate further positivity properties of the solution $u$ to (1.1)–(1.3). We show that $u(t, \cdot)$ becomes positive in $\Omega$ after a finite time if $\Omega$ satisfies an additional uniform interior sphere condition. Aside from this, $u$ may have a positive waiting time if the initial data are flat on the boundary of their support, namely the support of $u(t, \cdot)$ will be equal to that of $u_0$ for small times.

For further use, we introduce the following notation: Given $x \in \bar{\Omega}$, let $d(x, \partial \Omega) := \text{dist}(x, \partial \Omega)$ denote the distance to the boundary. Moreover, for $x \in \mathbb{R}^N$ and $r > 0$ we define $B(x, r) := \{y \in \mathbb{R}^N : |y - x| < r\}$ to be the ball of radius $r$ centered at $x$.

## 2 Uniqueness of the friendly giant

In this section we show that the friendly giant is unique. This will be a consequence of the following more general comparison lemma.

### Lemma 2.1

Let $w \in USC(\bar{\Omega})$ and $W \in LSC(\bar{\Omega})$ be respectively a bounded viscosity subsolution and a bounded viscosity supersolution to

$$-\Delta_\infty \zeta - \frac{\zeta}{2} = 0 \quad \text{in } \Omega$$

such that

$$w(x) = W(x) = 0 \quad \text{for } x \in \partial \Omega, \quad (2.2)$$

$$W(x) > 0 \quad \text{for } x \in \Omega. \quad (2.3)$$

Then

$$w \leq W \quad \text{in } \Omega. \quad (2.4)$$

**Proof.** We fix $N_0 \in \mathbb{N}$ large enough such that $\Omega_n := \{x \in \Omega : d(x, \partial \Omega) > 1/n\}$ is a nonempty open subset of $\Omega$ for all integer $n \geq N_0$. Let $n \geq N_0$. Since $\Omega_n$ is compact and $W \in LSC(\bar{\Omega})$, $W$ has a minimum in $\Omega_n$ and the positivity of $W$ in $\Omega_n$ implies that

$$\mu_n := \min_{\Omega_n} W > 0. \quad (2.5)$$

Similarly, the compactness of $\Omega \setminus \Omega_n$ and the upper semicontinuity and boundedness of $w$ ensure that $w$ has a point of maximum $x_n$ in $\Omega \setminus \Omega_n$ and we set

$$\eta_n := \max_{\Omega \setminus \Omega_n} w = w(x_n) \geq 0, \quad (2.6)$$

4
the nonnegativity of $\eta_n$ being a consequence of the fact that $w$ vanishes of $w$ on $\partial \Omega$. We next claim that
\[
\lim_{n \to \infty} \eta_n = 0.
\] (2.7)
Indeed, owing to the compactness of $\bar{\Omega}$ and the definition of $\Omega_n$ there are $y \in \partial \Omega$ and a subsequence of $(x_n)_{n \in \mathbb{N}}$ (not relabeled) such that $x_n \to y$ as $n \to \infty$. Since $w(y) = 0$, we deduce from the upper semicontinuity of $w$ that
\[
\limsup_{x \to y} w(x) = \limsup_{\varepsilon \downarrow 0} \sup \{ w(x) : x \in B(y, \varepsilon) \cap \Omega \} \leq 0.
\]
Given $\varepsilon > 0$, there is $n_\varepsilon$ such that $x_n \in B(y, \varepsilon) \cap \bar{\Omega}$ for all $n \geq n_\varepsilon$. Hence,
\[
\limsup_{n \to \infty} \eta_n \leq \sup \{ w(x) : x \in B(y, \varepsilon) \cap \Omega \}
\]
and letting $\varepsilon \downarrow 0$ and using (2.6) allow us to conclude that
\[
0 \leq \limsup_{n \to \infty} \eta_n \leq 0.
\]
This shows that a subsequence of $(\eta_n)_{n \geq N_0}$ converges to zero and the claim (2.7) follows by noticing that $(\eta_n)_{n \geq N_0}$ is a nonincreasing sequence.

Next, fix $s \in (0, \infty)$. For $\delta > 0$ and $n \geq N_0$, we define
\[
z_n(t, x) := (t + s)^{-1/2} w(x) - s^{-1/2} \eta_n, \quad (t, x) \in [0, \infty) \times \bar{\Omega},
\]
\[
Z_\delta(t, x) := (t + \delta)^{-1/2} W(x), \quad (t, x) \in [0, \infty) \times \bar{\Omega}.
\]
Then $z_n$ and $Z_\delta$ are respectively a bounded usc viscosity subsolution and a bounded lsc viscosity supersolution to (1.1) with
\[
Z_\delta(0, x) = 0 \geq -s^{-1/2} \eta_n = z_n(t, x), \quad (t, x) \in (0, \infty) \times \partial \Omega.
\]
In addition, if
\[
0 < \delta < \left( \frac{\mu_n}{1 + \|w\|_{L^\infty(\Omega)}} \right)^2 s
\] (2.8)
we have
\[
Z_\delta(0, x) = \delta^{-1/2} W(x) \geq \delta^{-1/2} \mu_n s^{-1/2} \|w\|_{L^\infty(\Omega)} \geq z_n(0, x) \quad \text{for } x \in \Omega_n
\]
and
\[
Z_\delta(0, x) \geq 0 \geq s^{-1/2} (w(x) - \eta_n) = z_n(0, x) \quad \text{for } x \in \bar{\Omega} \setminus \Omega_n.
\]
We are then in a position to apply the comparison principle [11, Theorem 8.2] to deduce that
\[
z_n(t, x) \leq Z_\delta(t, x), \quad (t, x) \in [0, \infty) \times \bar{\Omega},
\] (2.9)
for any $\delta > 0$ and $n \geq N_0$ satisfying (2.8). According to (2.8), the parameter $\delta$ can be taken arbitrarily small and we deduce from (2.9) that
\[
(t + s)^{-1/2} w(x) - s^{-1/2} \eta_n \leq t^{-1/2} W(x), \quad (t, x) \in (0, \infty) \times \bar{\Omega},
\]
for $n \geq N_0$. We next pass to the limit as $n \to \infty$ with the help of (2.7) to conclude that

$$(t + s)^{-1/2}w(x) \leq t^{-1/2}W(x), \quad (t, x) \in (0, \infty) \times \bar{\Omega}.$$ 

Finally, as $s > 0$ is arbitrary, we may let $s \searrow 0$ and take $t = 1$ in the above inequality to complete the proof. /////

Now the uniqueness of the friendly giant is a straightforward consequence of Lemma 2.1.

**Corollary 2.2** There is at most one positive viscosity solution to (1.10) in $C^0(\bar{\Omega})$.

### 3 Large time behaviour

In this section, we assume that $\Omega$ is a bounded domain fulfilling (1.6) and that $u_0$ satisfies (1.4). Let $u$ be the corresponding viscosity solution to (1.1)-(1.3). In order to investigate the asymptotic behaviour of $u$ as stated in Theorem 1.2 we introduce the scaling variable $s = \ln t$, $t > 0$, and the rescaled unknown function $v$ defined by

$$u(t, x) = t^{-1/2}v(\ln t, x), \quad (t, x) \in (0, \infty) \times \bar{\Omega}. \quad (3.1)$$

It is easy to check that $v$ is the viscosity solution to

$$\partial_s v = \Delta_\infty v - \frac{v^2}{2}, \quad (s, x) \in (0, \infty) \times \bar{\Omega}, \quad (s, x) \in (0, \infty) \times \partial \Omega, \quad v(0, x) = v_0(x) := u(1, x), \quad x \in \bar{\Omega}, \quad (3.2, 3.3, 3.4)$$

while it readily follows from (1.8) and (3.3) that

$$0 \leq v(s, x) \leq C_1, \quad (s, x) \in [0, \infty) \times \bar{\Omega}. \quad (3.5)$$

#### 3.1 Positivity and time monotonicity

A further property of $v$ is its time monotonicity which follows from the homogeneity of the operator $\Delta_\infty$ by a result from Bénilan & Crandall [10].

**Lemma 3.1** For $x \in \bar{\Omega}$, $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$ such that $s_1 \leq s_2$, we have

$$v(s_1, x) \leq v(s_2, x).$$

**Proof.** Theorem [10] provides the well-posedness of (1.1) in $C^0(\bar{\Omega})$ which is an ordered vector space. As the comparison principle is valid for (1.1)-(1.3) by [2, Theorem 2.3] and the infinity-Laplacian is homogeneous of degree 3, [10, Theorem 2] implies

$$u(t + h, x) - u(t, x) \geq \left(\left(\frac{t + h}{t}\right)^{-1/2} - 1\right)u(t, x) \quad \text{for } (t, x) \in (0, \infty) \times \bar{\Omega}, \ h > 0. \quad (3.6)$$
Hence, for any \((s, x) \in \mathbb{R} \times \Omega\) and \(h > 0\), we obtain
\[
v(s + h, x) - v(s, x) = e^{(s+h)/2} u(e^{s+h}, x) - e^{s/2} u(e^s, x) \\
\geq e^{(s+h)/2} \left( \frac{e^{s+h}}{e^s} \right)^{-1/2} u(e^s, x) - e^{s/2} u(e^s, x) = 0,
\]
which is the expected result. //

The monotonicity of \(v\) now enables us to prove that \(v\) eventually becomes positive inside \(\Omega\).

**Lemma 3.2** For any compact subset \(K \subset \Omega\) there are \(s_K > 0\) and \(\mu_K > 0\) such that
\[
v(s, x) \geq \mu_K > 0 \quad \text{in } [s_K, \infty) \times K.
\]

**Proof.** Three steps are needed to achieve the claimed result: we first prove that if \(v(s, \cdot)\) is positive at one point of \(\Omega\), then it becomes positive on a “large” ball centered around this point after a finite time. The second step is to prove that \(v(s, \cdot)\) becomes eventually positive in \(\Omega\) as \(s \to \infty\), from which we deduce (3.7) in a third step.

**Step 1:** Consider first \((t_0, x_0) \in (0, \infty) \times \Omega\) such that there are \(\varepsilon > 0\) and \(\delta > 0\) with \(B(x_0, \varepsilon) \subset \Omega\) and
\[
u(t_0, x) \geq \delta > 0 \quad \text{for } x \in B(x_0, \varepsilon). \tag{3.8}
\]
Then, choosing \(\alpha := \min\{4\delta^{1/3}, \varepsilon^{2/3}\}\), \(T := (d(x_0, \partial \Omega)\varepsilon^3/\alpha^3) - 1 \geq 0\), and defining
\[
B(t, x) := \frac{\alpha^3}{4} (t - t_0 + 1)^{-1/6} \left( 1 - \alpha^{-2}|x - x_0|^{4/3}(t - t_0 + 1)^{-2/3} \right)^{3/2}, \quad (t, x) \in [t_0, \infty) \times \mathbb{R}^N,
\]
we deduce from [1], Proposition 1 and Corollary 1, that \(B\) is a viscosity solution to (1.1) in \((t_0, t_0 + T) \times \Omega\). In addition, on the one hand, we have by (3.8)
\[
B(t_0, x) \leq \frac{\alpha^3}{4} \leq \delta \leq \nu(t_0, x) \quad \text{for } x \in B(x_0, \varepsilon)
\]
and
\[
B(t_0, x) = 0 \leq \nu(t_0, x) \quad \text{for } x \in \overline{\Omega} \setminus B(x_0, \varepsilon).
\]
On the other hand, we have \(\nu(t, x) = B(t, x) = 0\) for \((t, x) \in [t_0, t_0 + T] \times \partial \Omega\) thanks to the choice of \(T\), \(\alpha\) and the properties of \(B\). The comparison principle [1], Theorem 8.2] then implies \(\nu \geq B\) in \([t_0, t_0 + T] \times \Omega\). In particular, we have
\[
\nu(t_0 + T, x) > 0 \quad \text{for } x \in B(x_0, d(x_0, \partial \Omega)), \tag{3.9}
\]
where \(T\) only depends on \(\varepsilon\) and \(\delta\), but is independent of \(x_0\) and \(t_0\).

**Step 2:** We next define the positivity set \(\mathcal{P}(s)\) of \(v(s, \cdot)\) for \(s \geq 0\) by
\[
\mathcal{P}(s) := \{x \in \Omega : v(s, x) > 0\}.
\]
Owing to the time monotonicity of \(v\) (Lemma 3.1), \((\mathcal{P}(s))_{s \geq 0}\) is a non-decreasing family of open subsets of \(\Omega\) and
\[
\mathcal{P}_\infty := \bigcup_{s \geq 0} \mathcal{P}(s) \text{ is an open subset of } \Omega.
\]
Assume for contradiction that \( \partial \mathcal{P}_\infty \cap \Omega \neq \emptyset \). Then there is \( x_0 \in \partial \mathcal{P}_\infty \cap \Omega \). Since \( d(x_0, \partial \Omega) > 0 \) there is \( y_0 \in \mathcal{P}_\infty \) such that \( |y_0 - x_0| \leq d(x_0, \partial \Omega) / 2 < d(y_0, \partial \Omega) \). Next, since \( y_0 \in \mathcal{P}_\infty \), there is \( s_0 > 0 \) such that \( v(s_0, y_0) > 0 \), that is \( u_{s_0, y_0} > 0 \). The previous step then guarantees the existence of \( T \geq 0 \), such that \( u_{s_0, y_0} + T > 0 \) for \( x \in B(y_0, d(y_0, \partial \Omega)) \). As \( x_0 \in B(y_0, d(y_0, \partial \Omega)) \), we deduce from this that

\[
v(\ln(u_{s_0, y_0} + T), x_0) = (u_{s_0, y_0} + T)^{1/2} v(x_0) > 0,
\]

which contradicts the fact that \( x_0 \in \partial \mathcal{P}_\infty \). Therefore, \( \partial \mathcal{P}_\infty \cap \Omega = \emptyset \) and \( \Omega \) is the union of the two disjoint open sets \( \mathcal{P}_\infty \) and \( \Omega \setminus \mathcal{P}_\infty \). Since \( \mathcal{P}_\infty \neq \emptyset \) by (1.8), the connectedness of \( \Omega \) implies

\[
\Omega = \mathcal{P}_\infty.
\] (3.10)

**Step 3:** Let \( K \) be a compact subset of \( \Omega \) and assume for contradiction that \( K \not\subseteq \mathcal{P}(n) \) for each \( n \geq 1 \). Then there is a sequence \( (x_n)_{n \geq 1} \) in \( K \) such that \( v(n, x_n) = 0 \) for \( n \geq 1 \) and we may assume without loss of generality that \( x_n \) converges towards \( x_\infty \in K \) as \( n \to \infty \), thanks to the compactness of \( K \). Since \( x_\infty \in \Omega \), it follows from (3.10) that there is \( s_\infty > 0 \) such that \( v(s_\infty, x_\infty) > 0 \). Owing to the continuity of \( v(s_\infty, \cdot) \) there are \( \varepsilon > 0 \) and \( \delta > 0 \) such that \( v(s_\infty, x) \geq \delta \) for \( x \in B(x_\infty, \varepsilon) \subset \Omega \). But then for \( n \) large enough we have \( n \geq s_\infty \) and \( x_n \in B(x_\infty, \varepsilon) \) and it follows from Lemma 3.1 and the previous bound that

\[
0 = v(n, x_n) \geq v(s_\infty, x_n) \geq \delta
\]

and a contradiction. Consequently, there is \( n_K \) such that \( K \subset \mathcal{P}(n_K) \) and

\[
\mu_K := \min_{x \in K} v(n_K, x) > 0.
\]

Due to the time monotonicity of \( v \), this implies (3.7). /////

### 3.2 Convergence

Having studied the positivity properties of \( v \), we next turn to its behaviour near the boundary of \( \Omega \) and first show the following lemma which is a modification of [9, Lemma 10.1].

**Lemma 3.3** Consider \( x_0 \in \partial \Omega \), \( \alpha \in (0, 1/2) \), \( \delta > 0 \), \( B > 0 \), and define

\[
\psi_{\delta, B}(r) := \delta + B \left( r - \frac{r^2}{2} \right), \quad r \in \mathbb{R}.
\]

Let \( y_0 \in \mathbb{R}^N \) be such that \( |x_0 - y_0| = R \) and \( \Omega \cap B(y_0, R) = \emptyset \) (such a point \( y_0 \) exists according to the uniform exterior sphere condition (1.4)). Introducing

\[
U_{\alpha, x_0} := \{ x \in \Omega : R < |x - y_0| < R + \alpha \}
\]

and

\[
w(s, x) := \psi_{\delta, B}(|x - y_0| - R), \quad (s, x) \in [0, \infty) \times U_{\alpha, x_0},
\]

then \( w \) is a supersolution to (3.2) in \( (0, \infty) \times U_{\alpha, x_0} \) if \( B \geq 2(1 + \delta) \).
Lemma 3.4
Consider $\alpha$ then there is $\psi$ such that for any $\delta > 0$, we have
\[
\psi(r) = B^3 (1 - r^2) - \frac{B}{2} \left( r - \frac{r^2}{2} \right) - \frac{\delta}{2} \geq \frac{B - 2\delta}{4} \geq 0.
\]

As a consequence of Lemma 3.3, we have the following useful bound for $v$ on $\partial \Omega$.

**Lemma 3.4** Consider $\alpha \in (0, 1/2)$ and define
\[
\omega(\alpha) := \sup \{ v(0, x) : x \in \Omega \text{ and } d(x, \partial \Omega) < \alpha \}.
\]

Then there is $\alpha_0 \in (0, 1/2)$ such that, for any $\alpha \in (0, \alpha_0)$ and $x_0 \in \partial \Omega$, we have
\[
0 \leq v(s, x) \leq \omega(\alpha) + \frac{2C_1}{\alpha} |x - x_0|,
\]
for $(s, x) \in [0, \infty) \times \{0\} \cap B(x_0, \alpha)$, (3.13)

the constant $C_1$ being defined in (3.3).

**Proof.** Consider $x_0 \in \partial \Omega$ and let $y_0 \in \mathbb{R}^N$ be such that $|x_0 - y_0| = R$ and $\Omega \cap B(y_0, R) = \emptyset$, the existence of such a point $y_0$ being guaranteed by the uniform exterior sphere condition [16]. With the notations of Lemma 3.3, we define
\[
w(s, x) := \psi_{\omega(\alpha), 2C_1/\alpha}(|x - y_0| - R),
\]
the constant $C_1$ being defined in (3.3) and observe that
\[
B(x_0, \alpha) \cap \Omega \subset U_{\alpha, x_0} \subset \{ x \in \Omega : d(x, \partial \Omega) < \alpha \}. 
\]

(3.14)

On the one hand, it follows from (3.12) and (3.14) that
\[
w(0, x) \geq \omega(\alpha) \geq v(0, x), \quad x \in U_{\alpha, x_0}.
\]

On the other hand, if $(s, x) \in [0, \infty) \times \partial U_{\alpha, x_0}$, we have either $x \in \partial \Omega$ and $w(s, x) \geq 0 = v(s, x)$ or $|x - y_0| = R + \alpha$ and
\[
w(s, x) = \psi_{\omega(\alpha), 2C_1/\alpha}(\alpha - \alpha^2) \geq C_1 \geq v(s, x)
\]

On the other hand, if $(s, x) \in [0, \infty) \times \partial U_{\alpha, x_0}$, we have either $x \in \partial \Omega$ and $w(s, x) \geq 0 = v(s, x)$ or $|x - y_0| = R + \alpha$ and
\[
w(s, x) = \psi_{\omega(\alpha), 2C_1/\alpha}(\alpha - \alpha^2) \geq C_1 \geq v(s, x)
\]

9
by (3.5). Furthermore, since \( v(0, x) = 0 \) on \( \partial \Omega \), \( \omega(\alpha) \) converges to 0 as \( \alpha \searrow 0 \) and there is thus \( \alpha_0 \in (0, 1/2) \) such that \( 2C_1/\alpha \geq 2(1 + \omega(\alpha)) \) for \( \alpha \in (0, \alpha_0) \). This condition implies that \( w \) is a supersolution to (3.2) in \( (0, \infty) \times U_{\alpha,x_0} \) by Lemma 3.3. According to the above analysis, we are in a position to apply the comparison principle [11, Theorem 8.2] to conclude that

\[
v(s, x) \leq w(s, x), \quad (s, x) \in [0, \infty) \times U_{\alpha,x_0}.
\]

In particular, if \( (s, x) \in [0, \infty) \times (\Omega \cap B(x_0, \alpha)) \), the above inequality, (3.14), and the properties of \( y_0 \) entail that

\[
v(s, x) \leq \omega(\alpha) + \frac{2C_1}{\alpha} (|x - x_0| - R)
\]

whence (3.13).

Proof of Theorem 1.2. For \( \varepsilon \in (0, 1) \), we define

\[
V_\varepsilon(s, x) := v\left(\frac{s}{\varepsilon}, x\right), \quad (s, x) \in [0, \infty) \times \Omega,
\]

and the half-relaxed limits

\[
V_\ast(x) := \liminf_{(\sigma, y, \varepsilon) \to (s, x, 0)} V_\varepsilon(\sigma, y), \quad V^\ast(x) := \limsup_{(\sigma, y, \varepsilon) \to (s, x, 0)} V_\varepsilon(\sigma, y)
\]

for \( (s, x) \in (0, \infty) \times \Omega \). These functions are well-defined by (3.7), indeed do not depend on \( s > 0 \), and the stability result for (discontinuous) viscosity solutions ensures that

\[
V_\ast \text{ is a supersolution to } -\Delta_\infty z - \frac{z}{2} = 0 \quad \text{in } \Omega, \tag{3.15}
\]

\[
V^\ast \text{ is a subsolution to } -\Delta_\infty z - \frac{z}{2} = 0 \quad \text{in } \Omega. \tag{3.16}
\]

In addition, it follows from (3.4) and (3.13) that

\[
0 \leq V_\ast(x) \leq V^\ast(x) \leq C_1, \quad x \in \Omega, \tag{3.17}
\]

and, for all \( (x_0, \alpha) \in \partial \Omega \times (0, \alpha_0) \),

\[
0 \leq V_\ast(x) \leq V^\ast(x) \leq \omega(\alpha) + \frac{2C_1}{\alpha} |x - x_0|, \quad x \in \Omega \cap B(x_0, \alpha). \tag{3.18}
\]

In particular, (3.18) guarantees that \( 0 \leq V_\ast(x_0) \leq V^\ast(x_0) \leq \omega(\alpha) \) for all \( x_0 \in \partial \Omega \) and \( \alpha \in (0, \alpha_0) \). Since \( \omega(\alpha) \to 0 \) as \( \alpha \searrow 0 \), we end up with

\[
V_\ast(x) = V^\ast(x) = 0, \quad x \in \partial \Omega. \tag{3.19}
\]
We finally infer from Lemma 3.2 that
\[ V_*(x) > 0 \quad \text{for } x \in \Omega. \] (3.20)

We are then in the position to apply Lemma 2.1 to obtain that \( V_\ast \leq V_\ast \). Recalling (3.13), (3.14), (3.15), and (3.16) we conclude that \( V_\ast = V_\ast \in C_0(\Omega) \) is a viscosity solution to \( -\Delta_\infty z - z/2 = 0 \) in \( \Omega \). We have thus proved that \( f_\infty := V_\ast \) is a positive viscosity solution to (1.10) and it is the only one by Corollary 2.2. In addition, it follows from the identity \( V_\ast = V_\ast = f_\infty \) and \([7, \text{Lemme 4.1}]\) (see also \([6, \text{Lemma 5.1.9}]\)) that
\[ \lim_{\varepsilon \to 0} \| V_\varepsilon(2) - f_\infty \|_{L^\infty(\Omega)} = 0. \]

In other words,
\[ \lim_{s \to \infty} \| v(s) - f_\infty \|_{L^\infty(\Omega)} = 0, \] (3.21)
which is equivalent to (1.9) by (3.1). \\

**Proof of Corollary 1.3.** The claim now follows from Theorem 1.2 and Lemma 3.1. Indeed, we have \( v(s, \cdot) \leq v(\sigma, \cdot) \) for \( -\infty < s \leq \sigma < \infty \). Letting \( \sigma \to \infty \) and using (3.21) lead us to \( v(s, \cdot) \leq f_\infty \) for any \( s \in \mathbb{R} \), which is nothing but (1.11) once written in terms of \( u \).

---

### 4 Additional positivity properties

First we state an extension of Lemma 3.2 which shows that \( u_\ast \) is indeed positive in \( \Omega \) after a finite time provided that \( \Omega \) additionally satisfies a uniform interior sphere condition:

There is \( R_0 > 0 \) such that for any \( x_0 \in \partial \Omega \) there is \( y_0 \in \Omega \) such that \( |y_0 - x_0| = R_0 \) and \( B(y_0, R_0) \subset \Omega \). (4.1)

**Lemma 4.1.** Assume that \( \Omega \subset \mathbb{R}^N \) is a bounded domain satisfying (1.6) and (4.1) and that \( u_0 \) fulfills (3.2). If \( u \) denotes the viscosity solution to (1.1)-(1.3), then there is \( t_1 \in (0, \infty) \) such that
\[ u(t, x) > 0 \quad \text{in } [t_1, \infty) \times \Omega. \] (4.2)

**Proof.** Let \( v \) be defined by (3.1) and set
\[ K := \left\{ x \in \Omega : d(x, \partial \Omega) \geq \frac{R_0}{2} \right\} \quad \text{and} \quad M := \{ x \in \Omega : d(x, \partial \Omega) = R_0 \}. \]

Since \( K \) is a compact subset of \( \Omega \), we have
\[ v(s, x) \geq \mu_K > 0 \quad \text{in } [s_K, \infty) \times K \] (4.3)
for some \( s_K > 0 \) and \( \mu_K > 0 \) by Lemma 3.2. Thus, setting \( t_0 := e^{s_K} \), \( \varepsilon := R_0/2 \) and \( \delta := t_0^{-1/2} \mu_K \), (3.8) is valid for any \( x_0 \in M \). Then the first step of the proof of Lemma 3.2 implies the existence
of $T > 0$ which is independent of $x_0 \in M$ such that (3.1) is fulfilled for any $x_0 \in M$. Thus, we conclude that
\[
v(s_0, x) > 0 \quad \text{for } x \in \tilde{M} := \bigcup_{x_0 \in M} B(x_0, R_0),
\]
where $s_0 := \ln(t_0 + T) > s_K$. As (4.1) implies $\tilde{M} \cup K = \Omega$ (see e.g. [17, Section 14.6]), we deduce from Lemma 3.1 and (4.3) that
\[
v(s, x) > 0 \quad \text{in } [s_0, \infty) \times \Omega.
\]
By (3.1), this shows (4.2) with $t_1 := e^{s_0}$. ////

Having shown that $v$ is positive in $\Omega$ after a finite or infinite time, we next show that the expansion of the positivity set of $u(t, \cdot)$ may take some time to be initiated.

**Proposition 4.2** Consider $u_0 \in C_0(\bar{\Omega})$ and define its positivity set $\mathcal{P}_0$ by
\[
\mathcal{P}_0 := \{ x \in \Omega : u_0(x) > 0 \}.
\]
If $x_0 \in \Omega \cap \partial \mathcal{P}_0$ is such that
\[
u_0(x) \leq a \, |x - x_0|^2, \quad x \in B(x_0, \delta) \subset \Omega, \tag{4.4}
\]
for some $\delta > 0$ and $a > 0$, then there is $\tau(x_0) > 0$ such that $u(t, x_0) = 0$ for $t \in [0, \tau(x_0))$.

In other words, the so-called waiting time
\[
\tau_w(x_0) := \inf \{ t > 0 : u(t, x_0) > 0 \}
\]
of $u$ at $x_0 \in \Omega$ is positive if $u_0$ satisfies (4.4). In addition, it is finite by Lemma 3.2. This waiting time phenomenon is typical for degenerate parabolic equations, see [15, 21] and the references therein.

The proof of Proposition 4.2 relies on the construction of supersolutions as in [18, Theorem 8.2] which we describe now.

**Lemma 4.3** Consider $x_0 \in \Omega$ and $T > 0$ and define
\[
S_T(t, x) := \frac{|x - x_0|^2}{4(T - t)^{1/2}}, \quad (t, x) \in [0, T) \times \Omega.
\]
Then $S_T$ is a supersolution to (1.1) in $(0, T) \times \Omega$.

**Proof.** We first note that $S_T \in C^2([0, T) \times \Omega)$. For $(t, x) \in (0, T) \times \Omega$, we compute
\[
\partial_t S(t, x) - \Delta_\infty S(t, x) = \frac{|x - x_0|^2}{8(T - t)^{3/2}} - \frac{(x - x_0, x - x_0)}{8(T - t)^{3/2}} = 0
\]
and readily obtain the expected result. ///
Proof of Proposition 4.2. Define

\[ T := \min \left\{ \frac{1}{16a_2}, \frac{\delta^4}{16C_1^2} \right\}. \]

According to Lemma 4.3, the function \( S_T \) is a supersolution to (1.1) in \((0, T) \times B(x_0, \delta)\). In addition, the choice of \( T \) and (4.4) guarantee that

\[ S_T(0, x) = \frac{|x - x_0|^2}{4T^{1/2}} \geq a |x - x_0|^2 \geq u_0(x), \quad x \in B(x_0, \delta), \]

while we infer from the choice of \( T \) and (1.8) that, for \((t, x) \in (0, T) \times \partial B(x_0, \delta)\)

\[ S_T(t, x) = \frac{\delta^2}{4(T - t)^{1/2}} \geq \frac{\delta^2}{4T^{1/2}} \geq C_1 \geq u(t, x). \]

The comparison principle [11, Theorem 8.2] then entails that \( S_T(t, x) \geq u(t, x) \) for \((t, x) \in [0, T) \times B(x_0, \delta)\). In particular, \( 0 \leq u(t, x_0) \leq S_T(t, x_0) = 0 \) for \( t \in [0, T) \), and the proof of Proposition 4.2 is complete. \\

Acknowledgements

This work was done while C. Stinner held a one month invited position at the Institut de Mathématiques de Toulouse, Université Paul Sabatier - Toulouse III. He would like to express his gratitude for the invitation, support, and hospitality.

References


