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HOMOGENIZATION AND ENHANCEMENT FOR THE
G–EQUATION

P. CARDALIAGUET, J. NOLEN, AND P. E. SOUGANIDIS

Abstract. We consider the so-called G-equation, a level set Hamilton-Jacobi equation, used as a sharp interface model for flame propagation, perturbed by an oscillatory advection in a spatio-temporal periodic environment. Assuming that the advection has suitably small spatial divergence, we prove that, as the size of the oscillations diminishes, the solutions homogenize (average out) and converge to the solution of an effective anisotropic first-order (spatio-temporal homogeneous) level set equation. Moreover we obtain a rate of convergence and show that, under certain conditions, the averaging enhances the velocity of the underlying front. We also prove that, at scale one, the level sets of the solutions of the oscillatory problem converge, at long times, to the Wulff shape associated with the effective Hamiltonian. Finally we also consider advection depending on position at the integral scale.

1. Introduction

We study the limit, as $\varepsilon \to 0$, of the solution to the level-set equation

\[
\begin{align*}
(i) & \quad u_\varepsilon^t = |Du_\varepsilon^t| + \langle V(x_\varepsilon^t, t_\varepsilon^t), Du_\varepsilon^t \rangle \quad \text{in} \quad \mathbb{R}^N \times (0, T) \\
(ii) & \quad u_\varepsilon^t = u_0 \quad \text{on} \quad \mathbb{R}^N \times \{0\}.
\end{align*}
\]

Equation (1.1)(i) is referred to as the G-equation, and is used as a model for flame propagation in turbulent fluids ([23, 24]). In that setting, the level sets of the function $u_\varepsilon^t$ represent the evolving flame surface and $-V$ is the fluid velocity field. At points where $u_\varepsilon^t$ is differentiable and $|Du_\varepsilon^t| \neq 0$, the level sets of $u_\varepsilon^t$ move with normal velocity

$$
\nu = 1 - \langle V(x_\ell^t, \frac{t}{\varepsilon}), \hat{n} \rangle,
$$

where $\hat{n} = -Du_\varepsilon^t / |Du_\varepsilon^t|$ is the exterior normal vector of the front. When $V \equiv 0$, level sets move with constant speed $s_L = 1$, which is called the laminar speed of flame propagation.

We assume that the vector field $V \in C^{0,1}(\mathbb{R}^{N+1}; \mathbb{R}^N)$ is $\mathbb{Z}^{N+1}$-periodic in both $x$ and $t$, i.e., for all $(x, t) \in \mathbb{R}^{N+1}$, $k \in \mathbb{Z}^N$ and $s \in \mathbb{Z}$,

$$
V(x + k, t + s) = V(x, t).
$$

Our first result says that there exists a positively homogeneous of degree one, convex and continuous Hamiltonian $\bar{H}$ such that, as $\varepsilon \to 0$, the $u_\varepsilon^t$’s converge locally uniformly in $\mathbb{R}^N \times [0, \infty)$ to the solution $\bar{u}$ of the initial value problem

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\[
\begin{cases}
\bar{u}_t = \bar{H}(D\bar{u}) & \text{in } \mathbb{R}^N \times [0, \infty), \\
\bar{u} = u_0 & \text{on } \mathbb{R}^N \times \{0\}.
\end{cases}
\]

Although \( V \) is bounded, we do not assume that \(|V| < 1 \), and, hence, the Hamiltonian
\[ H(x,t,p) = |p| + \langle V(x,t), p \rangle \]

is not coercive in \(|p|\) at every point \((x,t)\). This lack of coercivity is the main mathematical challenge in the analysis. If either \(|V| < 1 \) or the nonlinearity \(|Du|\) were replaced with \(|Du|^2\), then \( H \) would be coercive in \(|p|\) and the problem would be within the scope of the theory developed in [13]. There are, however, relatively few homogenization results about noncoercive Hamiltonians [1, 3, 7, 11, 17], and none of them deals with the particular structure considered here.

The following simple example shows that, in the absence of coercivity, some additional assumption about the divergence of \( V \) is necessary in order for the \( u^\varepsilon \)'s to have a local uniform limit. To this end, let \( V = V(x) \) be a smooth \( \mathbb{Z}^N \)-periodic vector field such that, in the cube \( Q_1 = (-\frac{1}{2}, \frac{1}{2})^N \), \( V(x) = -10x \), if \(|x| < 1/6 \), and \( V(x) = 0 \) if \(|x| \geq 1/3 \). It is known that the \( u^\varepsilon \)'s have the control representation
\[ u^\varepsilon(x,t) = \sup(u_0(X_\varepsilon(t))), \]

where the supremum is over all functions \( X_\varepsilon \in W^{1,\infty}([0,t]; \mathbb{R}^N) \) such that \( X_\varepsilon(0) = x \) and \( X_\varepsilon'(s) = \kappa(s) + V(X_\varepsilon(s)) \) with the controls \( \kappa(\cdot) \) satisfying \(|\kappa| \leq 1 \). If \( u_0(x) = \langle p, x \rangle \) with \(|p| > 0 \), we see easily that \( \lim_{\varepsilon \to 0} u^\varepsilon(0,t) = 0 \) for all \( t > 0 \). However, \( \liminf_{\varepsilon \to 0} u^\varepsilon(x_, t) > 0 \) if \( t > 0 \) and \( \{x_\varepsilon\} \) is any point satisfying \(|x_\varepsilon| = \varepsilon/2 \). Roughly speaking, the problem with such a vector field \( V \) is that it traps the trajectories which start at the lattice points. If the divergence of \( V \) is sufficiently small, however, it is reasonable to expect that the controls are strong enough to overcome such traps.

We assume that \( V \) has “small divergence”, in the sense that, for all \( t \in \mathbb{R} \),
\[ \alpha(t) = \frac{1}{c_I} - \|\text{div} V(\cdot,t)\|_{L^N(Q_1)} \geq 0 \quad \text{and} \quad \alpha^* = \int_0^1 \alpha(s) \, ds > 0, \]

where \( c_I \) is the isoperimetric constant in the cube \( Q_1 \) (see, for instance, [13]), i.e., the smallest constant such that, for all measurable subsets \( E \) of \( Q_1 \),
\[ (|E| \wedge |Q_1 \setminus E|)^{(N-1)/N} \leq c_I \text{Per}(E, Q_1), \]

and also the optimal constant for the Poincaré inequality
\[ \|f - \langle f \rangle\|_{L^*} \leq c_I \|Df\|_{L^1(Q_1)}, \]

for \( f \in W^{1,1}(Q_1) \), \( 1^* = N/(N-1) \) and \( \langle f \rangle = \int_{Q_1} f(x) \, dx \).

To state the main results we introduce some additional notation. Throughout the paper we use \( Q^+_1 \) and \( BUC(\bar{U}) \) to denote respectively the space-time cube \( Q^+_1 = Q_1 \times [0,1] \subset \mathbb{R}^{N+1} \) and the space of bounded uniformly continuous functions on \( \bar{U} \), and we write
\[ \langle V \rangle = \int_{Q^+_1} V(x,t) \, dx \, dt \quad \text{and} \quad \langle x \text{div} V \rangle = \int_{Q^+_1} x \text{div} V(x,t) \, dx \, dt. \]

We have:
Theorem 1.1. Assume that \( V \in C^{0,1}(\mathbb{R}^{N+1};\mathbb{R}^{N}) \) satisfies \( (1.2) \) and \( (1.3) \) and that \( u_0 \in C^{0}(\mathbb{R}^{N}) \) is bounded. There exists a positively homogeneous of degree one, Lipschitz continuous, convex Hamiltonian \( \bar{H} : \mathbb{R}^{N} \to \mathbb{R} \) such that, if \( u^\varepsilon \in C^{0}(\mathbb{R}^{N} \times [0, +\infty)) \) and \( u \in C^{0}(\mathbb{R}^{N} \times [0, +\infty)) \) are the solutions to the initial value problems \( (1.1) \) and \( (1.3) \), respectively with initial datum \( u_0 \), then, as \( \varepsilon \to 0 \), the \( u^\varepsilon \)'s converge locally uniformly in \( \mathbb{R}^{N} \times [0, T] \) to \( u \). Moreover, for all \( P \in \mathbb{R}^{N} \),

\[
\bar{H}(P) \geq |P| \int_{0}^{1} \left( 1 - c_I \| \text{div}V(\cdot, t) \|_{L^\infty(Q_t)} \right) dt + \langle \langle V \rangle + \langle x \text{div}V \rangle, P \rangle. \tag{1.6}
\]

Finally, the convex map \( P \to \bar{H}(P) - \langle \langle V \rangle + \langle x \text{div}V \rangle, P \rangle \) is coercive.

For Lipschitz continuous initial datum \( u_0 \), we can actually estimate the convergence rate as \( \varepsilon \to 0 \). We have:

Theorem 1.2. Assume that \( u_0 \in C^{0,1}(\mathbb{R}^{N}) \) and let \( u^\varepsilon, \tilde{u} \in BUC(\mathbb{R}^{N} \times [0, T]) \), for all \( T > 0 \), be respectively the solutions to \( (1.1) \) and \( (1.3) \). Then, for all \( T > 0 \), there exists a positive constant \( C \) that depends only on \( T, N, V \) and the Lipschitz constant of \( u_0 \), such that, for all \( (x, t) \in \mathbb{R}^{N} \times [0, T] \),

\[
|u(x, t) - u^\varepsilon(x, t)| \leq C \varepsilon^{1/3}.
\]

In the case that, for all \( x \in \mathbb{R}^{N} \) and \( t \in \mathbb{R} \),

\[
\text{div}_{\nu}V(x; t) = 0, \tag{1.7}
\]

we derive some additional properties of the function \( \bar{H} \). To simplify the statement we also assume, without any loss of generality (see Lemma \( (3.4) \) below), that

\[
\int_{Q_1} V(x, t) dx = 0 \quad \forall t \in \mathbb{R}. \tag{1.8}
\]

Then, according to Theorem \( (1.1) \) the averaged Hamiltonian \( \bar{H} : \mathbb{R}^{N} \to \mathbb{R} \) satisfies, for all \( P \in \mathbb{R}^{N} \),

\[
\bar{H}(P) \geq |P|.
\]

We establish here a necessary and sufficient condition to have the strict inequality

\[
\bar{H}(P) > |P|,
\]

in which case we have enhancement of the speed due to averaging.

Recall that, since \( \bar{H} \) is homogeneous of degree one, the level sets of \( \tilde{u} \) move with speed \( \bar{H}(\tilde{u}) \) in the direction of the normal vector \( \tilde{n} = -D\tilde{u}/|D\tilde{u}| \). Therefore, we refer to the situation \( \bar{H}(P) > |P| \) as “enhancement”, because it implies that such velocity fields lead to faster propagation of interfaces compared to the case \( V \equiv 0 \).

First, we state the result in the case where \( V \) only depends on \( x \). We have:

Theorem 1.3. Assume that \( V \in C^{0,1}(\mathbb{R}^{N}) \) is \( \mathbb{Z}^{N} \)-periodic, \( \text{div}V = 0 \) and \( \langle V \rangle = 0 \), and let \( P \in \mathbb{R}^{N} \setminus \{0\} \). Then \( \bar{H}(P) = |P| \) if and only if, for all \( x \in \mathbb{R}^{N} \),

\[
\langle V(x), P \rangle = 0.
\]

In particular, if \( N = 2 \), then \( \bar{H}(P) = |P| \) if and only if the stream function \( E \) associated to \( V \) is of the form \( E = \bar{E}(\langle P, \cdot \rangle) \) for some \( \bar{E} : \mathbb{R} \to \mathbb{R} \), i.e., \( V \) is a shear drift in the direction orthogonal to \( P \).

When \( V \) is also time dependent, the characterization of equality \( \bar{H}(P) = |P| \) is provided by
Theorem 1.4. Assume that, for all $t \in \mathbb{R}$, $\text{div}_x V(\cdot, t) = 0$ and $\int_{Q^1} V(x, t)dx = 0$, and fix $P \in \mathbb{R}^N \setminus \{0\}$. Then $\hat{H}(P) = |P|$ if and only if there exists $\hat{z} \in BV_{loc}(\mathbb{R})$ such that $\hat{z}' \geq -|P|$ in the sense of distribution and the function $z(x, t) = \hat{z}(\frac{|P|x}{|P|} + t)$ is $\mathbb{Z}^{N+1}$-periodic and satisfies, for all $t \in \mathbb{R}$, in the sense of distributions
\[
\text{div} ((z(\cdot, t) + \langle P, \cdot \rangle)V(\cdot, t)) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.9}
\]

We continue with some observations about these results. Theorem 1.3 yields that, if $N = 2$, $\hat{H}(P) = |P|$ and $V$ is not constant, then $P = (P_1, P_2)$ must be a rational direction, i.e., either $P_2 = 0$ or $P_1/P_2 \in \mathbb{Q}$, since $V$ is $\mathbb{Z}^2$-periodic. For Theorem 1.4, observe that, if $z$ is not constant, then $P/|P|$ must be a rational vector.

Also (1.9) is equivalent to saying (see Lemma 3.2 below) that, for any fixed $t > 0$, the map $x \to z(x, t) + \langle P, x \rangle$ is constant along the flow of the differential equation $X'(s) = V(X(s), t)$.

We remark that it is possible to construct nontrivial examples of time-dependent flows for which $\hat{H}(P) = |P|$. Indeed when $N = 2$ for any smooth, $\mathbb{Z}^1$-periodic $(E_1, E_2)$ such that $E_1(0) = 0$, let $E(x_1, x_2, t) = E_1(x_1 + t)E_2(x_2)$, $V = \nabla^\perp E$ and $P = (1, 0)$. Then $\hat{H}(P) = |P|$ because the map $\hat{z}(s) = [s] - s$, where $[s]$ stands for the integer part of $s$, satisfies the condition of Theorem 1.4. For more analysis and numerical computation of $\hat{H}$ for specific flow structures, we refer to [12, 13, 21, 22].

The next result of the paper is about the long time behavior of the solution to (1.3) with $\varepsilon = 1$ and, in particular, the convergence, as $t \to \infty$, of its zero level set to the Wulff-shape associated with the effective $\hat{H}$, which is given by
\[
\mathcal{W} = \{y \in \mathbb{R}^N : \langle P, y \rangle + \hat{H}(P) \geq 0 \quad \text{for all} \quad P \in \mathbb{R}^N\}. \tag{1.10}
\]

We consider the initial value problem
\[
\begin{cases}
    u_t = |Du| + \langle V(x, t), Du \rangle & \text{in} \quad \mathbb{R}^N \times (0, \infty), \\
    u = u_0 & \text{on} \quad \mathbb{R}^N \times \{0\},
\end{cases} \tag{1.11}
\]
and set, for all $t \geq 0$,
\[
K(t) = \{x \in \mathbb{R}^N : u(x, t) \geq 0\}.
\]

Recall that, in the language of front propagation (see, for example, [4]), the family of closed sets $(K(t))_{t \geq 0}$ is solution of the front propagation problem
\[
\nu = 1 - \langle V(x, t), \eta \rangle
\]
staring from $K(0) = K_0$.

We have:

Theorem 1.5. Let $K_0$ be a non-empty compact subset of $\mathbb{R}^N$. There exist $C > 0$ and $T > 0$ such that, for all $t \geq T$,
\[
K(t) \subset (t + C)\mathcal{W}. \tag{1.12}
\]
Moreover, there exists a constant $C_0 > 0$, independent of $K_0$, such that, if $K_0$ contains a cube of side length $C_0$, then there exist $C > 0$ and $T > 0$ such that, for all $t \geq T$,
\[
(t - Ct^{2/3})\mathcal{W} \subset K(t). \tag{1.13}
\]
We note that we do not know whether the size condition on $K_0$ is actually necessary. The final result of the paper is about homogenization when $V$ depends on $x$ also at the integral scale, i.e., we are interested in the behavior as $\varepsilon \to 0$ of the solutions to the initial value problem

$$
\begin{align*}
\left\{ \begin{array}{ll}
    u_\varepsilon^t &= |Du_\varepsilon^x| + \langle V(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}), Du_\varepsilon^x \rangle & \text{in } \mathbb{R}^N \times (0, T), \\
    u_\varepsilon^x &= u_0 & \text{on } \mathbb{R}^N \times \{0\},
\end{array} \right.
\end{align*}
$$

(1.14)

where $V : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is smooth, bounded, $\mathbb{Z}^N$-periodic with respect to the last two variables, i.e., for all $(x, y, s) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R},$

$$
V(x, y + k, s + h) = V(x, y, s),
$$

(1.15)

is divergence free in the fast variable, i.e., for all $(x, y, s) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R},$

$$
div_y V(x, y, s) = 0,
$$

(1.16)

and satisfies, for all $x \in \mathbb{R}^N$ the “smallness” condition

$$
\left| \int_0^1 \int_{Q_1} V(x, y, s) dy ds \right| < 1.
$$

(1.17)

The homogenized initial value problem is

$$
\begin{align*}
\left\{ \begin{array}{ll}
    \bar{u}_t &= \bar{H}(x, D\bar{u}) & \text{in } \mathbb{R}^N \times (0, T), \\
    \bar{u} &= u_0 & \text{on } \mathbb{R}^N.
\end{array} \right.
\end{align*}
$$

(1.18)

We have:

**Theorem 1.6.** Assume (1.15), (1.16) and (1.18). There exists $\bar{H} \in C^0(\mathbb{R}^N \times \mathbb{R}^N)$, which is positively homogeneous of degree one and convex with respect to the second variable, such that, for any initial condition $u_0 \in BU(\mathbb{R}^N)$, the solution $u_\varepsilon^x$ to (1.14) converges, as $\varepsilon \to 0$, locally uniformly in $\mathbb{R}^N \times [0, T]$ to the solution $\bar{u}$ of (1.18). Moreover $\bar{H}$ satisfies, for all $(x, P) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$
\bar{H}(x, P) \geq |P| + \int_0^1 \int_{Q_1} V(x, y, s) dy ds, P).
$$

(1.19)

The paper is organized as follows. Theorem 1.1 and Theorem 1.2 are proved in Section 2. In Section 3 we prove Theorem 1.3 and Theorem 1.4, while Theorem 1.5 is proved in Section 4. In Section 5, we prove an extension of Theorem 1.1 to the case where $V = V(x, x/\varepsilon, t/\varepsilon)$ has large-scale spatial variation. The Appendix contains a proof of Lemma 2.3, which plays an important role in the proof of Theorem 1.1.

About the time this paper was completed, we learned about a similar but less general homogenization result obtained by different methods in [25]. In particular it is proved in [25] that homogenization takes place for time independent advection satisfying $V = V_1 + V_2$ with $\text{div} V_1 = 0$ and $|V_2| < 1$.

Finally we remark that throughout the paper we will need some basic results from the theory of viscosity solutions, like comparison principles, representation formulae, etc. All such results can be found, for instance, in [5].
2. Homogenization

We begin with some preliminary discussion and results to set the necessary background for the proofs of Theorem 1.1 and Theorem 1.2. First, we recall that for any $\lambda > 0$ and any $P \in \mathbb{R}^N$, the “penalized” cell problem

$$v_{\lambda,t} + \lambda v_{\lambda} = |Dv_{\lambda} + P| + \langle V, Dv_{\lambda} + P \rangle \quad \text{in} \quad \mathbb{R}^{N+1}, \quad (2.1)$$

has a unique $\mathbb{Z}^{N+1}$-periodic solution $v_{\lambda} \in BUC(\mathbb{R}^{N+1})$, which is actually Hölder continuous and satisfies, for all $(x,t) \in \mathbb{R}^{N+1}$, the bound

$$-\lambda^{-1}|P|(1 + \|V\|_\infty) \leq v_{\lambda} \leq \lambda^{-1}|P|(1 + \|V\|_\infty). \quad (2.2)$$

We also recall that, in the periodic setting, homogenization is equivalent to proving that the $(\lambda v_{\lambda})$’s converge uniformly in $\mathbb{R}^{N+1}$, as $\lambda \to 0$, to some constant $\bar{c}(P)$ and that $\bar{c} \in C^0(\mathbb{R}^N)$. In this case, $\bar{H}(P) = \bar{c}(P)$.

In view (2.2), to prove the convergence of the $\lambda v_{\lambda}$’s, we need to control their oscillations. We have:

**Lemma 2.1.** For all $P \in \mathbb{R}^N$ and $\lambda \in (0,1]$, $\text{osc}(\lambda v_{\lambda}) \leq C|P|\lambda$, (2.3)

where $C = 4(1 + \|V\|_\infty)(2^{1/N} \alpha^{*-1}N + 3)$.

Before we present the proof of Lemma 2.1, which is one of the most important parts of the paper, we point out its main consequence in the next

**Corollary 2.2.** Let $C$ be the constant given by Lemma 2.1. There exists some $\bar{H}(P) \in \mathbb{R}$ such that

$$\|\lambda v_{\lambda} - \bar{H}(P)\|_\infty \leq C|P|\lambda.$$  

**Proof:** The maps $\lambda \to \lambda \min v_{\lambda}$ and $\lambda \to \lambda \max v_{\lambda}$ are respectively nonincreasing and nondecreasing. For the sake of completeness we present a formal proof, which can be easily made rigorous using viscosity solutions arguments. Since the two claims are proved similarly, we present details only for the former.

To this end, for $0 < \lambda < \mu$, let $(x,t)$ be a maximum of $v_\mu - v_{\lambda}$. Then, at least formally, at $(x,t)$, we have $D_{x,t}v_\mu = D_{x,t}v_{\lambda}$, and

$$v_{\mu,t} + \mu v_{\mu} \leq |Dv_\mu + P| + \langle V, Dv_\mu + P \rangle$$

and

$$v_{\lambda,t} + \lambda v_{\lambda} \geq |Dv_{\lambda} + P| + \langle V, Dv_{\lambda} + P \rangle.$$  

It follows that

$$\mu v_{\mu}(x,t) \leq \lambda v_{\lambda}(x,t)$$

Let $(y,s)$ be a minimum point of $\lambda v_{\lambda}$. Then

$$\mu v_{\mu}(y,s) \leq \mu(v_\mu(x,t) - v_{\lambda}(x,t) + v_{\lambda}(y,s))$$

$$\leq \lambda v_{\lambda}(x,t) - \mu v_{\lambda}(x,t) + \mu v_{\lambda}(y,s)$$

$$\leq \lambda v_{\lambda}(x,t) + \lambda(v_{\lambda}(y,s) - v_{\lambda}(x,t)) \leq \lambda v_{\lambda}(y,s),$$

and, hence,

$$\min \mu v_{\mu} \leq \min \lambda v_{\lambda}.$$
The above remark combined with Lemma 2.1 implies that the \( \lambda v_\lambda \)'s converge uniformly to some constant \( \bar{H}(P) \) and that

\[
\lambda \min v_\lambda \leq \bar{H}(P) \leq \lambda \max v_\lambda.
\]

\( \square \)

We continue with the

**Proof of Lemma 2.1.** Without any loss of generality, we may assume that \( V \) is smooth. Indeed, if the result holds for any smooth \( V \), then it also holds by approximation for any \( V \in C^{0,1} \).

Recalling (2.2), \( w_\lambda = \lambda v_\lambda \) satisfies, in the viscosity sense,

\[
w_{\lambda,t} - |Dw_\lambda(x,t)| - (V(x,t), Dw_\lambda(x,t)) \geq -C_0 \lambda \quad \text{in } \mathbb{R}^{N+1}, \tag{2.4}
\]

where

\[
C_0 = 2(1 + \|V\|_\infty)|P|.
\]

It follows that \((x,t) \to w_\lambda(x,t) + C_0\lambda t \) is a viscosity super-solution of the level-set initial value problem

\[
\begin{align*}
(i) \quad z_t &= |Dz| + (V,Dz) \quad \text{in } \mathbb{R}^N \times (0,\infty), \\
(ii) \quad z &= w_\lambda \quad \text{on } \mathbb{R}^N \times \{0\}.
\end{align*} \tag{2.5}
\]

The standard comparison of viscosity solutions then implies that, for all \((x,t) \in \mathbb{R}^{N+1}, \)

\[
w_\lambda(x,t) + C_0\lambda t \geq z(x,t). \tag{2.6}
\]

The next step is to understand the evolution of the perimeter of the level-sets of \( z \). For this we need the following result, which is proved in the Appendix.

We have:

**Lemma 2.3.** Assume that \( V \in C^{1,1}(\mathbb{R}^{N+1}) \) and let \( z \in \text{BUC}(\mathbb{R}^{N+1}) \) be a solution of (2.2)/(i). Then, for any level \( \theta \in (\inf z(\cdot,0), \sup z(\cdot,0)) \) such that

\[
\{z(\cdot,0) = \theta\} = \partial \{z(\cdot,0) > \theta\} = \partial \{z(\cdot,0) < \theta\} \quad \text{and} \quad \{|z(\cdot,0) = \theta| \} = 0,
\]

and for all \( t > 0 \), we have

\[
\{z(\cdot,t) = \theta\} = \partial \{z(\cdot,t) > \theta\} = \partial \{z(\cdot,t) < \theta\}
\]

as long as \( \{z(\cdot,t) < \theta\} \neq \emptyset \). Moreover the sets \( \{z(\cdot,t) > \theta\} \) and \( \{z(\cdot,t) < \theta\} \) have locally of finite perimeter, and \( \{|z(\cdot,t) = \theta|\} = 0 \). Finally, for any compactly supported \( \varphi \in C^0(\mathbb{R}^N) \), the maps \( t \to I(t) = \int_{\{z(\cdot,t) > \theta\}} \varphi(x)dx \) and \( t \to J(t) = \int_{\{z(\cdot,t) < \theta\}} \varphi(x)dx \) are absolutely continuous and satisfy, for almost all \( t > 0 \),

\[
\frac{d}{dt}I(t) = \int_{\{z(\cdot,t) = \theta\}} \varphi(x)(1 - (V(x,t),\nu(x,t)))d\mathcal{H}^{N-1}(x),
\]

and

\[
\frac{d}{dt}J(t) = -\int_{\{z(\cdot,t) = \theta\}} \varphi(x)(1 + (V(x,t),\nu(x,t)))d\mathcal{H}^{N-1}(x),
\]

where \( \nu(x,t) \) denotes in the former identity the measure theoretic outward unit normal to \( \{z(\cdot,t) > \theta\} \) at \( x \in \partial\{z(\cdot,t) > \theta\} \), while in the latter is the measure theoretic outward unit normal to \( \{z(\cdot,t) < \theta\} \) at \( x \in \partial\{z(\cdot,t) < \theta\} \).
Continuing with the ongoing proof, suppose that there exists $\theta \in \mathbb{R}$ with
\[
\inf w_\lambda(\cdot, 0) < \theta < \sup w_\lambda(\cdot, 0),
\]
such that
\[
w_\lambda(\cdot, 0) = \partial\{w_\lambda(\cdot, 0) > \theta\} = \partial\{w_\lambda(\cdot, 0) < \theta\}
\]
and such that
\[
|\{x \in Q_1 : w_\lambda(x, 0) < \theta\}| < 1/2.
\]
For all $t \geq 0$, set
\[
\rho(t) = |\{x \in Q_1 : z(x, t) < \theta\}|
\]
Let $[0, T)$ be the maximal interval on which $\rho(t) < 1/2$ for any $t \in [0, T)$. Note that $T > 0$ because $\rho(0) < 1/2$. We claim that, for all $0 \leq t_1 \leq t_2 < T$,
\[
\rho(t_2) - \rho(t_1) \leq - \int_{t_1}^{t_2} \alpha(s)\rho(s)^{(N-1)/N} ds.
\]
Indeed fix a positive integer $R$ and let $Q_R = (-\frac{R}{2}, \frac{R}{2})^N$. The space periodicity of $z$ gives
\[
\rho(t) = R^{-N} |\{x \in Q_R : z(x, t) < \theta\}|.
\]
For $h > 0$ small, let $\chi_h \in C^{0,1}(\mathbb{R}^N; [0, 1])$ be such that
\[
\chi_h = 1 \text{ in } Q_R \quad \text{ and } \quad \chi_h = 0 \text{ in } \mathbb{R}^N \setminus Q_{R+h},
\]
and, for any $t \in [0, T]$, set
\[
\rho_{R,h}(t) = R^{-N} \int_{\{z(\cdot, t) < \theta\}} \chi_h(x)dx,
\]
and note that,
\[
\lim_{h \to 0} \rho_{R,h}(t) = \rho(t).
\]
It follows from Lemma 2.3 that, for almost all $t \in (0, T)$,
\[
\frac{d}{dt} \rho_{R,h}(t) = -R^{-N} \int_{\{z(\cdot, t) = \theta\}} \chi_h(x)(1 + \langle V(x, t), \nu(x, t)\rangle) d\mathcal{H}^{N-1}(x).
\]
Moreover
\[
\int_{\{z(\cdot, t) = \theta\}} \chi_h(x) d\mathcal{H}^{N-1}(x) \geq \mathcal{H}^{N-1}(\{z(\cdot, t) = \theta\} \cap Q_R),
\]
and, in view of the spatial periodicity of $z$,
\[
\mathcal{H}^{N-1}(\{z(\cdot, t) = \theta\} \cap Q_R) \geq R^N \mathcal{H}^{N-1}(\{z(\cdot, t) = \theta\} \cap Q_1).
\]
The isoperimetric inequality in the box $Q_1$ and the fact that $|\{z(\cdot, t) < \theta\} \cap Q_1| < 1/2$ give
\[
\mathcal{H}^{N-1}(\{z(\cdot, t) = \theta\} \cap Q_1) \geq \frac{1}{c_I} |\{z(\cdot, t) < \theta\} \cap Q_1|^{(N-1)/N}.
\]
Using once more the space periodicity of $z$ we get
\[
|\{z(\cdot, t) < \theta\} \cap Q_1|^{(N-1)/N} = (R + 1)^{(N-1)} |\{z(\cdot, t) < \theta\} \cap Q_{R+1}|^{(N-1)/N}
\]
\[
\geq (R(R+1)^{-1})^{N-1} \rho_{R,h}(t))^{(N-1)/N}. 
\]
Combining all the above we obtain
\[
R^{-N} \int_{\{z(t) = \theta\}} \chi_h(x) d\mathcal{H}^{N-1}(x) \geq \frac{1}{c_t} (R(R + 1)^{-1})^{N-1} (\rho_{R,h}(t))^{(N-1)/N}.
\]
Next we estimate the integral
\[
\int_{\{z(t) = \theta\}} \chi_h(x) \langle V(x, t), \nu(x, t) \rangle d\mathcal{H}^{N-1}(x).
\]
For some constant \(k\) depending only on \(N\) we have:
\[
- \int_{\{z(t) = \theta\}} \chi_h(x) \langle V(x, t), \nu(x, t) \rangle d\mathcal{H}^{N-1}(x)
\]
\[
= - \int_{\{z(t) < \theta\}} \text{div}(\chi_h V)(x, t) \, dx
\]
\[
\leq - \int_{\{z(t) < \theta\}} \chi_h(x) \text{div} V(x, t) dx + ||D\chi_h||_\infty ||V||_\infty |Q_{R+h} \setminus Q_R|
\]
\[
\leq (\int_{Q_{R+h}} |\text{div} V(x, t)|^N dx)^{1/N} \int_{\{z(t) < \theta\}} \chi_h(x)^{N/(N-1)} dx - \frac{1}{N} \int_{\{z(t) = \theta\}} \chi_h(x) ds
\]
\[
\leq (R + 1)R^{N-1} ||\text{div} V(\cdot, t)||_{L^N(Q_1)} (\rho_{R,h}(t))^{(N-1)/N} + kR^{N-1}||V||_\infty.
\]
Hence, for almost all \(t \in (0, T)\), we get
\[
\frac{d}{dt} \rho_{R,h}(t) \leq -\left(\frac{1}{c_t} \left(\frac{R}{R + 1}\right)^{N-1} - \frac{R}{R} ||\text{div} V(\cdot, t)||_{L^N(Q_1)} (\rho_{R,h}(t))^{(N-1)/N} + k\right)||V||_\infty.
\]
Integrating first over \([t_1, t_2]\) and then letting \(h \to 0\) and \(R \to +\infty\) gives (2.7).

Since \(\alpha(t) \geq 0\), \(\rho(t)\) is non increasing on \((0, T)\). Hence \(T = +\infty\). Integrating (2.7) over \((0, t)\) we obtain, for every \(t \geq 0\), that
\[
\rho(t) \leq \left[\rho_{1/N}(0) - \frac{1}{N} \int_0^t \alpha(s) ds \right]_+^N,
\]
where \([s]_+ = \max\{s, 0\}\).

From the assumption \(\rho(0) < 1/2\), it follows that
\[
t^* = 1 + \frac{N}{2^{1/N} \alpha^*} \geq \inf \left\{ t : \int_0^t \alpha(s) ds \geq \frac{N}{2^{1/N}} \right\},
\]
and, hence, \(\rho = 0\) in \([t^*, \infty)\). The continuity and the spatial periodicity of \(z\) then yield that
\[
z \geq \theta \quad \text{in} \quad \mathbb{R}^N \times [t^*, \infty).
\]
Let \(k\) be an integer in the interval \([t^*, t^* + 1]\). The space-time periodicity of \(w_\lambda\) and (2.6) give
\[
\inf_{t \in [0, 1]} \inf_{x \in \mathbb{R}^N} w_\lambda(x, t) = \inf_{t \in [k, k + 1]} \inf_{x \in \mathbb{R}^N} w_\lambda(x, t)
\]
\[
\geq \inf_{t \in [k, k + 1]} \inf_{x \in \mathbb{R}^N} z(x, t) - C_0 \lambda(t^* + 2)
\]
\[
\geq \theta - C_0 \lambda(t^* + 2).
\]
It follows that, if \( \theta \in \mathbb{R} \) is such that
\[
|\{x \in Q_1 : w_\lambda(x, 0) < \theta\}| < 1/2,
\]
then
\[
\inf_{t \in [0,1]} \inf_{x \in \mathbb{R}^N} w_\lambda(x, t) \geq \theta - C\lambda
\]
(2.10)
where
\[
C = C_0(t^* + 2).
\]

Now we derive an upper bound. To this end suppose that \( \theta \in \mathbb{R} \) with
\[
\inf w_\lambda(\cdot, 0) < \theta < \sup w_\lambda(\cdot, 0),
\]
such that
\[
\{ w_\lambda(\cdot, 0) = \theta \} = \partial \{ w_\lambda(\cdot, 0) > \theta \} = \partial \{ w_\lambda(\cdot, 0) < \theta \} \quad \text{and} \quad |\{ w_\lambda(\cdot, 0) = \theta \}| = 0,
\]
and such that
\[
|\{x \in Q_1 : w_\lambda(x, 0) > \theta\}| < 1/2.
\]
The claim is that
\[
\max_{x,t} w_\lambda(x, t) \leq \theta + C\lambda,
\]
(2.12)
where \( C = 2|\mathcal{P}|(1+\|V\|_\infty)(2+N2^{-1/N}(\alpha^*_{\text{max}})^{-1}) \). Indeed, arguing by contradiction, we assume that
\[
\max_{x,t} w_\lambda(x, t) > \theta + C\lambda.
\]

Then, by continuity and periodicity of \( w_\lambda \), there is some \( \tau \in [0,1] \) such that
\[
|\{x \in Q_1 : w_\lambda(x, \tau) > \theta + C\lambda\}| > 0.
\]
Let \( z \) satisfy (2.5-(i)) with initial condition \( z = w_\lambda \) on \( \mathbb{R}^N \times \{\tau\} \). As before, we have, for all \( (x,t) \in \mathbb{R}^N \times [\tau, +\infty) \),
\[
w_\lambda(x,t) \geq z(x,t) - C_0\lambda(t-\tau).
\]
Set
\[
\rho(t) = |\{x \in Q_1 : z(x,t) > \theta + C\lambda\}| \quad t \geq \tau,
\]
(2.13)
and observe that \( \rho \) is continuous with \( \rho(\tau) > 0 \). Then, arguing as before, we find that, for all \( \tau \leq t_1 \leq t_2 \),
\[
\rho(t_2) - \rho(t_1) \geq \int_{t_1}^{t_2} \alpha(s) \left( \min\{\rho(s), 1 - \rho(s)\} \right)^{(N-1)/N} ds.
\]
(2.14)
Since \( \alpha(t) \geq 0 \) for all \( t \geq 0 \), it follows that \( \rho \) is nondecreasing on \([\tau, +\infty)\). We claim that there is some \( T \leq t^* = \tau + 1 + N/(2^{1/N}\alpha^*) \) such that \( \rho \geq 1/2 \) for \( t \geq T \). Indeed, otherwise, \( \rho < 1/2 \) on \([\tau, t^*]\) and integrating (2.14) over \([\tau, t^*]\) gives
\[
\rho(t^*) \geq \left( \frac{1}{N} \int_{\tau}^{t^*} \alpha(s) ds \right)^N > \frac{1}{2},
\]
which contradicts our assumption. Let now \( k \) be an integer in \([T, T + 1]\). The time-periodicity of \( w_\lambda \) yields
\[
1/2 \leq \rho(k) = |\{x \in Q_1 : z(x,k) > \theta + C\lambda\}|
\leq |\{x \in Q_1 : w_\lambda(x,k) + C_0\lambda(k-\tau) > \theta + C\lambda\}|
= |\{x \in Q_1 : w_\lambda(x,0) + C_0\lambda(k-\tau) > \theta + C\lambda\}|.
\]
It follows from $k \leq (t^* + 1)$ that

$$C_0(k - \tau) \leq C$$

and $|\{x \in Q_1 : w_\lambda(x, 0) > \theta\}| \geq 1/2$,

which contradicts the definition of $\theta$, and, hence, (2.12) holds.

Finally set

$$\tilde{\theta} = \sup \left\{ \theta \in \mathbb{R} : |\{x \in Q_1 : w_\lambda(x, 0) < \theta\}| < \frac{1}{2} \right\}.$$  \hfill (2.15)

In view of the above, using (2.10) and (2.12), we get

$$\min_{t \in [0, 1]} \min_{x \in \mathbb{R}^N} w_\lambda(x, t) \geq \tilde{\theta} - C\lambda$$  \hfill (2.16)

and

$$\max_{t \in [0, 1]} \max_{x \in \mathbb{R}^N} w_\lambda(x, t) \leq \tilde{\theta} + C\lambda,$$  \hfill (2.17)

where $C = 2 |P| (1 + \|V\|_\infty) (3 + N2^{-1/N}/\alpha^*)$. It follows that $\text{osc}(w_\lambda) \leq 2C\lambda$, and, therefore, (2.3).

We proceed with the

**Proof of Theorem 1.1** : Let $\bar{H}(P)$ be defined by Corollary 2.2. The fact that the map $P \to \bar{H}(P)$ is positively homogeneous, convex and Lipschitz continuous follows easily from the properties of (2.1) and the comparison principle of viscosity solutions.

To prove (1.6), first we perturb (2.1) by a vanishing viscosity, i.e., for $\eta > 0$ we consider

$$v^\eta_{\lambda,t} + \lambda v^\eta_\lambda - \eta \Delta v^\eta_\lambda = |Dv^\eta_\lambda + P| + \langle V, Dv^\eta_\lambda + P \rangle$$ in $\mathbb{R}^{N+1}$, \hfill (2.18)

which has a unique $\mathbb{Z}^{N+1}$-periodic solution $v^\eta_\lambda \in BUC(\mathbb{R}^{N+1})$ which is at least in $C^1(\mathbb{R}^{N+1})$ and converges uniformly, as $\eta \to 0$, to $v_\lambda$.

Integrating (2.18) over $Q_1^+$ and using the periodicity, we find

$$\int_{Q_1^+} \lambda v^\eta_\lambda dx \ dt = \int_{Q_1^+} \langle V, Dv^\eta_\lambda + P \rangle dx \ dt + \int_{Q_1^+} \langle V, Dv^\eta_\lambda \rangle dx \ dt + \langle \langle V \rangle, P \rangle.$$  \hfill (2.19)

Set

$$\xi(x) = \langle P, x \rangle, \quad \langle v^\eta_\lambda(t) \rangle = \int_{Q_1} v^\eta_\lambda(x, t) dx \quad \text{and} \quad \langle \xi \rangle = \int_{Q_1} \xi(x) dx.$$  \hfill (2.20)

Since both $V$ and $v_\lambda$ are $\mathbb{Z}^{N+1}$-periodic, for each $t \in [0, 1]$, we have

$$\int_{Q_1} \langle V(x, t), Dv^\eta_\lambda(x, t) \rangle dx$$

$$= - \int_{Q_1} \left( v^\eta_\lambda(x, t) - \langle v^\eta_\lambda(t) \rangle \right) \text{div} V(x, t) \ dx$$

$$= - \int_{Q_1} \left( v^\eta_\lambda(x, t) - \langle v^\eta_\lambda(t) \rangle + \xi(x) - \langle \xi \rangle \right) \text{div} V(x, t) \ dx$$

$$+ \int_{Q_1} (\xi(x) - \langle \xi \rangle) \text{div} V(x, t) dx.$$
Applying, for each \( t \in [0, 1] \), Hölder’s and Poincaré’s inequalities yields

\[
\int_{Q_1} (v_\lambda^\eta(x, t) - \langle v_\lambda^\eta \rangle_t + \xi(x) - \langle \xi \rangle) \text{div} V(x, t) \, dx
\leq ( \int_{Q_1} |v_\lambda^\eta(x, t) - \langle v_\lambda^\eta \rangle_t + \xi(x) - \langle \xi \rangle|^{N/(N-1)} dx )^{(N-1)/N} \left( \int_{Q_1} |\text{div} V(x, t)|^{N} dx \right)^{1/N}
\leq c_I \left( \int_{Q_1} |Dv_\lambda^\eta(x, t) + P| \, dx \right) \| \text{div} V(\cdot, t) \|_{L^N(Q_1)} ,
\]

and, since

\[
\int_{Q_1^+} (\xi(x) - \langle \xi \rangle) \text{div} V(x, t) \, dx dt = \langle \langle x \text{div} V \rangle, P \rangle,
\]

\[
\int_{Q_1^+} \lambda v_\lambda^\eta(x, t) \, dx dt \geq \int_{Q_1} (1 - c_I \| \text{div} V(\cdot, t) \|_{L^N(Q_1)}) \left( \int_{Q_1} |Dv_\lambda^\eta(x, t) + P| \, dx \right) dt
+ \langle \langle V \rangle + \langle x \text{div} V \rangle, P \rangle .
\]

Finally, in view of (1.5), we have, for all \( t \geq 0 \),

\[
1 - c_I \| \text{div} V(\cdot, t) \|_{L^N(Q_1)} \geq 0 ,
\]

while the periodicity of \( v_\lambda \) implies, again for all \( t \geq 0 \), that

\[
|P| = \left| \int_{Q_1} (Dv_\lambda^\eta(x, t) + P) \, dx \right| \leq \int_{Q_1} |Dv_\lambda^\eta(x, t) + P| \, dx .
\]

Letting \( \eta \to 0 \) and then \( \lambda \to 0 \) in (2.19) we obtain (1.6).

Error estimates for the periodic homogenization of coercive Hamilton-Jacobi equations were obtained earlier in [10]. Although the proof of Theorem 1.2 is almost the same as the one of the analogous result in [10], we present it here for the sake of completeness. To simplify the presentation below we denote by \( C \) constants that may change from line to line but depend only on \( u_0, \| Du_0 \|_\infty \) and \( V \).

We have:

**Proof of Theorem 1.2:** For all \((x, t, P) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N\), set

\[
H(x, t, P) = |P| + \langle V(x, t), P \rangle .
\]

To avoid any technical difficulties due to the unboundedness of the domain, first we assume that \( u_0 \) is \((M\epsilon)\mathbb{Z}^N\)-periodic for some positive integer \( M \), which implies that \( u^\epsilon \) and \( \bar{u} \) are also \((M\epsilon)\mathbb{Z}^N\)-periodic, and we obtain the estimate with constant independent of \( M \). Then we use the finite speed of propagation property of the averaged initial value problem, to remove the restriction on \( u_0 \).

Let \( v_\lambda = v_\lambda(\cdot, ; P) \in BUC(\mathbb{R}^{N+1}) \) be the \( \mathbb{Z}^{N+1}\)-periodic solution to (2.1) and recall that the map

\[
P \to \nu_\lambda(\cdot, ; P) \text{ is } (1 + \| V \|_\infty) \text{-Lipschitz continuous;}
\]

(2.20)
to simplify statements heretofore we say the \( f \) is \( L \)-Lipschitz continuous if it is Lipschitz continuous with constant at most \( L \).

Fix \( T > 0 \) and consider \( \Phi : \mathbb{R}^N \times [0, T] \times \mathbb{R}^N \times [0, T] \to \mathbb{R} \) given by

\[
\Phi(x, t, y, s) = u^\epsilon(x, t) - u(y, s) - \varepsilon v_\lambda \left( x - \frac{t}{\varepsilon}, y - \frac{s}{\varepsilon}, t - s \frac{\varepsilon}{2} \right) - \frac{|x - y|^2}{2\varepsilon^2} - \frac{(t - s)^2}{2\varepsilon^2} - \delta s ,
\]
where $\beta \in (0,1)$, $\lambda \in (0,1)$ and $\delta > 0$ are to be chosen later.

Since $\Phi$ is periodic in the space variables, it has a maximum at some point $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$. The main part of the proof consists in showing that either $\hat{t} = 0$ or $\hat{s} = 0$ for a suitable choice of $\lambda$ and $\delta$.

We argue by contradiction and assume that $\hat{t} > 0$ and $\hat{s} > 0$. Since $\hat{u}$ is Lipschitz continuous and (2.20) holds, by standard arguments from the theory of viscosity solutions, we have

$$\frac{|\hat{x} - \hat{y}|}{\varepsilon^3} \leq C \left(1 + \frac{\varepsilon^{1-\beta}}{\lambda}\right) \quad \text{and} \quad \frac{|\hat{t} - \hat{s}|}{\varepsilon} \leq C. \quad (2.21)$$

We claim that

$$\frac{\hat{t} - \hat{s}}{\varepsilon} \leq \hat{H} \left(\frac{\hat{x} - \hat{y}}{\varepsilon^3}\right) + C \left(\frac{\varepsilon^{1-\beta}}{\lambda} + \lambda\right). \quad (2.22)$$

Indeed for $\alpha, \beta > 0$ small, let $(x_0, t_0, y_0, r_0, z_0)$ be a maximum point of $\Psi_1$ given by

$$\Psi_1(x,t,y,r,z) = u^\varepsilon(x,t) - \varepsilon v_\lambda(y, r, z - \hat{y}) - \frac{|x - \hat{y}|^2}{2\varepsilon^3} - \frac{(t - \hat{s})^2}{2\varepsilon} - \frac{1}{2\alpha}(|\varepsilon y - x|^2 + |z - x|^2 + |\varepsilon r - t|^2) - \frac{\beta^2}{2}(|x - \hat{x}|^2 + (t - \hat{t})^2).$$

Since $(\hat{x}, \hat{t})$ is the unique maximum point of $\Psi_1(x,t,x/\varepsilon,t/\varepsilon,x)$, we have that $(x_0, t_0, y_0, r_0, z_0)$ converges to $(\hat{x}, \hat{t}, \hat{x}/\varepsilon, \hat{t}/\varepsilon, \hat{x})$ as $\alpha \to 0$, with

$$\lim_{\alpha \to 0^+} \frac{1}{2\alpha} \left(|\varepsilon y_0 - x_0|^2 + |z_0 - x_0|^2 + |\varepsilon r_0 - t_0|^2\right) = 0, \quad (2.23)$$

while (2.20) implies that

$$\frac{|z_0 - x_0|}{\alpha} \leq C \varepsilon^{1-\beta} \lambda. \quad (2.24)$$

From the equation satisfied by $u^\varepsilon$ we have

$$\frac{t_0 - \hat{s}}{\varepsilon} + \frac{t_0 - \varepsilon r_0}{\alpha} + \beta(t_0 - \hat{t}) \leq H \left(\frac{x_0}{\varepsilon}, \frac{t_0}{\varepsilon}, \frac{x_0 - \hat{y}}{\varepsilon^3} + \frac{x_0 - \varepsilon y_0}{\alpha} + \frac{x_0 - z_0}{\alpha} + \beta(x_0 - \hat{x})\right), \quad (2.25)$$

while from the equation satisfied by $v_\lambda$ we also have

$$\frac{t_0 - \varepsilon r_0}{\alpha} + \lambda v_\lambda \left(\frac{y_0, r_0, z_0}{\varepsilon^3} - \frac{\hat{y}}{\varepsilon^3}\right) \geq H \left(\frac{y_0, r_0, x_0 - \varepsilon y_0}{\alpha} + \frac{z_0 - \hat{y}}{\varepsilon^3}\right).$$

Using the bound on the oscillation of the $\lambda v_\lambda$’s in Lemma 2.1, we get

$$\hat{H} \left(\frac{z_0 - \hat{y}}{\varepsilon^3}\right) + C\lambda \left|\frac{z_0 - \hat{y}}{\varepsilon^3}\right| \geq H \left(\frac{y_0, r_0, x_0 - \varepsilon y_0}{\alpha} + \frac{z_0 - \hat{y}}{\varepsilon^3}\right) - \frac{t_0 - \varepsilon r_0}{\alpha}. \quad \text{Using this inequality with (2.25) and using the regularity of } H \text{ gives}$$

$$\hat{H} \left(\frac{z_0 - \hat{y}}{\varepsilon^3}\right) + C\lambda \left|\frac{z_0 - \hat{y}}{\varepsilon^3}\right| \geq \frac{t_0 - \hat{s}}{\varepsilon} + \beta(t_0 - \hat{t}) - C\left|\frac{z_0 - x_0}{\varepsilon^3} + \frac{z_0 - x_0}{\varepsilon^3}\right| + \beta|x_0 - \hat{x}|)$$

$$- C (\frac{x_0 - y_0}{\varepsilon} + \frac{t_0}{\varepsilon} - r_0) (|\frac{x_0 - \varepsilon y_0}{\alpha}| + \frac{z_0 - \hat{y}}{\varepsilon^3}).$$
Using (2.23) and (2.24) we now let \( \alpha \to 0 \) to get
\[
\overline{H} \left( \frac{\hat{x} - \hat{y}}{\varepsilon^\beta} \right) + C \lambda \frac{|\hat{x} - \hat{y}|}{\varepsilon^\beta} \geq \frac{\hat{t} - \hat{s}}{\varepsilon} - C \frac{\varepsilon^{1-\beta}}{\lambda}.
\]
Recalling (2.21) we obtain
\[
\overline{H} \left( \frac{\hat{x} - \hat{y}}{\varepsilon^\beta} \right) + C \left( \lambda + \varepsilon^{1-\beta} + \frac{\varepsilon^{1-\beta}}{\lambda} \right) \geq \frac{\hat{t} - \hat{s}}{\varepsilon},
\]
and, since \( \lambda \in (0,1) \), we finally get (2.22).

We now show that
\[
\frac{\hat{t} - \hat{s}}{\varepsilon} - \delta \geq \overline{H} \left( \frac{\hat{x} - \hat{y}}{\varepsilon^\beta} \right) - C \frac{\varepsilon^{1-\beta}}{\lambda}.
\]
To this end, for \( \alpha, \beta > 0 \), we consider
\[
\Psi_2(y,s,z) = \bar{u}(y,s) + \varepsilon v_\lambda \left( \frac{\hat{x} - \hat{y}}{\varepsilon^\beta} \right) + \frac{|\hat{x} - y|^2}{2\varepsilon^\beta} + \frac{(\hat{t} - s)^2}{2\varepsilon} + \frac{|z - y|^2}{2\alpha} + \frac{\beta}{2} \left( |y - \hat{y}|^2 + |s - \hat{s}|^2 \right),
\]
which has a minimum at some point \( (y, s, z) \). Using (2.20) and the fact that \((\hat{y}, \hat{s})\) is the unique minimum of the map \((y, s) \to \Psi_2(y, s, y)\), we find that \( (y, s, z) \) converges to \((\hat{y}, \hat{s}, \hat{y})\) as \( \alpha \to 0 \), with
\[
\frac{|z - y|}{\alpha} \leq C \frac{\varepsilon^{1-\beta}}{\lambda} \quad \text{and} \quad \lim_{\alpha \to 0} \frac{|z - y|^2}{\alpha} = 0.
\]
Since \( \bar{u} \) solves (1.3) we have
\[
\frac{\hat{t} - s_0}{\varepsilon} - \delta \geq \overline{H} \left( \frac{\hat{x} - y_0}{\varepsilon^\beta} - \frac{y_0 - z_0}{\alpha} - \beta(y_0 - \hat{y}) \right),
\]
and, in view of the Lipschitz continuity of \( \overline{H} \), letting \( \alpha \to 0^+ \) gives (2.26).

Combining (2.22) and (2.26) we obtain
\[
\delta - C \left( \frac{\varepsilon^{1-\beta}}{\lambda} + \lambda \right) \leq 0,
\]
which for \( \lambda = \varepsilon^{(1-\beta)/2} \) and \( \delta = 3C\varepsilon^{(1-\beta)/2} \) gives a contradiction. So either \( \hat{t} = 0 \) or \( \hat{s} = 0 \).

Next we estimate \( \max_{x,y,s} \Phi(x,0,y,s) \) and \( \max_{x,t,y} \Phi(x,t,0) \). We have:
\[
\max_{x,y,s} \Phi(x,0,y,s) \leq \max_{x,y,s} \left\{ u_0(x) - u_0(y) + Cs + C \frac{\varepsilon^{1-\beta}}{\lambda} |x - y| - \frac{|x - y|^2}{2\varepsilon^\beta} - \frac{s^2}{2\varepsilon} - \delta s \right\}
\]
\[
\leq \max_{x,y} \left\{ C |y - x| + C \varepsilon - \frac{|x - y|^2}{2\varepsilon^\beta} \right\}
\]
\[
\leq C \left( \varepsilon^\beta + \varepsilon \right) \leq C \varepsilon^\beta,
\]
and, similarly,
\[
\max_{x,y,s} \Phi(x,t,0) \leq C \varepsilon^\beta.
\]
Therefore, for any \((x, t) \in \mathbb{R}^N \times [0, T]\), we have

\[
u^\varepsilon(x, t) - \bar{u}(x, t) \leq \delta t + C\varepsilon^\beta \leq CT\varepsilon^{(1-\beta)/2} + C\varepsilon^\beta.
\]

Choosing \(\beta = 1/3\) finally gives

\[
u^\varepsilon(x, t) - u(x, t) \leq C\varepsilon^{1/3}.
\]

This reverse inequality can be obtained in the same way.

\[
\square
\]

3. Enhancement of speed

To prove Theorem 1.3 and Theorem 1.4 we need three results which we formulate as Lemmas next but present their proof at the end of the section.

We begin with

**Lemma 3.1.** Let \(V\) and \(\bar{H}\) be as in Theorem 1.1. Then, for any \(Z\) periodic \(c \in C^1(\mathbb{R}; \mathbb{R}^N)\), the averaged Hamiltonian associated to \(V - c\) is \(\bar{H} - \langle \int_0^1 c(s)ds, \cdot \rangle\).

The second is

**Lemma 3.2.** The divergence zero condition (1.9) is equivalent to the fact that, for any fixed time \(t\), the map \(x \to z(x, t) + \langle P, x \rangle\) is constant along the flow of the ode \(X'(s) = V(X(s), t)\).

To state the final result we recall the notion of an \(\varepsilon\)-mollifier. To this end, let \(\phi \in C_\infty^\infty(\mathbb{R}^N + 1; [0, 1])\) be such that \(\phi(0) = 1\) and \(\int \phi = 1\) and define the \(\varepsilon\)-mollifier \(\phi_\varepsilon\) by \(\phi_\varepsilon = \varepsilon^{-N+1}\phi(x/\varepsilon, t/\varepsilon)\). Then \(\int \phi_\varepsilon = 1\) and, for any \(f \in L^1_{loc}(\mathbb{R}^{N+1})\), \(f_\varepsilon = f * \phi_\varepsilon\) is a smooth approximation of \(f\).

For the rest of the section we assume that, for each \(t \in \mathbb{R}\), \(\text{div}_x V(\cdot, t) = 0\) and \(\int_{Q_1} V(x, t)dx = 0\); (3.1)

recall that the average zero condition is actually not a restriction, since we can always replace \(V\) by \(V - \int_{Q_1} V(x, t)dx\) a fact, which, in view of Lemma 3.1, simply adds a translation to the effective Hamiltonian.

The final preliminary result is

**Lemma 3.3.** Assume (1.1) and fix \(P \in \mathbb{R}^N\). There exists a bounded, \(Z^{N+1}\) periodic \(z \in BV_{loc}(\mathbb{R}^N \times \mathbb{R})\) such that, for all \(\varepsilon > 0\) the smooth functions \(z_\varepsilon = z * \phi_\varepsilon\)'s satisfy in \(\mathbb{R}^{N+1}\)

\[
z_{\varepsilon, t} + \bar{H}(P) \geq |Dz_\varepsilon + P|
+ \int_{\mathbb{R}^{N+1}} (z(y, s) + \langle P, y \rangle) \langle D\phi_\varepsilon(\cdot - y, \cdot - s), V(y, s) \rangle dyds .
\]

We proceed with Theorem 1.3, which is a straightforward consequence of Theorem 1.4. We have:

**Proof of Theorem 1.3:** If \(\bar{H}(P) = |P|\), then let

\[
 z(x, t) = \hat{z}(\frac{\langle P, x \rangle}{|P|} + t) \quad \text{and} \quad z_1(x) = \int_0^1 z(x, s)ds,
\]

where \(\hat{z}\) is given by Theorem 1.4.
Since $V$ is independent of $x$, $z_1$ also satisfies (1.3) in the sense of distributions, and, since $z$ is periodic $z_1$ is actually a constant. In this case (1.3) reduced to $\langle V(x), P \rangle = 0$ for all $x \in \mathbb{R}^N$.

When $N = 2$ and $\langle V \rangle = 0$ there exists a $\mathbb{Z}^2$-periodic stream function $E : \mathbb{R}^2 \to \mathbb{R}$ such that $V = (-\frac{\partial E}{\partial x_2}, \frac{\partial E}{\partial x_1})$. In this case, if $P = (P_1, P_2)$ and $q = (P_1, P_2)$, $\langle V, P \rangle = 0$ in $\mathbb{R}^2$ becomes

$$0 = \langle DE(x), q \rangle .$$

If $q$ is an irrational direction, then the map $t \to x + tq$ is dense in $\mathbb{R}^2/\mathbb{Z}^2$. Since $E$ is constant along this trajectory, $E$ is constant and therefore $V$ is identically equal to 0. Otherwise, $t \to x + tq$ is periodic and $E$ has to be constant along this trajectory. This means that $E = \bar{E}(\cdot, P)$ for some smooth periodic map $\bar{E} : \mathbb{R} \to \mathbb{R}$, and, hence, $V$ is a shear advection.

Conversely, if $\langle V, P \rangle = 0$ in $\mathbb{R}^N$, then $v_\lambda = |P|/\lambda$ is the unique solution to (2.1), and $\lambda v_\lambda = |P|$ clearly converges uniformly to $\bar{H}(P) = |P|$. 

$\square$

We turn now to the Proof of Theorem 1.4: Assume that, for some $P \in \mathbb{R}^N \setminus \{0\}$, $\bar{H}(P) = P$ and let $z$ be given by Lemma 3.3. We first prove that $z$ is a function of only $\langle P, x \rangle$ and $t$. More precisely, we claim that there exists $\tilde{z} \in BV_{loc}(\mathbb{R}^2)$ such that

$$z = \tilde{z}(\langle P, x \rangle, t) \quad \text{and} \quad \tilde{z}_s(s, t) \geq -1 \quad \text{in the sense of distributions.} \quad (3.3)$$

To this end, let $\phi_\varepsilon$ and $z_\varepsilon = z * \phi_\varepsilon$ be as in Lemma 3.3; note that $z_\varepsilon$ is $\mathbb{Z}^{N+1}$-periodic. Then, for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$,

$$\int_{Q_1} \int_{\mathbb{R}^{N+1}} (z(y, s) + \langle P, y \rangle)(D\phi_\varepsilon(x - y, t - s), V(y, s))dyds dx = 0 . \quad (3.4)$$

Indeed the periodicity of $z$ and $V$ gives,

$$\int_{Q_1} \int_{\mathbb{R}^{N+1}} (z(y, s) + \langle P, y \rangle)(D\phi_\varepsilon(x - y, t - s), V(y, s))dyds dx = \int_{\partial Q_1} \int_{\mathbb{R}^{N+1}} (z(y, s) + \langle P, y \rangle)\phi_\varepsilon(x - y, t - s)\langle V(y, s), \nu_x \rangle dyds d\mathcal{H}^{N-1}(x)$$

$$= \int_{\mathbb{R}^N} \int_{\partial Q_1} \int_{\mathbb{R}^N} (z(x - y, t - s) + \langle P, y \rangle)\phi_\varepsilon(y, s)\langle V(x - y, t - s), \nu_x \rangle d\mathcal{H}^{N-1}(x) dyds$$

$$= \int_{\mathbb{R}^N} \int_{\partial Q_1} \langle P, x - y \rangle\phi_\varepsilon(y, s)\langle V(x - y, t - s), \nu_x \rangle d\mathcal{H}^{N-1}(x) dyds .$$

while (3.1) yields

$$\int_{\mathbb{R}^N} \int_{\partial Q_1} \langle P, x - y \rangle\phi_\varepsilon(y, s)\langle V(x - y, t - s), \nu_x \rangle d\mathcal{H}^{N-1}(x) dyds$$

$$= \int_{\mathbb{R}^N} \int_{\partial Q_1} \text{div}_x (\langle P, x - y \rangle\phi_\varepsilon(y, s)V(x - y, t - s)) dx dyds$$

$$= \int_{\mathbb{R}^N} \int_{Q_1} \langle P, V(x - y, t - s) \rangle \phi_\varepsilon(y, s) dx dyds = 0 ,$$

and, hence, (3.4) holds.
Next we integrate (3.2) over $Q_1 \times (0,1)$. Using (3.4) and the periodicity of $z_\varepsilon(x,t)$ we get
\[
|P| = \bar{H}(P) \geq \int_0^1 \int_{Q_1} |Dz_\varepsilon(y,t)+P|dydt \geq \int_0^1 \left| \int_{Q_1} (Dz_\varepsilon(y,t)+P)dy \right| dt = |P|.
\]

It follows that, for all $(x,t) \in \mathbb{R}^{N+1}$, there exists $\theta(x,t) \geq -1$ such that $Dz_\varepsilon(x) = \theta(x,t)P$. Thus $z_\varepsilon$ is of the form $z_\varepsilon(x,t) = \bar{z}_\varepsilon(\langle x,P \rangle,t)$, with $\bar{z}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\bar{z}_\varepsilon(s,t) \geq -1$.

Passing to the limit $\varepsilon \to 0$, we also find that $z = \bar{z}(\langle x,P \rangle,t)$ for some map $\bar{z} \in BV_{loc}(\mathbb{R}^2,\mathbb{R})$ satisfying $\bar{z}_s(s,t) \geq -1$ in the sense of distributions. Whence (3.3) holds.

Next we claim that $\bar{z}$ satisfies, in the sense of distributions,
\[
\bar{z}_t(s,t) = \bar{z}_s(s,t)|P| \quad \text{in } \mathbb{R}^2,
\]
and that (1.9) holds. Note that this proves the “if” part, since (3.5) implies the existence of a map $\bar{z} \in BV_{loc}(\mathbb{R},\mathbb{R})$ such that
\[
\bar{z}(s,t) = \bar{z}(|P|^{-1}s+t).
\]

Moreover we have $\bar{z}'(s) \geq -|P|$ in the sense of distributions because $\bar{z}_s(s,t) \geq -1$ in the same sense. Finally, $z(x,t) = \bar{z} \left( \frac{|P|}{|P|} + t \right)$ is periodic in space and time.

We continue with the proofs of (3.4) and (1.9). If $z$ is constant, then (3.3) is obvious and (1.9) just follows from (3.2) when $\varepsilon \to 0$.

Next we assume that $z$ is not constant. In this case $z = \bar{z}(\langle \cdot,P \rangle,t)$ is $\mathbb{Z}^{N+1}$-periodic and not constant. Therefore $P$ has to be a rational direction. So, up to a rational change of coordinates, we may assume without loss of generality that $P = 2\varepsilon c_1$ for some $\theta > 0$, while $V$ is still $\mathbb{Z}^{N+1}$-periodic.

Using the notation $x = (x_1,x')$ for each vector of $\mathbb{R}^N$ with $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$, for a fixed $(x_1,t) \in \mathbb{R}^2$ we integrate (3.2) over the cube $\{x_1\} \times Q'_1 \times \{t\}$, where $Q'_1 = (-1/2,1/2)^{N-1}$, and obtain
\[
\int_{Q_1} \int_{\mathbb{R}^{N+1}} (\bar{z}(\theta(x_1-y_1),t-s) + \theta(x_1-y_1))(D\phi_\varepsilon(y,s),V(x-y,t-s))dyds dx'.
\]

It turns out that the last integral in the right-hand side of the above inequality vanishes. Indeed
\[
\int_{Q'_1} \int_{\mathbb{R}^{N-1}} (D\phi_\varepsilon(y,s),V(x-y,t-s))dy' dx' = \int_{\mathbb{R}^{N-1}} \frac{\partial \phi_\varepsilon(y,s)}{\partial x_1} V_1(x-y,t-s)dx' dy' + \sum_{j=2}^N \int_{Q'_1} \int_{\mathbb{R}^{N-1}} \phi_\varepsilon(y,s) \frac{\partial V_j}{\partial x_j}(x-y,t-s)dy'dx'.
\]

The periodicity of $V$ yields, for any $j = 2,\ldots,N$,
\[
\int_{\mathbb{R}^{N-1}} \phi_\varepsilon(y,s) \int_{Q'_1} \frac{\partial V_j}{\partial x_j}(x-y,t-s)dy'dx' = 0,
\]
while the divergence free condition and, again, the periodicity give
\[ \frac{\partial}{\partial x_1} \int_{Q'_1} V_1(x-y,t-s)dx' = \int_{Q'_1} \frac{\partial V_1}{\partial x_1}(x-y,t-s)dx' = -\sum_{j=2}^N \int_{Q'_1} \frac{\partial V_1}{\partial x_j}(x-y,t-s)dx' = 0. \]

On the other hand,
\[ \int_{-1/2}^{1/2} \int_{Q'_1} V_1(x-y,t-s)dx'dx_1 = \int_{Q_1} V_1(x,t-s)dx = 0, \]
and, hence, for all \( x_1 \in \mathbb{R} \),
\[ \int_{Q'_1} V_1(x-y,t-s)dx' = 0. \]

Therefore
\[ \int_{Q'_1} \int_{\mathbb{R}^{N-1}} (D\phi_\varepsilon(y,s), V(x-y,t-s))dy' dx' = 0, \]
which, going back to (3.6), proves that \( x \) and, hence, for all \( \varepsilon \)
shows that in fact it must be an equality. Letting \( \varepsilon \to 0 \) then gives (3.5).

Since \( \hat{z} \) is periodic, integrating the above inequality over \((-1/2,1/2) \times (0,1)\) shows that in fact it must be an equality. Letting \( \varepsilon \to 0 \) then gives (3.5).

To prove (1.9), we first combine (3.2), (3.3) and (3.5) to get, for all \( (x,t) \in \mathbb{R}^{N+1} \),
\[ 0 \geq \int_{\mathbb{R}^{N+1}} (z(y,s) + \langle P,y \rangle)(D\phi_\varepsilon(x-y,t-s),V(y,s))dyds. \]

Averaging over the cube \( Q_1 \) we see that, as a matter of fact, equality must hold for all \( (x,t) \in \mathbb{R}^{N+1} \).

Integrating the resulting equality against any compactly supported smooth function \( \psi : \mathbb{R}^N \to \mathbb{R} \) we get
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^{N+1}} \psi(x)(z(y,s) + \langle P,y \rangle)(D\phi_\varepsilon(x-y,t-s),V(y,s))dyds dx = 0, \]
and after integrating again by parts and letting \( \varepsilon \to 0 \), we obtain, for all \( t \in \mathbb{R} \),
\[ \int_{\mathbb{R}^N} (D\psi(x), (z(x,t) + \langle P,x \rangle)V(x,t))dx = 0, \]
which is exactly (1.3).

To prove the “only if” part let \( \hat{z} \) be as claimed. Since \( \hat{z} \in BV_{loc}(\mathbb{R},\mathbb{R}) \), we may assume without loss of generality that \( \hat{z} \), and, hence, \( z(x,t) = \hat{z}(\frac{\langle P,x \rangle}{|P|} + t) \) are lower semi-continuous.

We show next that \( z \) satisfies, in the viscosity sense,
\[ \partial_t z + |P| \geq |Dz + P| + \langle V, Dz + P \rangle \quad \text{in} \quad \mathbb{R}^{N+1}. \quad (3.7) \]

To this end, let \( \phi \) be a smooth test function such that \( z \geq \phi \) with equality at \( (\bar{x},\bar{t}) \). It follows from equality \( z(x,t) = \hat{z}(\frac{\langle P,x \rangle}{|P|} + t) \) that \( D\phi(\bar{x},\bar{t}) = \theta |P| |P| \) where \( \theta = \phi_t(\bar{x},\bar{t}) \).

Since \( \hat{z}' \geq -|P| \) in the sense of distribution, it follows that \( \theta \geq -|P| \). Moreover recalling (1.5) and Lemma 3.2, we have, for a fixed \( t \), that the function \( x \to V(x,t) + \langle P,x \rangle \) is constant under the flow of the ode \( X'(s) = V(X(s),t) \).
Let now $X$ be a solution with $X(0) = \bar{x}$ and $t = \bar{t}$. Then, for any $s \in \mathbb{R}$,
\[ \phi(\bar{x}, \bar{t}) + \langle P, \bar{x} \rangle = z(\bar{x}, \bar{t}) + \langle P, \bar{z} \rangle = z(X(s), \bar{t}) + \langle P, X(s) \rangle \geq \phi(X(s), \bar{t}) + \langle P, X(s) \rangle, \]
and, therefore,
\[ 0 = \frac{d}{ds}_{|s=0} \left[ \phi(X(s), \bar{t}) + \langle X(s), P \rangle \right] = \langle D\phi(\bar{x}, \bar{t}) + P, V(\bar{x}, \bar{t}) \rangle. \]

Combining the above relations gives
\[ \phi_t(\bar{x}, \bar{t}) + |P| = |D\phi(\bar{x}, \bar{t}) + P| + \langle V(\bar{x}, \bar{t}), D\phi(\bar{x}, \bar{t}) + P \rangle, \]
i.e., $z$ is a super-solution of (3.7).

Then it is easy to check that $\tilde{z}(x, t) = z(x, t) + \frac{|P|}{\lambda} + \|z\|_{\infty}$ is a super-solution of (2.1). It then follows from the comparison principle that $v_\lambda \leq z + \lambda^{-1}|P| + \|z\|_{\infty}$.

Recalling that $\lambda v_\lambda$ converges uniformly, as $\lambda \to 0$, to $\tilde{H}(P)$, we obtain $\tilde{H}(P) \leq |P|$. Since the reverse inequality always holds, the proof of the implication is complete.

We continue with the proofs of the lemmas. We begin with the

**Proof of Lemma 3.1:** Fix $\lambda > 0$ and $P \in \mathbb{R}^N$, let $v_\lambda$ be the solution of (2.1) and we recall that $\lambda v_\lambda$ converges uniformly, as $\lambda \to 0$, to $H(P)$.

Set $c = \int_0^1 c(s) ds$ and consider $w_\lambda \in BUC(\mathbb{R}^{N+1})$ given by
\[ w_\lambda(x, t) = v_\lambda(x - \int_0^t c(s) ds, t) - \left( \int_0^t (c(s) - c) ds + \frac{c}{\lambda}, P \right) + 2\|c\|_{\infty} |P|. \]

It follows that $w_\lambda$ is a super-solution of
\[ z_{\lambda,t} + \lambda z_\lambda = |Dz_\lambda + P| + \langle V - c, Dz_\lambda + P \rangle \quad \text{in} \quad \mathbb{R}^{N+1}. \quad (3.8) \]

We only present a formal proof, which can be easily justified using viscosity solution arguments. To this end observe that it is immediate from the definition of $w_\lambda$ that
\[ w_{\lambda,t} + \lambda w_\lambda \geq -\langle Dv_\lambda, c(t) \rangle + v_{\lambda,t} - \langle c - \bar{c}, P \rangle + \lambda v_\lambda - (\bar{c}, P), \]
while
\[ |Dw_\lambda + P| + \langle V - c, Dw_\lambda + P \rangle = |Dv_\lambda + P| + \langle V - c, Dw_\lambda + P \rangle; \]
the comparison principle now gives $w_\lambda \geq z_\lambda$ where $z_\lambda$ is the solution of (3.8).

Since the $z_{\lambda,t}$’s and the $\lambda w_\lambda$’s converge uniformly, as $\lambda \to 0$, to the averaged Hamiltonian $\tilde{H}_c(P)$ associated to $V - c$ and to $\tilde{H}(P) - (\bar{c}, P)$ respectively, we get $\tilde{H}_c(P) \leq \tilde{H}(P) - (\bar{c}, P)$. The opposite inequality is proved similarly by considering $-c$ instead of $c$.

We continue with the

**Proof of Lemma 3.2:** Let $X_x(\cdot)$ be the solution of $X'(s) = V(X(s), t)$ with initial condition $x$ at time $s = 0$. Then, for any $h \in \mathbb{R}$ and $\psi : C_c^\infty(\mathbb{R}^N)$ the divergence zero property of $V$ yields
\[ \int_{\mathbb{R}^N} \psi(x)(z(X_x(h), t) + \langle P, X_x(h) \rangle) dx = \int_{\mathbb{R}^N} \psi(x)(z(-h)(z(x, t) + \langle P, x \rangle) dx. \]
Therefore, in view of (1.9),
\[
\frac{d}{dh}|_{h=0} \int_{\mathbb{R}^N} \psi(x) (z(X_x(h), t) + \langle P, X_x(h) \rangle) dx
= \int_{\mathbb{R}^N} \langle D\psi(x), V(x, t) \rangle \psi(x) dx = 0 ,
\]
Applying this last equality to the test function \( \psi \circ X_x(-s) \) we get
\[
\frac{d}{dh}|_{h=0} \int_{\mathbb{R}^N} \psi(x) (z(X_x(s+h), t) + \langle P, X_x(s+h) \rangle) dx
= \frac{d}{dh}|_{h=0} \int_{\mathbb{R}^N} \psi(X_x(-s)) (z(X_x(h), t) + \langle P, X_x(h) \rangle) dx = 0 .
\]
Hence \( \int_{\mathbb{R}^N} \psi(x) (z(X^x(s), t) + \langle P, X^x(s) \rangle) dx \) is constant in time, which means that \( z(\cdot, t) \) is constant along the flow.

We conclude with the
**Proof of Lemma 3.3**: For \( \lambda > 0 \), let \( v_\lambda \) be the solution of of (2.1) and set
\[
z_\lambda(x, t) = v_\lambda(x, t) - v_\lambda(0, 0).
\]
It follows from Lemma 2.1 and Corollary 2.2 that, for some \( C > 0 \) independent of \( \lambda \),
\[
\|z_\lambda\|_{\infty} \leq C|P|.
\]
Next we show that the \( z_\lambda \)'s are also bounded in \( BV_{loc}(\mathbb{R}^{N+1}) \). Indeed, for \( \alpha > 0 \), consider the \( \mathbb{Z}^{N+1} \)-periodic solution \( v_{\lambda, \alpha} \) to
\[
v_{\lambda, \alpha, t} + \lambda v_{\lambda, \alpha} = \alpha \Delta v_{\lambda, \alpha} + |Dv_{\lambda, \alpha} + P| + \langle V, Dv_{\lambda, \alpha} + P \rangle \quad \text{in } \mathbb{R}^{N+1} , \quad (3.9)
\]
which is at least in \( C^{1,1} \) and, moreover, converges uniformly, as \( \alpha \to 0 \), to \( v_\lambda \).
Integrating (3.9) over a cylinder of the form \( Q_R \times (-R, R) \) for some positive integer \( R \), we obtain, using the periodicity, that
\[
\int_{Q_R \times (-R, R)} \lambda v_{\lambda, \alpha} \geq \int_{Q_R \times (-R, R)} |Dv_{\lambda, \alpha} + P| + \int_{Q_R \times (-R, R)} \langle V(x, t), Dv_{\lambda, \alpha}(x, t) + P \rangle dx dt .
\]
Since, in view of (3.1),
\[
\int_{Q_R} \langle V(x, t), Dv_{\lambda, \alpha}(x, t) + P \rangle dx = 0,
\]
it follows that
\[
\frac{d}{dh}|_{h=0} \int_{Q_R \times (-R, R)} |Dv_{\lambda, \alpha} + P| \leq 2R^{N+1} \|\lambda v_{\lambda, \alpha}\|_{\infty} . \quad (3.10)
\]
Let $\Phi \in C^\infty_c(Q_R \times (-R,R); \mathbb{R}^{N+1})$ be such that $|\Phi(x,t)| \leq 1$ for all $(x,t) \in \mathbb{R}^{N+1}$. From (3.9) and (3.10) we get

$$\int_{Q_R \times (-R,R)} v_{\lambda,\alpha} \text{div}_x \Phi dxdt = - \int_{Q_R \times (-R,R)} [v_{\lambda,\alpha} \Phi + (Dv_{\lambda,\alpha}, \Phi)]dxdt =$$

$$- \int_{Q_R \times (-R,R)} [((\lambda \nu_{\lambda,\alpha} - \alpha \Delta v_{\lambda,\alpha} - |Dv_{\lambda,\alpha} + P| - (V, Dv_{\lambda,\alpha} + P))\Phi + (Dv_{\lambda,\alpha}, \Phi)]dxdt \leq C_R(\lambda \|v_{\lambda,\alpha}\|_{\infty} + \alpha \int_{Q_R \times (-R,R)} v_{\lambda,\alpha} \Delta \Phi dxdt),$$

where $C_R(P)$ depends only on $N$, $R$, $\|V\|_{\infty}$ and $P$.

Letting $\alpha \to 0$ yields

$$\int_{Q_R \times (-R,R)} v_{\lambda} \text{div}_x \Phi dxdt \leq C_R(P)\|v_{\lambda}\|_{\infty} \leq C_R(P)(\tilde{\Phi}(P) + CL), \quad (3.11)$$

which in turn implies, in view of the assumptions on $\Phi$, that the $z_{\lambda}$’s are bounded in $L^\infty$ and in $BV_{loc}$ uniformly with respect to $\lambda$. Hence the $z_{\lambda}$’s converge, up to a subsequence and in $L^1_{loc}$, to some $z \in BV_{loc}(\mathbb{R}^{N+1})$.

Let $\varepsilon > 0$, $\phi_\varepsilon$ and $z_\varepsilon$ as in statement of the lemma, set $z_{\lambda,\varepsilon} = \phi_\varepsilon \ast z_\lambda$, and $z_{\lambda,\alpha,\varepsilon} = \phi_\varepsilon \ast z_{\lambda,\alpha}$ and fix $(x,t) \in \mathbb{R}^{N+1}$.

It follows from (3.1) that

$$z_{\lambda,\alpha,\varepsilon,\theta} + \lambda z_{\lambda,\alpha,\varepsilon} + \lambda \nu_{\lambda,\alpha}(0,0) \geq \alpha \Delta z_{\lambda,\alpha,\varepsilon} + |Dz_{\lambda,\alpha,\varepsilon} + P| + \int_{\mathbb{R}^{N+1}} \phi_\varepsilon(x-y,t-s)(V(y,s), Dz_{\lambda,\alpha}(y,s) + P)dyds.$$

Integrating by parts and using (3.1) we find

$$z_{\lambda,\alpha,\varepsilon,\theta} + \lambda z_{\lambda,\alpha,\varepsilon} + \lambda \nu_{\lambda,\alpha}(0,0) \geq \alpha \Delta z_{\lambda,\alpha,\varepsilon} + |Dz_{\lambda,\alpha,\varepsilon} + P| + \int_{\mathbb{R}^{N+1}} \langle z_{\lambda,\alpha}(y,s) + \langle P, y \rangle \rangle(D\phi_\varepsilon(x-y,t-s), V(y,s))dyds.$$

Letting first $\alpha \to 0$ and then $\lambda \to 0$ gives (3.2).

4. CONVERGENCE TO THE WULFF SHAPE

We begin recalling some important facts from the theory of front propagation. The first is (see, for instance, [3] and the references therein), which we will use repetitively in the sequel, is the following superposition principle of the geometric flow. If $(K_0^t)_{t \geq 0}$ is an arbitrary family of non-empty closed subsets of $\mathbb{R}^N$, with corresponding solution $(K^t)_{t \geq 0}$, then the solution starting from $\bigcup_{t \geq 0} K_0^t$ is given by $(\bigcup_{t \geq 0} K^t(t))_{t \geq 0}$. This can be seen either by using the control representation of the geometric flow or the stability and comparison properties of viscosity solutions. A consequence is the well known inclusion principle. If $K_0 \subset K'_0$ are two non-empty closed subsets of $\mathbb{R}^N$, then the corresponding solutions $(K(t))_{t \geq 0}$ and $(K'(t))_{t \geq 0}$ satisfy $K(t) \subset K'(t)$ for all $t \geq 0$.

We also remark that the convergence, as $t \to \infty$, of the level sets of geometric equations without spatio-temporal inhomogeneities was considered in [18]. The results of [18] do not, however, apply to the problem at hand.
The proof of Theorem 1.5 is long. We formulate two important steps as separate lemmas which we prove at the end of the section.

The first is about some "controllability" estimates. We have:

**Lemma 4.1.** Assume (3.1). There exist a positive integer $n_0$ and $T > 0$ such that, for all $x \in \mathbb{R}^N$, the solution $(\hat{K}(t))_{t \geq 0}$ of the front propagation problem starting from the set $x + \{k \in \mathbb{Z}^N : |k| \leq n_0\}$ contains $Q_1(x)$ at time $T$.

The second is about some growth property for fronts.

**Lemma 4.2.** There exists a positive integer $\tilde{R}$ and positive constants $r$ and $T_1$ such that, if the initial compact set $K_0 \subset \mathbb{R}^N$ contains a set of the form $Q_{\tilde{R}}(k)$ for some $k \in \mathbb{Z}^N$, then the solution $(K(t))_{t \geq 0}$ of the front propagation problem starting from $K_0$ satisfies, for all $t \geq 0$,

$$Q_{rt}(k) \subset K(t + T_1) \quad \forall t \geq 0.$$ 

We continue with the

**Proof of Theorem 1.5**:

We first show (1.12). It is well known (see [6]) that the characteristic function $1_{K(t)}$ of $K(t)$ is a solution to the geometric equation

$$\partial_t u = |Du| + \langle V, Du \rangle \quad \text{in } \mathbb{R}^N \times (0, +\infty) .$$

(4.1)

Fix next a direction $\nu \in \mathbb{R}^N$ with $|\nu| = 1$. For $\lambda > 0$, let $v_\lambda$ be the solution of

$$v_{\lambda,t} + \lambda v_\lambda = |Dv_\lambda + \nu| + \langle V, Dv_\lambda + \nu \rangle \quad \text{in } \mathbb{R}^{N+1},$$

and recall that there is some constant $\bar{C}$ independent of $\nu$ and $\lambda$ such that

$$\text{osc}(v_\lambda) \leq \bar{C} \quad \text{and} \quad \|\lambda v_\lambda - \bar{H}(\nu)\|_\infty \leq \bar{C} \lambda .$$

It is immediate that

$$z(x, t) = v_\lambda(x, t) - v_\lambda(0) + \langle \nu, x \rangle + (\bar{H}(\nu) + \bar{C} \lambda)t + C ,$$

with

$$C = \bar{C} - \min_{\nu \in K_0} \langle \nu, x \rangle + 1 ,$$

is a super-solution of (4.1). Since (4.1) geometric, it is immediate that $\max(z, 0)$ is also a super-solution. Moreover, we clearly have $\max(z(\cdot, 0), 0) \geq 1_{K_0}$.

The standard comparison gives, for all $(x, t) \in \mathbb{R}^N \times [0, +\infty)$,

$$\max(z(x, t), 0) \geq 1_{K(t)}(x) ,$$

which, in turn, implies that, for all $t \geq 0$,

$$K(t) \subset \{ x \in \mathbb{R}^N : z(x, t) \geq 1 \} .$$

Recalling that $\bar{H}(\nu) \geq |\nu| = 1$, we find, for all $(x, t) \in \mathbb{R}^N \times [1, \infty)$,

$$z(x, t) \leq \langle \nu, x \rangle + (\bar{H}(\nu) + \bar{C} \lambda)t + C + \bar{C} \leq \langle \nu, x \rangle + \bar{H}(\nu)((1 + \bar{C} \lambda)t + C + \bar{C}) ,$$

and, therefore, for all $t \geq 0$,

$$K(t) \subset \{ x \in \mathbb{R}^N : \langle \nu, x \rangle + \bar{H}(\nu)((1 + \bar{C} \lambda)t + C + \bar{C} - 1) \geq 0 \} \quad \forall t \geq 1 .$$

Letting $\lambda \to 0$, we get, for all $t \geq 0$ and a new positive constant $C$,

$$K(t) \subset \{ x \in \mathbb{R}^N : \langle \nu, x \rangle + \bar{H}(\nu)(t + C) \geq 0 \} \quad \forall t \geq 1 .$$

Taking the intersection of the right-hand side over all $\nu$ we obtain, by the definition of $\mathcal{W}$, that, for all $t \geq 0$,

$$K(t) \subset (t + C)\mathcal{W} .$$
The proof of (4.2) is more intricate. To this end, let $\tilde{R}$, $r$ and $T_1$ be defined by Lemma 1.2 and let $(K(t))_{t \geq 0}$ be the solution of the front propagation problem starting at $K_0 \subset \mathbb{R}^N$, a compact set which contains $Q_{\tilde{R}}(k)$ for some $k \geq \mathbb{Z}^N$. Then, for all $t \geq 0$,

$$B_{r+t} \subset K(t+T_1).$$

Recall that $u^\varepsilon(x,t) = 1_{K(t/\varepsilon)}(x/\varepsilon)$ is the solution to

$$\begin{cases}
  u_t^\varepsilon = |Du^\varepsilon| + (V(x/\varepsilon, t/\varepsilon), Du^\varepsilon) & \text{in } \mathbb{R}^N \times (0, \infty), \\
  u^\varepsilon(x,0) = 1_{K_0}(x/\varepsilon),
\end{cases}$$

and, in view of (4.2), for all $(x,t) \in \mathbb{R}^{N+1}$,

$$u^\varepsilon(x,t) \geq 1_{B_{r_t(t/\varepsilon-T_1)}(x/\varepsilon)}(x/\varepsilon).$$

Fix next $\delta \in (0, 1)$ such that $T_1 + \delta/(r\varepsilon)$ is an integer and, for $w_\delta(x) = (\delta - |x|) \vee 0$, let $w_\delta^\varepsilon$ be the solution of (4.3) with initial datum $w_\delta$. Then we know from Theorem 1.2 that there exists a constant $\bar{C} > 0$ such that, for all $t \in (0, 1)$,

$$\|w_\delta^\varepsilon - \bar{w}_\delta\|_\infty \leq \bar{C}\varepsilon^{1/3}$$

where $\bar{w}_\delta$ is the solution of the homogenized problem

$$\begin{cases}
  \bar{w}_{\delta,t} = \bar{H}(Du_{\delta}) & \text{in } \mathbb{R}^N \times (0, 1), \\
  \bar{w}_\delta(\cdot, 0) = w_\delta & \text{on } \mathbb{R}^N.
\end{cases}$$

Note that the constant $\bar{C}$ is independent of $\delta$ because the $w_\delta$’s have Lipschitz constant which are bounded uniformly in $\delta$.

From the Lax-Oleinik formula, $\bar{w}_\delta$ is given, for $(x,t) \in \mathbb{N} \times [0, 1)$ by

$$\bar{w}_\delta(x,t) = \sup_{y \in W} w_\delta(x-ty).$$

Since $w_\delta \leq 1_{B_\delta} \leq u^\varepsilon(\cdot, \varepsilon T_1 + \delta/r)$, it follows, from the time-periodicity of $V$ and the choice of $\delta$, that, for all $(x,t) \in \mathbb{N} \times [0, 1)$,

$$w_\delta^\varepsilon(x,t) \leq u^\varepsilon(x,t + \varepsilon(T_1 + \delta/r)).$$

Hence, for all $t \in [0, 1]$,

$$\{w_\delta^\varepsilon(\cdot, t) \geq \delta/2\} \supset \{u^\varepsilon(\cdot, t + \varepsilon T_1 + \delta/r) \geq \delta/2\},$$

while

$$\{w_\delta^\varepsilon(\cdot, t) \geq \delta/2\} \supset \{w_\delta(\cdot, t) \geq \delta/2 + \bar{C}\varepsilon^{1/3}\} \supset \{\sup_{y \in W} w_\delta(\cdot - ty) \geq \delta/2 + \bar{C}\varepsilon^{1/3}\}. $$

If we choose $\delta/2 - \bar{C}\varepsilon^{1/3} > 0$, then

$$\{\sup_{y \in W} w_\delta(\cdot - ty) \geq \delta/2 + \bar{C}\varepsilon^{1/3}\} \supset \{x : \sup_{y \in W} (\delta - |x - ty|) \geq \delta/2 + \bar{C}\varepsilon^{1/3}\} \supset tW.$$

Therefore, for all $t \in [0, 1]$,

$$\varepsilon K((t + \varepsilon T_1 + \delta/r)/\varepsilon) = \{u^\varepsilon(\cdot, t + \varepsilon T_1 + \delta/r) \geq \delta/2\} \supset tW,$$

i.e., for all $t \in [0, 1/\varepsilon]$,

$$K(t + T_1 + \delta/(r\varepsilon)) \supset tW. \tag{4.4}$$

Finally, for $t$ sufficiently large, choose $\varepsilon = 1/(t - 4\bar{C}t^{2/3}/r)$ and $\delta = (n - T_1)r\varepsilon$ where $n$ is the integer part of $4\bar{C}t^{2/3}/r + 1$. Then $n = T_1 + \delta/(r\varepsilon)$ is an integer,
\( \delta/2 - \tilde{C}\varepsilon^{1/3} \) is positive and, applying inclusion (4.4) to \( t - n \) which belongs to \([0,1/\varepsilon]\), we get

\[
K(t) \supset (t - Ct^{2/3})W
\]

for some new constant \( C \).

We conclude the section with the proofs of the two lemmas used in the proof. We have:

**Proof of Lemma 4.1**: Fix \( x \in \mathbb{R}^N \) and let \( (K(t))_{t \geq 0} \) be the solution of the front propagation \( V_{x,t} = 1 - V(x,t) \) starting from \( K_0 = \{x\} + Z^N \), i.e.,

\[
K(t) = \{y \in \mathbb{R}^N : u(y, t) = 0\}
\]

where \( u \) is the solution to

\[
\begin{cases}
\partial_t u = |Du| + (V, Du) & \text{in } \mathbb{R}^N \times (0, \infty) \\
u = -d_{K_0} & \text{on } \mathbb{R}^N,
\end{cases}
\]

where \( d_{K_0} \) is the distance function to the set \( K_0 \).

Then, for each \( t > 0 \), the set \( K(t) \) is \( Z^N \)-periodic and a non empty interior (because it has an interior ball property, as recalled in Appendix). So \( \rho(t) = |K(t) \cap Q_1(x)| \) is positive for positive time and, following the computation in the proof of Theorem 1.1, it satisfies for all \( t_2 > t_1 \geq 0 \),

\[
\rho(t_2) - \rho(t_1) \geq \frac{1}{c_1} \int_{t_1}^{t_2} \min\{\rho(t), 1 - \rho(t)\} \right(N-1\right)/N dt .
\]

Hence there exists a time \( T \) depending only on \( N \) such that \( |K(T) \cap Q_1(x)| = 1 \). This means that \( Q_1(x) \subset K(T) \).

It follows from the finite speed of propagation, that there exits a positive integer \( n_0 \) such that the solution \( \tilde{K}(t) \) starting from \( \{x + k \in Z^N : |k| \leq n_0\} \) coincides with \( K(t) \) on \( Q_1(x) \times [0,T] \). Then \( Q_1(x) \subset \tilde{K}(T) \).

\( \square \)

**Proof of Lemma 4.2**: Consider the solution \( \tilde{u} \) to

\[
\begin{cases}
\tilde{u}_t = H(D\tilde{u}) & \text{in } \mathbb{R}^N \times (0, \infty), \\
\tilde{u}(x,0) = (-|x|) \vee (-1) & \text{on } \mathbb{R}^N \times \{0\}.
\end{cases}
\]

Since, \( H(p) \geq |p| \), given \( 0 < \theta_1 < \theta_0 < 1 \) and \( \delta > 0 \) small, there exists \( \bar{\ell} \in (0,1) \) such that

\[
\{\bar{u}(\cdot,0) \geq -\theta_0\} + B(0,\delta) \subset \{\bar{u}(\cdot,\bar{\ell}) \geq -\theta_1\}.
\]

The fact that the solution \( u^\varepsilon \) of

\[
\begin{cases}
\partial_t u^\varepsilon = |Du^\varepsilon| + \langle V, Du^\varepsilon \rangle & \text{in } \mathbb{R}^N \times (0, \infty), \\
u^\varepsilon(x,0) = (-|x|) \vee (-1) & \text{on } \mathbb{R}^N \times \{0\},
\end{cases}
\]

converges, as \( \varepsilon \to 0 \), locally uniformly to \( \tilde{u} \), yields an \( \varepsilon \in (0,\delta/4) \) such that

\[
\{u^\varepsilon(\cdot,0) \geq -\theta_0\} + B(0,\delta \varepsilon) \subset \{u^\varepsilon(\cdot,\bar{\ell}) \geq -\theta_0\}. \tag{4.5}
\]

Next fix \( n_0 \) and \( T \) as in Lemma 4.1, choose \( \varepsilon \) such that \( T_0 = \frac{T}{\varepsilon} \) is an integer and

\[
Q_{n_0+1}(0) \subset B(0,\frac{\delta}{2\varepsilon}), \tag{4.6}
\]

and set

\[
\tilde{K}_0 = \{x : u^\varepsilon(\varepsilon x,0) \geq -\theta_0\}.
\]
The solution of the front propagation problem starting from $\tilde{K}_0$ is given by

$$\tilde{K}_0(t) = \{ x : u'(x, t) \geq -\theta_0 \}.$$

From (4.5) we have

$$\tilde{K}_0 + B(0, \frac{\delta}{2\epsilon}) \subset \tilde{K}(T_0),$$

while from (4.6), for any $k \in \mathbb{Z}^N$ with $|k| \leq n_0$, we have $\tilde{K}_0 + k \subset \tilde{K}(T_0)$.

The periodicity of $V$ also implies that the solution of the front propagation problem starting from $\tilde{K}_0$ is just $\tilde{K}(t)$ while the solution starting from $\tilde{K}(T_0)$ is $K(t + T_0)$.

From the inclusion principle we get, for all $k \in \mathbb{Z}^N$ with $|k| \leq n_0$ and all $t \geq 0$,

$$\tilde{K}(t) + k \subset \tilde{K}(t + T_0). \quad (4.7)$$

Then Lemma 4.1 implies that, for all $t \geq 0$,

$$\tilde{K}(t) + Q_1 \subset \tilde{K}(t + T_0 + T).$$

In particular, by induction, we get, for all positive integers $n$ and all $t \geq 0$,

$$\tilde{K}(t) + Q_n \subset \tilde{K}(t + n(T_0 + T)).$$

Choose a positive integer $M$ such that, for all $t \in [0, T_0 + T]$, $\tilde{K}(t) \subset Q_M$. Then, for all positive integers $n$ such that $n \geq M$ and all $t \in [0, T_0 + T]$,

$$Q_{n-M} \subset \tilde{K}(t) + Q_n \subset \tilde{K}(t + n(T_0 + T)),$$

and, hence, there exist $r > 0$ and $T_1 > 0$ such that, for all $t \geq 0$,

$$Q_{rt} \subset \tilde{K}(t + T_1).$$

Finally choose a positive integer $R$ such that $\tilde{K}_0 \subset Q_R$. Then, for any compact initial set $K_0$ such that $Q_R(k) \subset K_0$ for some $k \in \mathbb{Z}^N$, we have $\tilde{K}_0 + k \subset K_0$. Therefore the solution of the front propagation problem $K(t)$ starting from $K_0$ satisfies, for all $t \geq 0$,

$$Q_{rt}(k) \subset \tilde{K}(t + T_1) + k \subset K(t + T_1).$$

5. Homogenization for $x$-dependent velocities at scale one

Before we begin the proof, we remark that, since we are only able to prove that $\tilde{H}$ is continuous with respect to the $x$-variable, uniqueness of the solution to equation (1.18) could be an issue. This is not, however, the case because, in view of (1.17) and (1.19), $\tilde{H}$ is coercive. Note that this is the reason we do not consider $V$ that also depends on a slow time variable, because then the coercivity of the averaged Hamiltonian would no longer ensure a comparison principle for the limit problem.

We continue with the Proof of Theorem 1.4. For any fixed $(x, P) \in \mathbb{R}^N \times \mathbb{R}^N$, let $v_{\lambda}^{P,x} = v_{\lambda}^{P,x}(y, t)$ be the solution to

$$v_{\lambda}^{P,x} + \lambda v_{\lambda}^{P,x} = |Dv_{\lambda}^{P,x}| + \langle V(x, y, t), Dv_{\lambda}^{P,x} + P \rangle$$

in $\mathbb{R}^N \times \mathbb{R}$. 
From Lemma 2.1 and Corollary 2.2 we know that there exist \( \bar{H}(x, P) \) and, for any \( M > 0 \), a constant \( C_M > 0 \) independent of \( \lambda \) and \( P \), such that, for all \( x \in \mathbb{R}^N \) such that \( |x| \leq M \),

\[
\| \bar{H}(x, P) - \lambda v^{P,x}_\lambda \|_\infty \leq C|P|\lambda . \tag{5.1}
\]

Arguing as in the proof of Theorem 1.1, one easily checks that \( \bar{H} \) is positively homogeneous of degree one and convex in \( P \), and that (1.19) holds.

To complete the proof, it only remains to show that \( \bar{H} \) is continuous in \( (x, P) \). To this end, observe that the standard comparison arguments imply \( \bar{H} \) is \( (1 + \|V\|_\infty) \)-Lipschitz continuous in \( P \).

Fix next \( M > 0 \), \( x_1, x_2 \in \mathbb{R}^N \) with \( |x_1|, |x_2| \leq M \). Once again the standard comparison of viscosity solutions (see [3]) gives

\[
\| v^{P,x_1}_\lambda - v^{P,x_2}_\lambda \|_\infty \leq C_M|P|\frac{|x_1 - x_2|^\lambda/L}{\lambda} \tag{5.2}
\]

where \( L = \sup_{|x| \leq M} \| D_y z(x, \cdot, \cdot) \|_\infty \) and \( C_M \) depends only on \( M \).

Combining (5.1) with (5.2), we obtain, for all \( x_1, x_2 \in \mathbb{R}^N \) such that \( |x_1|, |x_2| \leq M \) and \( |x_1 - x_2| \leq 1 \),

\[
|\bar{H}(x_1, P) - \bar{H}(x_2, P)| \leq C_M|P|\omega(|x_1 - x_2|)
\]

with (the modulus) \( \omega \) given, for \( r \in (0, 0] \), by

\[
\omega(r) = \inf_{\lambda \in [0,1]} (\lambda^{-1} r^{\lambda/L} + \lambda) .
\]

The proof of the continuity of \( \bar{H} \) is now complete.

\[\Box\]

6. Appendix

We present here the

**Proof of Lemma 2.3:** Fix \( \theta \in (\inf z(\cdot, 0), \sup z(\cdot, 0)) \) and let \( K_\theta(t) = \{ z(\cdot, t) \geq \theta \} \). Since \( z \) is a solution to (2.3)-(i), \( K_\theta(t) \) is given by

\[
K_\theta(t) = \left\{ x \in \mathbb{R}^N : \exists \xi : [0,t] \to \mathbb{R}^N \text{ absolutely continuous such that } \xi(t) = x, \quad z(\xi(0),0) \geq \theta, \quad |\xi'(s) + V(\xi(s), s)| \leq 1 \text{ a.e. } s \in (0,t) \right\}
\]

i.e., \( K_\theta(t) \) is the reachable set for the controlled system

\[
\xi'(s) = f(\xi(s), s, \alpha(s)), \quad |\alpha(s)| \leq 1 ,
\]

where \( f(x, s, \alpha) = \alpha + V(x, s) \) and \( \alpha \in \mathbb{R}^N \) are such that \( |\alpha| \leq 1 \).

Since \( V \) is of class \( C^{1,1} \), it follows from [3] that, for any \( 0 < \tau < T \), there exits a constant \( r = r(\tau, T) > 0 \) such that, for all \( t \in [\tau, T] \), the set \( K_\theta(t) \) has the interior ball property of radius \( r \), i.e.,

for all \( x \in \partial K_\theta(t) \) there exists \( y \in \mathbb{R}^N \) such that \( x \in \overline{B}(y, r) \subset K_\theta(t) \),

where \( \overline{B}(y, r) \) stands for the closed ball of radius \( r \) centered at \( y \), and, hence, for all \( t > 0 \), \( K_\theta(t) \) is a set of finite perimeter (see, for instance, [4]).

Note that [3] deals with time independent dynamics, but the proofs can be easily adapted to the time-dependent framework considered here.

Next set \( d(x, t) = d_{K_\theta(t)}(x) \). It follows that \( d \) is Lipschitz continuous in \( (x,t) \) and, moreover,

\[
d_t = -1 + \langle V(x - dDd, t), Dd(x, t) \rangle \quad \text{a.e in } \quad \{ d > 0 \} . \tag{6.1}
\]
It is, of course, clear that \( x \to d(x, t) \) is 1–Lipschitz continuous. The \( \|V\|_{\infty} + 1 \)-Lipschitz continuity of the map \( t \to d(x, t) \) comes from the above representation formula of \( K_{\theta,t} \).

To check (6.1), recall that, since the map \( P \to |P| + \langle V, P \rangle \) is convex, \( z \) is a sub-solution of (2.5)(i) with an equality (see [8] and [16]), i.e., for any test function \( \varphi \) and any local maximum point \((x, t)\) of \( z - \varphi \), we have

\[
\varphi_t = |D\varphi| + \langle V, D\varphi \rangle .
\]

The invariance property of the geometric equation (2.5)-(i) (see [6]) then implies that the map \((x, t) \to 1_{K_{\theta,t}}(x)\) is also a sub-solution of (2.5)(i) with equality.

Assume now that \( d \) is differentiable at some point \((\bar{x}, \bar{t})\) and let \( \bar{y} \) be the unique projection of \( \bar{x} \) onto \( K_{\theta}(\bar{t}) \) and note that \( w(y, t) := 1_{K_{\theta}(t)}(y) + d(y + \bar{x} - \bar{y}, t) \) has a local maximum at \((\bar{y}, \bar{t})\). Indeed, if \( y \notin K_{\theta}(t) \), then, for all \((y, t)\) sufficiently close to \((\bar{y}, \bar{t})\),

\[
w(y, t) = d(y + \bar{x} - \bar{y}, t) \leq 1 + d(\bar{x}, \bar{t}) = w(\bar{y}, \bar{t}),
\]

and, if \( y \in K_{\theta,t} \), then

\[
w(y, t) = 1 + d(y + \bar{x} - \bar{y}, t) \leq 1 + |y + \bar{x} - \bar{y} - y| = 1 + |\bar{x} - \bar{y}| = w(\bar{y}, \bar{t}) .
\]

Since \((x, t) \to 1_{K_{\theta}(t)}(x)\) is a sub-solution of (2.5)(i) with equality and \( |Dd(\bar{x}, \bar{t})| = 1 \) we obtain

\[-d_t(\bar{x}, \bar{t}) = 1 + \langle V(\bar{y}, \bar{t}) , -Dd(\bar{x}, \bar{t}) \rangle ,
\]

and (6.1) holds because \( \bar{y} = \bar{x} - d(\bar{x}, \bar{t})Dd(\bar{x}, \bar{t}) \).

Next we assume that \( \theta \) is such that \( \partial \{z(\cdot, 0) \geq \theta \} = \{z(\cdot, 0) = \theta \} \), which is true for almost all \( \theta \in (\inf z(\cdot, 0), \sup z(\cdot, 0)) \). It then follows from [1] that, for all \( t \geq 0 \), \( \partial \{z(\cdot, t) \geq \theta \} = \{z(\cdot, t) = \theta \} \) for any \( t \geq 0 \).

We now prove that, for all \( \varphi \in C^\infty_c(\mathbb{R}^N) \) and all \( t \in (0, T) \),

\[
\int_{K_{\theta}(t)} \varphi(x)dx - \int_{K_{\theta}(0)} \varphi(x)dx = \int_0^t \int_{\partial K_{\theta}(s)} \varphi(x)(1-V(x,s),V(x,s)))dH^{N-1}(x)ds .
\]

(6.2)

To this end, for \( h > 0 \) small, let \( \zeta_h : \mathbb{R} \to [0, 1] \) be such that \( \zeta_h(\rho) = 1 \) if \( \rho \leq 0 \), \( \zeta_h(\rho) = 1 - \rho/h \) if \( \rho \in [0, h] \) and \( \zeta_h(\rho) = 0 \) if \( \rho \geq h \).

Multiplying (6.1) by \( \varphi \zeta_h'(d) \) (which makes sense because the sets \( \{d(\cdot, s) = 0\} = \partial K_t(t) \) and \( \{d(\cdot, s) = h\} \) have a zero measure since \( K_t(t) \) and \( \{d(\cdot, s) \leq h\} \) have the interior ball property) and integrating over \( \mathbb{R}^N \times (0, t) \) gives

\[
\int_{\mathbb{R}^N} \varphi(x)\zeta_h(d(x,t))dx - \int_{\mathbb{R}^N} \varphi(x)\zeta_h(d(x,0))dx \\
= \int_0^t \int_{\{0<d(\cdot,s)<h\}} \varphi(x)(-1 + V(x-d(\cdot,s)Dd(x,s),Dd(x,s))) dx ds .
\]

(6.3)

Note that

\[
\lim_{h \to 0} \int_{\mathbb{R}^N} \varphi(x)\zeta_h(d(x,t))dx - \int_{\mathbb{R}^N} \varphi(x)\zeta_h(d(x,0))dx = \int_{K_{\theta}(t)} \varphi(x)dx - \int_{K_{\theta}(0)} \varphi(x)dx .
\]
Lemma 6.1. Let $r$ be such that $r > d$ where $d > 0$, the co-area formula implies

$$\int_0^t \frac{1}{h} \int_{\{0 < d(x,s) < h\}} \varphi(x)(-1 + (V(x-d(x,s)Dd(x,s), Dd(x,s))) \, dx \, ds$$

$$= \int_0^t \frac{1}{h} \int_0^h \varphi(x)(1 - (V(x - \sigma \varrho(x,s), \varrho(x,s))) \, d\mathcal{H}^{N-1}(x) \, d\sigma, \, ds$$

where $\varrho(x,s)$ is the measure theoretic outward unit normal to the set $\{d(\cdot,s) < \sigma\}$, which has finite perimeter since it satisfies the interior ball property with radius $r + \sigma$.

In order to complete the proof of (6.2) we just need to use the following

**Lemma 6.1.** Let $E$ be a closed subset of $\mathbb{R}^N$ with the interior ball property of radius $r > 0$. Then, for all compactly supported in $x \in C(\mathbb{R}^N \times S^{N-1})$,

$$\lim_{\sigma \to 0} \int_{\{d_E(\cdot) = \sigma\}} \Phi(x, \varrho(x)) d\mathcal{H}^{N-1}(x) = \int_{\partial E} \Phi(x, \varrho(x)) d\mathcal{H}^{N-1}(x),$$

where $d_E(x)$ stands for the distance of $x$ to $E$ and $\varrho(x)$ (resp. $\varrho(x)$) is the measure theoretic outward unit normal to $\{d_E(\cdot) < \sigma\}$ (resp. to $E$) at $x \in \partial E$.

**Proof:** Set $E_\sigma = \{d_E(\cdot) \leq \sigma\}$ and denote by $\Pi_\sigma$ the projection of $\partial E_\sigma$ onto $\partial E$. It is known that $\Pi_\sigma$ is uniquely defined for $\mathcal{H}^{N-1}$-a.e. point $x \in \partial E_\sigma$. Let $\mu_\sigma = \mathcal{H}^{N-1}(\partial E_\sigma)$ and $\mu_\sigma = \Pi_\sigma \sharp \mu_\sigma$, $\mu_0 = \mu_0 = \mathcal{H}^{N-1}(\partial E)$.

The first claim is that $\mu_\sigma$ is absolutely continuous with respect to $\mu_0$. For this let us first recall that, since $E$ has the interior ball property of radius $r$, the map $\Pi_\sigma^{-1}$ is well-defined on $F_\sigma := \{x \in \partial E : \exists y \in \partial E_\sigma \text{ with } |x - y| = \sigma\}$ and that $\Pi_\sigma^{-1}$ is Lipschitz continuous, with constant at most $(r + \sigma)/r$, on $F_\sigma$ (see [2]). So, if $Z$ is a Borel subset of $\partial E$, then

$$\bar{\mu}_\sigma(Z) = \mathcal{H}^{N-1}(\Pi_\sigma^{-1}(Z)) \leq \text{Lip}(\Pi_\sigma^{-1}) \mathcal{H}^{N-1}(Z) = \frac{r + \sigma}{r} \mu_0(Z).$$

In particular, $\bar{\mu}_\sigma$ is absolutely continuous with respect to $\bar{\mu}_0$ and, if $f_\sigma = \frac{d\bar{\mu}_\sigma}{d\bar{\mu}_0}$, then $f_\sigma$ is bounded by $\mathcal{H}^{N-1}$-a.e. in $\partial E$.

Next we note that, for any $\Phi \in C(\mathbb{R}^N \times S^{N-1})$,

$$\int_{\partial E_\sigma} \Phi(x, \varrho(x)) d\mathcal{H}^{N-1}(x) = \int_{\partial E} \Phi(y + \sigma \varrho(y), \varrho(y)) f_\sigma(y) d\mathcal{H}^{N-1}(y). \quad (6.4)$$

Indeed, the definitions of $\bar{\mu}_\sigma$ and $f_\sigma$, give

$$\int_{\partial E} \Phi(y + \sigma \varrho(y), \varrho(y)) f_\sigma(y) d\mathcal{H}^{N-1}(y) = \int_{\partial E} \Phi(y + \sigma \varrho(y), \varrho(y)) d(\Pi_\sigma \sharp \mu_\sigma)(y)$$

$$= \int_{\partial E_\sigma} \Phi(\Pi_\sigma(x) + \sigma \varrho(\Pi_\sigma(x)), \varrho(\Pi_\sigma(x))) d\mu_\sigma(x),$$

which implies (6.4) because $\varrho(\Pi_\sigma(x)) = \varrho(x)$ and $\Pi_\sigma(x) + \sigma \varrho(\Pi_\sigma(x)) = x$.

In view of (6.4), to complete the proof, it only remains to show that the $(f_\sigma)$’s converge, as $\sigma \to 0$, to 1 in $L^1(\partial E, \mathcal{H}^{N-1})$. 

Applying (6.4) with $\Phi = 1$ gives
$$\text{Per}(E_\sigma) = \int_{\partial E} f_\sigma(y) d\mathcal{H}^{N-1}(y).$$
Since $f_\sigma \leq \frac{r + \sigma}{r} \mathcal{H}^{N-1}$-a.e. in $\partial E$, the lower-continuity of the perimeter implies that $\lim_{\sigma \to 0} \text{Per}(E_\sigma) = \text{Per}(E)$. Using again the inequality $f_\sigma \leq \frac{r}{r} + \sigma$ which holds $\mathcal{H}^{N-1}$-a.e. in $\partial E$, we obtain, in the limit $\sigma \to 0$, 
$$\int_{\partial E} |1 - f_\sigma(y)| d\mathcal{H}^{N-1}(y) \leq \left| 1 - \frac{r + \sigma}{r} \right| \text{Per}(\partial E) + \int_{\partial E} \frac{r + \sigma}{r} - f_\sigma(y)) d\mathcal{H}^{N-1}(y) \to 0. \quad \square$$

References


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