Viscosity solutions for a polymer crystal growth model

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VISCOSITY SOLUTIONS FOR A POLYMER CRYSTAL GROWTH MODEL

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ABSTRACT. We prove existence of a solution for a polymer crystal growth model describing the movement of a front $(\Gamma(t))$ evolving with a nonlocal velocity. In this model the nonlocal velocity is linked to the solution of a heat equation with source δ_Γ. The proof relies on new regularity results for the eikonal equation, in which the velocity is positive but merely measurable in time and with Hölder bounds in space. From this result, we deduce *a priori* regularity for the front. On the other hand, under this regularity assumption, we prove bounds and regularity estimates for the solution of the heat equation.

1. Introduction

The paper is devoted to the analysis of following system of equations:

$$\begin{cases} i) & u_t(x,t) = \bar{g}(v(x,t))|Du(x,t)| \qquad \text{in } \mathbb{R}^N \times (0,+\infty) \\ ii) & v_t(x,t) - \Delta v(x,t) + \kappa \bar{g}(v(x,t))\mathcal{H}^{N-1}\lfloor\{u(\cdot,t)=0\} = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R}^N \times (0,+\infty) \\ iii) & v(x,0) = v_0(x),\ u(x,0) = u_0(x) \qquad \text{in } \mathbb{R}^N . \end{cases} \tag{1.1}$$

Following [10, 11, 12, 18], the 3-dimensional version of this system modelizes the growth of the surface $\Gamma(t)$ of a polymer crystal in a nonhomogeneous temperature field $v(x,t)$. In this model one describes the evolving surface $\Gamma(t)$ of the crystal as the 0-level-set of an auxiliary function u:

$$\{x \in \mathbb{R}^N \ ;\ u(x,t) = 0\} = \Gamma(t) .$$

(This is the level-set approach, see [19] and references therein). It has experimentaly been observed that the normal velocity V_n of the crystal is a known, positive function of the temperature: $V_n = \bar{g}(v(x,t))$, where $\bar{g}$ is a bell-shaped function depending on the specific polymer ([16]). Expressing the normal velocity V_n in terms of the function u gives the eikonal equation (1.1)-i), which holds at least on the set $\{u(\cdot,t)=0\}$. As for the temperature field v it has to follow a heat equation with a (negative) heat source proportional to $V_n\mathcal{H}^{N-1}\lfloor\Gamma(t)$. Whence (1.1)-$ii$).

Similar systems, coupling eikonal and diffusion equations, appear in many applications: shape optimization, image segmentation, etc. (see for instance [25, 26] and the references therein). However the mathematical analysis of such couplings is delicate and few existence or uniqueness results are available in the

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literature. Most of them are concerned with classical solutions on a short time interval. For instance short time existence and uniqueness of smooth solutions are obtained for system (1.1) in [18].

The point is that, in general, one cannot expect such a system to have classical solutions when the time becomes large: indeed the front $\Gamma(t)$ usually develops singularities in finite time. For this reason a good description of this front is obtained by its representation as the 0-level-set of the solution of an eikonal equation, which has to be understood in the sense of viscosity solutions. However this approach (which is satisfactory from a numerical view point) raises severe mathematical difficulties. Such issues have been overcome in only a very few number of situations: for a dislocation dynamics model, introduced in [1] and analyzed in [2, 4, 5], or for a system arising in the study of the asymptotics of a Fitzhugh-Nagumo model [6, 20, 27]. In this later framework, the associated heat equation is of the form

$$v_t(x,t) - \Delta v(x,t) - \bar{g}(v(x,t))\mathbf{1}_{\{u(\cdot,t)\geq 0\}} = 0\,, \tag{1.2}$$

where $\mathbf{1}_E$ is the indicator function of a set E. In [6, 20, 27] existence of generalized solutions for this Fitzhugh-Nagumo system is proved, while [7] contains some uniqueness results. However, system (1.1) turns out to be much more challenging than the coupling in the Fitzhuch-Nagumo system. Indeed the surface term $\mathcal{H}^{N-1}\lfloor\{u(\cdot,t)=0\}$ in (1.1)-*ii*) is more singular than the volume one $\mathbf{1}_{\{u(\cdot,t)\geq 0\}}$ in (1.2). For this reason, up to now, only the long time existence in space dimension $N=2$ is known [29, 28].

The aim of our paper is to obtain a similar existence result for the physical dimension $N=3$ (and in fact in any dimension). In order to state precisely our main result, let us introduce the definition of a solution to (1.1).

Definition 1.1. *A solution* (u,v) *of* (1.1) *on the time interval* $[0,T]$ *is a map* $(u,v):\mathbb{R}^N\times[0,T]\to\mathbb{R}^2$ *which is bounded, uniformly continuous, such that* u *satisfies the equation*

$$u_t(x,t) = \bar{g}(v(x,t))|Du(x,t)| \text{ in } \mathbb{R}^N\times(0,T), \qquad u(x,0)=u_0(x) \text{ in } \mathbb{R}^N$$

in the viscosity sense, with

$$\int_0^T \mathcal{H}^{N-1}(\{u(\cdot,t)=0\}) < +\infty\ ,$$

and such that $v(\cdot,0)=v_0$ *and* v *satisfies in the sense of distributions*

$$v_t(x,t) - \Delta v(x,t) + \kappa\bar{g}(v(x,t))\mathcal{H}^{N-1}\lfloor\{u(\cdot,t)=0\} = 0 \qquad \text{in } \mathbb{R}^N\times(0,T)\ .$$

We introduce the following set of assumptions, denoted by **(A)** in the rest of the paper.

(A1) κ is a fixed real number (κ is positive in the case of a negative heat source and negative otherwise), $\bar{g}:\mathbb{R}^N\to\mathbb{R}$ is Lipschitz continuous, bounded, and there exist $A,B>0$ such that

$$A\leq \bar{g}(z)\leq B \qquad \text{for all } z\in\mathbb{R}\ .$$

(A2) $v_0:\mathbb{R}^N\to\mathbb{R}$ is Lipschitz continuous and bounded.

(A3) $u_0 : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and satisfies $\{u_0 = 0\} = \partial\{u_0 > 0\}$. Moreover, we assume that $\{u_0 \geq 0\}$ is compact and has the interior ball property of radius $r_0 > 0$, that is,

$$\text{For all } x \in K_0,\ \text{there exists } y \in K_0,\ \text{with } x \in \overline{B}(y, r_0) \subset K_0\ , \tag{1.3}$$

where $\overline{B}(y, r_0)$ is the closed ball of radius r_0 centered at y.

Our main result states that, under the above assumptions, system (1.1) has a solution. More precisely:

Theorem 1.2. *Under Assumption* **(A)***, for any* $T > 0$*, there exists at least one solution to System* (1.1)*. This solution is bounded on* $\mathbb{R}^N \times [0, T]$ *and satisfies, for all* $x, y \in \mathbb{R}^N$, $0 \leq s, t \leq T$,

$$|v(x,t) - v(y,t)| \leq C|x - y|(1 + |\log|x - y||),$$

and

$$|v(x,t) - v(x,s)| \leq C|t - s|^{\frac{1}{2}}(1 + |\log|t - s||).$$

for some constant C *which only depends on the data appearing in Assumption* **(A)** *and* T.

Note that uniqueness of the solution is an open problem (even in dimension 2).

Let us now briefly describe the method of proof. The main difficulty in System (1.1) is the singular surface term in the heat equation: to deal with this term, one has to obtain fine regularity estimates for the level-sets of u. Such estimates, which cannot be derived from the usual regularity results on the eikonal equation, have been investigated through several works. When the velocity $x \mapsto \bar{g}(v(x,t))$ is positive of class $\mathcal{C}^{1,1}$, the front enjoys the interior ball property (1.3) [13] (see also [2, 5]); it has an interior cone property when the velocity is positive and Lipschitz continuous [7]. Unfortunately, for System (1.1), the interior cone property is not sufficient for guarantying the stability of the surface term $\mathcal{H}^{N-1}\lfloor\{u(\cdot,t) = 0\}$. Moreover we were only able to prove that the map $x \mapsto v(x,t)$ has a modulus of continuity of the form $\omega(\rho) = \rho(1 + |\log(\rho)|)$ (even when the front is smooth this map is at most Lipschitz continuous [18]). Our main and new estimate on the eikonal equation is an interior paraboloid property for the level-sets of u. We call paraboloid a solid deformation of the set

$$\left\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}\ ;\ x_N \geq c|x'|^{1+\gamma}\right\}, \quad c > 0,\ \gamma \in (0,1).$$

This property is obtained under the (weak) assumption that the velocity $x \mapsto \bar{g}(v(x,t))$ is of class $\mathcal{C}^{0,\alpha}$. For this, we use a representation formula for the solutions of (1.1)-i) in terms of optimal control as well as sharp regularity properties of optimal solutions for this control problem. As a direct consequence of the interior paraboloid property one obtains that the front has an interior cone property. These interior paraboloid and cone properties are the two key ingredients which allow us to obtain *a priori* estimates on the heat flow: indeed, because of the cone property, the front $\Gamma(t)$ can be covered by a finite (and controlled) number of Lipschitz graphs. The stability result on the surface term $\mathcal{H}^{N-1}\lfloor\{u(\cdot,t) = 0\}$ (see Lemma 4.1) is a consequence of the interior paraboloid property. Let us finally point out that, although the cone and paraboloid properties do not appear in [29, 28], we use several arguments from these papers: in particular the regularity of the optimal solutions of some control problem is borrowed from [29, 28] and some of our estimates on the heat flow are related with those of [29, 28].

The paper is organized as follows: Section 2 is dedicated to estimates on the eikonal equation, while the *a priori* estimates for the heat flow are the object of Section 3. We prove the main result in Section 4.

Notations: For any integer $k \geq 1$ we denote by $B_k(x,r)$ (resp. $\overline{B}_k(x,r)$) the open (resp. closed) ball of radius $r>0$ and of center x in $\mathbb{R}^k$. For $k=N$ (the ambiant space), we simply abbreviate to $B(x,r)$. We also denote by $\mathbb{S}^{N-1}$ the unit sphere of $\mathbb{R}^N$.

2. Representation formula and a priori estimates for the eikonal equation

Throughout this section, we investigate the eikonal equation

$$\begin{cases} u_t = c(x,t)|Du| & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.1}$$

We assume that the velocity c is Borel measurable on $\mathbb{R}^N \times [0,T]$ and satisfies

$$A \leq c(x,t) \leq B \qquad \text{for all } (x,t) \in \mathbb{R}^N \times [0,T] \tag{2.2}$$

for some $A, B > 0$. We also assume that there exist $\alpha \in (0,1)$, $\omega \in L^p(0,T)$ with $p \in (1,+\infty]$ and $C>0$ such that for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T]$,

$$|c(x,t) - c(y,t)| \leq C|y-x| \left(1 + |\log|x-y||\right), \tag{2.3}$$

and

$$|c(x,t) - c(y,t)| \leq \omega(t)|y-x|^\alpha . \tag{2.4}$$

Finally, the initial datum u_0 is Lipschitz continuous on $\mathbb{R}^N$. Our aim is to prove existence and uniqueness for the solution of (2.1) under assumptions (2.2) and (2.3), and give some estimates depending only on assumption (2.4). Note that the first two parts are quite classical: they are given here for sake of completeness and also because we are working in a framework (assumption (2.3)) which slightly differs from the standard one. In constrast, the regularity results on the optimal solutions for the controlled system associated with equation (2.1) and its consequence on the level-sets of the solution of (2.1) are new. Their proofs borrow some ideas of [28, 29], as for instance Lemma 2.7.

2.1. Existence, uniqueness, stability and representation formula.

Let us recall some known results for Equation (2.1). The notion of L^1-viscosity solution provides a framework for equations such as (2.1) where the dependance on the time variable is merely measurable. We refer to [5, Appendix] for the definition and properties of L^1-viscosity solutions that we need here, and to [21, 23, 24, 8, 9] for a complete overview of the theory.

Let us introduce the following controlled system: for any $b \in L^\infty([0,T), \mathbb{R}^N)$,

$$x'(s) = c(x(s),s)\, b(s) \qquad |b(s)| \leq 1, \quad \text{for a.e. } s \geq 0. \tag{2.5}$$

We start by recalling that, for a given initial data and a given control, equation (2.5) has a unique solution (this is Osgood's Theorem, see [15] for instance):

Lemma 2.1. *Assume that the function $c : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is Borel measurable, bounded and satisfies* (2.3)*. For any fixed $b \in L^\infty([0,T), \mathbb{R}^N)$, with $|b(s)| \leq 1$ a.e., Equation*

$$\begin{cases} x'(s) = c(x(s),s)\, b(s) & \text{for a.e. } s \in [0,T], \\ x(0) = x_0 \end{cases}$$

has a unique absolutely continuous solution on $[0,T]$. Moreover, if x and y are two solutions of (2.5)*, associated to the same control $b \in L^\infty([0,T), \mathbb{R}^N)$, then*

$$|x(t) - y(t)| \leq \tilde{\omega}(|x(0) - y(0)|) \tag{2.6}$$

for some modulus $\tilde{\omega}$ which only depends on the constant C in Assumption (2.3)*.*

Proposition 2.2 (Existence, uniqueness and stability for (2.1))**.**
Assume that the velocity $c : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is Borel measurable and satisfies (2.2) *and* (2.3)*. Let $u_0 : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz continuous function. Then:*

(i) (Existence and uniqueness) Equation (2.1) *has a unique L^1-viscosity solution satisfying*

$$u_0(x) \leq u(x,t) \leq u_0(x) + B\|Du_0\|_\infty t\,, \tag{2.7}$$

for any $(x,t) \in \mathbb{R}^N \times [0,T]$.

(ii) (Properties and representation formula) This solution is nondecreasing in time, uniformly continuous on $\mathbb{R}^N \times [0,T]$ and given by the formula

$$u(x,t) = \sup\{u_0(y);\ \exists\, \bar{x} \text{ solution of (2.5) with } \bar{x}(0) = y \text{ and } \bar{x}(t) = x\}\,. \tag{2.8}$$

In particular,

$$\begin{aligned} K(t) :&= \{x \in \mathbb{R}^N;\ u(x,t) \geq 0\} \\ &= \left\{x \in \mathbb{R}^N;\ \exists\, \bar{x} \text{ solution of (2.5) with } \bar{x}(0) \in K(0) \text{ and } \bar{x}(t) = x\right\}. \end{aligned} \tag{2.9}$$

(iii) (Stability) If (c_n) is a sequence of measurable functions satisfying (2.2) *and* (2.3) *with the same constants $A, B, C > 0$ and such that (c_n) converges a.e. to some $c : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, then the sequence of solutions (u_n) of* (2.1) *associated to the velocities (c_n) converges locally uniformly to the solution u associated to c.*

Proof: The existence of a solution u which satisfies (2.7) is a consequence of the general theory (see [24, Propositions 2.1 and 2.2]). To prove that this solution is unique and given by (2.8), we proceed by approximation: let $(\rho_n)_{n\geq 1}$ be a mollifier on $\mathbb{R}^N$ such that $\text{supp}(\rho_n) \subset \overline{B}(0, 1/n)$, $\rho_n \geq 0$ and $\|\rho_n\|_1 = 1$. Let $(\tilde{c}_n)_{n\geq 1}$ be the sequence of approximate velocities defined by

$$\tilde{c}_n(x,t) = \int_{\mathbb{R}^N} c(x-y,t)\, \rho_n(y)\, dy.$$

Then $\tilde{c}_n$ is Borel measurable on $\mathbb{R}^N \times [0,T]$, Lipschitz continuous in space (with a n-dependant constant), satisfies (2.2) and (2.3), and $(\tilde{c}_n)$ converges to c as $n \to +\infty$. More precisely, using (2.3), we have for any $(x,t) \in \mathbb{R}^N \times [0,T]$,

$$|\tilde{c}_n(x,t) - c(x,t)| \leq \int_{\overline{B}(0,1/n)} |c(x-y,t) - c(x,t)|\, \rho_n(y)\, dy \leq C\, \frac{1}{n}(1 + \log n).$$

Let

$$c_n^-(x,t) = \tilde{c}_n(x,t) - \frac{C}{n}(1+\log n) \quad \text{and} \quad c_n^+(x,t) = \tilde{c}_n(x,t) + \frac{C}{n}(1+\log n),$$

so that $c_n^- \le c \le c_n^+$ and $c_n^\pm$ satisfies (2.2) with $A/2$ and $2B$ for n large enough. By the comparison principle for (2.1) with a velocity which is Lipschitz continuous in space (see [24, Theorem 3.1]), we obtain that $u_n^- \le u \le u_n^+$, where u_n^- (resp. u_n^+) is the solution of (2.1) associated to the velocity c_n^- (resp. c_n^+). Moreover (2.7) (with $2B$) and (2.8) hold for both u_n^- and u_n^+. To conclude, it only remains to prove that, if a sequence of velocities (c_n) satisfies (2.2) and (2.3), and converges almost everywhere to c as $n \to +\infty$, then the representation formulae for the corresponding solutions u_n converge to the representation formula for u.

First of all, fix $(x,t) \in \mathbb{R}^N \times [0,T]$ and let (y_n) be a sequence of points in $\mathbb{R}^N$ such that $u_0(y_n) \to z \in \mathbb{R}$ as $n \to +\infty$ and for any n, there exists an absolutely continuous function $\bar{x}_n : [0,t] \to \mathbb{R}^N$ such that $\bar{x}_n(0) = y_n$, $\bar{x}_n(t) = x$ and $|\bar{x}_n'(s)| \le c_n(\bar{x}_n(s), s)$ on $[0,t]$. Since $|c_n| \le B$ for any n, up to an extraction, $(\bar{x}_n)$ converges uniformly to some $\bar{x} : [0,t] \to \mathbb{R}^N$. As a consequence, $\bar{x}(t) = x$, $u_0(\bar{x}(0)) = z$ and, using the a.e. convergence of (c_n) to c as well as (2.2) and (2.3), we obtain $|\bar{x}'(s)| \le c(\bar{x}(s), s)$ on $[0,t]$. This proves that

$$\limsup u_n(x,t) \le \sup\{u_0(y);\ \exists\, \bar{x} \text{ solution of (2.5) with } \bar{x}(0) = y \text{ and } \bar{x}(t) = x\}.$$

Conversely, let $y \in \mathbb{R}^N$ such that there exists a solution $\bar{x}$ of (2.5) with $\bar{x}(0) = y$ and $\bar{x}(t) = x$. Let b be the control associated by $\bar{x}$ and $\bar{x}_n$ be the solution of $\bar{x}_n'(s) = c_n(\bar{x}_n(s), s)b(s)$ with $\bar{x}_n(t) = x$. Then we must have $u_n(x,t) \ge u_0(\bar{x}_n(0))$ for any n. By the same argument as above, $(\bar{x}_n)$ must converge uniformly to a solution of $x'(s) = c(x(s), s)b(s)$, and by uniqueness of such a solution (Lemma 2.1), the limit (x_n) must be $\bar{x}$. Therefore

$$u_0(y) = \lim u_0(\bar{x}_n(0)) \le \liminf u_n(x,t),$$

and

$$\sup\{u_0(y);\ \exists\, \bar{x} \text{ solution of (2.5) with } \bar{x}(0) = y \text{ and } \bar{x}(t) = x\} \le \liminf u_n(x,t).$$

This concludes the proof of the representation formula (2.8) for the unique solution of (2.1). This representation formula implies that u is nondecreasing in time. We also point out that the proof of uniqueness can be easily adapted to prove that, in fact, comparison holds for (2.1).

To prove the stability property (*iii*), let (c_n) be a sequence of functions satisfying (2.2) and (2.3) with the same constants A, B and C, and such that (c_n) converges a.e. to some $c : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, and let (u_n) be the sequence of solutions of (2.1) associated to the velocities (c_n). Using the same arguments as above and the representation formula (2.8), we can actually prove that the half-relaxed limits

$$\liminf{}_* u_n : (x,t) \mapsto \liminf_{n \to +\infty} \{u_n(x_n, t_n);\ x_n \to x,\ t_n \to t\}$$

and

$$\limsup{}^* u_n : (x,t) \mapsto \limsup_{n \to +\infty} \{u_n(x_n, t_n);\ x_n \to x,\ t_n \to t\}$$

coincide and are equal to the solution u of (2.1) associated to c. This is known to imply the locally uniform convergence of (u_n) to u, and proves the stability property.

Finally, let us prove the uniform continuity of the solution u of (2.1), starting with the regularity in space: fix $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T]$, and let $\bar{x}$ be a solution of (2.5) with control $\bar{b}$, $\bar{x}(t) = x$ and $u(x,t) = u_0(\bar{x}(0))$ (notice that the supremum is achieved in (2.8)). Let $\bar{y}$ be the solution of (2.5) associated to the same control $\bar{b}$ and satisfying $\bar{y}(t) = y$. Applying (2.6) for System (2.5) with reverse time, we have

$$|\bar{x}(0) - \bar{y}(0)| \le \tilde{\omega}(|\bar{x}(t) - \bar{y}(t)|) \quad \text{for all } t \in [0,T].$$

Using that $\bar{y}$ is a solution of (2.5) and $u_0(\bar{y}(0)) \le u(y,t)$ thanks to (2.8), we obtain

$$\begin{aligned} u(x,t) = u_0(\bar{x}(0)) &\le u_0(\bar{y}(0)) + \|Du_0\|_\infty |\bar{x}(0) - \bar{y}(0)| \\ &\le u(y,t) + \bar{\omega}(|x-y|), \end{aligned}$$

where $\bar{\omega} = \|Du_0\|_\infty \tilde{\omega}$ is still a modulus of continuity. Exchanging the roles of x and y, we obtain the uniform continuity of u in space.

Now let us fix $t \in [0,T]$. The map $(x,s) \mapsto u(x,t+s)$ is a sub-solution of $\bar{u}_t = B\,|D\bar{u}|$ in $\mathbb{R}^N \times [0, T-t]$ with uniformly continuous initial datum $u(\cdot,t)$. By the Lax formula, for any $0 \le s \le T-t$,

$$u(x,t) \le u(x,t+s) \le \sup\{u(y,t);\ |x-y| \le Bs\}\,.$$

Using the uniform continuity of $u(\cdot,t)$ in space, we deduce that for any $0 \le s \le T-t$,

$$u(x,t) \le u(x,t+s) \le u(x,t) + \bar{\omega}(Bs)\,.$$

This proves the uniform continuity of u in time.

□

2.2. **Properties of the minimal time function.** Let us now introduce the function

$$z : x \mapsto \min\{t \in [0,T];\ u(x,t) \ge 0\}\,,$$

which by definition is well-defined on $K(T) = \cup_{t\in[0,T]} K(t)$ (see (2.9) for the definition of $K(t)$) and is such that $K(t) = \{x \in \mathbb{R}^N;\ z(x) \le t\}$.

We say that a solution $\bar{x}$ of (2.5) on $[0,t]$ is extremal if

$$\bar{x}(0) \in K(0) \quad \text{and} \quad z(\bar{x}(t)) = t.$$

Lemma 2.3. *Assume that the velocity* $c : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ *is Borel measurable and satisfies* (2.2) *and* (2.3)*.*

(1) *Let* $\bar{x}$ *be an extremal solution on* $[0,t]$*. Then:*

  (*i*) *For any* $s \in [0,t]$, $z(\bar{x}(s)) = s$.

  (*ii*) *For almost every* $s \in [0,t]$, $|\bar{x}'(s)| = c(\bar{x}(s), s)$.

(2) *If* $\{x \in \mathbb{R}^N;\ u_0(x) = 0\} = \partial\{x \in \mathbb{R}^N;\ u_0(x) > 0\}$*, then for any* $t \in (0,T]$,

$$\{x \in \mathbb{R}^N;\ u(x,t) = 0\} = \{x \in \mathbb{R}^N;\ z(x) = t\}\,.$$

Proof: (1) (*i*) By definition of $\bar{x}$ and z, we have for any $s \in [0,t]$, $z(\bar{x}(s)) \le s$. To prove the converse inequality, we argue by contradiction: let $s_0 \in [0,t)$ be such that $\theta := z(\bar{x}(s_0)) < s_0$. Let us first prove that for $\delta > 0$ small enough,

$$\overline{B}(\bar{x}(s_0), A(s_0 - \theta - \delta)) \subset \{y \in \mathbb{R}^N;\ z(y) \le s_0 - \delta\}\,.$$

Let y be such that $|y-\bar{x}(s_0)|<A(s_0-\theta-\delta)$, and let x_θ be a solution of (2.5) on $[0,\theta]$ such that $x_\theta(0)\in K(0)$ and $x_\theta(\theta)=\bar{x}(s_0)$. We extend x_θ to $[0,s_0-\delta]$ by setting

$$x_\theta(s)=\bar{x}(s_0)+\frac{y-\bar{x}(s_0)}{s_0-\theta-\delta}\,(s-\theta)\quad \text{for all } s\in[\theta,s_0-\delta]\,.$$

The bound $c\geq A$ shows that x_θ is a solution of (2.5) on $[0,s_0-\delta]$ with $x_\theta(0)\in K(0)$ and $x_\theta(s_0-\delta)=y$, which means that $z(y)=z(x_\theta(s_0-\delta))\leq s_0-\delta$.

Now, for any $\delta>0$ small enough, let us solve

$$\begin{cases} x_\delta'(s)=c(x_\delta(s),s)\,b(s) \quad \text{on } [s_0-\delta,t-\delta]\,,\\ x_\delta(t-\delta)=\bar{x}(t)\,. \end{cases}$$

where b is the control associated to $\bar{x}$. Applying (2.6) for System (2.5) with reverse time, we have

$$\begin{aligned} |x_\delta(s_0-\delta)-\bar{x}(s_0-\delta)| &\leq \tilde{\omega}(|x_\delta(t-\delta)-\bar{x}(t-\delta)|)\\ &= \tilde{\omega}(|\bar{x}(t)-\bar{x}(t-\delta)|)\\ &\leq \tilde{\omega}(B\delta) \end{aligned}$$

because $|\bar{x}'|\leq B$. In particular, for δ small enough,

$$|x_\delta(s_0-\delta)-\bar{x}(s_0-\delta)|<\frac{1}{2}A(s_0-\theta-\delta)\,,$$

while

$$|\bar{x}(s_0-\delta)-\bar{x}(s_0)|\leq B\delta<\frac{1}{2}A(s_0-\theta-\delta)\,.$$

For such a choice of δ,

$$x_\delta(s_0-\delta)\in\overline{B}(\bar{x}(s_0),A(s_0-\theta-\delta))\subset\{y\in\mathbb{R}^N;\ z(y)\leq s_0-\delta\}.$$

Therefore $z(x_\delta(s_0-\delta))\leq s_0-\delta$. In particular, there exists a solution $\tilde{x}$ of (2.5) on $[0,s_0-\delta]$ with $\tilde{x}(0)\in K(0)$ and $\tilde{x}(s_0-\delta)=x_\delta(s_0-\delta)$. The reunion of the paths associated to $\tilde{x}$ on $[0,s_0-\delta]$ and x_δ on $[s_0-\delta,t-\delta]$ gives a solution x of (2.5) on $[0,t-\delta]$ with $x(0)\in K(0)$ and $x(t-\delta)=x_\delta(t-\delta)=\bar{x}(t)$. In particular, $z(\bar{x}(t))\leq t-\delta<t$, which is absurd.

(1) *(ii)* Now, let us prove that $|\bar{x}'(s)|=c(\bar{x}(s),s)$ for almost every $s\in[0,t]$: for $s_0\in(0,t)$ and $h>0$ be small enough, let $y:[s_0-h,s_0+h]$ be the solution of

$$\begin{cases} y'(s)=c(y(s),s)\frac{\bar{x}(s_0+h)-\bar{x}(s_0-h)}{|\bar{x}(s_0+h)-\bar{x}(s_0-h)|}\,,\\ y(s_0-h)=\bar{x}(s_0-h)\,. \end{cases}$$

($\bar{x}$ is injective from (1) *(i)*). Note that y remains in the segment $[\bar{x}(s_0-h),\bar{x}(s_0+h)]$ on $[s_0-h,s_0+h]$ because $|y'(s)|\leq c(y(s),s)$, which means that y is suboptimal. Moreover y is monotonous on this segment. In particular we have

$$|\bar{x}(s_0+h)-\bar{x}(s_0-h)|\geq|y(s_0+h)-y(s_0-h)|=\int_{s_0-h}^{s_0+h}c(y(s),s)\,ds.$$

Using the bound $c\leq B$, we have

$$|y(s)-\bar{x}(s)|\leq 4Bh\quad \text{for all } s\in[s_0-h,s_0+h]\,.$$

Therefore, thanks to (2.3), we get

$$\int_{s_0-h}^{s_0+h} c(y(s),s)\,ds \geq \int_{s_0-h}^{s_0+h} c(\bar{x}(s),s)\,ds - 8BCh^2(1+|\log(4Bh)|).$$

If s_0 is a Lebesgue point of $s \mapsto c(\bar{x}(s),s)$ such that $\bar{x}$ is differentiable at s_0, which is the case of almost every $s_0 \in [0,t]$, then we obtain

$$\begin{aligned}|\bar{x}'(s_0)| &= \lim_{h\to 0} \frac{|\bar{x}(s_0+h)-\bar{x}(s_0-h)|}{2h} \\ &\geq \lim_{h\to 0} \frac{1}{2h}\int_{s_0-h}^{s_0+h} c(\bar{x}(s),s)\,ds = c(\bar{x}(s_0),s_0)\,.\end{aligned}$$

(2) Let $(x,t) \in \mathbb{R}^N \times (0,T]$ be such that $z(x)=t$; by definition of z, we know that $u(x,t) \geq 0$ and for any $h>0$ enough, $u(x,t-h)<0$. By continuity of u, we must have $u(x,t)=0$.

Conversely, let $(x,t) \in \mathbb{R}^N \times (0,T]$ be such that $u(x,t)=0$. We argue by contradiction and assume that $\theta = z(x) < t$. Since u is nondecreasing in t, one necessarily has $u(x,\theta)=0$. Let $\bar{x}$ be a solution of (2.5) such that $u(x,\theta) = u_0(\bar{x}(0)) = 0$ and $\bar{x}(\theta)=x$. By our assumption on u_0, there exists y such that $u_0(y)>0$ and

$$\tilde{\omega}(|y-\bar{x}(0)|) < A(t-\theta)$$

(recall that $\tilde{\omega}$ is defined in (2.6)). Let $\bar{y}$ be the solution of (2.5) on $[0,\theta]$ with the control b associated to $\bar{x}$, and such that $\bar{y}(0)=y$. Then, from (2.6), we have

$$|\bar{y}(\theta)-x| = |\bar{y}(\theta)-\bar{x}(\theta)| \leq \tilde{\omega}(|\bar{y}(0)-\bar{x}(0)|) < A(t-\theta).$$

We extend $\bar{y}$ to $[0,t]$ by setting for any $s\in[\theta,t]$,

$$\bar{y}(s) = \bar{y}(\theta) + \frac{x-\bar{y}(\theta)}{t-\theta}\,(s-\theta)\,.$$

The bound $c \geq A$ implies that $\bar{y}$ is a solution of (2.5) with $\bar{y}(0)=y$ and $\bar{y}(t)=x$. By (2.8), we have $u(x,t) \geq u_0(\bar{y}(0)) = u_0(y) > 0$, which is absurd. Therefore $z(x)=t$, and this concludes the proof.

□

Proposition 2.4. *Under the assumptions of Proposition 2.2, the map z satisfies*

$$\frac{1}{B} \leq |Dz| \leq \frac{1}{A}$$

in the viscosity sense and therefore almost everywhere in $\{x \in \mathbb{R}^N;\ 0 < z(x) < T\}$.

Proof: The proof of the right-hand side inequality follows along the same lines as the beginning of the proof of [7, Theorem 5.9], and shows that z is Lipschitz continuous. For the left-hand side inequality, let $\phi : \{x \in \mathbb{R}^N;\ 0<z(x)<T\} \to \mathbb{R}$ be a function of class C^1 such that $z-\phi$ has a local minimum equal to 0 at some x. Let $\bar{x}$ be an extremal on $[0,t]$ with $\bar{x}(t)=x$. For any $s\in[0,t]$, $z(\bar{x}(s))=s$ by Lemma 2.3. Then for any $h>0$ small enough,

$$z(\bar{x}(t-h)) \geq \phi(\bar{x}(t-h))\,,$$

whence, by definition of ϕ,

$$\phi(\bar{x}(t)) - h = z(\bar{x}(t)) - h = t-h = z(\bar{x}(t-h)) \geq \phi(\bar{x}(t-h))\,.$$

In particular,

$$h \leq \phi(\bar{x}(t)) - \phi(\bar{x}(t-h)) = \int_{t-h}^{t} \langle D\phi(\bar{x}(s)), \bar{x}'(s)\rangle \, ds \leq B \int_{t-h}^{t} |D\phi(\bar{x}(s))| \, ds$$

thanks to the the bound $c \leq B$. Dividing this expression by h and letting $h \to 0$, we get $|D\phi(x)| \geq 1/B$. Since z is Lipschitz continuous, the viscosity inequality $|Dz| \geq 1/B$ also holds almost everywhere.

□

Remark 2.5. A consequence of the inequality $|Dz| \geq 1/B$ and Lemma 2.3 (2) is that for any $t \in [0, T]$, the front $\{x \in \mathbb{R}^N;\ u(x,t) = 0\}$ has measure 0 and coincides with $\partial K(t)$. Indeed, $\{x \in \mathbb{R}^N;\ u(x,t) = 0\} = \{x \in \mathbb{R}^N;\ z(x) = t\}$, and Stampacchia's theorem (see for instance [17]) states that $Dz = 0$ almost everywhere on the set $\{x \in \mathbb{R}^N;\ z(x) = t\}$. Moreover, the viscosity decrease principle (see [22]) shows that

$$\partial K(t) = \partial\{x \in \mathbb{R}^N;\ z(x) \leq t\} = \{x \in \mathbb{R}^N;\ z(x) = t\} = \{x \in \mathbb{R}^N;\ u(x,t) = 0\}\,.$$

In particular, a solution $\bar{x}$ of (2.5) is extremal on $[0, t]$ if $x(t) \in \partial K(t)$; in this case, it satisfies $\bar{x}(s) \in \partial K(s)$ for any $s \in [0, t]$ and $|\bar{x}'(s)| = c(\bar{x}(s), s)$ for a.e. $s \in [0, t]$.

2.3. **Regularity of extremal solutions.** From now on we assume that c satisfies (2.2), (2.3) and (2.4). Our first result is the following:

Proposition 2.6. *Under the above assumptions, if $\bar{x}$ is extremal on $[0, \bar{t}\,]$ for some $\bar{t} \in (0, T]$ and if $\beta := \alpha - 1/p > 0$, then the map $t \to \bar{x}'(t)/|\bar{x}'(t)|$ is of class $\mathcal{C}^{\beta/2}(0, \bar{t}\,)$. Namely*

$$\left| \frac{\bar{x}'(s_2)}{|\bar{x}'(s_2)|} - \frac{\bar{x}'(s_1)}{|\bar{x}'(s_1)|} \right| \leq C\|\omega\|_p^{1/2}\, |s_2 - s_1|^{\beta/2} \qquad \textit{for all } s_1, s_2 \in [0, \bar{t}\,]\,,$$

where C only depends on the constants A, B, α and p introduced in (2.2)–(2.4).

Proof: Throughout the proof C denotes a constant which depends on A, B, α and p only.

By Lemma 2.3 (1)(ii), we have $|\bar{x}'(t)| = c(\bar{x}(t), t)$ a.e. on $[0, \bar{t}]$. We reparametrize the path $\bar{x}$ with speed 1 as follows. Let θ be a solution of

$$\begin{cases} \theta'(s) = \frac{1}{c(\bar{x}(\theta(s)), \theta(s))} & s \in [0, \theta^{-1}(\bar{t}\,)], \\ \theta(0) = 0. \end{cases} \tag{2.10}$$

Let us set $\bar{s} = \theta^{-1}(\bar{t}\,)$ and $\bar{y}(s) = \bar{x}(\theta(s))$ on $[0, \bar{s}]$. Then

$$|\bar{y}'(s)| = |\bar{x}'(\theta(s))\theta'(s)| = 1 \quad \text{for any } s \in [0, \bar{s}]\,. \tag{2.11}$$

Let us introduce

$$\bar{c}(y, s) = \frac{c(y, \theta(s))}{c(\bar{y}(s)), \theta(s))}. \tag{2.12}$$

From our assumptions (2.2)–(2.4), we have

$$\big|\bar{c}(y, s) - \bar{c}(y', s)\big| \leq \frac{\omega(\theta(s))}{A} |y - y'|^{\alpha} \quad \text{for all } (y, y', s) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, \bar{s}] \tag{2.13}$$

and

$$\frac{A}{B} \leq \bar{c}(y, s) \leq \frac{B}{A} \quad \text{for all } (y, s) \in \mathbb{R}^N \times [0, \bar{s}]. \tag{2.14}$$

In order to proceed we need the following lemma:

Lemma 2.7. *For any* $0 \le s_1 < s_2 \le \bar{s}$,

$$|\bar{y}(s_2)-\bar{y}(s_1)| \le s_2-s_1 = \int_{s_1}^{s_2} |\bar{y}'(s)|ds \le |\bar{y}(s_2)-\bar{y}(s_1)|+C(s_2-s_1)^\alpha \int_{s_1}^{s_2} \omega(\theta(s))ds\ .$$

Proof: First of all, $|\bar{y}(s_2)-\bar{y}(s_1)| \le s_2 - s_1 = \int_{s_1}^{s_2} |\bar{y}'(s)|ds$ because $|\bar{y}'| = 1$. Let $y : [s_1, s_2] \to \mathbb{R}^N$ solve

$$\begin{cases} y'(s) = \bar{c}(y(s), s)\frac{\bar{y}(s_2)-\bar{y}(s_1)}{|\bar{y}(s_2)-\bar{y}(s_1)|}\ , \\ y(s_1) = \bar{y}(s_1)\ . \end{cases} \tag{2.15}$$

Note that y remains in the segment $[\bar{y}(s_1), \bar{y}(s_2)]$ on $[s_1, s_2]$ because y is admissible for (2.5), and so is sub-optimal. Moreover y is monotonous on the segment. From the bounds (2.14) on $\bar{c}$, we have

$$|y(s) - \bar{y}(s)| \le \frac{2B}{A}(s_2 - s_1) \quad \text{for all } s \in [s_1, s_2]\ .$$

Since $\bar{c}(\bar{y}(s), s) = 1$ and $\bar{c}$ satisfies (2.13), we have

$$s_2 - s_1 = \int_{s_1}^{s_2} \bar{c}(\bar{y}(s), s)dt \le \int_{s_1}^{s_2} \bar{c}(y(s), s)dt + \left(\frac{2B}{A}\right)^\alpha (s_2 - s_1)^\alpha \int_{s_1}^{s_2} \frac{\omega(\theta(s))}{A}ds.$$

On the other hand, y lives in the segment $[\bar{y}(s_1), \bar{y}(s_2)]$ and is monotonous on this segment, so that, from (2.15), we get

$$\int_{s_1}^{s_2} \bar{c}(y(s), s)ds = \int_{s_1}^{s_2} |y'(s)|ds = |y(s_2) - y(s_1)| \le |\bar{y}(s_2) - \bar{y}(s_1)|\ .$$

Putting together the last two estimates proves the Lemma.

□

Next we claim the following result:

Lemma 2.8. *For any* $0 \le s_1 < s_2 \le \bar{s}$, *we have*

$$\left|\bar{y}\left(\frac{s_1+s_2}{2}\right) - \frac{\bar{y}(s_1)+\bar{y}(s_2)}{2}\right| \\ \le C\left\{(s_2-s_1)^\alpha \int_{s_1}^{s_2} \omega(\theta(s))ds + (s_2-s_1)^{(1+\alpha)/2}\left(\int_{s_1}^{s_2} \omega(\theta(s))ds\right)^{\frac{1}{2}}\right\}.$$

Proof: Let us set $s_0 = (s_1+s_2)/2$, $a = \bar{y}(s_0) - \bar{y}(s_1)$, $b = \bar{y}(s_2) - \bar{y}(s_0)$ and $\tau = s_2 - s_1$. Then, from Lemma 2.7 we have

$$|a| + |b| \ \le \int_{s_1}^{s_0} |\bar{y}'(s)|\, ds + \int_{s_0}^{s_2} |\bar{y}'(s)|\, ds \ \le \ \int_{s_1}^{s_2} |\bar{y}'(s)|ds \ \le \ |a+b| + \varepsilon\ ,$$

where $\varepsilon := C\tau^\alpha \int_{s_1}^{s_2} \omega(\theta(s))ds$. Taking the square in the above inequality and expanding this expression, we get

$$2|a||b| \le 2\langle a, b\rangle + 2|a+b|\varepsilon + \varepsilon^2\ .$$

Hence

$$\left|\frac{a}{|a|} - \frac{b}{|b|}\right|^2 \le \frac{2|a+b|\varepsilon + \varepsilon^2}{|a||b|}\ .$$

From (2.11) and (2.14), we have

$$\frac{A}{B}\frac{\tau}{2} \le |a|, |b| \le \frac{\tau}{2}\ .$$

It follows that

$$\left|\frac{a}{|a|}-\frac{b}{|b|}\right|^2 \leq 8\left(\frac{B}{A}\right)\frac{\varepsilon}{\tau}+4\left(\frac{B}{A}\right)^2\frac{\varepsilon^2}{\tau^2}\,.$$

Let us estimate $||a|-|b||$: from Lemma 2.7 we have

$$|a| \leq \int_{s_1}^{s_0} |\bar{y}'(s)|ds = \frac{\tau}{2} = \int_{s_0}^{s_2} |\bar{y}'(s)|ds \leq |\bar{y}(s_2)-\bar{y}(s_0)|+\varepsilon = |b|+\varepsilon\,.$$

We obtain the inequality $|b| \leq |a|+\varepsilon$ in the same way, which proves that $||a|-|b|| \leq \varepsilon$. Then we write

$$|a-b| = |a|\left|\frac{a}{|a|}-\frac{b}{|a|}\right| \leq |a|\left|\frac{a}{|a|}-\frac{b}{|b|}\right|+||a|-|b||\,.$$

Therefore, since $|a| \leq \tau/2$, we have

$$|a-b| \leq C(\sqrt{\varepsilon\tau}+\varepsilon)\,,$$

which is the desired result from the definition of ε.

□

We are now ready to complete the proof of Proposition 2.6. Since $1/B \leq \theta' \leq 1/A$, we have

$$\int_{s_1}^{s_2} \omega(\theta(s))ds = \int_{\theta(s_1)}^{\theta(s_2)} \frac{\omega(s)}{\theta'(\theta^{-1}(s))}ds \leq B\int_{\theta(s_1)}^{\theta(s_2)} \omega(s)ds$$

where, from Hölder's inequality,

$$\int_{\theta(s_1)}^{\theta(s_2)} \omega(s)ds \;\leq\; |\theta(s_2)-\theta(s_1)|^{1-1/p}\|\omega\|_p \;\leq\; A^{-1+1/p}|s_2-s_1|^{1-1/p}\|\omega\|_p.$$

This shows that

$$\int_{s_1}^{s_2} \omega(\theta(s))ds \leq C\|\omega\|_p\,|s_2-s_1|^{1-1/p}\,. \tag{2.16}$$

If $\beta=\alpha-1/p>0$, then, combining Lemma 2.8 with (2.16), we get

$$\left|\bar{y}\left(\frac{s_1+s_2}{2}\right)-\frac{\bar{y}(s_1)+\bar{y}(s_2)}{2}\right| \;\leq\; C\|\omega\|_p^{1/2}\,|s_2-s_1|^{1+\beta/2}$$

as soon as $s_2-s_1 \leq \|\omega\|_p^{-1/\beta}$. Theorem 2.1.10 of [14] then states that each component of $\bar{y}$ is semi-convex and semi-concave with a modulus m of the form $m(\rho)=C\|\omega\|_p^{1/2}\rho^{\beta/2}$. Moreover, from Theorem 3.3.7 of [14], we know that $\bar{y}$ is $\mathcal{C}^{1,\beta/2}$ with constant $C\|\omega\|_p^{1/2}$. Therefore

$$|\bar{y}'(s_2)-\bar{y}'(s_1)| \leq C\|\omega\|_p^{1/2}\,|s_2-s_1|^{\beta/2}$$

which completes the proof since θ^{-1} is $B-$Lipschitz continuous and

$$\frac{\bar{x}'(t)}{|\bar{x}'(t)|} = \bar{y}'(\theta^{-1}(t)).$$

□

Remark 2.9. We have actually proved that $\bar{y}$ is $\mathcal{C}^{1,\beta/2}$, $\beta=\alpha-1/p$, with constant $C\|\omega\|_p^{1/2}$, where C depends only on A, B, α and p.

2.4. **A priori regularity of the moving front.** We consider a solution u to (2.1) for a velocity c which satisfies (2.2), (2.3) and (2.4). We set, as before,

$$K(t) = \{x \in \mathbb{R}^N \, ; \, u(x,t) \geq 0\} \quad \text{for all } t \in [0,T] \, .$$

We introduce cone-like sets and interior cone properties as follows.

Definition 2.10. *Let $x \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{N-1}$ be a unit vector.*

- *For any $0 < \rho < \theta$, the* cone *of vertex x, axis ν and parameters (ρ,θ) is defined by*

$$\begin{aligned} \widehat{\mathcal{C}}^{\,\rho,\theta}_{\nu,x} &:= \bigcup_{t\in[0,\theta]} B\left(x+t\nu, t\frac{\rho}{\theta}\right) \\ &= \{x + t\nu + t\frac{\rho}{\theta}\xi \, : \, t \in [0,\theta], \, \xi \in \overline{B}(0,1)\}. \end{aligned}$$

- *For $C > 0$, $\delta \in (0,1)$, we define the* paraboloid

$$\begin{aligned} \widehat{\mathcal{C}}^{\,\delta,C}(x,\nu) &= \bigcup_{t\in[0,C^{-1/\delta}]} B\left(x+t\nu, t - Ct^{1+\delta}\right) \\ &= \{x + t\nu + (t - Ct^{1+\delta})\xi \, : \, t \in [0,C^{-1/\delta}], \, \xi \in \overline{B}(0,1)\}. \end{aligned}$$

We recall from [7] that a compact subset K of $\mathbb{R}^N$ is said to have the interior cone property of parameters (ρ,θ) if, for any $x \in \partial K$, there exists $\nu \in \mathbb{S}^{N-1}$ such that the cone $\widehat{\mathcal{C}}^{\,\rho,\theta}_{\nu,x}$ is contained in K.

In the same way, we say that K satisfies the interior $\widehat{\mathcal{C}}^{\,\delta,C}$-property if for any $x \in \partial K$, there exists $\nu \in \mathbb{S}^{N-1}$ such that $\widehat{\mathcal{C}}^{\,\delta,C}(x,\nu)$ is contained in K.

The set $\widehat{\mathcal{C}}^{\,\rho,\theta}_{\nu,x}$ is a classical cone (see Figure 1). Since the map $t \to t - Ct^{1+\delta}$ is concave, a tedious but straightforward computation shows that the set $\widehat{\mathcal{C}}^{\,\delta,C}(x,\nu)$ is convex. We shall see below (Lemma 2.13) that it has a $\mathcal{C}^{1,\gamma}$ boundary in a neighbourhood of x for some $\gamma \in (0,1)$ and contains a paraboloid-like subset. This motivates the name *paraboloid* (see Figure 1 for an illustration). Notice that $\widehat{\mathcal{C}}^{\,\rho,\theta}_{\nu,x} \subset \widehat{\mathcal{C}}^{\,\delta,C}(x,\nu)$ as soon as $\theta \leq C^{-1/\delta}$ and $\rho \leq \theta - C\theta^{1+\delta}$.

Lemma 2.11. *Let us still assume that $\beta = \alpha - 1/p > 0$. There exist positive constants C_0, C_1 depending only on A, B, α and p, such that, setting $C(\omega) = C_0\|\omega\|_p^{1/2}$, for any extremal solution $\bar{x}$ on $[0,\bar{t}]$ with $\bar{t} \geq C_1 C(\omega)^{-2/\beta}$, the set $\widehat{\mathcal{C}}^{\,\beta/2,C(\omega)}(x,\nu)$ is contained in $K(\bar{t})$, where*

$$x = \bar{x}(\bar{t}) \, , \quad \nu = -\frac{\bar{x}'(\bar{t})}{|\bar{x}'(\bar{t})|}.$$

Proof: As in the proof of Proposition 2.6, we reparametrize $\bar{x}$ with speed 1 by introducing $\bar{y}(s) = \bar{x}(\theta(s))$ on $[0,\bar{s}]$ where $\bar{y}$ and $\bar{s}$ are defined by (2.10). Notice that $\bar{y}'(s) = \bar{x}'(\theta(s))/|\bar{x}'(\theta(s))|$ for a.e. $s \in [0,\bar{s}]$.

Next we define $\bar{c}$ by (2.12) and, for $s \in (0,\bar{s})$ and $b \in \overline{B}(0,1)$, we consider the solution $y : [s,\bar{s}] \to \mathbb{R}^N$ to

$$\begin{cases} y'(\sigma) = \bar{c}(y(\sigma),\sigma)b, & \sigma \in [s,\bar{s}], \\ y(s) = \bar{y}(s). \end{cases}$$

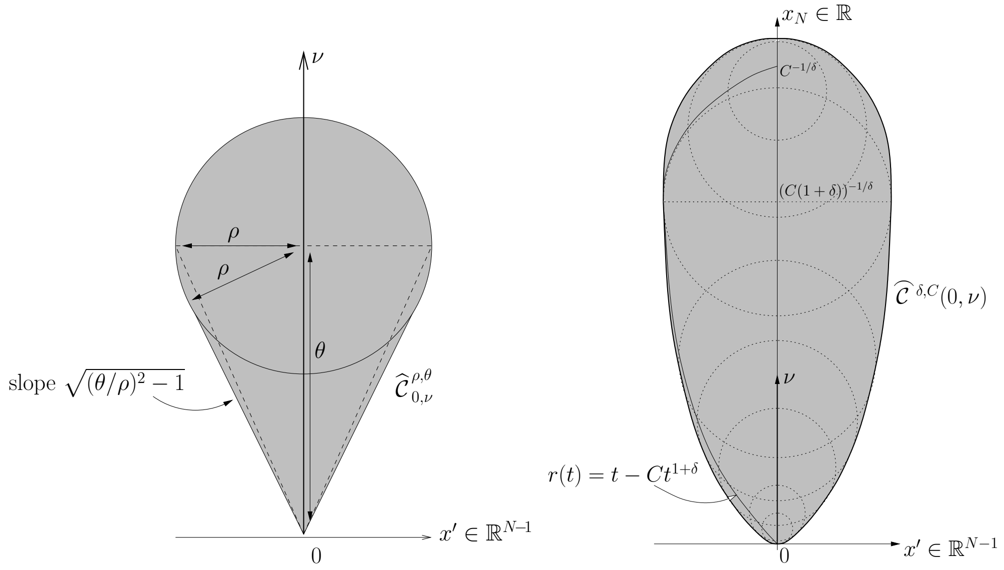

FIGURE 1. *Classical cone and paraboloid.*

Arguing as in the proof of Lemma 2.7, we obtain that y is sub-optimal and monotonous on the segment $[\bar{y}(s), y(\bar{s})]$. In particular, this whole segment lies in $K(\bar{t}\,)$.

From the bound (2.14) on $\bar{c}$, we have

$$|y(\sigma) - \bar{y}(\sigma)| \leq \frac{2B}{A}(\bar{s} - s) \quad \text{for all } \sigma \in [s, \bar{s}].$$

Hence, by (2.13),

$$\bar{s} - s = \int_s^{\bar{s}} \bar{c}(\bar{y}(\sigma), \sigma) d\sigma \leq \int_s^{\bar{s}} \bar{c}(y(\sigma), \sigma) d\sigma + \frac{(2B)^\alpha}{A^{1+\alpha}}(\bar{s} - s)^\alpha \int_s^{\bar{s}} \omega(\theta(\sigma)) d\sigma \ ,$$

where

$$\int_s^{\bar{s}} \omega(\theta(\sigma)) d\sigma \leq BA^{-1+1/p} \left\|\omega\right\|_p \, (\bar{s} - s)^{1-1/p} \ .$$

Since y lives in the segment $[\bar{y}(s), y(\bar{s})] \subset [\bar{y}(s), \bar{y}(s) + \frac{B}{A}(\bar{s} - s)b]$ and is monotonous on this segment, we have

$$\int_s^{\bar{s}} \bar{c}(y(\sigma), \sigma) d\sigma = |y(\bar{s}) - \bar{y}(s)| \ .$$

It follows that

$$|y(\bar{s}) - \bar{y}(s)| \geq (\bar{s} - s)(1 - \tilde{C}\|\omega\|_p(\bar{s} - s)^\beta), \text{ where } \tilde{C} = \frac{2^\alpha B^{1+\alpha}}{A^{2+\alpha-1/p}} \text{ and } \beta = \alpha - \frac{1}{p}.$$

Moreover, any point in the segment $[\bar{y}(s), y(\bar{s})]$ also belongs to $K(\bar{t}\,)$. We have therefore proved that

$$\bar{y}(s) + (\bar{s} - s)(1 - \tilde{C}\|\omega\|_p(\bar{s} - s)^\beta)b \in K(\bar{t}\,) \ .$$

This holds true for any $b \in \overline{B}(0,1)$ and any s such that $(\bar{s}-s) \le \tilde{C}^{-1/\beta}\|\omega\|_p^{-1/\beta}$. In particular, as soon as $\bar{s} \ge \tilde{C}^{-1/\beta}\|\omega\|_p^{-1/\beta}$, we have, setting $t = \bar{s}-s$,

$$\bigcup_{t\in[0,\tilde{C}^{-1/\beta}\|\omega\|_p^{-1/\beta}]} B\left(\bar{y}(\bar{s}-t), t(1-\tilde{C}\|\omega\|_p t^\beta)\right) \subset K(\bar{t}\,) ,$$

where $\beta = \alpha - 1/p > 0$. From the $\mathcal{C}^{1,\beta/2}$ regularity of $\bar{y}$ (see Remark 2.9), using that $\nu = -\dfrac{\bar{x}'(\bar{t}\,)}{|\bar{x}'(\bar{t}\,)|} = -\bar{y}'(\bar{s})$, we have

$$|\bar{y}(\bar{s}-t) - (x+t\nu)| \le C\|\omega\|_p^{1/2}\int_0^t s^{\beta/2}ds \le C\|\omega\|_p^{1/2}\, t^{1+\beta/2},$$

where C only depends on A, B, α and p. Let us set

$$C_0 = C + B^{\beta/2}\tilde{C}^{1/2}, \quad C_1 = A^{-1}\tilde{C}^{-1/\beta}C_0^{2/\beta}$$

and

$$C(\omega) = C_0\|\omega\|_p^{1/2}.$$

Then, going back to the expression of $\bar{x}$, we obtain that, if $\bar{t} \ge C_1 C(\omega)^{-2/\beta}$,

$$\widehat{\mathcal{C}}^{\,\beta/2,C(\omega)}(x,\nu) = \bigcup_{t\in[0,C(\omega)^{-2/\beta}]} B\left(x+t\nu, t(1-C(\omega)t^{\beta/2})\right) \subset K(\bar{t}\,).$$

□

The above results have the following consequence:

Corollary 2.12. *Let us assume that K_0 has the interior ball property of radius r_0:*

$$\textit{For all } x\in K_0,\ \textit{there exists } y \in K_0,\ \textit{with } x \in \overline{B}(y,r_0) \subset K_0\ . \tag{2.17}$$

Then there is a positive constant C_0 depending only on A, B, α and p such that for any $t \in [0,T]$, $K(t)$ has the interior $\widehat{\mathcal{C}}^{\,\beta/2,C(\omega)}$-property , where $C(\omega) = C_0\|\omega\|_p^{1/2}$.

In particular, there is a constant

$$\rho = \frac{1}{2}(2C(\omega))^{-2/\beta} = \frac{1}{2}(2C_0)^{-2/\beta}\|\omega\|_p^{-1/\beta}$$

such that for any $t\in[0,T]$, the set $K(t)$ has the interior cone property of parameters $(\rho, 2\rho)$.

Proof: Let us prove the first part of the corollary. Let K_1 be such that $K_0 = K_1 + r_0\overline{B}(0,1)$. Then $K(t)$ is the reachable set at time r_0+t for the system

$$x'(t) = \tilde{c}(x(t),t)b(t) \qquad |b(t)| \le 1\ ,$$

starting from K_1, where $\tilde{c}(x,t) = 1$ if $t \in [0,r_0]$, and $\tilde{c}(x,t) = c(x,t-r_0)$ if $t \in (r_0, T+r_0]$ (notice that $\tilde{c}$ satisfies (2.2)–(2.3)–(2.4)). For this system, Lemma 2.11 shows the result as soon as $t \ge C_1C(\omega)^{-2/\beta}$. Therefore, if we assume that $C_1C(\omega)^{-2/\beta} \le r_0$, which is always possible by increasing $\|\omega\|_p$, then the result holds for $K(t)$, for any $t\in[0,T]$.

For the second part of the result, let $\theta = 2\rho = (2C(\omega))^{-2/\beta}$, $t\in(0,T]$, $x \in \partial K(t)$ and $\nu \in \mathbb{S}^{N-1}$ be such that $\widehat{\mathcal{C}}^{\,\beta/2,C(\omega)}(x,\nu) \subset K(t)$.

Since $\theta = (2C(\omega))^{-2/\beta}$, we have $\theta \le C(\omega)^{-2/\beta}$ and $\rho \le \theta - C(\omega)\theta^{1+\beta/2}$, so that $\widehat{\mathcal{C}}^{\,\rho,\theta}_{x,\nu} \subset \widehat{\mathcal{C}}^{\,\beta/2,C(\omega)}$. This proves that the cone $\widehat{\mathcal{C}}^{\,\rho,\theta}_{x,\nu}$, with $\theta = 2\rho$, is contained in $K(t)$.

□

We now show that the convex set $\widehat{\mathcal{C}}^{\,\delta,C}(\bar{x},\nu)$ has a boundary of class $\mathcal{C}^{1,\gamma}$ in a neighborhood of $\bar{x}$ for some $\gamma > 0$. Let us fix a frame $\{e_1,\dots,e_N\}$ of $\mathbb{R}^N$ such that $\bar{x}=0$, $\nu=e_N$. We denote by (x',x_N) a generic element of $\mathbb{R}^N$, with $x'\in\mathbb{R}^{N-1}$, $x_N\in\mathbb{R}$.

Lemma 2.13. *Let $C>0$ and $\delta>0$ be fixed. There are constants $\gamma=\delta/(2+\delta)$, $c=2(2C)^{1/(2+\delta)}$, $\tau_0=(2C)^{-\frac{1}{\delta}}$ and $r_0=(\sqrt{3}-1)^{\frac{2+\delta}{\delta}}\tau_0$ such that the set*

$$\{(x',x_N)\in\mathbb{R}^N\ ;\ |x'|\le r_0,\ c|x'|^{1+\gamma}\le x_N\le\tau_0\}$$

is contained in $\widehat{\mathcal{C}}^{\,\delta,C}(0,\nu)$.

Proof: Note that, by choice of τ_0, the map $\tau\to r(\tau)=\tau(1-C\tau^\delta)$ is nondecreasing on $[0,\tau_0]$. For any $\tau\in(0,\tau_0]$, the ball $\overline{B}(\tau e_N,r(\tau))$ is contained in $\widehat{\mathcal{C}}^{\,\delta,C}(0,\nu)$, which is convex. Let us set $\psi_\tau(x')=\tau-(r^2(\tau)-|x'|^2)^{1/2}$. Since the set $\widehat{\mathcal{C}}^{\,\delta,C}(0,\nu)$ is convex, the set

$$\{(x',x_N)\in\mathbb{R}^N\ ;\ |x'|\le r(\tau),\ \psi_\tau(x')\le x_N\le\tau_0\} \tag{2.18}$$

is contained in $\widehat{\mathcal{C}}^{\,\delta,C}(0,\nu)$. Indeed, if $|x'|\le r(\tau)$, then $(x',\psi_\tau(x'))\in\overline{B}(\tau e_N,r(\tau))$ while $(x',\tau_0)\in\overline{B}(\tau_0 e_N,r(\tau))$. Let $|x'|\le r_0$ and let us choose

$$\tau=(2C)^{-1/(2+\delta)}|x'|^{2/(2+\delta)}.$$

Then $\tau\in(0,\tau_0)$ and $|x'|\le r(\tau)$ (here we use the fact that $|x'|\le r_0$). Moreover, since $|x'|^2=2C\tau^{2+\delta}$, we get

$$\begin{aligned}\psi_\tau(x')\le&\ \ \tau-\left(\tau^2(1-C\tau^\delta)^2-2C\tau^{2+\delta}\right)^{1/2}\\ \le&\ \ \tau\left[1-\left(1-4C\tau^\delta\right)^{1/2}\right]\\ \le&\ \ 2C\tau^{1+\delta}\ =\ (2C)^{1/(2+\delta)}|x'|^{1+\gamma}\ .\end{aligned}$$

Using (2.18), we get that any point of the form (x',x_N) with

$$|x'|\le r_0\quad\text{and}\quad c|x'|^{1+\gamma}\le x_N\le\tau_0,\quad\text{where } c=2(2C)^{1/(2+\delta)},$$

belongs to $\widehat{\mathcal{C}}^{\,\delta,C}(0,\nu)$.

□

Let us now state a stability property for sets satisfying an interior $\widehat{\mathcal{C}}^{\,\delta,C}$-property:

Lemma 2.14. *Let (z_n) be a sequence of Lipschitz continuous real-valued maps on $\mathbb{R}^N$ which converges uniformly to some z. We assume that $\{z_n\le 0\}=\{z\le 0\}$, that there exist constants $A,B>0$ such that the following inequality holds in the viscosity sense: for any $n\in\mathbb{N}$,*

$$\frac{1}{B}\le|Dz_n(x)|\le\frac{1}{A}\qquad\text{in }\{0<z_n<T\},$$

and that there exist $C,\delta>0$ such that for any $x\in\{0<z<T\}$ and any n sufficiently large, there is some $\nu\in\mathbb{S}^{N-1}$ with $\widehat{\mathcal{C}}^{\,\delta,C}(x,\nu)\subset\{z_n\le z_n(x)\}$. Then

$$\frac{Dz_n(x)}{|Dz_n(x)|}\to\frac{Dz(x)}{|Dz(x)|}\qquad a.e.\ in\ \{0<z<T\}$$

and $(|Dz_n|)$ converges to $|Dz|$ in $L^\infty-$weak$-*$ in $\{0<z<T\}$.

Proof: By standard stability property of viscosity solutions we have that

$$\frac{1}{B} \le |Dz(x)| \le \frac{1}{A} \qquad \text{in } \{0 < z < T\},$$

in the viscosity and a.e. sense. Note also that, in view of Remark 2.5, the indicator function of the set $\{0 < z_n < T\}$ converges a.e. to the indicator function of $\{0 < z < T\}$. Let x be such that z_n and z are positive and differentiable at x for any n. Then $|Dz_n(x)| > 0$ for any n and $|Dz(x)| > 0$. From the regularity assumption on z_n there exists $\nu_n \in \mathbb{S}^{N-1}$ such that $\widehat{\mathcal{C}}^{\delta,C}(x, \nu_n) \subset \{z_n \le z_n(x)\}$. Since $Dz_n(x)$ exists and is nonzero and since the set $\widehat{\mathcal{C}}^{\delta,C}(x, \nu_n)$ is of class $\mathcal{C}^1$ at x (thanks to Lemma 2.13), one must have $\nu_n = -Dz_n(x)/|Dz_n(x)|$. Let ν be the limit of a subsequence of the (ν_n). Then $\widehat{\mathcal{C}}^{\delta,C}(x, \nu) \subset \{z \le z(x)\}$, so that by the same argument as above, $\nu = -Dz(x)/|Dz(x)|$. Accordingly any converging subsequence of $Dz_n(x)/|Dz_n(x)|$ converges to $Dz(x)/|Dz(x)|$, which shows the a.e. convergence of $(Dz_n/|Dz_n|)$ to $Dz/|Dz|$.

Since the (z_n) are uniformly Lipschitz continuous and (z_n) converges uniformly to z, (Dz_n) converges to Dz in $L^\infty-$weak$-*$ in $\{0 < z < T\}$. Let $a \in L^1(\mathbb{R}^N, \mathbb{R}^N)$. Then we have on the one hand

$$\lim_{n\to+\infty} \int_{\{0<z<T\}} \langle a, Dz_n \rangle = \int_{\{0<z<T\}} \langle a, Dz \rangle \ .$$

On the other hand, if we denote by ξ any weak$-*$ limit of a subsequence $(|Dz_{n_k}|)$, we have, from the a.e. convergence of $(Dz_n/|Dz_n|)$ to $Dz/|Dz|$,

$$\lim_{k\to+\infty} \int_{\{0<z<T\}} \langle a, Dz_{n_k} \rangle = \lim_{k\to+\infty} \int_{\{0<z<T\}} \langle a, \frac{Dz_{n_k}}{|Dz_{n_k}|} \rangle |Dz_{n_k}| = \int_{\{0<z<T\}} \langle a, \frac{Dz}{|Dz|} \rangle \xi \ .$$

This implies that

$$Dz(x) = \frac{Dz(x)}{|Dz(x)|}\xi(x) \qquad \text{a.e. in } \{0 < z < T\} \ ,$$

and shows that $\xi = |Dz|$. Hence $(|Dz_n|)$ converges to $|Dz|$ weakly$-*$ in $\{0 < z < T\}$.

□

We complete the section by proving that a set with the interior cone property is the union of a finite number of Lipschitz graphs.

Proposition 2.15. *Let $(K(t))_{t\in[0,T]}$ be a nondecreasing family of compact subsets of $\mathbb{R}^N$, each $K(t)$ having the interior cone property of parameter $(\rho, 2\rho)$ for some $\rho > 0$. Then for any $\bar{x} \in \mathbb{R}^N$ and any $r \ge \rho$, there is an integer $C(r, \rho) \le C(N)r/\rho$ (where $C(N)$ only depends on N) and, for each $i \in \{1, \dots, C(r, \rho)\}$,*

- *a Borel measurable map $\Psi_i : B_{N-1}(0, r) \times [0, T] \to \mathbb{R}$, which is $\sqrt{15}-$Lipschitz continuous with respect to the space variable,*
- *and a change of coordinates $O_i : \mathbb{R}^N \to \mathbb{R}^N$ (i.e., the composition of a rotation and a translation), with $O_i(0) = \bar{x}$,*

such that, for all $t \in [0, T]$,

$$\partial K(t) \cap B(\bar{x}, r) \subset \bigcup_{i=1,\dots,C(r,\rho)} \left\{ O_i(x', \Psi_i(x', t)) \ , \ x' \in B_{N-1}(0, r) \right\} .$$

If furthermore the family $(K(t))$ is contained in some ball $\overline{B}(0,M)$, then we can take $r=+\infty$ and $C(\rho)\leq C(N)M/\rho$ and we have, for all $t\in[0,T]$,

$$\partial K(t)\subset \bigcup_{i=1,\ldots,C(r,\rho)} \left\{O_i(x',\Psi_i(x',t))\ ,\ x'\in B_{N-1}(0,M)\right\}.$$

An important and straightforward consequence of the fact that $\partial K(t)$ is piecewise Lipschitz continuous is that the sets $K(t)$ are of (locally) finite perimeter.

Proof: We closely follow several arguments of [7]. We first observe that if $x\in\partial K$ and $\widehat{\mathcal{C}}^{\,\rho,2\rho}_{x,\nu}\subset K(t)$, then for all $\nu'\in\mathbb{S}^{N-1}$ verifying $|\nu-\nu'|\leq 1/4$, we have $\widehat{\mathcal{C}}^{\,\rho/2,2\rho}_{x,\nu'}\subset K(t)$. By compactness of $\mathbb{S}^{N-1}$, we can cover $\mathbb{S}^{N-1}$ with the traces on $\mathbb{S}^{N-1}$ of at most p balls of radius $1/4$ centered at ν_i, for some positive constant $p=p(N)$ and $1\leq j\leq p$. Therefore, for any $x\in\partial K(t)$, there exists $1\leq j\leq p$ such that $\widehat{\mathcal{C}}^{\,\rho/2,2\rho}_{x,\nu_j}\subset K(t)$.

Let us now fix $\bar{x}$ and $1\leq j\leq p$. Up to a translation and a rotation of the space, we can assume that $\bar{x}=0$, $\nu_j=(0,\ldots,0,1)$. For any $x\in\mathbb{R}^N$, we write $x=(x',x_N)$ with $x'\in\mathbb{R}^{N-1}$ and $x_N\in\mathbb{R}$. For any $t\in[0,T]$ and any integer k with $|k|\leq r/\rho+1$, we set

$$U_k = B_{N-1}(0,r)\times[k\rho,(k+1)\rho]\ ,$$

$$A_{j,k}(t)=\left\{x=(x',x_N)\in\partial K(t)\cap\overline{U_k}\ ;\ \widehat{\mathcal{C}}^{\,\rho/2,2\rho}_{x,\nu_j}\subset K(t)\right\}\ ,$$

and, for all $y'\in B_{N-1}(0,r)$,

$$\Psi_{j,k}(y',t)=\min\left\{(k+1)\rho\ ,\ \inf_{x\in A_{j,k}(t)}\psi_x(y')\right\},$$

where $\psi_x(y')=\sqrt{15}|y'-x'|+x_N$ is such that $(\operatorname{graph}\psi_x)\cap U_k=\widehat{\mathcal{C}}^{\,\rho/2,2\rho}_{x,\nu_j}\cap U_k$ (see Figure 2 for an illustration). We claim that

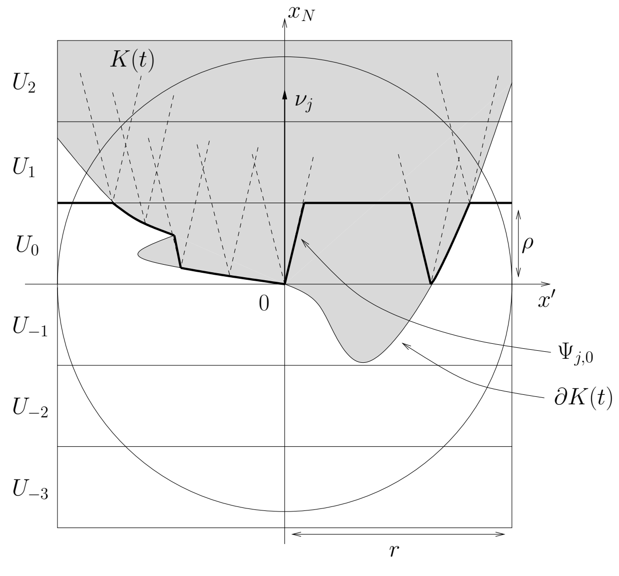

FIGURE 2.

$$A_{j,k}(t) \cap U_k \subset \operatorname{graph} \Psi_{j,k}(\cdot, t).$$

Indeed, let $x \in A_{j,k}(t) \cap U_k$. If $x \notin \operatorname{graph} \Psi_{j,k}(\cdot,t)$, then $\Psi_{j,k}(x', t) < \psi_x(x') = x_N$. Therefore, there exists $z \in A_{j,k}(t)$ such that $\psi_z(x') < x_N$. It follows that $x \in \operatorname{int} \widehat{\mathcal{C}}^{\rho/2,2\rho}_{z,\nu_j} \subset \operatorname{int} K(t)$ and x cannot belong to $\partial K(t)$, which is a contradiction. This proves the claim. Then we remark that $\Psi_{j,k}(\cdot, t)$ is a Lipschitz continuous map with constant $\sqrt{15}$ as the infimum of a family of maps having this property.

This means that $\partial K(t) \cap B(\bar{x}, r)$ is contained in at most $p(2r/\rho + 2)$ Lipschitz graphs with constant $\sqrt{15}$, which concludes the proof since $r \geq \rho$; indeed this implies that $p(2r/\rho + 2) \leq 4p\, r/\rho =: C(r, \rho)$.

□

3. Representation and a priori estimates for the heat equation

The aim of this section is to provide estimates for the following heat equations

$$\begin{cases} v_t - \Delta v + g(x,t)\mathcal{H}^{N-1} \lfloor \Gamma(t) = 0 & \text{in } \mathbb{R}^N \times (0,T) \ , \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N \ , \end{cases} \tag{3.1}$$

and

$$\begin{cases} v_t - \Delta v + \kappa \bar{g}(v(x,t))\mathcal{H}^{N-1} \lfloor \Gamma(t) = 0 & \text{in } \mathbb{R}^N \times (0,T) \ , \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N \ , \end{cases} \tag{3.2}$$

for a given evolving front $(\Gamma(t))_{t \geq 0}$.

Throughout the section we work under the following conditions on the data:

(H1) $g : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is continuous and bounded by a constant $M > 0$.

(H2) $\kappa \in \mathbb{R}$ and $\bar{g} : \mathbb{R} \to \mathbb{R}$ is bounded by M and Lipschitz continuous.

(H3) v_0 is Lipschitz continuous and bounded.

(H4) The evolving family $(\Gamma(t))_{t \in [0,T]}$ can be represented as

$$\Gamma(t) = \{x \in \mathbb{R}^N \ ; \ z(x) = t\} \qquad \text{for all } t \in (0,T) \ . \tag{3.3}$$

where $z : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and satisfies

$$\frac{1}{B} \leq |Dz(x)| \leq \frac{1}{A} \qquad \text{in } \{0 < z < T\} \tag{3.4}$$

in the viscosity sense for some $A, B > 0$. Furthermore we assume that there is some $\bar{\rho} > 0$ such that the set

$$K(t) = \{x \in \mathbb{R}^N \ ; \ z(x) \leq t\}$$

has the interior cone property of parameter $(\bar{\rho}, 2\bar{\rho})$ for all $t \in (0,T)$, and that there exists $M > 0$ such that

$$K(t) \subset \overline{B}(0, M).$$

Let us recall that, thanks to the interior cone condition, $K(t)$ is a set of finite perimeter and, moreover, its boundary $\Gamma(t)$ is contained in the union of a finite number of Lipschitz graphs (Proposition 2.15).

Throughout the section we denote by C a constant which only depends on A, B, N, T, M, κ and may vary from line to line in the computations.

3.1. **Representation and L^∞ bounds for the solution of** (3.1)**.**

Lemma 3.1. *There exists a unique solution to* (3.1)*. This solution is given, for all* $(x,t) \in \mathbb{R}^N \times [0,T]$*, by*

$$v(x,t) = \int_{\mathbb{R}^N} G(x-y,t)v_0(y)\,dy - \int_0^t \int_{\Gamma(s)} G(x-y,t-s)g(y,s)\,d\mathcal{H}^{N-1}(y)ds,$$

where $G(x,t) = (4\pi t)^{-N/2}e^{-|x|^2/(4t)}$ *is the kernel of the heat equation, and satisfies the uniform bound*

$$|v(x,t)| \le C(1+|\log(\bar\rho)|) \qquad for\ all\ (x,t) \in \mathbb{R}^N \times [0,T], \tag{3.5}$$

where $\bar\rho$ *is the cone paramater which appears in* **(H4)***.*

Proof: Uniqueness of the solution is clear. The term $\int_{\mathbb{R}^N} G(x-y,t)v_0(y)\,dy$ corresponds to the initial datum and satisfies the bound

$$\left|\int_{\mathbb{R}^N} G(x-y,t)v_0(y)\,dy\right| \le \|v_0\|_\infty\,.$$

In order to prove the representation formula and the bound for v, we can therefore assume that $v_0 = 0$. Let us set $f_\varepsilon(x,t) = \mathbf{1}_{K(t)} * G(\cdot,\varepsilon)$ (where the convolution is only made with respect to the space variable). Then f_ε is smooth in space and strictly converges in the BV sense to $\mathbf{1}_{K(t)}$ (see [3, Def. 3.14] and [17, Sect. 5.2]). In particular, since $\partial K(t)$ is piecewise Lispchitz continuous, the measure $|Df_\varepsilon(\cdot,t)|dx$ weakly-$*$ converges to $\mathcal{H}^{N-1}\lfloor\Gamma(t)$ ([3, Prop. 3.62]). For all $(x,t) \in \mathbb{R}^N \times [0,T]$, let

$$v_\varepsilon(x,t) = -\int_0^t \int_{\mathbb{R}^N} g(y,s)G(x-y,t-s)|Df_\varepsilon(y,s)|dyds.$$

Since $|Df_\varepsilon(\cdot,t)|$ is Lipschitz continuous, it is well-known that v_ε is a solution of

$$(v_\varepsilon)_t - \Delta v_\varepsilon + g(y,s)|Df_\varepsilon(x,t)| = 0 \ \text{ in } \mathbb{R}^N \times (0,T)\,. \tag{3.6}$$

The key step in the proof of (3.5) is the following uniform bound on (v_ε):

$$|v_\varepsilon(x,t)| \le C(1+|\log(\bar\rho)|) \ \text{ for all } (x,t) \in \mathbb{R}^N \times [0,T]\,, \tag{3.7}$$

which holds for any $\varepsilon > 0$. Let us assume for a while that this is true. Then, by the weak-$*$ convergence of $|Df_\varepsilon|dx$ to $\mathcal{H}^{N-1}\lfloor\Gamma$, (v_ε) converges pointwise to v in $(\mathbb{R}^N \times (0,T))\backslash\Gamma$, hence in $L^1_{loc}(\mathbb{R}^N \times [0,T])$ since it is uniformly bounded in L^∞ thanks to the bound (3.7), and Γ has zero measure in $\mathbb{R}^N \times (0,T)$. By (3.6) v is a solution of (3.1).

It remains to prove (3.7). To do this we note that, since $K(t)$ is a set of finite perimeter, we have

$$|Df_\varepsilon(y,s)| \le \int_{\Gamma(t)} G(y-x',\varepsilon)d\mathcal{H}^{N-1}(x') \quad \text{for all } (y,s) \in \mathbb{R}^N \times (0,T),\ y \notin \Gamma(s)\,.$$

Therefore, since $G(x-x',t-s+\varepsilon) = \int_{\mathbb{R}^N} G(x-y,t-s)G(y-x',\varepsilon)dy$, we get

$$\begin{aligned}|v_\varepsilon(x,t)| \ &\le M \int_0^t \int_{\mathbb{R}^N} \int_{\Gamma(s)} G(x-y,t-s)G(y-x',\varepsilon)\,d\mathcal{H}^{N-1}(x')dyds \\ &\le C \int_0^t \int_{\Gamma(s)} G(x-x',t+\varepsilon-s)d\mathcal{H}^{N-1}(x')ds.\end{aligned}$$

Let us split this last integral in two parts, the first one denoted by I_1 being the integral between 0 and $t-\tau$ and the other one, denoted by I_2, between $t-\tau$ and t for some $\tau \in (0,t]$. Let us first estimate

$$I_1 = C\int_0^{t-\tau}\int_{\Gamma(s)} G(x-y,t+\varepsilon-s)d\mathcal{H}^{N-1}(y)ds\,.$$

From (3.3) and Lemma 3.2 below, we have

$$\begin{aligned} I_1 &= C\int_0^{t-\tau}\int_{\{z=s\}} G(x-y,t+\varepsilon-s)\ d\mathcal{H}^{N-1}(y)ds \\ &\le \frac{C}{A}\left[\int_{\{0<z<t-\tau\}} G(x-y,\varepsilon+\tau)\,dy + \int_0^{t-\tau}\int_{\{0<z<s\}} |G_t(x-y,t+\varepsilon-s)|\ dyds\right]. \end{aligned}$$

Note that

$$\int_{\{0<z<t-\tau\}} G(x-y,\varepsilon+\tau)\ dy \le \int_{\mathbb{R}^N} G(x-y,\varepsilon+\tau)\ dy\ =\ 1\,.$$

Moreover we have

$$\begin{aligned} &\int_{\{0<z<s\}} |G_t(x-y,t+\varepsilon-s)|\ dy \\ \le\ & C\int_{\mathbb{R}^N}\left(\frac{1}{(t+\varepsilon-s)^{(N+2)/2}}+\frac{|y-x|^2}{(t+\varepsilon-s)^{(N+4)/2}}\right)e^{-|y-x|^2/(4(t+\varepsilon-s))}\,dy \\ \le\ & C\int_0^{\infty}\left(\frac{r^{N-1}}{(t+\varepsilon-s)^{(N+2)/2}}+\frac{r^{N+1}}{(t+\varepsilon-s)^{(N+4)/2}}\right)e^{-r^2/(4(t+\varepsilon-s))}\,dr \\ \le\ & \frac{C}{t+\varepsilon-s}\int_0^{\infty}(r^{N-1}+r^{N+1})e^{-r^2}\,dr\ \le\ \frac{C}{t+\varepsilon-s}\ \le\ \frac{C}{t-s}. \end{aligned}$$

Therefore we get

$$I_1\ \le\ C(1+\log(t/\tau))\,.$$

We now estimate

$$I_2 = C\int_{t-\tau}^{t}\int_{\Gamma(s)} G(x-y,t+\varepsilon-s)\,d\mathcal{H}^{N-1}(y)ds\,.$$

From the structure condition on $K(s)$ and Proposition 2.15, there exists an integer $C(\bar\rho) \le C_1/\bar\rho$ (where C_1 only depends on N, M) and, for each $i \in \{1,\dots,C(\bar\rho)\}$,

- a Borel measurable map $\Psi_i : B_{N-1}(0,M)\times[0,T] \to \mathbb{R}$, which is $\sqrt{15}-$Lipschitz continuous with respect to the space variable,
- and a change of coordinates $O_i : \mathbb{R}^N \to \mathbb{R}^N$, where $O_i(0) = x$,

such that, for all $s \in [0,T]$,

$$\Gamma(s) \subset \bigcup_{i=1,\dots,C(\bar\rho)} \left\{O_i(y',\Psi_i(y',s)),\ y' \in B_{N-1}(0,M)\right\}.$$

Therefore, using that

$$\mathcal{H}^{N-1}\lfloor\{(y',\Psi_i(y',s)),y'\in B_{N-1}(0,M)\} = \sqrt{1+|D\Psi_i(y',s)|^2}\,\mathcal{L}^{N-1}\lfloor B_{N-1}(0,M),$$

we have

$$I_2 \le C\sum_{i=1}^{C(\bar\rho)}\int_{t-\tau}^{t}\int_{B_{N-1}(0,M)} G((y',\Psi_i(y',s)),t+\varepsilon-s)\sqrt{1+|D\Psi_i(y',s)|^2}\,dy'ds\,.$$

We deduce that

$$\begin{aligned} I_2 &\leq \frac{C}{\bar\rho}\int_{t-\tau}^t\int_{\mathbb{R}^{N-1}}\frac{1}{(t+\varepsilon-s)^{N/2}}e^{-|y'|^2/(4(t+\varepsilon-s))}\,dy'ds\\ &\leq \frac{C}{\bar\rho}\int_{t-\tau}^t\int_0^{+\infty}\frac{r^{N-2}}{(t+\varepsilon-s)^{N/2}}e^{-r^2/(4(t+\varepsilon-s))}\,drds\\ &\leq \frac{C}{\bar\rho}\int_{t-\tau}^t\int_0^{+\infty}\frac{r^{N-2}}{(t-s)^{1/2}}e^{-r^2}\,drds\ \leq \frac{C\sqrt{\tau}}{\bar\rho}. \end{aligned}$$

Putting together the estimates for I_1 and I_2 gives

$$|v_\varepsilon(x,t)|\leq C\left(1+\log\left(\frac{t}{\tau}\right)+\frac{\sqrt{\tau}}{\bar\rho}\right),$$

which holds for any $\tau\in(0,t]$. Choosing $\tau=\bar\rho^2$ if $t\geq\bar\rho^2$ and $\tau=t$ otherwise (in which case the decomposition reduces to I_2), we finally obtain (3.7).

□

The following Lemma, which was used in the proof, is a simple consequence of the Coarea formula.

Lemma 3.2. *Let $T>0$, $z:\mathbb{R}^N\to\mathbb{R}$ be Lipschitz continuous and such that*

$$\frac{1}{B}\leq|Dz|\leq\frac{1}{A}\qquad a.e.\ in\ \{0<z<T\}\,.$$

Let $0\leq s_1<s_2\leq T$ and assume that $\phi:\mathbb{R}^N\times(s_1,s_2)\to\mathbb{R}$ is nonnegative and such that ϕ and ϕ_t are integrable on $\{s_1<z<s_2\}$. Then

$$\int_{s_1}^{s_2}\int_{\{z=s\}}\phi(x,s)\ d\mathcal{H}^{N-1}(x)ds \leq \frac{1}{A}\left[\int_{\{s_1<z<s_2\}}\phi(x,s_2)\ dx+\int_{s_1}^{s_2}\int_{\{s_1<z<s\}}|\phi_t(x,s)|\ dxds\right].$$

Proof : Let us first assume that ϕ is smooth and bounded. From the Coarea formula [17, Sect. 3.4.4] we have

$$\int_{s_1}^{s_2}\int_{\{z=s\}}\frac{\phi(x,s)}{|Dz(x)|}\,d\mathcal{H}^{N-1}(x)ds\ =\ \int_{\{s_1<z<s_2\}}\phi(x,z(x))\,dx$$

while, by Fubini's Theorem, we get

$$\begin{aligned} \int_{s_1}^{s_2}\int_{\{s_1<z<s\}}\phi_t(x,s)\,dxds\ &=\int_{\{s_1<z<s_2\}}\int_{z(x)}^{s_2}\phi_t(x,s)\,dsdx\\ &=\int_{\{s_1<z<s_2\}}\phi(x,s_2)\,dx-\int_{\{s_1<z<s_2\}}\phi(x,z(x))\,dx\,. \end{aligned}$$

So

$$\int_{s_1}^{s_2}\int_{\{z=s\}}\frac{\phi(x,s)}{|Dz(x)|}\ d\mathcal{H}^{N-1}(x)ds \leq\int_{\{s_1<z<s_2\}}\phi(x,s_2)\ dx+\int_{s_1}^{s_2}\int_{\{s_1<z<s\}}|\phi_t(x,s)|\ dxds\,.$$

Since $|Dz|\leq 1/A$, this gives the result for ϕ smooth and bounded. The general case follows by regularization.

□

We shall need two types of space regularity estimates for the solution v to (3.1). The first one is a continuity estimate with a modulus $\omega(s) = s(1 + |\log(s)|)$: it is required in order to solve unambiguously the eikonal equation with a velocity $\bar{g}(v(x,t))$, but is very crude with respect to the $\bar{\rho}$ dependance; we prove it in Subsection 3.2. The second one is merely a Hölder estimate, but it is much sharper with respect to the $\bar{\rho}$ dependance. It is the aim of Subsection 3.3.

3.2. Modulus of continuity in space for the solution of (3.1).

Lemma 3.3. *Let v be the solution of* (3.1) *given by Lemma 3.1. Then, for any $x, y \in \mathbb{R}^N$, $t \in [0,T]$,*

$$|v(x,t) - v(y,t)| \leq \frac{C}{\bar{\rho}} |x-y| \, (1 + |\log |x-y||) \, . \tag{3.8}$$

Proof : We prove the result for $N \geq 3$, the case $N = 2$ being similar but simpler.

The term $x \mapsto \int_{\mathbb{R}^N} G(x-y,t) v_0(y) \, dy$ is Lipschitz continuous with constant $\|Dv_0\|_\infty$; we can therefore assume that $v_0 = 0$ and $t > 0$.

Using again the structure condition on $K(s)$ and Proposition 2.15, for any $x \in \mathbb{R}^N$, there is an integer $C(\bar{\rho}) \leq C_1/\bar{\rho}$ (where C_1 only depends on N, M) and, for each $i \in \{1, \dots, C(\bar{\rho})\}$,

- a Borel measurable map $\Psi_i : B_{N-1}(0,M) \times [0,T] \to \mathbb{R}$, which is $\sqrt{15}-$Lipschitz continuous with respect to the space variable,
- and a change of coordinates $O_i = R_i \circ \tau_x : \mathbb{R}^N \to \mathbb{R}^N$, where $\tau_x(z) = z + x$, R_i is a rotation, such that $O_i(0) = x$ and

$$\Gamma(s) \subset \bigcup_{i=1,\dots,C(\bar{\rho})} \left\{ O_i(z', \Psi_i(z',s)) \; , \; z' \in B_{N-1}(0,M) \right\} \quad \text{for all } s \in [0,T] \, .$$

Setting

$$E_i(s) = \left\{ z = (z', \Psi_i(z',s)) \; , \; z' \in B_{N-1}(0,M) \right\} = \mathrm{graph}(\Psi_i(\cdot, s)_{|B_{N-1}(0,M)}),$$

for any $h \in \mathbb{R}^N$, we have

$$\begin{aligned}
&|v(x+h,t) - v(x,t)| \\
\leq\; & M \int_0^t \int_{\Gamma(s)} |G(x+h-y, t-s) - G(x-y,t-s)| \, d\mathcal{H}^{N-1}(y) ds \\
\leq\; & C \sum_{i=1}^{C(\bar{\rho})} \int_0^t \int_{O_i(E_i(s))} |G(x+h-y, t-s) - G(x-y,t-s)| \, d\mathcal{H}^{N-1}(y) ds \\
\leq\; & C \sum_{i=1}^{C(\bar{\rho})} \int_0^t \int_{E_i(s)} |G(x+h-O_i(z), t-s) - G(x-O_i(z),t-s)| \, d\mathcal{H}^{N-1}(z) ds.
\end{aligned}$$

Let us set $h_i = (h_i', h_{iN}) := R_i^{-1} h$, where $h_i' \in \mathbb{R}^{N-1}$ and $h_{iN} \in \mathbb{R}$. We note that, for any $z \in \mathbb{R}^N$, $R_i^{-1}(x + h - O_i z) = h_i - z$, so that

$$G(x+h-O_i(z), t-s) = G(h_i - z, t-s)$$

because $G(\cdot, t-s)$ has rotational invariance. It follows that

$$\begin{aligned}
&|v(x+h,t)-v(x,t)| \\
\leq\ & C\sum_{i=1}^{C(\bar\rho)}\int_0^t\int_{E_i(s)} |G(h_i-z,t-s)-G(-z,t-s)|\ d\mathcal{H}^{N-1}(z)ds \\
\leq\ & C\sum_{i=1}^{C(\bar\rho)}\int_0^t\int_{B_{N-1}(0,M)} \big|G((h_i'-z', h_{iN}-\Psi_i(z',s)),t-s) \\
&\qquad\qquad -G((-z',-\Psi_i(z',s)),t-s)\big| \sqrt{1+|D\Psi_i(z',s)|^2}\, dz'ds
\end{aligned}$$

since $\mathcal{H}^{N-1}\lfloor E_i(s) = \sqrt{1+|D\Psi_i(y',s)|^2}\mathcal{L}^{N-1}\lfloor B_{N-1}(0,M)$.

We recall that $|D\Psi_i(z',s)| \leq \sqrt{15}$ and introduce

$$D_i(s) = \bigcup_{\sigma\in[0,1]} B_{N-1}(\sigma h_i', |h|(t-s)^{1/4})$$

in order to split the latter integral into two parts. We get

$$\begin{aligned}
&|v(x+h,t)-v(x,t)| \\
\leq\ & C\sum_{i=1}^{C(\bar\rho)}\int_0^t\int_{D_i(s)} \big|G((h_i'-z', h_{iN}-\Psi_i(z',s)),t-s) \\
&\qquad\qquad -G((-z',-\Psi_i(z',s)),t-s)\big|\ dz'ds \\
& +|h|\int_0^1\int_0^t\int_{\mathbb{R}^{N-1}\setminus D_i(s)} \big|DG((\sigma h_i'-z', \sigma h_{iN}-\Psi_i(z',s)),t-s)\big|\, dz'dsd\sigma \\
=\ & C\sum_{i=1}^{C(\bar\rho)} (I_i + |h|\, J_i)\ .
\end{aligned}$$

Let us fix $i \in \{1,\dots,C(\bar\rho)\}$ and estimate I_i. Without loss of generality we can assume that h_i belongs to the plane spanned by e_1 and e_N. Then,

$$D_i(s) \subset \mathbb{R}\times B_{N-2}(0,|h|(t-s)^{1/4})\ ,$$

and setting $z' = (z_1, z'')$ with $z_1 \in \mathbb{R}$, $z'' \in \mathbb{R}^{N-2}$, we have

$$\begin{aligned}
I_i &\leq C\int_0^t \int_{\mathbb{R}} \int_{B_{N-2}(0,|h|(t-s)^{1/4})} \frac{1}{(t-s)^{N/2}} e^{-\frac{|h_1-z_1|^2+|h''-z''|^2+|h_{iN}-\Psi_i(z',s)|^2}{4(t-s)}}\, dz''dz_1ds \\
&\quad +C\int_0^t \int_{\mathbb{R}} \int_{B_{N-2}(0,|h|(t-s)^{1/4})} \frac{1}{(t-s)^{N/2}} e^{-\frac{|z_1|^2+|z''|^2+|\Psi_i(z',s)|^2}{4(t-s)}}\, dz''dz_1ds \\
&\leq C\int_0^t \int_0^{|h|(t-s)^{1/4}} \left(\int_{\mathbb{R}} \left(e^{-\frac{|h_1-z_1|^2}{4(t-s)}} + e^{-\frac{|z_1|^2}{4(t-s)}}\right) dz_1\right) \frac{r^{N-3}}{(t-s)^{N/2}} e^{-\frac{r^2}{4(t-s)}}\, drds \\
&\leq C\int_0^t \int_0^{|h|(t-s)^{-1/4}} \frac{r^{N-3}}{(t-s)^{1/2}} e^{-r^2/4}\, drds \\
&\leq C\int_0^{+\infty} \int_{0\vee(t-(|h|/r)^4)}^t \frac{r^{N-3}}{(t-s)^{1/2}} e^{-r^2/4}\, dsdr \\
&\leq C\left(\int_0^{|h|/t^{1/4}} r^{N-3}e^{-r^2/4}\, t^{1/2}\, dr + \int_{|h|/t^{1/4}}^{+\infty} r^{N-5}e^{-r^2/4}|h|^2\, dr\right) .
\end{aligned}$$

Let $M_N = \sup_{[0,+\infty)} r^{N-3}e^{-r^2/4}$ (recall that $N \geq 3$ by assumption). Then

$$\begin{aligned}
I_i &\leq CM_N\left(|h|\, t^{1/4} + \int_{|h|/t^{1/4}}^{+\infty} \frac{|h|^2}{r^2}\, dr\right) \\
&\leq CM_N T^{1/4}|h| \\
&= C|h|. \qquad (3.9)
\end{aligned}$$

We now estimate J_i. We have

$$\begin{aligned}
&\left|DG((\sigma h_i' - z', \sigma h_{iN} - \Psi_i(z',s)), t-s)\right| \\
&\leq C\frac{|\sigma h_i' - z'| + |\sigma h_{iN} - \Psi_i(z',s)|}{(t-s)^{(N+2)/2}} e^{-|\sigma h_i'-z'|^2/(4(t-s))} e^{-|\sigma h_{iN}-\Psi_i(z',s)|^2/(4(t-s))},
\end{aligned}$$

with

$$|\sigma h_{iN} - \Psi_i(z',s)|e^{-|\sigma h_{iN}-\Psi_i(z',s)|^2/(4(t-s))} \leq C(t-s)^{1/2} .$$

Since $\mathbb{R}^N \setminus D_i(s) \subset \mathbb{R}^N \setminus B_{N-1}(0, |h|(t-s)^{1/4})$, we get

$$\begin{aligned}
J_i &\leq C\int_0^1 \int_0^t \int_{|h|(t-s)^{1/4}}^{+\infty} \left(\frac{r^{N-1}}{(t-s)^{(N+2)/2}} + \frac{r^{N-2}}{(t-s)^{(N+1)/2}}\right) e^{-r^2/(4(t-s))}\, drdsd\sigma \\
&\leq C\int_{|h|t^{-1/4}}^{+\infty} \int_0^{t-(|h|/r)^4} \frac{r^{N-1}+r^{N-2}}{t-s} e^{-r^2/4}\, dsdr \\
&\leq C\int_{|h|t^{-1/4}}^{+\infty} (r^{N-1}+r^{N-2}) \log\left(\frac{tr^4}{|h|^4}\right) e^{-r^2/4}\, dr \\
&\leq C\int_0^{+\infty} (r^{N-1}+r^{N-2}) \left(|\log(T)| + |\log(r)| + |\log(|h|)\right) e^{-r^2/4}\, dr \\
&\leq C(1+|\log|h||). \qquad (3.10)
\end{aligned}$$

Finally, combining (3.9), (3.10) and the bound $C(\bar\rho) \leq C_1/\bar\rho$, we obtain (3.8).

□

3.3. **Hölder estimate for the solution of** (3.1)**.**

Lemma 3.4 (Hölder bounds)**.** *Let v be the solution of* (3.1) *given by Lemma 3.1. Then, for any $t \in [0,T]$, $x, y \in \mathbb{R}^N$,*

$$|v(x,t) - v(y,t)| \le C(1 + |\log(\bar\rho)|)(\bar\rho)^{-\frac{1}{4}}\,|x-y|^{\frac{1}{2}}\,. \tag{3.11}$$

Proof: The main part of the proof consists in showing the following local Hölder inequality: for any $t \in [0,T]$, $x, h \in \mathbb{R}^N$ with $|h| \le \sqrt{\bar\rho}/4$, we have

$$|v(x+h,t) - v(x,t)| \le C(\bar\rho)^{-\frac{1}{4}}\,|h|^{\frac{1}{2}}\,.$$

We will complete the proof of (3.11) by using Lemma 3.1.

The term $x \mapsto \int_{\mathbb{R}^N} G(x-y,t)v_0(y)\,dy$ is Lipschitz continuous with constant $\|Dv_0\|_\infty$, and therefore locally 1/2-Hölder continuous; we can assume that $v_0 = 0$ and $t > 0$. Then

$$\begin{aligned}
&|v(x+h,t) - v(x,t)| \\
\le\ & M\Bigg[\,|h|\int_0^1\int_0^{t-\tau}\int_{\Gamma(s)} |DG(x+\sigma h - y, t-s)|\,d\mathcal{H}^{N-1}(y)\,dsd\sigma \\
&+ \int_{t-\tau}^{t}\int_{\Gamma(s)\backslash B(x,\bar r)} (G(x-y,t-s) + G(x+h-y,t-s))\,d\mathcal{H}^{N-1}(y)\,dsd\sigma \\
&+ \int_{t-\tau}^{t}\int_{\Gamma(s)\cap B(x,\bar r)} (G(x-y,t-s) + G(x+h-y,t-s))\,d\mathcal{H}^{N-1}(y)\,dsd\sigma\Bigg] \\
=\ & \|g\|_\infty\,[\,|h|J_1 + J_2 + J_3\,]
\end{aligned}$$

where $\bar r, \tau > 0$ are chosen such that

$$\bar r = \sqrt{\bar\rho} \qquad \text{and} \qquad \tau = |h|\sqrt{\bar\rho}\,.$$

Since $|h| \le \sqrt{\bar\rho}/4$, we have $\tau \le \bar\rho/4$ and $\bar r/\sqrt{\tau} \ge 2$. If $\tau > t$, the decomposition reduces to $J_2 + J_3$ with $\tau = t$.

In order to estimate J_1, we argue as for I_1 in the proof of the estimate (3.7): we have

$$\begin{aligned}
&\int_0^1\int_0^{t-\tau}\int_{\Gamma(s)} |DG(x+\sigma h - y, t-s)|\,d\mathcal{H}^{N-1}(y)\,dsd\sigma \\
&\qquad \le C\int_0^1\int_0^{t-\tau}\int_{\Gamma(s)} \frac{|y-x-\sigma h|}{(t-s)^{(N+2)/2}} e^{-|y-x-\sigma h|^2/(4(t-s))} d\mathcal{H}^{N-1}(y)\,dsd\sigma,
\end{aligned}$$

where, using Lemma 3.2, we have for any $\sigma \in (0,1)$:

$$\begin{aligned}
&\int_0^{t-\tau}\int_{\Gamma(s)} \frac{|y-x-\sigma h|}{(t-s)^{(N+2)/2}} e^{-|y-x-\sigma h|^2/(4(t-s))}\,d\mathcal{H}^{N-1}(y)ds \\
\le\ & \frac{1}{A}\Bigg[\int_{K(t-\tau)} \frac{|y-x-\sigma h|}{\tau^{(N+2)/2}} e^{-|y-x-\sigma h|^2/(4\tau)}\,dy \\
&+ C\int_0^{t-\tau}\int_{K(s)} \left(\frac{|y-x-\sigma h|}{(t-s)^{(N+4)/2}} + \frac{|y-x-\sigma h|^3}{(t-s)^{(N+6)/2}}\right) e^{-|y-x-\sigma h|^2/(4(t-s))}\,dyds\Bigg].
\end{aligned}$$

Since, for any $\sigma \in (0,1)$, we have

$$\int_{K(t-\tau)} \frac{|y-x-\sigma h|}{\tau^{(N+2)/2}} e^{-|y-x-\sigma h|^2/(4\tau)}\, dy \le \int_0^{+\infty} \frac{r^N}{\tau^{(N+2)/2}} e^{-r^2/(4\tau)} dr \le C\tau^{-\frac{1}{2}}$$

and

$$\begin{aligned} &\int_0^{t-\tau}\int_{K(s)} \left(\frac{|y-x-\sigma h|}{(t-s)^{(N+4)/2}} + \frac{|y-x-\sigma h|^3}{(t-s)^{(N+6)/2}}\right) e^{-|y-x-\sigma h|^2/(4(t-s))} dyds \\ \le\ & C\int_0^{t-\tau}\int_0^{+\infty} \frac{r^N + r^{N+2}}{(t-s)^{3/2}} e^{-r^2/4}\, drds \\ \le\ & C\,\tau^{-1/2}, \end{aligned}$$

we get

$$J_1 \le C\,\tau^{-1/2}.$$

For J_2 we use the same strategy of proof: from Lemma 3.2 we have, for any $\epsilon \in (0,\tau)$,

$$\begin{aligned} &\int_{t-\tau}^{t-\epsilon}\int_{\Gamma(s)\setminus B(x,\bar r)} \left(G(x-y,t-s) + G(x+h-y,t-s)\right) d\mathcal{H}^{N-1}(y)\, dsd\sigma \\ \le\ & \frac{1}{A}\Bigg[\int_{\{t-\tau<z<t-\epsilon\}} \mathbf{1}_{\mathbb{R}^N\setminus B(x,\bar r)}(y)(G(x-y,\epsilon) + G(x+h-y,\epsilon))\, dy \\ & + \int_{t-\tau}^{t-\epsilon}\int_{\{t-\tau<z<s\}} \mathbf{1}_{\mathbb{R}^N\setminus B(x,\bar r)}(y)\,|G_t(x-y,t-s) + G_t(x+h-y,t-s)|\ dyds\Bigg]. \end{aligned}$$

It is easily seen that

$$\lim_{\epsilon\to 0}\int_{\{t-\tau<z<t-\epsilon\}} \mathbf{1}_{\mathbb{R}^N\setminus B(x,\bar r)}(y)(G(x-y,\epsilon) + G(x+h-y,\epsilon))\, dy = 0,$$

because $\bar r$ is larger than $4|h|$. On the other hand

$$\begin{aligned} &\int_{t-\tau}^{t-\epsilon}\int_{\{t-\tau<z<s\}} \mathbf{1}_{\mathbb{R}^N\setminus B(x,\bar r)}(y)\,|G_t(x-y,t-s) + G_t(x+h-y,t-s)|\ dyds \\ &\le C\int_{t-\tau}^{t}\int_{\bar r/(2(t-s)^{1/2})}^{+\infty} \frac{r^{N-1}+r^{N+1}}{t-s} e^{-r^2/4}\, drds \\ &\le C\int_{\bar r/(2\sqrt{\tau})}^{+\infty}\int_{t-\tau}^{t-\bar r^2/(4r^2)} \frac{r^{N-1}+r^{N+1}}{t-s} e^{-r^2/4}\, dsdr \\ &\le C\int_{\bar r/(2\sqrt{\tau})}^{+\infty} (r^{N-1}+r^{N+1})\log\left(\frac{4\tau r^2}{\bar r^2}\right) e^{-r^2/4}\, dr \\ &\le C\frac{\sqrt{\tau}}{\bar r}\int_1^{+\infty} (r^N + r^{N+2})\log(r^2) e^{-r^2/4}\, dr \end{aligned}$$

because $\bar r/\sqrt{\tau}$ is larger than 2. So $J_2 \le C\sqrt{\tau}/\bar r$.

In order to estimate J_3 we use the structure of $K(s)$: from Proposition 2.15, there exists an integer $C(\bar r,\bar\rho) \le C_1\bar r/\bar\rho$ (where C_1 only depends on N) and, for each $i \in \{1,\dots,C(\bar r,\bar\rho)\}$,

- a Borel measurable map $\Psi_i : B_{N-1}(0,\bar r)\times[0,T] \to \mathbb{R}$, which is Lipschitz continuous with constant $\sqrt{15}$ with respect to the space variable,

- and a change of coordinates $O_i = R_i \circ \tau_i : \mathbb{R}^N \to \mathbb{R}^N$ (where R_i is a rotation and τ_x is a translation), with $O_i(0) = x$,

such that, for all $s \in [0,T]$,

$$\Gamma(s) \cap B(x, \bar r) \subset \bigcup_{i=1,\dots,C(\bar r, \bar\rho)} \left\{ O_i(z', \Psi_i(z',t)) \ , \ z' \in B_{N-1}(0, \bar r) \right\}.$$

Let us set, for any $i \in \{1, \dots, C(\bar r, \bar\rho)\}$, $h_i = (h_i', h_{iN}) := R_i^{-1} h$ where $h_i' \in \mathbb{R}^{N-1}$ and $h_{iN} \in \mathbb{R}$. Then

$$\begin{aligned} J_3 \leq \ & \sum_{i=1}^{C(\bar r, \bar\rho)} \int_{t-\tau}^{t} \int_{B_{N-1}(0,\bar r)} [G((-z', \Psi_i(z',s)), t-s) \\ & \qquad + G((h_i' - z', h_{iN} - \Psi_i(z',s)), t-s)] \sqrt{1 + |D\Psi_i(z',s)|^2}\, dz' ds \\ = \ & \sum_{i=1}^{C(\bar r, \bar\rho)} J_{3,i} \ . \end{aligned}$$

Let us fix $i \in \{1, \dots, C(\bar r, \bar\rho)\}$. Since $|h| \leq \sqrt{\bar\rho}/4 = \bar r/4$, we have

$$\begin{aligned} J_{3,i} \leq \ & C \int_{t-\tau}^{t} \int_{B_{N-1}(0,\bar r)} \frac{e^{-|z'|^2/(4(t-s))} + e^{-|h_i'-z'|^2/(4(t-s))}}{(t-s)^{N/2}}\, dz' ds \\ \leq \ & C \int_{t-\tau}^{t} \int_{B_{N-1}(0,2\bar r)} \frac{e^{-|z'|^2/(4(t-s))}}{(t-s)^{N/2}}\, dz' ds \ . \end{aligned}$$

It follows that

$$\begin{aligned} J_{3,i} \leq \ & C \int_{t-\tau}^{t} \int_0^{2\bar r/(t-s)^{1/2}} \frac{r^{N-2}}{(t-s)^{1/2}} e^{-r^2/4}\, dr ds \\ \leq \ & C \int_0^{+\infty} \int_{(t-\tau)\vee(t-(2\bar r)^2/r^2)}^{t} \frac{r^{N-2}}{(t-s)^{1/2}} e^{-r^2/4}\, ds dr \\ \leq \ & C \left(\sqrt{\tau} \int_0^{2\bar r/\sqrt{\tau}} r^{N-2} e^{-r^2/4}\, ds dr + 2\bar r \int_{2\bar r/\sqrt{\tau}}^{+\infty} r^{N-3} e^{-r^2/4}\, ds dr \right) \\ \leq \ & C \left(\sqrt{\tau} \int_0^{+\infty} r^{N-2} e^{-r^2/4}\, ds dr + \sqrt{\tau} \int_4^{+\infty} r^{N-2} e^{-r^2/4}\, ds dr \right) \end{aligned}$$

since $\bar r/\sqrt{\tau} \geq 2$. Accordingly

$$J_3 \leq C \frac{\bar r}{\bar\rho} \sqrt{\tau}.$$

Therefore

$$|h| J_1 + J_2 + J_3 \leq C \left(\frac{|h|}{\sqrt{\tau}} + \frac{\sqrt{\tau}}{\bar r} + \frac{\bar r \sqrt{\tau}}{\bar\rho} \right).$$

With the choice of $\bar r = \sqrt{\bar\rho}$ and $\tau = |h|\sqrt{\bar\rho}$ we get

$$|v(x+h,t) - v(x,t)| \leq C(\bar\rho)^{-\frac14} |h|^{\frac12} \ \text{ for all } (h,t) \in \mathbb{R}^N \times [0,T] \text{ with } |h| \leq \sqrt{\bar\rho}/4. \tag{3.12}$$

Now recall that, according to Lemma 3.1, we have

$$|v(x,t)| \leq C(1 + |\log(\bar\rho)|) \ \text{ for all } (x,t) \in \mathbb{R}^N \times (0,T). \tag{3.13}$$

Combining (3.12) and (3.13) then implies (3.11).

□

3.4. **Existence, bounds and Hölder estimate for the solution of** (3.2).

Lemma 3.5. *Equation* (3.2) *has a unique solution* $v : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, *given by*

$$v(x,t) = \int_{\mathbb{R}^N} G(x-y,t)v_0(y)dy - \kappa \int_0^t \int_{\Gamma(s)} G(x-y,t-s)\bar{g}(v(y,s))d\mathcal{H}^{N-1}(y)ds.$$

For all $x, y \in \mathbb{R}^N$, $t, s \in [0,T]$, v *satisfies the following estimates.*

(i) *Uniform* L^∞ *bound:*

$$|v(x,t)| \le C(1 + |\log(\bar{\rho})|), \tag{3.14}$$

(ii) *Space modulus of continuity:*

$$|v(x,t) - v(y,t)| \le \frac{C}{\bar{\rho}}\,|x-y|(1 + |\log|x-y||), \tag{3.15}$$

(iii) *Space-time Hölder continuity:*

$$|v(x,t) - v(y,t)| \le C(1 + |\log(\bar{\rho})|)\,(\bar{\rho})^{-1/4}\,|x-y|^{1/2}, \tag{3.16}$$

$$|v(x,t) - v(x,s)| \le \frac{C}{\bar{\rho}}\,(1 + |\log|h||)\,|t-s|^{1/2}. \tag{3.17}$$

Proof: The existence, uniqueness, representation and space estimates for the solution of (3.2) follow from Banach fixed point theorem and Lemmata 3.1–3.4.

Let us now check the time estimate; we fix $0 \le s \le t \le T$ and set $h = t - s$. We note that, from the uniqueness of the solution we have, for any $x \in \mathbb{R}^N$,

$$v(x,t+h) = \int_{\mathbb{R}^N} G(x-y,h)v(y,t)dy - \kappa \int_0^h \int_{\Gamma(t+s)} G(x-y,h-s)\,\bar{g}(v(y,t+s))d\mathcal{H}^{N-1}(y)ds\ .$$

Since v satisfies (3.15), we get from standard estimates on the heat flow that

$$\left| \int_{\mathbb{R}^N} G(x-y,h)v(y,t)dy - v(x,t) \right| \le \frac{C}{\bar{\rho}}(1 + |\log|h||)h^{\frac{1}{2}}.$$

From the structure condition on $K(s)$ and Proposition 2.15 (see the computations in the proof of Lemma 3.1 for details), we have

$$\begin{aligned} &\left| \int_0^h \int_{\Gamma(t+s)} G(x-y,h-s)\,\bar{g}(v(y,t+s))\,d\mathcal{H}^{N-1}(y)ds \right| \\ \le\ & \frac{C}{\bar{\rho}} \int_0^h \int_{\mathbb{R}^{N-1}} \frac{1}{(h-s)^{N/2}} e^{-|y'-x'|^2/(4(h-s))}\,dy'ds \\ \le\ & \frac{C}{\bar{\rho}} \int_0^h \int_0^{+\infty} \frac{r^{N-2}}{(h-s)^{1/2}} e^{-r^2/4}\,drds \\ \le\ & \frac{C\sqrt{h}}{\bar{\rho}}. \end{aligned}$$

Putting together the above estimates gives (3.17).

□

4. Stability and existence of solutions for the system (1.1)

We start with an *a priori* stability property for the solution and then prove our main result.

4.1. **A stability property.**

We first investigate the convergence of the solution of

$$\begin{cases} (u_n)_t = c_n(x,t)|Du_n| & \text{in } \mathbb{R}^N \times (0,T) \\ (v_n)_t - \Delta v_n + \kappa \bar{g}(v_n)\mathcal{H}^{N-1}\lfloor\{u_n(\cdot,t)=0\} = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ v_n(x,0) = v_0(x),\ u_n(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

to the solution of

$$\begin{cases} u_t = c(x,t)|Du| & \text{in } \mathbb{R}^N \times (0,T) \\ v_t - \Delta v + \kappa \bar{g}(v)\mathcal{H}^{N-1}\lfloor\{u(\cdot,t)=0\} = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ v(x,0) = v_0(x),\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

as (c_n) converges to c.

Lemma 4.1. *Let us assume that*

- *For any $n \in \mathbb{N}$, the velocity $c_n : \mathbb{R}^N \to [0,T]$ satisfies* (2.2)–(2.3)–(2.4) *with fixed $\alpha > 1/p$ and modulus ω.*
- *The sequence (c_n) converges a.e. to some $c : \mathbb{R}^N \times [0,T] \to \mathbb{R}$.*

Then (v_n) converges locally uniformly to v in $\mathbb{R}^N \times [0,T]$.

Proof: Without loss of generality we can assume that $v_0 = 0$. Let us set as usual

$$K_n(t) = \{u_n(\cdot,t) \geq 0\},\ \Gamma_n(t) = \{u_n(\cdot,t) = 0\},\ z_n(x) = \inf\{t \geq 0\ ;\ x \in K_n(t)\},$$

and

$$K(t) = \{u(\cdot,t) \geq 0\},\ \Gamma(t) = \{u(\cdot,t) = 0\},\ z(x) = \inf\{t \geq 0\ ;\ x \in K(t)\}.$$

From Proposition 2.2 we know that (u_n) converges locally uniformly to u.

We claim that this implies that (z_n) converges uniformly to z in $\{0 < z < t\}$. Indeed, $u_n(x, z_n(x)) = 0$ for all n and, passing to the limit, we get $u(x, \liminf z_n(x)) = 0$. Thus $\liminf z_n(x) \geq z(x)$. Now, let $x \in \{0 < z < t\}$. From Proposition 2.4, for every ϵ, there exists x_ϵ such that $|x - x_\epsilon| < \epsilon$ and $u(x_\epsilon, z(x)) > 0$. For n sufficiently large, we also have $u_n(x_\epsilon, z(x)) > 0$ and therefore $z_n(x_\epsilon) < z(x)$. It follows that $\limsup z_n(x_\epsilon) \leq z(x)$. Applying again Proposition 2.4, we get $-|x - x_\epsilon|/A + \limsup z_n(x) \leq z(x)$. We conclude by sending ϵ to 0.

Corollary 2.12 states that there is some $\bar{\rho} > 0$ such that each $K_n(t)$ has the interior cone property of parameter $(\bar{\rho}, 2\bar{\rho})$ and that, for any $x \in \partial K_n(t)$, there is a vector $\nu \in \mathbb{R}^N$ such that $|\nu| = 1$ and the set $\widehat{\mathcal{C}}^{\beta/2,C}(x,\nu)$ is contained in $K_n(t)$, where $C = C_0\|\omega\|_p^{1/2}$ and $\beta = \alpha - 1/p$. Then Lemma 2.14 implies that $|Dz_n|$ weakly-$*$ converges to $|Dz|$ in $\{0 < z < T\}$.

By the representation formula for the solution of (3.2) (Lemma 3.5) and Lemma 2.3 (2),

$$v_n(x,t) \ = \ -\kappa \int_0^t \int_{\{z_n = s\}} G(x-y, t-s)\bar{g}(v_n(y,s))d\mathcal{H}^{N-1}(y)ds\,.$$

From the estimates of Lemma 3.5 we know that the v_n are uniformly bounded and uniformly Hölder continuous. So, up to some subsequence, we can assume that (v_n) uniformly converges to some $\bar{v}$. Our aim is to show that $\bar{v} = v$.

Fix $x \in \mathbb{R}^N$ and let $\theta \in (0,t)$ be small. Then, following for instance the estimates obtained for the proof of (3.17), one easily checks that

$$\left| v_n(x,t) + \kappa \int_0^{t-\theta} \int_{\{z_n=s\}} G(x-y,t-s)\bar{g}(v_n(y,s))\, d\mathcal{H}^{N-1}(y)ds \right| \le |\kappa| \|\bar{g}\|_\infty \int_{t-\theta}^{t} \int_{\{z_n=s\}} G(x-y,t-s)\, d\mathcal{H}^{N-1}(y)ds \le C(\bar{\rho})\,\theta^{1/2}\ .$$

By the Coarea formula, we have

$$\int_0^{t-\theta} \int_{\{z_n=s\}} G(x-y,t-s)\,\bar{g}(v_n(y,s))d\mathcal{H}^{N-1}(y)\,ds = \int_{\{0<z_n<t-\theta\}} G(x-y,t-z_n(y))\,\bar{g}(v_n(y,z_n(y)))|Dz_n(y)|dy.$$

In this expression,

$$G(x-\cdot,t-z_n(\cdot))\,\bar{g}(v_n(\cdot,z_n(\cdot))) \underset{n\to+\infty}{\longrightarrow} G(x-\cdot,t-z(\cdot))\,\bar{g}(\bar{v}(\cdot,z(\cdot)))$$

uniformly in $\{0<z<t-\theta\}$ while $(|Dz_n|)$ converges weakly-$*$ to $|Dz|$. Moreover, by Remark 2.5, the front $\Gamma(s)$ has zero measure for any s. Therefore, the indicator function of $\{0<z_n<t-\theta\}$ converges to the indicator function of $\{0<z<t-\theta\}$ almost everywhere. It follows that

$$\lim_{n\to+\infty} \int_0^{t-\theta} \int_{\{z_n=s\}} G(x-y,t-s)\,\bar{g}(v_n(y,s))\, d\mathcal{H}^{N-1}(y)ds = \int_{\{0<z<t-\theta\}} G(x-y,t-z(y))\,\bar{g}(\bar{v}(y,z(y)))|Dz(y)|\, dy = \int_0^{t-\theta} \int_{\{z=s\}} G(x-y,t-s)\,\bar{g}(\bar{v}(y,s))\, d\mathcal{H}^{N-1}(y)ds\ .$$

Since, as above,

$$\left| \bar{v}(x,t) + \kappa \int_0^{t-\theta} \int_{\{z=s\}} G(x-y,t-s)\bar{g}(\bar{v}(y,s))d\mathcal{H}^{N-1}(y)ds \right| \le C(\bar{\rho})\,\theta^{1/2}\ ,$$

we have proved that $\bar{v}$ satisfies

$$\bar{v}(x,t) = -\kappa \int_0^{t} \int_{\{z=s\}} G(x-y,t-s)\bar{g}(\bar{v}(y,s))d\mathcal{H}^{N-1}(y)ds\ ,$$

i.e., $\bar{v}$ is a solution to

$$\begin{cases} v_t - \Delta v + \kappa\bar{g}(v)\mathcal{H}^{N-1}\lfloor\{u(\cdot,t)=0\} = 0 & \text{in } \mathbb{R}^N\times(0,T), \\ v(x,0) = 0 & \text{in } \mathbb{R}^N. \end{cases}$$

The solution of this equation being unique, we have $\bar{v} = v$, which proves the convergence of (v_n) to v.

□

4.2. **Proof of the existence Theorem 1.2.** We are now ready to prove Theorem 1.2. Throughout the proof, C denotes a constant which depends only the data of the problem: N, T, κ, $\bar{g}$, u_0 and v_0. Let us fix some constants $\bar{C}, R, C_1 > 0$ to be chosen later and let $\mathcal{V} = \mathcal{V}(\bar{C}, R, C_1)$ be the set of maps $v : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ such that v is measurable, 1/2-Hölder continuous in space with constant $\bar{C}$, bounded by a constant $R > \|v_0\|_\infty$ and such that

$$|v(x,t) - v(y,t)| \leq C_1 |x-y|(1 + |\log|x-y||) \quad \text{for all } x, y \in \mathbb{R}^N,\ t \in [0,T].$$

Notice that $\mathcal{V}$ is a closed convex subset of the Banach space $L^\infty(\mathbb{R}^N \times [0,T])$.

To any $v \in \mathcal{V}$ we associate a map $\tilde{v}$ defined in the following way: let u be the solution to

$$\begin{cases} u_t(x,t) = \bar{g}(v(x,t))|Du(x,t)| \\ u(x,0) = u_0(x), \end{cases}$$

and let us set

$$K(t) = \{u(\cdot,t) \geq 0\}, \quad \Gamma(t) = \partial K(t) \quad \text{and} \quad z(x) = \inf\{t \geq 0\ ;\ x \in K(t)\}.$$

Since the velocity $c(x,t) := \bar{g}(v(x,t))$ satisfies (2.2), (2.3) and is 1/2-Hölder continuous in space with constant $\|\bar{g}'\|_\infty \bar{C}$, and since the initial condition enjoys the interior ball property, we know from Corollary 2.12 with $\beta = \alpha - 1/p = 1/2$ that each $K(t)$ has the interior cone property of parameter $(\bar{\rho}, 2\bar{\rho})$, where $\bar{\rho} = C_0 \bar{C}^{-2}$. Moreover, by (2.9) there exists $M > 0$ depending only on the data such that for any $t \in [0;T]$, $K(t) \subset \overline{B}(0,M)$, while (3.4) holds thanks to Proposition 2.4.

By Lemma 3.5 we can therefore define the unique solution $\tilde{v}$ to

$$\begin{cases} \tilde{v}_t(x,t) - \Delta \tilde{v}(x,t) + \bar{g}(\tilde{v}(x,t))\mathcal{H}^{N-1}\lfloor\{u(\cdot,t) = 0\} = 0 \\ \tilde{v}(x,0) = v_0(x). \end{cases}$$

From Lemma 3.5 we also have, for all $x, y \in \mathbb{R}^N$, $0 \leq t \leq t+h \leq T$,

$$|\tilde{v}(x,t)| \leq C(1 + |\log(\bar{\rho})|) \leq C(1 + |\log(\bar{C})|),$$

$$|\tilde{v}(x,t) - \tilde{v}(y,t)| \leq C(1 + |\log(\bar{\rho})|)(\bar{\rho})^{-1/4}\,|x-y|^{1/2} \leq C(1 + |\log(\bar{C})|)\bar{C}^{1/2}\,|x-y|^{1/2},$$

$$|\tilde{v}(x,t) - \tilde{v}(y,t)| \leq \frac{C}{\bar{\rho}}\,|x-y|(1 + |\log(|x-y|)|) \leq C\,\bar{C}^2\,|x-y|(1 + |\log(|x-y|)|),$$

and

$$|\tilde{v}(x,t+h) - \tilde{v}(x,t)| \leq \frac{C}{\bar{\rho}}(1 + |\log|h||)\,h^{1/2} \leq C\,\bar{C}^2\,(1 + |\log|h||)\,h^{1/2}.$$

So, if we choose $\bar{C}$ such that

$$C(1 + |\log(\bar{C})|)\bar{C}^{1/2} \leq \bar{C}$$

and then R and C_1 such that

$$R \geq C(1 + |\log(\bar{C})|) \text{ and } C_1 \geq C\,\bar{C}^2\ ,$$

we obtain that $\tilde{v} \in \mathcal{V}$. Let us now fix $\bar{C}, R$ and C_1 as above. Then the map Φ, which associates to $v \in \mathcal{V}$ the map $\tilde{v}$, is compact because of the L^∞ and Hölder bounds on $\tilde{v}$ recalled above. Since, from Lemma 4.1, Φ is also continuous, we can complete the proof thanks to Schauder's fixed point theorem.

□

References

[1] O. Alvarez, P. Hoch, Y. Le Bouar, and R. Monneau. Dislocation dynamics: short-time existence and uniqueness of the solution. *Arch. Ration. Mech. Anal.*, 181(3):449–504, 2006.

[2] O. Alvarez, P. Cardaliaguet, and R. Monneau. Existence and uniqueness for dislocation dynamics with nonnegative velocity. *Interfaces Free Bound.*, 7(4):415–434, 2005.

[3] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[4] G. Barles and O. Ley. Nonlocal first-order Hamilton-Jacobi equations modelling dislocations dynamics. *Comm. Partial Differential Equations*, 31(8):1191–1208, 2006.

[5] G. Barles, P. Cardaliaguet, O. Ley, and R. Monneau. Global existence results and uniqueness for dislocation equations. *SIAM J. Math. Anal.*, 40(1):44–69, 2008.

[6] G. Barles, P. Cardaliaguet, O. Ley, and A. Monteillet. Existence of weak solutions for general nonlocal and nonlinear second-order parabolic equations. *Nonlinear Anal.*, 71(7-8):2801–2810, 2009.

[7] G. Barles, P. Cardaliaguet, O. Ley, and A. Monteillet. Uniqueness results for nonlocal Hamilton-Jacobi equations. *J. Funct. Anal.*, 257(5):1261–1287, 2009.

[8] M. Bourgoing. Viscosity solutions of fully nonlinear second order parabolic equations with L^1 dependence in time and Neumann boundary conditions. *Discrete Contin. Dyn. Syst.*, 21(3):763–800, 2008.

[9] M. Bourgoing. Viscosity solutions of fully nonlinear second order parabolic equations with L^1 dependence in time and Neumann boundary conditions. Existence and applications to the level-set approach. *Discrete Contin. Dyn. Syst.*, 21(4):1047–1069, 2008.

[10] M. Burger. Iterative regularization of a parameter identification problem occurring in polymer crystallization. *SIAM J. Numer. Anal.*, 39(3):1029–1055, 2001.

[11] M. Burger. Growth of multiple crystals in polymer melts. *European J. Appl. Math.*, 15(3):347–363, 2004.

[12] M. Burger, V. Capasso, and C. Salani. Modelling multi-dimensional crystallization of polymers in interaction with heat transfer. *Nonlinear Anal. Real World Appl.*, 3(1):139–160, 2002.

[13] P. Cannarsa and H. Frankowska. Interior sphere property of attainable sets and time optimal control problems. *ESAIM Control Optim. Calc. Var.*, 12(2):350–370 (electronic), 2006.

[14] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*. Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA, 2004.

[15] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[16] G. Eder. Mathematical modelling of crystallization processes as occurring in polymer processing. Proceedings of the Second World Congress of Nonlinear Analysts, Part 6 (Athens, 1996). *Nonlinear Anal.*, 30(6):3807–3815, 1997.

[17] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. CRC Press, Boca Raton, FL, 1992.

[18] A. Friedman and J. L. Velázquez. A free boundary problem associated with crystallization of polymers in a temperature field. *Indiana Univ. Math. J.*, 50(4):1609–1649, 2001.

[19] Y. Giga. *Surface evolution equations*, volume 99 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006. A level set approach.

[20] Y. Giga, S. Goto, and H. Ishii. Global existence of weak solutions for interface equations coupled with diffusion equations. *SIAM J. Math. Anal.*, 23(4):821–835, 1992.

[21] H. Ishii. Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets. *Bull. Fac. Sci. Eng. Chuo Univ.*, 28:33–77, 1985.

[22] O. Ley. Lower-bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts. *Adv. Differential Equations*, 6(5):547–576, 2001.

[23] D. Nunziante. Uniqueness of viscosity solutions of fully nonlinear second order parabolic equations with discontinuous time-dependence. *Differential Integral Equations*, 3(1):77–91, 1990.

[24] D. Nunziante. Existence and uniqueness of unbounded viscosity solutions of parabolic equations with discontinuous time-dependence. *Nonlinear Anal.*, 18(11):1033–1062, 1992.

[25] Osher, S. and Fedkiw, R. *Level set methods and dynamic implicit surfaces.* Applied Mathematical Sciences, 153. Springer-Verlag, New York, 2003.

[26] Sethian, J. A. *Level set methods and fast marching methods. Evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science.* Second edition. Cambridge Monographs on Applied and Computational Mathematics, 3. Cambridge University Press, Cambridge, 1999.

[27] P. Soravia and P. E. Souganidis. Phase-field theory for FitzHugh-Nagumo-type systems. *SIAM J. Math. Anal.*, 27(5):1341–1359, 1996.

[28] B. Su and M. Burger. Weak solutions of a polymer crystal growth model. *CAM-Report 06-40 (UCLA)*, 2006.

[29] B. Su and M. Burger. Global weak solutions of non-isothermal front propagation problem. *Electron. Res. Announc. Amer. Math. Soc.*, 13:46–52 (electronic), 2007.

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