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To cite this version:

HAL Id: hal-00459423
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Submitted on 23 Feb 2010

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Phi-entropy inequalities and Fokker-Planck equations

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November, 2010

Abstract

We present new Φ-entropy inequalities for diffusion semigroups under the curvature-dimension criterion. They include the isoperimetric function of the Gaussian measure. Applications to the long time behaviour of solutions to Fokker-Planck equations are given.

Keywords: Functional inequalities, logarithmic Sobolev inequality, Poincaré inequality, Φ-entropies, Bakry-Emery criterion, diffusion semigroups, Fokker-Planck equation.

AMS subject classification: 35B40, 35K10, 60J60.

1 Introduction

We consider a Markov semigroup \((P_t)_{t \geq 0}\) on \(\mathbb{R}^n\), acting on functions on \(\mathbb{R}^n\) by

\[P_t f(x) = \int_{\mathbb{R}^n} f(y) p_t(x,dy)\]

for \(x \in \mathbb{R}^n\). The kernels \(p_t(x,dy)\) are probability measures on \(\mathbb{R}^n\) for all \(x\) and \(t \geq 0\), called transition kernels. We assume that the Markov infinitesimal generator \(L = \frac{\partial P_t}{\partial t} \bigg|_{t=0^+}\) is given by

\[L f(x) = \sum_{i,j=1}^n D_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \sum_{i=1}^n a_i(x) \frac{\partial f}{\partial x_i}(x)\]

where \(D(x) = (D_{ij}(x))_{1 \leq i,j \leq n}\) is a symmetric \(n \times n\) matrix, nonnegative in the sense of quadratic forms on \(\mathbb{R}^n\) and with smooth coefficients, and where the \(a_i, 1 \leq i \leq n\), are smooth. Such a semigroup or generator is called a diffusion, and we refer to Refs. [5], [6], [13] for backgrounds on them.

If \(\mu\) is a Borel probability measure on \(\mathbb{R}^n\) and \(f\) a \(\mu\)-integrable map on \(\mathbb{R}^n\) we let \(\mu(f) = \int_{\mathbb{R}^n} f(x) \mu(dx)\). If, moreover, \(\Phi\) is a convex map on an interval \(I\) of \(\mathbb{R}\) and \(f\) an \(I\)-valued map with \(f\) and \(\Phi(f)\) \(\mu\)-integrable, we let

\[\text{Ent}_\mu^\Phi(f) = \mu(\Phi(f)) - \Phi(\mu(f))\]

be the \(\Phi\)-entropy of \(f\) under \(\mu\) (see Ref. [10] for instance). Two fundamental examples are \(\Phi(x) = x^2\) on \(\mathbb{R}\), for which \(\text{Ent}_\mu^\Phi(f)\) is the variance of \(f\), and \(\Phi(x) = x \ln x\) on \([0, +\infty[\), for which \(\text{Ent}_\mu^\Phi(f)\) is the Boltzmann entropy of \(f\). By Jensen’s inequality, \(\text{Ent}_\mu^\Phi(f)\) is always nonnegative and, if \(\Phi\) is strictly convex, it is positive unless \(f\) is a constant, equal to \(\mu(f)\). The semigroup \((P_t)_{t \geq 0}\) is said \(\mu\)-ergodic if \(P_t f\) tends to \(\mu(f)\) as \(t\) tends to infinity in \(L^2(\mu)\), for all \(f\).

In Section 2 we shall derive bounds on \(\text{Ent}_\mu^\Phi(f)\) and \(\text{Ent}_{P_t}^\Phi(f)(x)\) which will measure the convergence of \(P_t f\) to \(\mu(f)\) in the ergodic setting. This is motivated by the study of the long time behaviour of solutions to Fokker-Planck equations, which will be discussed in Section 3.

Some results of this note with their proofs are detailed in Ref. [3].
2 Phi-entropy inequalities

Bounds on $\text{Ent}_\Phi^p(f)$ and assumptions on $L$ will be given in terms of the carré du champ and $\Gamma_2$ operators associated to $L$, defined by

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf), \quad \Gamma_2(f) = \frac{1}{2}(L\Gamma(f) - 2\Gamma(f, Lf)).$$

If $\rho$ is a real number, we say that the semigroup $(P_t)_{t \geq 0}$ satisfies the $CD(\rho, \infty)$ curvature-dimension (or Bakry-Émery) criterion (see Ref. [7]) if

$$\Gamma_2(f) \geq \rho \Gamma(f)$$

for all functions $f$, where $\Gamma(f) = \Gamma(f, f)$.

The carré du champ is explicitly given by

$$\Gamma(f, g)(x) = \langle \nabla f(x), D(x) \nabla g(x) \rangle.$$

Expressing $\Gamma_2$ is more complex in the general case but, for instance, if $D$ is constant, then $L$ satisfies the $CD(\rho, \infty)$ criterion if and only if

$$\frac{1}{2}(Ja(x)D + (Ja(x)D)^*) \geq \rho D$$

for all $x$, as quadratic forms on $\mathbb{R}^n$, where $Ja$ is the Jacobian matrix of $a$ and $M^*$ denotes the transposed matrix of a matrix $M$ (see Ref. [2, 3]).

Poincaré and logarithmic Sobolev inequalities for the semigroup $(P_t)_{t \geq 0}$ are known to be implied by the $CD(\rho, \infty)$ criterion. More generally, and following Ref. [8, 9, 10], let $\rho > 0$ and $\Phi$ be a $C^4$ strictly convex function on an interval $I$ of $\mathbb{R}$ such that $-1/\Phi''$ is convex. If $(P_t)_{t \geq 0}$ is $\mu$-ergodic and satisfies the $CD(\rho, \infty)$ criterion, then $\mu$ satisfies the $\Phi$-entropy inequality

$$\text{Ent}_\Phi^p(f) \leq \frac{1}{2p} \mu(\Phi''(f) \Gamma(f))$$

for all $I$-valued functions $f$.

The main instances of such $\Phi$’s are the maps $x \mapsto x^2$ on $\mathbb{R}$ and $x \mapsto x \ln x$ on $[0, +\infty[$ or more generally, for $1 \leq p \leq 2$

$$\Phi_p(x) = \begin{cases} \frac{x^p - x}{p(p-1)}, & x > 0 \quad \text{if } p \in [1, 2] \\ x \ln x, & x > 0 \quad \text{if } p = 1. \end{cases}$$

(3)

For this $\Phi_p$ with $p$ in $[1, 2]$ the $\Phi$-entropy inequality (2) becomes

$$\frac{\mu(g^2) - \mu((g^2)^p)}{p-1} \leq \frac{2}{pp} \mu(\Gamma(g))$$

(4)

for all positive functions $g$. For given $g$ the map $p \mapsto \frac{\mu(g^2) - \mu((g^2)^p)}{p-1}$ is nonincreasing with respect to $p > 0, p \neq 1$. Moreover its limit for $p \to 1$ is $\text{Ent}_\mu(g^2)$, so that the so-called Beckner inequalities (2) for $p$ in $[1, 2]$ give a natural monotone interpolation between the weaker Poincaré inequality (for $p = 2$), and the stronger logarithmic Sobolev inequality (for $p \to 1$).
Long time behaviour of the semigroup

The $\Phi$-entropy inequalities provide estimates on the long time behaviour of the associated diffusion semigroups. Indeed, let $(P_t)_{t \geq 0}$ be such a semigroup, ergodic for the measure $\mu$. If $\Phi$ is a $C^2$ function on an interval $I$, then

$$\frac{d}{dt} \text{Ent}_\mu^\Phi(P_t f) = -\mu(\Phi'(P_t f) \Gamma(P_t f))$$

for all $t \geq 0$ and all $I$-valued functions $f$. As a consequence, if $C$ is a positive number, then the semigroup converges in $\Phi$-entropy with exponential rate:

$$\text{Ent}_\mu^\Phi(P_t f) \leq e^{-t/\rho} \text{Ent}_\mu^\Phi(f)$$

for all $t \geq 0$ and all $I$-valued functions $f$, if and only if the measure $\mu$ satisfies the $\Phi$-entropy inequality for all $I$-valued functions $f$,

$$\text{Ent}_\mu^\Phi(f) \leq C \mu(\Phi''(f) \Gamma(f)).$$

2.1 Refined $\Phi$-entropy inequalities

We now give and study improvements of (2) for the $\Phi_p$ maps given by (3):

**Theorem 1** ([9]) Let $\rho \in \mathbb{R}$ and $p \in [1,2]$. Then the following assertions are equivalent, with $(1 - e^{-2pt})/\rho$ and $(e^{2pt} - 1)/\rho$ replaced by $2t$ if $\rho = 0$:

(i) the semigroup $(P_t)_{t \geq 0}$ satisfies the $CD(\rho, \infty)$ criterion;

(ii) $(P_t)_{t \geq 0}$ satisfies the refined local $\Phi_p$-entropy inequality

$$\frac{1}{(p-1)^2} \left[ P_t(f^p) - P_t(f)^p \left( \frac{P_t(f^p)}{P_t(f)^p} \right)^{\frac{2}{p}-1} \right] \leq \frac{1 - e^{-2pt}}{\rho} P_t(f^{p-2} \Gamma(f))$$

for all positive $t$ and all positive functions $f$;

(iii) $(P_t)_{t \geq 0}$ satisfies the reverse refined local $\Phi_p$-entropy inequality

$$\frac{1}{(p-1)^2} \left[ P_t(f^p) - P_t(f)^p \left( \frac{P_t(f^p)}{P_t(f)^p} \right)^{\frac{2}{p}-1} \right] \geq \frac{e^{2pt} - 1}{\rho} \left( \frac{P_t(f^p)}{P_t(f)^p} \right)^{\frac{2}{p}-1} (P_t f)^{p-2} \Gamma(P_t f)$$

for all positive $t$ and all positive functions $f$.

If, moreover, $\rho > 0$ and the measure $\mu$ is ergodic for the semigroup $(P_t)_{t \geq 0}$, then $\mu$ satisfies the refined $\Phi_p$-entropy inequality

$$\frac{p^2}{(p-1)^2} \left[ \mu(g^2) - \mu(g^{2/p}) \mu(g^{2/p}) \left( \frac{\mu(g^2)}{\mu(g^{2/p})^p} \right)^{\frac{2}{p}-1} \right] \leq \frac{4}{p} \mu(\Gamma(g))$$

for all positive maps $g$.

The bound (3) has been obtained in Ref. [3] for the generator $L = \text{div}(D \nabla f) - \langle D \nabla V, \nabla f \rangle$ with $D(x)$ a scalar matrix and for the ergodic measure $\mu = e^{-V}$, and under the corresponding $CD(\rho, \infty)$ criterion.

It improves on the Beckner inequality (4) since

$$\frac{\mu(g^2) - \mu(g^{2/p})^p}{p-1} \leq \frac{p}{2(p-1)^2} \left[ \mu(g^2) - \mu(g^{2/p})^p \left( \frac{\mu(g^2)}{\mu(g^{2/p})^p} \right)^{\frac{2}{p}-1} \right].$$
We have noticed that for all \(g\) the map \(p \mapsto \frac{\mu(g^2) - \mu(g^{2/p})^p}{p-1}\) is continuous and nonincreasing on \([0, +\infty[\), with values \(\text{Ent}_\mu(g^2)\) at \(p = 1\) and \(\text{Var}_\mu(g)\) at \(p = 2\). Similarly, for the larger functional introduced in (3), the map

\[
p \mapsto \frac{p}{2(p-1)^2} \left[ \mu(g^2) - \mu(g^{2/p})^p \left( \frac{\mu(g^2)}{\mu(g^{2/p})} \right)^{\frac{2}{p-1}} \right]
\]

is nonincreasing on \([1, +\infty[\) (see [9, Prop. 11]). Moreover its value is \(\text{Var}_\mu(g)\) at \(p = 2\) and it tends to \(\text{Ent}_\mu(g^2)\) as \(p \to 1\), hence providing a new monotone interpolation between Poincaré and logarithmic Sobolev inequalities.

The pointwise \(CD(\rho, \infty)\) criterion can be replaced by the integral criterion

\[
\mu \left( g^{\frac{2-p}{p-2}} \Gamma_2(g) \right) \geq \rho \mu \left( g^{\frac{2-p}{p-1}} \Gamma(g) \right)
\]

for all positive functions \(g\), and one can still get the refined \(\Phi_{\rho}\)-entropy inequality (3), even in the case of non-reversible semigroups (see [9, Prop. 14]).

**Remark 2** For \(\rho = 0\), and following Ref. [3], the convergence of \(P_t f\) towards \(f\) can be measured on \(H(t) = \text{Ent}_\mu^\Phi(P_t f)\) as

\[
|H'(t)| \leq \frac{|H'(0)|}{1 + ct^\alpha}, \quad t \geq 0
\]

where \(\alpha = \frac{2-p}{p} |H'(0)| / H(0)\). This illustrates the improvement offered by (3) instead of (4), which does not give here any convergence rate.

### 2.2 The case of the Gaussian isoperimetry function

Let \(F\) be the distribution function of the one-dimensional standard Gaussian measure. The map \(U = F^\rho \circ F^{-1}\), which is the isoperimetry function of the Gaussian distribution, satisfies \(U'' = -1/U\) on the set \([0, 1]\), so that the map \(\Phi = -U\) is convex with \(-1/\Phi''\) also convex on \([0, 1]\).

**Theorem 3** Let \(\rho\) be a real number. Then the following three assertions are equivalent, with \((1 - e^{-2\rho t})/\rho\) and \((e^{2\rho t} - 1)/\rho\) replaced by \(2t\) if \(\rho = 0\):

(i) the semigroup \((P_t)_{t \geq 0}\) satisfies the \(CD(\rho, \infty)\) criterion;

(ii) the semigroup \((P_t)_{t \geq 0}\) satisfies the local \(\Phi\)-entropy inequality

\[
\text{Ent}_{P_t}^\Phi(f) \leq \frac{1}{\Phi''(P_t f)} \log \left( 1 + \frac{1 - e^{-2\rho t}}{2 \rho} \Phi''(P_t f) P_t (\Phi''(f) \Gamma(f)) \right)
\]

for all positive \(t\) and all \([0,1]\)-valued functions \(f\);

(iii) the semigroup \((P_t)_{t \geq 0}\) satisfies the reverse local \(\Phi\)-entropy inequality

\[
\text{Ent}_{P_t}^\Phi(f) \geq \frac{1}{\Phi''(P_t f)} \log \left( 1 + \frac{e^{2\rho t} - 1}{2 \rho} \Phi''(P_t f)^2 \Gamma(P_t f) \right)
\]

for all positive \(t\) and all \([0,1]\)-valued functions \(f\).

If, moreover, \(\rho > 0\) and the measure \(\mu\) is ergodic for the semigroup \((P_t)_{t \geq 0}\), then \(\mu\) satisfies the \(\Phi\)-entropy inequality for all \([0,1]\)-valued functions \(f\):

\[
\text{Ent}_{\mu}^\Phi(f) \leq \frac{1}{\Phi''(\mu(f))} \log \left( 1 + \frac{\Phi''(\mu(f))}{2 \rho} \mu(\Phi''(f) \Gamma(f)) \right).
\]
The proof is based on \cite[Lemma 4]{[3]. For $\Phi = -\mathcal{H}$ it improves on the general $\Phi$-entropy inequality \cite{[2] since $\log(1 + x) \leq x$. Links with the isoperimetric bounds of Ref. \cite{[8] for instance will be addressed elsewhere.

3 Long time behaviour for Fokker-Planck equations

Let us consider the linear Fokker-Planck equation

$$
\frac{\partial u_t}{\partial t} = \text{div}[D(x)(\nabla u_t + u_t(\nabla V(x) + F(x)))], \quad t \geq 0, \quad x \in \mathbb{R}^n
$$

\label{eq:10}

where $D(x)$ is a positive symmetric $n \times n$ matrix and $F$ satisfies

$$
\text{div}(e^{-V}DF) = 0.
$$

\label{eq:11}

It is one of the purposes of Refs. \cite{[2] and \cite{[4]} to rigorously study the asymptotic behaviour of solutions to (\ref{eq:10})-(\ref{eq:11}). Let us formally rephrase the argument.

Assume that the Markov diffusion generator $L$ defined by

$$
Lf = \text{div}(DVf) - < D(VF - F), \nabla f >
$$

satisfies the $CD(\rho, \infty)$ criterion with $\rho > 0$, that is \cite{[1]} if $D$ is constant, etc.

Let $u_t = e^{-V}P_t(e^V u_0)$. Then we can deduce the convergence of the solution $u_t$ towards the stationary state $e^{-V}$ (up to a constant) from the convergence estimate \cite{[2]} for the semigroup, in the form

$$
\text{Ent}_\mu^\Phi\left(\frac{u_t}{e^{-V}}\right) \leq e^{-2\rho t} \text{Ent}_\mu^\Phi\left(\frac{u_0}{e^{-V}}\right), \quad t \geq 0.
$$

\label{eq:13}

In fact such a result holds for the general Fokker-Planck equation

$$
\frac{\partial u_t}{\partial t} = \text{div}[D(x)(\nabla u_t + u_t a(x))], \quad t \geq 0, \quad x \in \mathbb{R}^n
$$

\label{eq:14}

where again $D(x)$ is a positive symmetric $n \times n$ matrix and $a(x) \in \mathbb{R}^n$. Its generator is the dual (for the Lebesgue measure) of the generator

$$
Lf = \text{div}(DVf) - < Da, \nabla f >.
$$

\label{eq:15}

Assume that the semigroup associated to $L$ is ergodic and that its invariant probability measure $\mu$ satisfies a $\Phi$-entropy inequality \cite{[5]} with a constant $C \geq 0$: this holds for instance if $L$ satisfies the $CD(1/(2C), \infty)$ criterion.

In this setting when $a(x)$ is not a gradient, the invariant measure $\mu$ is not explicit. Moreover the relation $u_t = e^{-V}P_t(e^V u_0)$ between the solution of \cite{[4]} and the semigroup associated to $L$ does not hold, so that the asymptotic behaviour \cite{[3]} for solutions to \cite{[4]} can not be proved by using \cite{[5]}. However, this relation can be replaced by the following argument, for which the ergodic measure is only assumed to have a positive density $u_\infty$ with respect to the Lebesgue measure.

Let $u$ be a solution of \cite{[4]} with initial datum $u_0$. Then, by \cite[Lemma 7]{[5]},

$$
\frac{d}{dt} \text{Ent}_\mu^\Phi\left(\frac{u_t}{u_\infty}\right) = \int \Phi'\left(\frac{u_t}{u_\infty}\right)L^* u_t dx = \int L\left[\Phi'\left(\frac{u_t}{u_\infty}\right)\right] u_t u_\infty d\mu = -\int \Phi''\left(\frac{u_t}{u_\infty}\right) \Gamma\left(\frac{u_t}{u_\infty}\right) d\mu.
$$

Then a $\Phi$-Entropy inequality \cite{[5]} for $\mu$ implies the exponential convergence:
Theorem 4 With the above notation, assume that a $\Phi$-entropy inequality \cite{7} holds for $\mu$ and with a constant $C$. Then all solutions $u = (u_t)_{t \geq 0}$ to the Fokker-Planck equation \cite{14} converge to $u_\infty$ in $\Phi$-entropy, with

$$\text{Ent}_\mu^\Phi \left( \frac{u_t}{u_\infty} \right) \leq e^{-t/C} \text{Ent}_\mu^\Phi \left( \frac{u_0}{u_\infty} \right), \quad t \geq 0.$$ 

Acknowledgment: This work was presented during the 7th ISAAC conference held in Imperial College, London in July 2009. It is a pleasure to thank the organizers for giving us this opportunity.

References


