Optimal investment with bounded VaR for power utility functions

Bénamar Chouaf, Serguei Pergamenchtchikov

▶ To cite this version:

Bénamar Chouaf, Serguei Pergamenchtchikov. Optimal investment with bounded VaR for power utility functions. 2010. <hal-00457354v2>

HAL Id: hal-00457354

https://hal.archives-ouvertes.fr/hal-00457354v2

Submitted on 20 Feb 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Optimal investment with bounded VaR for power utility functions*

Bénamar Chouaf and Serguei Pergamenchtchikov

Abstract We consider the optimal investment problem for Black-Scholes type financial market with bounded VaR measure on the whole investment interval $[0, T]$. The explicit form for the optimal strategies is found.

Key words: Portfolio optimization, Stochastic optimal control, Risk constraints, Value-at-Risk

Mathematical Subject Classification (2000) 91B28, 93E20

1 Introduction

We consider an investment problem aiming at optimal terminal wealth at maturity T. The classical approach to this problem goes back to Merton [12] and involves utility functions, more precisely, the expected utility serves as the functional which has to be optimized.

We adapt this classical utility maximization approach to nowadays industry practice: investment firms customarily impose limits on the risk of trading portfolios. These limits are specified in terms of downside Value-at-Risk (VaR) risk measures.

As Jorion [5], p. 379 points out, VaR creates a common denominator for the comparison of different risk activities. Traditionally, position limits of traders are

Bénamar Chouaf
Université de Sidi Bel Abbes Laboratoire de Mathématiques Appliquées
e-mail: bchouaf@univ-sba.dz

Serguei Pergamenchtchikov
Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, BP.12, 76801 Saint Etienne du Rouvray, France, e-mail: Serge.Pergamenchtchikov@univ-rouen.fr

* This work was supported by the scinetific cooperation CNRS/DPGRF, Projet DZAC 19856, France-Algérie. The second author is partially supported by the RFBR-Grant 09-01-00172-a.

set in terms of notional exposure, which may not be directly comparable across treasuries with different maturities. In contrast, VaR provides a common denominator to compare various asset classes and business units. The popularity of VaR as a risk measure has been endorsed by regulators, in particular, the Basel Committee on Banking Supervision, which resulted in mandatory regulations worldwide.

Our approach combines the classical utility maximization with risk limits in terms of VaR. This leads to control problems under restrictions on uniform versions of VaR, where the risk bound is supposed to be intact throughout the duration of the investment. To our knowledge such problems have only been considered in dynamic settings, which reduce intrinsically to static problems. Emmer, Klüppelberg and Korn [4] consider a dynamic market, but maximize only the expected wealth at maturity under a downside risk bound at maturity. Basak and Shapiro [2] solve the utility optimization problem for complete markets with bounded VaR at maturity. Gabih, Gretsch and Wunderlich [3] solve the utility optimization problem for constant coefficients markets with bounded ES at maturity. Klüppelberg and Pergamenshchikov [8]-[9] considered the optimisation problems with bounded Var and ES risk measure on the whole time interval in the class of the nonrandom financial stratedies. In this paper we consider the optimal investment problem with the bounded VaR uniformly on whole time interval $[0,T]$ for all admissible financial strategies (nonrandom or random). It should be noted that it is immpossible to calculate the explicit form of the VaR risk measure for the random financial strategies. This is the main difficulty in such problems. In this paper we propose some method to overcome this difficulty by applying optimisations methods in the Hilbert spaces. We find the explicit form for the optimal strategies.

Our paper is organised as follows. In Section 2 we formulate the Black-Scholes model for the price processes. In Section 3 all optimization problems and their solutions are given. All proofs are summarized in Section 4 with the technical lemma postponed to the Appendix 5.

2 The model

We consider a Black-Scholes type financial market consisting of one *riskless bond* and several *risky stocks*. Their respective prices $(S_0(t))_{t\ge 0}$ and $(S_i(t))_{t\ge 0}$ for $i = 1,\ldots,d$ evolve according to the equations:

$$\begin{cases} \mathrm{d}S_0(t) = r_t\, S_0(t)\,\mathrm{d}t\,, & S_0(0) = 1\,, \\ \mathrm{d}S_i(t) = S_i(t)\,\mu_i(t)\,\mathrm{d}t + S_i(t)\,\sum_{j=1}^{d}\sigma_{ij}(t)\,\mathrm{d}W_j(t)\,, & S_i(0) = s_i > 0\,, \end{cases} \tag{2.1}$$

Here $W_t = (W_1(t),\ldots,W_d(t))'$ is a standard d-dimensional Brownian motion; $r_t \in \mathbb{R}$ is the *riskless interest rate*, $\mu_t = (\mu_1(t),\ldots,\mu_d(t))' \in \mathbb{R}^d$ is the vector of *stock-appreciation rates* and $\sigma_t = (\sigma_{ij}(t))_{1\le i,j\le d}$ is the matrix of *stock-volatilities*. We assume that the coefficients r_t, μ_t and σ_t are deterministic functions, which are right continuous with left limits (càdlàg). We also assume that the matrix σ_t is non-singular for Lebesgue-almost all $t \ge 0$.

We denote by $\mathscr{F}_t = \sigma\{W_s, s \le t\}$, $t \ge 0$, the filtration generated by the Brownian motion (augmented by the null sets). Furthermore, $|\cdot|$ denotes the Euclidean norm for vectors and the corresponding matrix norm for matrices.

For $t \ge 0$ let $\phi_t \in \mathbb{R}$ denote the amount of investment into bond and

$$\varphi_t = (\varphi_1(t), \ldots, \varphi_d(t))' \in \mathbb{R}^d$$

the amount of investment into risky assets. We recall that a *trading strategy* is an $\mathbb{R}^{d+1}$-valued $(\mathscr{F}_t)_{t\ge 0}$-progressively measurable process $(\phi_t, \varphi_t)_{t\ge 0}$ and that

$$X_t = \phi_t S_0(t) + \sum_{j=1}^{d} \varphi_j(t) S_j(t), \quad t \ge 0,$$

is called the *wealth process*.

The trading strategy $((\phi_t, \varphi_t))_{t\ge 0}$ is called *self-financing*, if the wealth process satisfies the following equation

$$X_t = x + \int_0^t \phi_u \, \mathrm{d}S_0(u) + \sum_{j=1}^{d} \int_0^t \varphi_j(u) \, \mathrm{d}S_j(u), \quad t \ge 0, \tag{2.2}$$

where $x > 0$ is the initial endowment.

In this paper we work with relative quantities, i.e., we define for $j = 1, \ldots, d$

$$\pi_j(t) := \frac{\varphi_j(t) S_j(t)}{\phi_t S_0(t) + \sum_{j=1}^{d} \varphi_i(t) S_i(t)}, \quad t \ge 0.$$

Then $\pi_t = (\pi_1(t), \ldots, \pi_d(t))'$, $t \ge 0$, is called the *portfolio process* and we assume throughout that it is $(\mathscr{F}_t)_{t\ge 0}$-progressively measurable. We assume that for the fixed investment horizon $T > 0$

$$\|\pi\|_T^2 := \int_0^T |\pi_t|^2 \mathrm{d}t < \infty \quad \text{a.s.}.$$

We also define with $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^d$ the quantities

$$y_t = \sigma_t' \pi_t \quad \text{and} \quad \theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}), \quad t \ge 0, \tag{2.3}$$

where it suffices that these quantities are defined for Lebesgue-almost all $t \ge 0$. Taking these definitions into account we rewrite equation (2.2) for X_t as

$$\mathrm{d}X_t = X_t (r_t + y_t' \theta_t) \mathrm{d}t + X_t y_t' \mathrm{d}W_t, \quad X_0 = x > 0. \tag{2.4}$$

This implies in particular that any optimal investment strategy is equal to

$$\pi_t^* = \sigma_t'^{-1} y_t^* ,$$

where y_t^* is the optimal control process for equation (2.4). We also require for the investment horizon $T > 0$

$$\|\theta\|_T^2 = \int_0^T |\theta_t|^2 \, \mathrm{d}t < \infty . \tag{2.5}$$

We assume that $(y_t)_{0\le t\le T}$ is any $(\mathscr{F}_t)_{0\le t\le T}$ - adapted a.s. square integrated process, i.e.

$$\|y\|_T^2 = \int_0^T |y_t|^2 \, \mathrm{d}t < \infty \quad \text{a.s.},$$

such that the stochastic equation (2.4) has a unique strong solution. We denote by $\mathscr{Y}$ the class of all such processes $y = (y_t)_{0\le t\le T}$. Note that for every $y \in \mathscr{Y}$, through Itô's formula, we represent the equation (2.4) in the following form (to emphasize that the wealth process corresponds to some control process y we write X^y)

$$X_t^y = x \, e^{R_t + (y,\theta)_t} \, \mathscr{E}_t(y) , \tag{2.6}$$

where $R_t = \int_0^t r_u \mathrm{d}u$, $(y,\theta)_t = \int_0^t y_u' \, \theta_u \mathrm{d}u$ and the process $(\mathscr{E}_t(y))_{0\le t\le T}$ is the stochastic exponent for y, i.e.

$$\mathscr{E}_t(y) = \exp\left(\int_0^t y_u' \mathrm{d}W_u - \frac{1}{2} \int_0^t |y_u|^2 \mathrm{d}u \right) .$$

Therefore, for every $y \in \mathscr{Y}$ the process $(X_t^y)_{t\ge 0}$ is a.s. positive and continuous.

For initial endowment $x > 0$ and a control process $y = (y_t)_{t\ge 0}$ in $\mathscr{Y}$, we introduce the *cost function*

$$J(x,y) := \mathbf{E}_x \left(X_T^y \right)^\gamma , \tag{2.7}$$

where $\mathbf{E}_x$ is the expectation operator conditional on $X_0^y = x$.

For $0 < \gamma < 1$ the utility function $U(z) = z^\gamma$ is concave and is called a power (or HARA) utility function. We include the case of $\gamma = 1$, which corresponds to simply optimizing expected consumption and terminal wealth. In combination with a downside risk bound this allows us in principle to disperse with the utility function, where in practise one has to choose the parameter γ.

3 Optimisation problems

3.1 The Unconstrained Problem

We consider two regimes with cost functions (2.7) for $0 < \gamma < 1$ and for $\gamma = 1$.

$$\max_{y\in\mathscr{Y}} J(x,y)\,. \tag{3.1}$$

First we consider Problem 3.1 for $0<\gamma<1$. The following result can be found in Example 6.7 on page 106 in Karatzas and Shreve [7]; it's proof there is based by the martingale method.

Theorem 1. *Consider Problem 3.1 for $0<\gamma<1$. The optimal value of $J(x,y)$ is given by*

$$J^*(x) \,=\, \max_{y\in\mathscr{Y}} J(x,y) \,=\, J(x,y^*) \,=\, x^\gamma \exp\{\gamma R_T + \frac{\gamma}{2(1-\gamma)}\|\theta\|_T^2\}\,,$$

where the optimal control $y^*=(y_t^*)_{0\le t\le T}$ is for all $0\le t\le T$ of the form

$$y_t^* = \frac{\theta_t}{1-\gamma} \quad \left(\pi_t^* = \frac{(\sigma_t\sigma_t')^{-1}(\mu_t - r_t\mathbf{1})}{1-\gamma}\right)\,. \tag{3.2}$$

The optimal wealth process $(X_t^*)_{0\le t\le T}$ is given by

$$\mathrm{d}X_t^* \,=\, X_t^*\left(r_t + \frac{|\theta_t|^2}{1-\gamma}\right)\mathrm{d}t + X_t^*\frac{\theta_t'}{1-\gamma}\mathrm{d}W_t\,, \quad X_0^* = x\,. \tag{3.3}$$

Let now $\gamma=1$.

Theorem 2. *[8] Consider the problem 3.1 with $\gamma=1$. Assume a riskless interest rate $r_t\ge 0$ for all $t\in[0,T]$. If $\|\theta\|_T>0$ then*

$$\max_{y\in\mathscr{Y}} J(x,y) \,=\, \infty\,.$$

If $\|\theta\|_T=0$ then a solution exists and the optimal value of $J(x,y)$ is given by

$$\max_{y\in\mathscr{Y}} J(x,y) \,=\, J(x,y^*) \,=\, x\,e^{R_T}\,,$$

corresponding to arbitrary deterministic square integrable function $(y_t^*)_{0\le t\le T}$. In this case the optimal wealth process $(X_t^*)_{0\le t\le T}$ satisfies the following equation

$$\mathrm{d}X_t^* = X_t^* r_t \mathrm{d}t + X_t^* y_t^{*\prime}\mathrm{d}W_t\,, \quad X_0^* \,=\, x\,. \tag{3.4}$$

3.2 The Constrained Problem

As risk measures we use modifications of the Value-at-Risk as introduced in Emmer, Klüppelberg and Korn [4]. They can be summarized under the notion of Capital-at-Risk as they reflect the required capital reserve. To avoid non-relevant cases we consider only $0<\alpha<1/2$. We use here the definition as in [8]-[9].

Definition 1. *[Value-at-Risk (VaR)]*
Define for initial endowment $x > 0$, a control process $y \in \mathscr{Y}$ and $0 < \alpha \le 1/2$ the *Value-at-Risk (VaR)* by

$$\mathrm{VaR}_t(x,y,\alpha) := x e^{R_t} - Q_t\,, \quad t \ge 0\,,$$

where $Q_t = Q_t(x,y,\alpha)$ is the $(\mathscr{F}_t^y) = \sigma\{y_s\,, 0 \le s \le t\}$ measurable random variable such that

$$\alpha \quad \text{quantile of the ratio} \quad \widehat{X}_t^y = \frac{X_t^y}{Q_t} \quad \text{is equal to} \quad 1 \tag{3.5}$$

i.e.

$$\inf\{z \ge 0\,:\, \mathbf{P}(\widehat{X}_t^y \le z) \ge \alpha\} = 1\,.$$

Remark 1. Note that for the nonrandom financial strategies $(y_t)_{0\le t\le T}$ the process Q_t is the usual α- quantile for the process X_t^y. To define the “random“ quantile for the process X_t^y we consider the ratio process $\widehat{X}_t^y$ for which the α- quantile is equal to 1.

Corollary 1. *For every $y \in \mathscr{Y}$ with $\|y\|_t > 0$ the process Q_t defined in Definition 1, is given by*

$$Q_t = x \exp\left(R_t + (y,\theta)_t - \frac{1}{2}\|y\|_t^2 + \tau_t \|y\|_t\right)\,, \quad t \ge 0\,,$$

where $\tau_t = \tau_t(\alpha,y)$ is the α-quantile of the normalized stochastic integral

$$\xi_t(y) = \frac{1}{\|y\|_t}\int_0^t y_u' \mathrm{d}W_u\,,$$

i.e.

$$\tau_t = \inf\{z \ge -\infty : \mathbf{P}\,(\xi_t(y) \le z) \ge \alpha\}\,. \tag{3.6}$$

It is clear that for any nonrandom function $(y_t)_{0\le t\le T}$ the random variable

$$\xi_t \sim \mathscr{N}(0,1)\,,$$

i.e. in this case $\tau_t = -|z_\alpha|$, where z_α is the α-quantile of the standard normal distribution.

Indeed, to obtain the explicit form for the optimal solutions in this paper we work with a upper bound for VaR risk measure, i.e. we consider the

$$\mathrm{VaR}_t^*(x,y,\alpha) := x e^{R_t} - Q_t^*\,, \quad t \ge 0\,, \tag{3.7}$$

where

$$Q_t^* = x \exp\left(R_t + (y,\theta)_t - \frac{1}{2}\|y\|_t^2 + \tau_t^* \|y\|_t\right) \quad \text{with} \quad \tau_t^* = \min(z_\alpha, \tau_t)\,.$$

Obviously,

$$\mathrm{VaR}_t(x,y,\alpha) \le \mathrm{VaR}^*_t(x,y,\alpha)\,.$$

We define the *level risk function* for some coefficient $0 < \zeta < 1$ as

$$\zeta_t(x) \,=\, \zeta\, x e^{R_t}\,, \quad t \in [0,T]\,. \tag{3.8}$$

We consider only controls $y \in \mathscr{Y}$ for which the Value-at-Risk is a.s. bounded by this level function over the interval $[0,T]$; i.e. we require

$$\sup_{0\le t\le T} \frac{\mathrm{VaR}^*_t(x,y,\alpha)}{\zeta_t(x)} \le 1 \quad \text{a.s..} \tag{3.9}$$

The optimisation problem is

$$\max_{y\in\mathscr{Y}} J(x,y) \quad \text{subject to} \quad \sup_{0\le t\le T} \frac{\mathrm{VaR}^*_t(x,y,\alpha)}{\zeta_t(x)} \le 1 \quad \text{a.s..} \tag{3.10}$$

To describe the optimal strategies we need the following function

$$g(a) := \sqrt{2a+\widetilde{z}^2_\alpha} - \widetilde{z}_\alpha \tag{3.11}$$

with

$$\widetilde{z}_\alpha = |z_\alpha| - \|\theta\|_T \quad \text{and} \quad 0 \le a \le a_{max} := -\ln(1-\zeta)\,.$$

Moreover, we set

$$a_0 = \frac{\|\theta\|^2_T}{2(1-\gamma)^2} + \widetilde{z}_\alpha\, \frac{\|\theta\|_T}{1-\gamma}\,. \tag{3.12}$$

Theorem 3. *Consider the problem* (3.10) *for* $0<\gamma<1$. *Assume that* $|z_\alpha| \ge 2\|\theta\|_T$. *Then the optimal value for the cost function is given by*

$$J(x,y^*) = x^\gamma\, e^{\gamma R_T + \gamma G(g^*)}\,, \tag{3.13}$$

where $G(g) = g\|\theta\|_T + (1-\gamma)g^2/2$, $g^* = g(a^*)$ *with*

$$a^* = \min(a_0, a_{max})\,, \tag{3.14}$$

and the optimal control y^* *is for all* $0 \le t \le T$ *of the form*

$$y^*_t = \frac{g^*}{\|\theta\|_T}\,\theta_t\,\mathbf{1}_{\{\|\theta\|_T>0\}}\,. \tag{3.15}$$

Moreover, if $\|\theta\|_T > 0$ *then the optimal wealth process* $(X^*_t)_{0\le t\le T}$ *is given by*

$$\mathrm{d}X^*_t \,=\, X^*_t\left(r_t + \frac{g^*|\theta_t|^2}{\|\theta\|_T}\right)\mathrm{d}t + X^*_t\,\frac{g^*}{\|\theta\|_T}\,\theta'_t\,\mathrm{d}W_t \quad \textit{with} \quad X^*_0 = x\,; \tag{3.16}$$

and if $\|\theta\|_T = 0$ *then* $X^*_t = x e^{R_t}$ *for* $0 \le t \le T$.

Theorem 4. *Consider the problem* (3.10) *for* $\gamma = 1$. *Assume that* $|z_\alpha| \ge 2\|\theta\|_T$. *Then the optimal value for the cost function is given by*

$$J(x,y^*) = x e^{R_T + g(a_{max})\|\theta\|_T}\,, \tag{3.17}$$

where and the optimal control y^* *is for all* $0 \le t \le T$ *of the form*

$$y^*_t = \frac{g(a_{max})}{\|\theta\|_T}\theta_t\,\mathbf{1}_{\{\|\theta\|_T > 0\}}\,. \tag{3.18}$$

Moreover, if $\|\theta\|_T > 0$ *then the optimal wealth process* $(X^*_t)_{0\le t\le T}$ *is given by*

$$\mathrm{d}X^*_t = X^*_t\left(r_t + \frac{g(a_{max})|\theta_t|^2}{\|\theta\|_T}\right)\mathrm{d}t + X^*_t\frac{g(a_{max})}{\|\theta\|_T}\,\theta'_t\mathrm{d}W_t \quad \textit{with} \quad X^*_0 = x; \tag{3.19}$$

and if $\|\theta\|_T = 0$ *then* $X^*_t = x e^{R_t}$ *for* $0 \le t \le T$.

4 Proofs

4.1 Proof of Theorem 3

Let now $0 < \gamma < 1$. By (2.6) we represent the γ power of the wealth process as

$$(X^y_T)^\gamma = x^\gamma e^{\gamma R_T + \gamma F_T(y)}\,\mathscr{E}_T(\gamma y)\,,$$

where

$$F_T(y) = (\theta, y)_T - \frac{1-\gamma}{2}\|y\|^2_T\,. \tag{4.1}$$

Moreover, we introduce the measure (generally non probability) by the following Radon-Nikodym density

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}}{\mathrm{d}\mathbf{P}} = \mathscr{E}_T(\gamma y)\,.$$

By denoting $\widetilde{\mathbf{E}}$ the expectation with respect to this measure we get that

$$\mathbf{E}(X^y_T)^\gamma = x^\gamma e^{\gamma R_T}\,\widetilde{\mathbf{E}} e^{\gamma F_T(y)}\,. \tag{4.2}$$

Note that, if $\|\theta\|_T = 0$ then

$$\mathbf{E}(X^y_T)^\gamma = x^\gamma e^{\gamma R_T}\,\widetilde{\mathbf{E}}\,e^{-\frac{\gamma(1-\gamma)}{2}\|y\|^2_T}\,.$$

Taking into account that for any process y from $\mathscr{Y}$

$$\mathbf{E}\mathscr{E}_T(\gamma y) \le 1$$

we get for any $y \in \mathscr{Y}$

$$\mathbf{E}(X_T^y)^\gamma \le x^\gamma e^{\gamma R_T}$$

with the equality if and only if $y_t = 0$.

Therefore, in the sequel we assume that $\|\theta\|_T > 0$. Now we shall consider the almost sure optimisation problem for the function $F_T(\cdot)$. First, we consider this contrained the last time moment $t = T$, i.e.

$$\sup_{y\in\mathscr{Y}} F_T(y) \quad \text{subject to} \quad \frac{\mathrm{VaR}_T^*(x,y,\alpha)}{\zeta_T(x)} \le 1 \quad \text{a.s..} \tag{4.3}$$

This constraint is equivalent to

$$\frac{1}{2}\|y\|_T^2 - \tau_T^*\|y\|_T - (\theta,y)_T \le -\ln(1-\zeta) =: a_{max}\,.$$

By fixing the the quantile as $\tau_T^* = -\beta$ for some $\beta \ge |z_\alpha|$ and denoting

$$K_T(y) = \frac{1}{2}\|y\|_T^2 + \beta\|y\|_T - (\theta,y)_T$$

we will consider more general problem than (4.3), i.e. we will find the optimal solution in the Hilbert space $\mathbf{L}_2[0,T]$, i.e.

$$\sup_{y\in\mathbf{L}_2[0,2]} F_T(y) \quad \text{subject to} \quad K_T(y) \le a_{max}\,.$$

To resolve this problem we have to resolve the following one

$$\sup_{y\in\mathbf{L}_2[0,T]} F_T(y) \quad \text{subject to} \quad K_T(y) = a \tag{4.4}$$

for some parameter $0 \le a \le a_{max}$. We use the Lagrange multiplicators method, i.e. we pass to the Lagrange cost function $H_\lambda(y) = F_T(y) - \lambda K_T(y)$ and we have to resolve the optimisation problem for this function, i.e.

$$\max_{y\in\mathbf{L}_2[0,T]} H_\lambda(y)\,. \tag{4.5}$$

In this case

$$H_\lambda(y) = -\frac{\lambda+1-\gamma}{2}\|y\|_T^2 + (1+\lambda)(\theta,y)_T - \lambda\beta\|y\|_T\,,$$

where λ is Lagrange multiplicator. It is clear that $\lambda > \gamma - 1$. Since the problem (4.5) has no finite solution for $\lambda \le \gamma - 1$, i.e.

$$\max_{y\in\mathbf{L}_2[0,T]} H_\lambda(y) = +\infty.$$

to this end we calculate the Gâteau derivative, i.e.

$$\mathbf{D}_\lambda(y,h) = \lim_{\delta\to 0} \frac{H_\lambda(y+\delta h) - H_\lambda(y)}{\delta}.$$

It is easy to check directly that for any function y from $\mathbf{L}_2[0,T]$ with $\|y\|_T > 0$

$$\mathbf{D}_\lambda(y,h) = \int_0^T h'_t\left((1+\lambda)\theta_t - (1-\gamma+\lambda)y_t - \lambda\beta\overline{y}_t\right)\mathrm{d}t$$

with $\overline{y}_t = y_t/\|y\|_T$. Moreover, if $\|y\|_T = 0$, then

$$\mathbf{D}_\lambda(y,h) = (1+\lambda)\int_0^T h'_t\theta_t\mathrm{d}t - \lambda\beta\|h\|_T.$$

It is clear that $D_\lambda(y,h) \neq 0$ for $h_t = -\mathrm{sign}(\lambda)\theta_t$. Therefore, to resolve the equation

$$D_\lambda(y,h) = 0 \tag{4.6}$$

for all $h \in \mathbf{L}_2[0,T]$ we assume that $\|y\|_T > 0$. This implies

$$(1+\lambda)\theta_t - (1-\gamma+\lambda)y_t - \lambda\beta\overline{y}_t = 0,$$

i.e.

$$y_t = \frac{(1+\lambda)\|y\|_T}{\lambda\beta + (1+\lambda-\gamma)\|y\|_T}\theta_t.$$

Therefore,

$$y^\lambda_t = \frac{\psi(\lambda)}{\|\theta\|_T}\theta_t \quad \text{with} \quad \psi(\lambda) = \frac{\|\theta\|_T + \lambda(\|\theta\|_T - \beta)}{1-\gamma+\lambda}. \tag{4.7}$$

The coefficient ψ must be positive, i.e.

$$\gamma - 1 < \lambda < \frac{\|\theta\|_T}{(\beta - \|\theta\|_T)_+}. \tag{4.8}$$

Now we have to verify that the solution of the equation (4.6) gives the maximum solution for the problem (4.5). To end this for any function y from $\mathbf{L}_2[0,T]$ with $\|y\|_T > 0$ we set

$$\Delta_\lambda(y,h) = H_\lambda(y+h) - H_\lambda(y) - D_\lambda(y,h).$$

Moreover, by putting

$$\delta(y,h) = \|y+h\|_T - \|y\|_T - (h,\overline{y})_T, \tag{4.9}$$

we obtain

$$\Delta_\lambda(y,h) = -\frac{\lambda+1-\gamma}{2}\|h\|_T^2 - \lambda\beta\delta(y,h),$$

Now Lemma 1 implies that the function $\Delta(y,h) \le 0$ for all $h \in \mathbf{L}_2[0,T]$. Therefore the solution of the equation (4.6) gives the solution for the problem (4.5).

Now we chose the lagrange multiplicator λ to satisfy the condition in (4.4), i.e.

$$K_T(y^\lambda) = a,$$

i.e.

$$\psi^2(\lambda) + 2\psi(\lambda)(\beta - \|\theta\|_T) = 2a,$$

i.e.

$$\widetilde{\psi}(a) = \psi(\lambda(a)) = \sqrt{2a + (\beta - \|\theta\|_T)^2} - (\beta - \|\theta\|_T)$$

with

$$\lambda = \lambda(a) = \frac{\|\theta\|_T + (1-\gamma)(\beta - \|\theta\|_T)}{\sqrt{2a + (\beta - \|\theta\|_T)^2}} - 1 + \gamma.$$

One can check directly that the function $\lambda(a)$ satisfies the condition (4.8) for any $a > 0$. This means that the solution for the problem (4.4) is given by the function

$$\widetilde{y}_t^a = y_T^{\lambda(a)} = \frac{\widetilde{\psi}(a)}{\|\theta\|_T}\theta_t.$$

Now to chose the parameter $0 < a \le a_{max}$ in (4.4) we have to maximize the function (4.1), i.e.

$$\max_{0 \le a \le a_{max}} F_T(\widetilde{y}^a).$$

Note that

$$F_T(\widetilde{y}^a) = G(\widetilde{\psi}(a)) \quad \text{with} \quad G(\psi) = \psi\|\theta\|_T - (1-\gamma)\frac{\psi^2}{2}.$$

Moreover, note that for any $a > 0$ and $\beta \ge |z_\alpha|$

$$\widetilde{\psi}(a) \le g(a),$$

where the function g is defined in (3.11). Therefore,

$$\max_{0 \le a \le a_{max}} F_T(\widetilde{y}^a) \le \max_{0 \le a \le a_{max}} G(g(a)) = G(g(a^*)),$$

where a^* is defined in (3.14). To obtain here the equality we take in (4.7) $\beta = |z_\alpha|$. Thus, the function (3.15) is the solution of the problem (4.3). Now to pass to the problem (3.10) we have to check the condition (3.9) for the function (3.15). To this end note that

$$\frac{1}{2}\|y^*\|_t^2 + |z_\alpha|\|y^*\|_t - (\theta, y^*)_t = \int_0^t \omega(s)\,\mathrm{d}s,$$

where

$$\omega_s = |\theta_s|^2 \left(\frac{(g^*)^2}{2\|\theta\|_T^2} + \frac{g^*(|z_\alpha| - 2\|\theta\|_s)}{2\|\theta\|_T \|\theta\|_s} \right) .$$

taking into account here the condition $|z_\alpha| \geq \|\theta\|_T$ we obtain $\omega_t \geq 0$, i.e.

$$\begin{aligned} &\frac{1}{2}\|y^*\|_t^2 + |z_\alpha| \|y^*\|_t - (\theta, y^*)_t \\ &\leq \frac{1}{2}\|y^*\|_T^2 + |z_\alpha| \|y^*\|_T - (\theta, y^*)_T \\ &= a^* \leq -\ln(1-\zeta) . \end{aligned}$$

This implies immediately that the function (3.15) is a solution of the problem (3.10).

□

4.2 Proof of Theorem 4

Let now $\gamma = 1$. Note that in this case we can obtain the following upper bound:

$$\mathbf{E} X_T^y \leq x e^{R_T} \mathbf{E} e^{\|\theta\|_T \|y\|_T} \mathscr{E}_T(y) .$$

Obviously, that if $\|\theta\|_T = 0$ than we obtain here equality if and only if $y = 0$. Let now $\|\theta\|_T > 0$. Note that the condition

$$K_T(y) \leq a_{max} \tag{4.10}$$

implies $\|y\|_T \leq g(a_{max})$. Thus, for any function $(y_t)_{0 \leq t \leq T}$ satisfying the condition we have

$$\mathbf{E} X_T^y \leq x e^{R_T + g(a_{max}) \|\theta\|_T} .$$

Moreover, the function (3.18) transforms this inequality in the equality. By the same way as in the proof of Theorem 4 we check that the function (3.18) satisfies the condition (3.9).

□

5 Appendix

A.1 Properties of the function (4.9)

Lemma 1. *Assume that* $y \in \mathbf{L}_2[0,T]$ *with* $\|y\|_T > 0$*. Then for every* $h \in \mathbf{L}_2[0,T]$ *the function* (4.9) *is positive, i.e.* $\delta(y,h) \geq 0$*.*

Proof. Obviously, if $h \equiv ay$ for some $a \in \mathbb{R}$, then $\delta(y,h) = (|1+a| - 1 - a)\|y\|_T \geq 0$. Let now the functions h and y be linearly independent. Then

$$\delta(y,h) = \frac{2(y\,,h)_T + \|h\|_T^2}{\|y+h\|_T + \|y\|_T} - (\overline{y},h)_T = \frac{\|h\|_T^2 - (\overline{y},h)_T((\overline{y},h)_T + \delta(y,h))}{\|y+h\|_T + \|y\|_T}\,.$$

It is clear that for all h

$$\|y+h\|_T + \|y\|_T + (\overline{y},h)_T \geq 0$$

with equality if and only if $h \equiv ay$ for some $a \leq -1$.
Therefore, if the functions h and y are linearly independent, then

$$\delta(y,h) = \frac{\|h\|_T^2 - (\overline{y}\,,h)_T^2}{\|y+h\|_T + \|y\|_T + (\overline{y},h)_T} \geq 0\,.$$

□

References

1. Artznèr, P., Delbaen, F., Eber, J.-M., Heath, D.: Coherent measures of risk. Math. Finance. **9**, 203–228 (1999)
2. Basak, S., Shapiro, A. : Value at Risk based risk management: optimal policies and asset prices. Review of Financial Studies. **14**(2), 371–405 (1999)
3. Gabih, A., Grecksch, W., Wunderlich, R. : Dynamic portfolio optimization with bounded shortfall risks. Stoch. Anal. Appl. **23**, 579–594 (2005)
4. Emmer, S., Klüppelberg, C., Korn, R. : Optimal portfolios with bounded Capital-at-Risk. Math. Finance. **11**, 365–384 (2001)
5. orion, P. : Value at Risk. McGraw-Hill, New York (2001)
6. Karatzas, I. and Shreve, S.E. : Brownian Motion and Stochastic Calculus. Springer, Berlin (1988)
7. Karatzas, I. and Shreve, S.E. : Methods of Mathematical Finance. Springer, Berlin (2001)
8. Klüppelberg, C. and Pergamenshchikov, S. M. (2009) Optimal consumption and investment with bounded downside risk for power utility functions. *In: F. Delbaen, M. Rásonyi and C. Stricker (Eds.) : Optimality and Risk - Modern Trends in Mathematical Finance*, Springer-Heidelberg-Dordrecht-London-New York, 133-169.
9. Klüppelberg, C. and Pergamenshchikov, S. M. (2009) Optimal consumption and investment with bounded downside risk measures for logarithmic utility functions. - *In: H. Albrecher, W. Runggaldier and W. Schachermayer (Eds.): Advanced Financial Modelling. Radon Ser. Comput. Appl. Math.*, Walter de Gruyter, Berlin, **8**, p. 245 - 273.
10. Korn, R. : Optimal Portfolios. World Scientific, Singapore (1997)
11. Liptser, R.S., Shirayev, A.N. : Statistics of Random Processes I. General Theory. Springer, New York (1977)
12. Merton, R.C. : Continuous-Time Finance. Blackwell, Cambridge MA (1990)