Some flows in shape optimization

Pierre Cardaliaguet, Olivier Ley

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Abstract. Geometric flows related to shape optimization problems of Bernoulli type are investigated. The evolution law is the sum of a curvature term and a nonlocal term of Hele-Shaw type. We introduce generalized set solutions, the definition of which is widely inspired by viscosity solutions. The main result is an inclusion preservation principle for generalized solutions. As a consequence, we obtain existence, uniqueness and stability of solutions. Asymptotic behavior for the flow is discussed: we prove that the solutions converge to a generalized Bernoulli exterior free boundary problem.

Résumé. On étudie des flots géométriques liés à des problèmes d'optimisation de forme du type "problème de Bernoulli". La loi d'évolution considérée a la forme d'une somme d'un terme de courbure et d'un terme non-local de type Hele-Shaw. Notre définition de solution généralisée est fortement inspirée de la notion de solutions de viscosité. Le résultat central est un principe d'inclusion pour les ensembles solutions. Nous en déduisons l'existence, une unicité générique et des propriétés de stabilité des solutions. Enfin, nous étudions le comportement asymptotique des solutions en montrant qu'elles convergent vers la solution d'un problème à frontière libre de Bernoulli.

1 Introduction

In recent years several works have been devoted to the study of viscosity solution for moving boundary problems whose evolution law is governed by a nonlocal equation. See in particular [2, 7, 8, 9, 12, 20, 21]. In this paper, we consider subsets $\Omega(t)$ of $I\!R^N$ (with $N \geq 2$) whose boundary $\partial\Omega(t)$ evolves with a normal velocity of the type

$$V^{\Omega}_{(t,x)} = F(\nu^{\Omega(t)}_x, H^{\Omega(t)}_x) + \lambda \bar{h}(x, \Omega(t)) \tag{1}$$

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where $\lambda \geq 0$, $\nu_x^{\Omega(t)}$ is the outward unit normal to $\partial\Omega(t)$ at x, $H_x^{\Omega(t)}$ is the curvature matrix of $\partial\Omega(t)$ at x (nonpositive for convex sets), F is continuous and elliptic, i.e., nondecreasing with respect to the curvature matrix. The nonlocal term $\bar{h}$ is of Hele-Shaw type:

$$\bar{h}(x, \Omega(t)) = |Du(x)|^2 ,$$

where $u : \Omega(t) \to I\!\!R$ is the solution to the following p.d.e.

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega(t)\backslash S, \\ \quad u = g & \text{on } \partial S, \\ \quad u = 0 & \text{on } \partial\Omega(t). \end{cases} \tag{2}$$

The set $S \neq \emptyset$ is a fixed source with a smooth boundary and $g : \partial S \to I\!\!R$ is positive and smooth. We always assume that $S \subset\subset \Omega(t)$.

The motivation to study such problems comes from several numerical works using the "level-set approach" in shape optimization [1, 23, 24, 25, 28]. The idea of these papers is to use formally a gradient method for the minimization of an objective function $J(\Omega)$ where Ω is a subset of $I\!\!R^N$. The use of the level-set method for building the gradient flow has then the major advantage to allow topological changes. Let us underline that this technique is up to now purely heuristic. One of the goals of this paper is to justify it for some simple shape optimisation problems.

In order to make our purpose more transparent, a brief description of the level-set approach to shape optimization problem is now in order (see also the discussion in [1] for a more detailed presentation concerning more realistic shape optimization problems). Consider the problem of minimizing the capacity of a set under volume constraints:

$$\min_{S\subset\subset\Omega\subset\subset R^N} \{\text{cap}(\Omega) \quad \text{with} \quad \text{vol}(\Omega) = \text{ constant}\} , \tag{3}$$

where

$$\text{cap}(\Omega) = \int_{\Omega\backslash S} |Du(x)|^2 dx \quad \text{and} \quad \text{vol}(\Omega) = \int_{\Omega\backslash S} dx$$

and u is the solution of (2) with Ω instead of $\Omega(t)$. For any local diffeomorphism θ, we can compute the shape derivatives with respect to θ of the capacity and of the volume. By Hadamard formulas we get

$$\text{cap}'(\Omega)(\theta) = \int_{\partial\Omega} |Du(\sigma)|^2 \langle\theta(\sigma), \nu_\sigma^\Omega\rangle d\sigma \text{ and } \text{vol}'(\Omega)(\theta) = \int_{\partial\Omega} \langle\theta(\sigma), \nu_\sigma^\Omega\rangle d\sigma .$$

Assuming that the optimal shape Ω is smooth, the necessary conditions of optimality states that there is a Lagrange multiplier $\Lambda > 0$ such that

$$\mathrm{cap}'(\Omega)(\theta) + \Lambda\, \mathrm{vol}'(\Omega)(\theta) = 0 \ .$$

So it is natural to set

$$J_\lambda(\Omega) = \mathrm{vol}(\Omega) + \lambda\, \mathrm{cap}(\Omega) \ ,$$

where $\lambda = 1/\Lambda$. If we choose $\theta(x) = (-1 + \lambda |Du(x)|^2)\nu_x^\Omega$ on $\partial\Omega$, then, at least formally, we get

$$J'_\lambda(\Omega)(\theta) = -\int_{\partial\Omega} (-1 + \lambda |Du(\sigma)|^2)^2 d\sigma \le 0 \ .$$

Therefore the velocity $\theta(x) = (-1 + \lambda |Du(x)|^2)\nu_x^\Omega$ appears as a descent direction for the optimization problem (3) and for the set Ω. The heuristic method for solving (3) is now clear: fix an initial position Ω_0, consider the evolution $(\Omega(t))_{t\ge 0}$ with normal velocity given by (1) and $F \equiv -1$, and compute the limit of $\Omega(t)$ as $t \to +\infty$: this limit is the natural candidate minimizer for (3).

It is worth noticing that problem (3) has for necessary condition the classical Bernoulli exterior free boundary problem

$$\text{Find a set } K \subset\subset I\!\!R^N, \text{ with } S \subset\subset K \text{ and } |Du(x)| = k \text{ for all } x \in \partial K, \quad (4)$$

where $k > 0$ is a fixed constant and u is the solution of (2). We refer the reader to the survey paper [15] for a complete description of this problem.

If one considers a perimeter constraint instead of a volume constraint:

$$\min_{S\subset\subset\Omega\subset\subset I\!\!R^N} \{\mathrm{cap}(\Omega) \quad \text{with} \quad \mathrm{per}(\Omega) = \ \text{constant}\} \ ,$$

one is naturally lead to consider the evolution equation (1) with $F(\nu, A) = \frac{1}{N-1} Tr(A)$ (i.e., the mean curvature). The flow is then formally a descent direction for

$$J_\lambda(\Omega) = \mathrm{per}(\Omega) + \lambda \mathrm{cap}(\Omega) \ .$$

Let us underline that this problem has for necessary condition the generalization of the free boundary problem (4) with curvature dependance (see (53)).

Of course all the above computations are only formal: in general, solutions to the evolution equation do not remain smooth, even when starting from smooth initial data. Numerically, this difficulty is overcome by using the level-set approach, which allows to define the solution after the onset of singularities. The aim of this paper is to define and study generalized solutions of the evolution equation, and to investigate the asymptotic behavior of the solution as $t \to +\infty$.

Our concept of solutions is widely inspired by the definition of viscosity solution for the mean curvature motion, which corresponds to equation (1) with $F(\nu, A) = \frac{1}{N-1}Tr(A)$ and $\lambda = 0$. Motivated by the numerical work of Osher and Sethian [22], a weak notion of solution for this motion was introduced in the articles of Chen, Giga and Goto [10] and Evans and Spruck [13]. In this so-called level-set method, the evolution is described as the level set of the solution of an auxiliary pde, the level set equation. This equation is solved in the sense of viscosity solutions (see [11]). This powerful method leads to plenty of results, we refer for instance to the survey book of Giga [16]. Note that the level-set approach in shape optimization is a natural–but up to now formal–generalization of these ideas.

As pointed out in [3, 4, 26], the generalized solutions obtained by the level set approach can also be defined in more geometric and intric ways (see also the related notion of barrier solutions introduced by De Giorgi). We use here a definition introduced in [2], and used repetitively in [7, 8, 9]. In the case of the mean curvature motion, Giga [16] proved this definition is equivalent to the level-set one. Compared with the already quoted studies on viscosity solutions of front propagation problems with nonlocal terms, the main novelty of this paper is the fact that we are able to treat signed velocities which also involve curvature terms. We learnt recently that a similar result (for a Stefan problem) has been obtained by Kim in [21].

Our main result is an inclusion principle, which is the equivalent of the maximum principle for geometric evolutions. It states that viscosity subsolutions for the flow remain included into viscosity supersolutions, provided the initial positions are. For this we have to generalize Ilmanen interposition Lemma, which was already the key tool of [7, 9]. This Lemma allows to separate disjoint sets by a smooth (that is $\mathcal{C}^{1,1}$) surface in a clever way. We improve this result in two directions (see Theorem 3.3). At first we show that, when dealing with subsets of $\mathbb{R} \times \mathbb{R}^N$, the smooth separating hypersurfaces in $\mathbb{R} \times \mathbb{R}^N$ can be chosen to be smoothly evolving hypersurfaces of $\mathbb{R}^N$. Secondly, we build in a carefull way a $\mathcal{C}^2$ approximation of these evolving hypersurfaces which allows to treat problems with curvature as in (1).

Let us finally explain how this paper is organized. In Section 2, we define the notion of generalized solutions and state the main properties of the velocity law. Section 3 is devoted to the interposition Theorems. In Section

4 we state and prove the inclusion principle for our generalized solutions. As a consequence, we derive results about existence, uniqueness and stability of generalized solutions. Finally, Section 5 is devoted to the asymptotic behaviour of the solutions in terms of a generalized Bernoulli exterior free boundary problem.

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2 Definitions and preliminary results

2.1 Definition of the solutions

Let us first fix some notations: throughout the paper $|\cdot|$ denotes the euclidean norm (of $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, depending on the context) and $B(x,R)$ the open ball centered at x and of radius R. If K is a subset of $\mathbb{R}^N$ and $x \in \mathbb{R}^N$, then $d_K(x)$ denotes the usual distance from x to K: $d_K(x) = \inf_{y\in K} |y-x|$ and $\mathbf{d}_K$ is the signed distance to ∂K defined by

$$\mathbf{d}_K(x) = \begin{cases} d_K(x) & \text{if} \quad x \notin K, \\ -d_{\partial K}(x) & \text{if} \quad x \in K. \end{cases} \tag{5}$$

Finally, in the whole paper, if K_1 and K_2 are subset of $\mathbb{R}^M$ for $N \geq 1$, then

$$K_1 \subset\subset K_2$$

means that K_1 is bounded and, either $\overline{K_1} \subset \operatorname{int}(K_2)$ or equivalently $\overline{K_1} \cap \overline{\mathbb{R}^N \backslash K_2} = \emptyset$.

We intend to study the evolution of compact hypersurfaces $\Sigma(t) = \partial\Omega(t)$ of $\mathbb{R}^N$, where $\Omega(t)$ is an open set, evolving with the following law:

$$\forall t \geq 0\ ,\ x \in \Sigma(t), \ \ V^{\Omega}_{(t,x)} = h_\lambda(x, \Omega(t)) \tag{6}$$

where $V^{\Omega}_{(t,x)}$ is the normal velocity of the evolving set, $h_\lambda = h_\lambda(x,\Omega)$ is given, for any set $\Omega \subset \mathbb{R}^N$ with smooth boundary by

$$h_\lambda(x,\Omega) = F(\nu^\Omega_x, H^\Omega_x) + \lambda \bar{h}(x,\Omega) \tag{7}$$

where ν^Ω_x is the outward unit normal to Ω at x, H^Ω_x the curvature matrix. Throughout this paper we assume that $(\nu, A) \in S^{N-1} \times \mathcal{S}_N \mapsto F(\nu, A) \in \mathbb{R}$ is continuous and elliptic, i.e., nondecreasing with respect to the matrix. Here S^{N-1} denotes the $(N-1)-$dimensional unit sphere, and $\mathcal{S}_N$ the space of $N-$dimensional symmetric matrices. Typical examples for F are $F(\nu, A) = -1$ (this corresponds to the flow associated to Bernoulli problem in the introduction) or $F(\nu, A) = \mathrm{Tr}(A)$ (for the flow arising in the minimization of the capacity under perimeter constraints). As for $\bar{h}$, it is a nonlocal evolution term of Hele-Shaw type: the example we consider here is

$$\bar{h}(x, \Omega) = |Du(x)|^2 , \tag{8}$$

where $u : \Omega \to \mathbb{R}$ is the solution of the following p.d.e.

$$\left\{ \begin{array}{lll} i) & -\Delta u = 0 & \text{in } \Omega \backslash S, \\ ii) & u = g & \text{on } \partial S, \\ iii) & u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{9}$$

The set $S \neq \emptyset$ is a fixed source and we always assume above that $S \subset\subset \Omega(t)$. Here and throughout the paper, we suppose that

$$\left\{ \begin{array}{ll} i) & S \subset \mathbb{R}^N \text{ is bounded and equal to the closure of an open set} \\ & \text{with a } \mathcal{C}^2 \text{ boundary,} \\ ii) & g : \partial S \to (0, +\infty) \text{ is } \mathcal{C}^{1,\alpha} \text{ (for some } \alpha \in (0,1)). \end{array} \right. \tag{10}$$

Let us underline that $\bar{h}(x, \Omega)$ is well defined as soon as Ω has a "smooth" (say for instance $\mathcal{C}^{1,\alpha}$) boundary and that $S \subset\subset \Omega$. In the sequel, we set

$$\mathcal{D} = \{K \subset \mathbb{R}^N \ : \ K \text{ is bounded and } S \subset \mathrm{int}(K)\} , \tag{11}$$

where $\mathrm{int}(K)$ denotes the interior of K.

From now on, we consider the graph

$$\mathcal{K} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N \ : \ x \in \Omega(t)\} .$$

of the evolving sets $\Omega(t)$. Note that $\mathcal{K}$ is a subset of $\mathbb{R}^+ \times \mathbb{R}^N$. The set $\mathcal{K}$ is our main unknown. We denote by (t, x) an element of such a set, where $t \in \mathbb{R}^+$ denotes the time and $x \in \mathbb{R}^N$ denotes the space. We set

$$\mathcal{K}(t) \ = \ \{x \in \mathbb{R}^N \ : \ (t, x) \in \mathcal{K}\} .$$

The closure of the set $\mathcal{K}$ in $\mathbb{R}^{N+1}$ is denoted by $\overline{\mathcal{K}}$. The closure of the complementary of $\mathcal{K}$ is denoted $\widehat{\mathcal{K}}$:

$$\widehat{\mathcal{K}} = \overline{(\mathbb{R}^+ \times \mathbb{R}^N) \backslash \mathcal{K}}$$

and we set

$$\widehat{\mathcal{K}}(t) = \{x \in \mathbb{R}^N \ : \ (t,x) \in \widehat{\mathcal{K}}\} .$$

We use here repetitively the terminology and the notations introduced in [8, 9] and [7]:

- A *tube* $\mathcal{K}$ is a subset of $\mathbb{R}^+ \times \mathbb{R}^N$, such that $\overline{\mathcal{K}} \cap ([0,t] \times \mathbb{R}^N)$ is a compact subset of $\mathbb{R}^{N+1}$ for any $t \geq 0$.
- A set $\mathcal{K} \subset \mathbb{R}^+ \times \mathbb{R}^N$ is *left lower semicontinuous* if

$$\forall t > 0, \ \forall x \in \mathcal{K}(t), \text{ if } t_n \to t^-, \text{ then } \exists x_n \in \mathcal{K}(t_n) \text{ such that } x_n \to x .$$

- If $s = 1, 2$ or $(1,1)$, a $\mathcal{C}^s$ *regular tube* $\mathcal{K}_r$ is a tube with a nonempty interior and whose boundary has a $\mathcal{C}^s$ regularity, and is such that at any point $(t,x) \in \partial\mathcal{K}_r$, the outward unit normal $\nu^{\mathcal{K}_r}_{(t,x)} = (\nu_t, \nu_x)$ to $\mathcal{K}_r$ at (t,x) satisfies

$$\nu_x \neq 0. \tag{12}$$

- The *normal velocity* $V^{\mathcal{K}_r}_{(t,x)}$ of a $\mathcal{C}^1$ regular tube $\mathcal{K}_r$ at the point $(t,x) \in \partial\mathcal{K}_r$ is defined by

$$V^{\mathcal{K}_r}_{(t,x)} = -\frac{\nu_t}{|\nu_x|}, \tag{13}$$

  where $\nu^{\mathcal{K}_r}_{(t,x)} = (\nu_t, \nu_x)$ is the outward unit normal to $\mathcal{K}_r$ at (t,x).

- A $\mathcal{C}^1$ regular tube $\mathcal{K}_r$ is *externally tangent* to a tube $\mathcal{K}$ at $(t,x) \in \mathcal{K}$ if

$$\mathcal{K} \subset \mathcal{K}_r \text{ and } (t,x) \in \partial\mathcal{K}_r .$$

  It is *internally tangent* to $\mathcal{K}$ at $(t,x) \in \widehat{\mathcal{K}}$ if

$$\mathcal{K}_r \subset \mathcal{K} \text{ and } (t,x) \in \partial\mathcal{K}_r .$$

- We say that a sequence of $\mathcal{C}^{1,1}$ tubes $(\mathcal{K}_n)$ converges to some $\mathcal{C}^{1,1}$ tube $\mathcal{K}$ *in the* $\mathcal{C}^{1,\mathrm{b}}$ *sense* if $(\mathcal{K}_n)$ converges to $\mathcal{K}$ and $(\partial\mathcal{K}_n)$ converges to $\partial\mathcal{K}$ for the Hausdorff distance, and if there is an open neighborhood $\mathcal{O}$ of $\partial\mathcal{K}$ such that, if $\mathbf{d}_{\mathcal{K}}$ (respectively $\mathbf{d}_{\mathcal{K}_n}$) is the signed distance (5) to $\mathcal{K}$ (respectively to $\mathcal{K}_n$), then $(\mathbf{d}_{\mathcal{K}_n})$ and $(D\mathbf{d}_{\mathcal{K}_n})$ converge uniformly to $\mathbf{d}_{\mathcal{K}}$ and $D\mathbf{d}_{\mathcal{K}}$ on $\mathcal{O}$ and $\|D^2\mathbf{d}_{\mathcal{K}_n}\|_\infty$ are uniformly bounded on $\mathcal{O}$.

Remark 2.1
1. The reason to introduce $\mathcal{C}^{1,1}$ and $\mathcal{C}^2$ tubes is clear when looking at (6) and (7): $\bar{h}$ is well defined if $\mathcal{K}_r$ is a $\mathcal{C}^{1,1}$ tube (see Section 2.2) and F is well defined if $\mathcal{K}_r$ is a $\mathcal{C}^2$ tube (to be able to compute the curvature of $\mathcal{K}_r$). Therefore, (6) is well defined when $\mathcal{K}_r$ is a $\mathcal{C}^2$ regular tube and, following the ideas of viscosity solutions (see [11]), $\mathcal{C}^2$ regular tubes will play the role of "test-functions" in the following definition.
2. For simplicity, we gave all the above definitions with tubes defined for all time $t \geq 0$. But this is not the point, everything in the sequel is local in time, so we use the same definitions with tubes $\mathcal{K}$ where $\mathcal{K}(s)$ is only defined in a neighborhood of some fixed t. Note that smooth tubes will always be defined locally in time.

Definition 2.1 *Let $\mathcal{K}$ be a tube and $K_0 \in \mathcal{D}$ be an initial set.*

1. *$\mathcal{K}$ is* a viscosity subsolution *to the front propagation problem (in short FPP) (6) if $\mathcal{K}$ is left lower semicontinuous and $\mathcal{K}(t) \in \mathcal{D}$ for any t, and if, for any $\mathcal{C}^2$ regular tube $\mathcal{K}_r$ externally tangent to $\mathcal{K}$ at some point (t,x), with $\mathcal{K}_r(t) \in \mathcal{D}$ and $t > 0$, we have*
$$V^{\mathcal{K}_r}_{(t,x)} \leq h_\lambda(x, \mathcal{K}_r(t))$$
where $V^{\mathcal{K}_r}_{(t,x)}$ is the normal velocity of $\mathcal{K}_r$ at (t,x).
We say that $\mathcal{K}$ is a subsolution to the FPP (6) with initial position K_0 if $\mathcal{K}$ is a subsolution and if $\overline{\mathcal{K}}(0) \subset \overline{K_0}$.

2. *$\mathcal{K}$ is a* viscosity supersolution *to the FPP (6) if $\widehat{\mathcal{K}}$ is left lower semicontinuous, and $\mathcal{K}(t) \subset \mathcal{D}$ for any t, and if, for any $\mathcal{C}^2$ regular tube $\mathcal{K}_r$ internally tangent to $\mathcal{K}$ at some point (t,x), with $\mathcal{K}_r(t) \in \mathcal{D}$ and $t > 0$, we have*
$$V^{\mathcal{K}_r}_{(t,x)} \geq h_\lambda(x, \mathcal{K}_r(t)) \ .$$
We say that $\mathcal{K}$ is a supersolution to the FPP (6) with initial position K_0 if $\mathcal{K}$ is a supersolution and if $\widehat{\mathcal{K}}(0) \subset \overline{I\!\!R^N \backslash K_0}$.

3. *Finally, we say that a tube $\mathcal{K}$ is a* viscosity solution *to the front propagation problem (with initial position K_0) if $\mathcal{K}$ is a sub- and a super-solution to the FPP (with initial position K_0).*

Remark 2.2 The operator h_λ defined in (7) is the sum of a local operator F and a nonlocal one $\bar{h}$. As in the theory of viscosity solutions, we can localize arguments related to the local part of the operator. More precisely, $\mathcal{C}^2$

regularity of the boundary of the tube is required to compute the curvature in F, but only $\mathcal{C}^{1,1}$ regularity is needed to compute the nonlocal part $\bar{h}$. Therefore, the above definition is equivalent if we replace "for any $\mathcal{C}^2$ regular tube $\mathcal{K}_r$ internally (respectively externally) tangent to $\mathcal{K}$ at some point (t,x)..." by "for any $\mathcal{C}^{1,1}$ regular tube $\mathcal{K}_r$ internally (respectively externally) tangent to $\mathcal{K}$ at some point (t,x) such that $\partial\mathcal{K}_r$ is $\mathcal{C}^2$ in a neighborhood of (t,x)..." We will use this equivalent definition in the proof of Theorem 4.1.

2.2 Regularity properties of the velocity $\bar{h}$

We complete this part by recalling the regularity properties of the nonlocal term $\bar{h}$ defined by (8) and (9). These results were already given in [7], so we omit the proofs. Here we assume that the set S and the function g satisfy assumptions (10).

Because of the maximum principle, the function $\bar{h}$ is nonnegative and nondecreasing: if $K_1 \in \mathcal{D}$ and $K_2 \in \mathcal{D}$ are closed and with a $\mathcal{C}^{1,1}$ boundary, if $K_1 \subset K_2$ and if $x \in \partial K_1 \cap K_2$, then $0 \leq \bar{h}(x, K_1) \leq \bar{h}(x, K_2)$.

Furthermore, $\bar{h}$ is continuous in the following sense: If K_n and $K \in \mathcal{D}$ are closed subsets of $I\!\!R^N$ with $\mathcal{C}^{1,1}$ boundary such that K_n converge to K in the $\mathcal{C}^{1,\mathrm{b}}$ sense, if $x_n \in \partial K_n$ converge to $x \in \partial K$, then

$$\lim_n \bar{h}(x_n, K_n) = \bar{h}(x, K).$$

This is a straightforward application of [17, Theorem 8.33].

Next we give a result describing the behaviour of $\bar{h}$ for large ball:

Lemma 2.2 *For any $x_0 \in I\!\!R^N$, there are constants $r_0 > 0$ and $\alpha > 0$ such that*

$$\forall r \geq r_0,\ \forall x \in \partial B(x_0, r), \quad \bar{h}(x, B(x_0, r)) \leq \begin{cases} \alpha r^{2-2N} & \text{if } N \neq 2, \\ \dfrac{\alpha}{r^2|\log(r)|^2} & \text{if } N = 2. \end{cases}$$

Moreover, the constants r_0 and α only depend on S and on $\|g\|_\infty$.

The proof is based on standard construction of supersolutions to (9) for $\Omega = B(0, r)$, and so we omit it.

Lemma (2.2) states that $\bar{h}$ is small when Ω is a large ball. On the contrary, the following lemma means that $\bar{h}$ is large when "Ω is close to S." For all $\gamma \geq 0$, we introduce

$$S_\gamma = \{x \in I\!\!R^N\ ,\ d_S(x) \leq \gamma\}. \tag{14}$$

Then, we have

Lemma 2.3 *There exist $\gamma_0 > 0$ and a constant $\alpha > 0$ which depends only on g and S such that, for all $\gamma \in (0, \gamma_0)$,*

$$\bar{h}(x, S_\gamma) \geq \frac{\alpha}{\gamma^2} \quad \forall x \in \partial S_\gamma.$$

Proof of Lemma 2.3. Since S has a $\mathcal{C}^2$ boundary, we can fix $\gamma_0 > 0$ small enough such that $\mathbf{d}_S$ defined by (5) is $\mathcal{C}^2$ in $S_{2\gamma_0} \backslash \{\mathbf{d}_S < -2\gamma_0\}$. We fix $\gamma \in (0, \gamma_0)$ and set $K = \{\mathbf{d}_S \leq -\gamma\}$. We note that $d_K = \mathbf{d}_S + \gamma$ is $\mathcal{C}^2$ on $S_{2\gamma_0} \backslash \{\mathbf{d}_S < -2\gamma_0\}$. Moreover $d_K = \gamma$ on ∂S and $d_K = 2\gamma$ on ∂S_γ. Set

$$M = \max\{|\Delta d_K^2(x)| \mid 0 \leq \mathbf{d}_S(x) \leq \gamma\} \text{ and } m = \min\{g(x) \mid x \in \partial S\} \ . \quad (15)$$

Finally we set $\Omega = S_\gamma$ and, for $\beta = \mathrm{e}^{-3M/4} m$, we define

$$\varphi(r) = \beta(\mathrm{e}^{-Mr/(4\gamma^2)} - 1) \quad \forall r \in I\!R \ .$$

We claim that

$$u(x) = \varphi(d_K^2(x) - (2\gamma)^2)$$

is a subsolution of (9). Indeed, since $\varphi(0) = 0$, for all $x \in \partial S_\gamma$, $u(x) = 0$. From the definition of β, for all $x \in \partial S$, $u(x) = \varphi(-3\gamma^2) \leq m \leq g$. Setting $r_x = d_K^2(x) - (2\gamma)^2$, an easy computation gives

$$-\Delta u(x) = -\varphi''(r_x)|D(d_K^2)|^2 - \varphi'(r_x)\Delta(d_K^2).$$

But $D(d_K^2) = 2d_K D d_K$ and $|Dd_K| = 1$. From (15), we get

$$-\Delta u(x) \leq -4\varphi''(r_x)d_K^2 + M|\varphi'(r_x)|.$$

A computation of the derivatives of φ gives

$$-\Delta u(x) \leq \frac{\beta M^2}{4\gamma^2} \mathrm{e}^{-Mr_x/(4\gamma^2)} \left(1 - \frac{d_K^2(x)}{\gamma^2}\right).$$

For $x \in \Omega \backslash S$, we have $d_K(x) \geq \gamma$ and therefore we obtain $-\Delta u(x) \leq 0$. Finally u is a subsolution with $u \geq 0$ in Ω and $u = 0$ on ∂S_γ. Thus, for all $x \in \partial S_\gamma$,

$$\bar{h}(x, S_\gamma) \geq |Du(x)|^2 = \frac{M^2 \mathrm{e}^{-3M/2} m^2}{4\gamma^2}.$$

QED

We now recall the main regularity property of the map $\bar{h}$:

Lemma 2.4 *Let $R > 0$ be some large constant and $\gamma > 0$ be sufficiently small such that S_γ defined by (14) has a $\mathcal{C}^2$ boundary. There is a constant $\theta > 1/\gamma$ such that, for any compact set K with $\mathcal{C}^{1,1}$ boundary such that $S_\gamma \subset \mathrm{int}(K)$ and $K \subset B(0, R-\gamma)$, for any $v \in \mathbb{R}^N$ with $|v| < 1/\theta$ and any $x \in \partial K$, we have*

$$\bar{h}(x+v, K+v) \geq (1-\theta|v|)^2 \bar{h}(x, K). \tag{16}$$

For the proof, see [7, Proposition 2.4].

3 Interposition theorems

This part is devoted to interposition theorems in space and in space-time. Such results are fondamental in the proof of the inclusion principle. They play the same role as Jensen's maximum principle (see [19]) or Ishii's lemma (see [11, Theorem 8.3]) in the standard theory of viscosity solutions.

3.1 An interposition theorem in $\mathbb{R}^N$

Let us start with an interposition result for subsets of $\mathbb{R}^N$. The following proposition is a direct consequence of Ilmanen interposition lemma [18] and can be found in [7, Proposition 3.7].

Proposition 3.1 (Interposition) *Let K_1 and K_2 be two closed subsets of $\mathbb{R}^N$, with K_1 compact and such that $K_1 \subset\subset K_2$. Let $y_1 \in K_1$ and $y_2 \in \partial K_2$ be such that*

$$|y_1 - y_2| = \min_{z_1 \in K_1, z_2 \in \partial K_2} |z_1 - z_2| \,.$$

Then there is some open subset Σ_1 of $\mathbb{R}^N$ with a $\mathcal{C}^{1,1}$ boundary, such that Σ_1 is externally tangent to K_1 at y_1 (i.e., $K_1 \subset \overline{\Sigma_1}$ and $y_1 \in \partial\Sigma_1$) and such that $\Sigma_2 := \Sigma_1 + y_2 - y_1$ is internally tangent to K_2 at y_2 (i.e., $\Sigma_2 \subset K_2$ and $y_2 \in \partial\Sigma_2$).

See Figure 1 for an illustration of this proposition. The key point in this result is that the smooth set Σ_2 internally tangent to K_2 is just a translation of the smooth set Σ_1 externally tangent to K_1.

The $\mathcal{C}^{1,1}$ regularity of the sets Σ_1 and Σ_2 turns out to be optimal: one cannot expect Σ_1 and Σ_2 to be $\mathcal{C}^2$ in general. Unfortunately the $\mathcal{C}^2$ regularity will be required in the sequel to be able to deal with curvature terms. In order to overcome this difficulty, one can approximate the sets Σ_1 and Σ_2 in the following way:

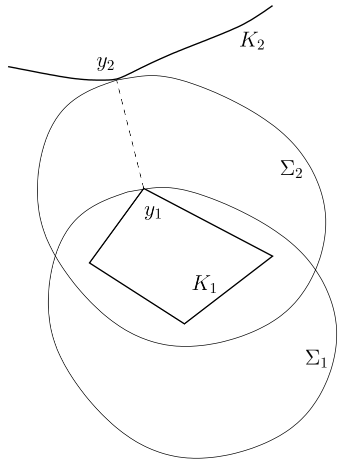

Figure 1: *Illustration of the result of Proposition 3.1.*

Theorem 3.2 (Approximation) *Let K_1, K_2, y_1, y_2, Σ_1 and Σ_2 be as in Proposition 3.1 and $\delta > 0$ be sufficiently small. Then there exists $\Sigma_{1,n}$ and $\Sigma_{2,n}$ open subsets of $\mathbb{R}^N$ with $\mathcal{C}^{1,1}$ boundary, converging respectively to Σ_1 and Σ_2 in the $\mathcal{C}^{1,\mathrm{b}}$ sense, there exists $y_{1,n} \in K_1$ and $y_{2,n} \in \partial K_2$ converging respectively to y_1 and y_2, and there exists $(N-1) \times (N-1)$ matrices X_1, X_2 such that*

(i) $\Sigma_{1,n}$ is externally tangent to K_1 at $y_{1,n}$ and $\Sigma_{2,n}$ is internally tangent to K_2 at $y_{2,n}$.

(ii) For $i = 1$ and 2, $\Sigma_{i,n}$ is of class $\mathcal{C}^2$ in a neighbourhood of $y_{i,n}$ with $\lim_n H^{\Sigma_{i,n}}_{y_{i,n}} \to X_i$, and

$$-\frac{1}{\delta} I_{2(N-1)} \leq \begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq \frac{1}{\delta} \begin{pmatrix} I_{N-1} & -I_{N-1} \\ -I_{N-1} & I_{N-1} \end{pmatrix} . \qquad (17)$$

Remark 3.1
1. Note carefully that the two approximations are not independent because of the inequalities (17), which implies in particular that $X_1 \leq X_2$.
2. By "$\delta > 0$ sufficiently small", we mean $\delta \in (0, |y_1 - y_2|/(2 + |y_1 - y_2|))$.

The proof of Theorem 3.2 is very similar to the (more difficult) proof of the second part of Theorem 3.3 below, so we omit it.

3.2 Interposition by regular tubes

The aim of this part is to extend previous results for subsets $\Sigma_1, \Sigma_2 \subset \mathbb{R} \times \mathbb{R}^N$ which are regular tubes. The point here is to be able to construct tubes satisfying the regularity assumption (12).

For this we introduce some notations. In $\mathbb{R} \times \mathbb{R}^N$ we work with the norm (where $\sigma > 0$ is fixed)

$$|(t,x)|_\sigma = \left(\frac{1}{\sigma^2} t^2 + |x|^2\right)^{\frac{1}{2}} .$$

For any subset E of $\mathbb{R}^{N+1}$, we note the distance to E for this norm

$$d_E^\sigma(t,x) = \inf_{(s,y)\in E} |(s,y) - (t,x)|_\sigma .$$

For any two subsets A_1 and A_2 of $\mathbb{R}^{N+1}$, we define the minimal distance between A_1 and A_2 by

$$e(A_1, A_2) = \inf_{(t_1,x_1)\in A_1,\ (t_2,x_2)\in A_2} |(t_2,x_2) - (t_1,x_1)|_\sigma .$$

We consider the following transversality condition:

$$\begin{array}{c} \text{for } C_1 \subset\subset C_2 \subset \mathbb{R} \times \mathbb{R}^N \text{ with } C_1 \text{ compact and } C_2 \text{ closed,} \\ \text{and for any } (\bar{s}_1, \bar{y}_1) \in C_1 \text{ and any } (\bar{s}_2, \bar{y}_2) \in \widehat{C_2}, \\ \text{if } |(\bar{s}_1, \bar{y}_1) - (\bar{s}_2, \bar{y}_2)|_\sigma = e(C_1, \widehat{C_2}), \text{ then } \bar{s}_1 > 0, \bar{s}_2 > 0 \text{ and } \bar{y}_1 \neq \bar{y}_2. \end{array} \tag{18}$$

Theorem 3.3 *Let C_1 and C_2 be such that (18) holds. Let us fix $(\bar{s}_1, \bar{y}_1) \in C_1$ and $(\bar{s}_2, \bar{y}_2) \in \widehat{C_2}$ with*

$$|(\bar{s}_1, \bar{y}_1) - (\bar{s}_2, \bar{y}_2)|_\sigma = e(C_1, \widehat{C_2}) .$$

1. **Interposition:** *There exists a $\mathcal{C}^{1,1}$ regular tube Σ_1, defined on an open interval I (see Remark 2.1.2), such that Σ_1 is externally tangent to C_1 at $(\bar{s}_1, \bar{y}_1)$, with $\bar{s}_1 \in I$, and $\Sigma_2 := \Sigma_1 + (\bar{s}_2, \bar{y}_2) - (\bar{s}_1, \bar{y}_1)$ is internally tangent to C_2 at $(\bar{s}_2, \bar{y}_2)$.*

2. **Joint approximation by $\mathcal{C}^2$ tubes:** *Futhermore, for any $\delta > 0$ sufficiently small, there exists $\mathcal{C}^{1,1}$ regular tubes $\Sigma_{1,n}$ and $\Sigma_{2,n}$ converging respectively to Σ_1 and Σ_2 in the $\mathcal{C}^{1,\mathrm{b}}$ sense, there exists $(\bar{s}_{1,n}, \bar{y}_{1,n}) \in C_1$ and $(\bar{s}_{2,n}, \bar{y}_{2,n}) \in \widehat{C_2}$ converging respectively to $(\bar{s}_1, \bar{y}_1)$ and $(\bar{s}_2, \bar{y}_2)$, and there exists $(N-1) \times (N-1)$ matrices X_1, X_2 such that*

(i) $\Sigma_{1,n}$ *is externally tangent to* C_1 *at* $(\bar{s}_{1,n}, \bar{y}_{1,n})$ *and* $\Sigma_{2,n}$ *is internally tangent to* C_2 *at* $(\bar{s}_{2,n}, \bar{y}_{2,n})$.

(ii) For $i = 1$ *and* 2, $\Sigma_{i,n}$ *is of class* $\mathcal{C}^2$ *in a neighbourhood of* $(\bar{s}_{i,n}, \bar{y}_{i,n})$ *with* $\lim_n H^{\Sigma_{i,n}(\bar{s}_{i,n})}_{\bar{y}_{i,n}} \to X_i$ *and*

$$-\frac{1}{\delta} I_{2(N-1)} \leq \begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq \frac{1}{\delta} \begin{pmatrix} I_{N-1} & -I_{N-1} \\ -I_{N-1} & I_{N-1} \end{pmatrix} . \tag{19}$$

The proof of this theorem is done in Section 3.5.

Remark 3.2
1. Inequality (19) implies that $X_1 \leq X_2$. Although we only use this latter inequality in the sequel, inequality (19) allows to treat equations with F depending on x (see for instance [11]). Let us once again point out that the two approximations are not independent because of (19).
2. Thanks to the $\mathcal{C}^{1,\mathrm{b}}$ convergence of $\Sigma_{1,n}$ and $\Sigma_{2,n}$ to Σ_1 and Σ_2 respectively, one also has:

$$\lim_n \nu^{\Sigma_{1,n}(\bar{s}_{1,n})}_{\bar{y}_{1,n}} = \lim_n \nu^{\Sigma_{2,n}(\bar{s}_{2,n})}_{\bar{y}_{2,n}} = \nu^{\Sigma_1(\bar{s}_1)}_{\bar{y}_1} = \nu^{\Sigma_2(\bar{s}_2)}_{\bar{y}_2}, \tag{20}$$

$$\lim_n V^{\Sigma_{1,n}}_{(\bar{s}_{1,n}, \bar{y}_{1,n})} = \lim_n V^{\Sigma_{2,n}}_{(\bar{s}_{2,n}, \bar{y}_{2,n})} = V^{\Sigma_1}_{(\bar{s}_1, \bar{y}_1)} = V^{\Sigma_2}_{(\bar{s}_2, \bar{y}_2)} . \tag{21}$$

3. By "$\delta > 0$ sufficiently small, we mean: $\delta \in (0, e(C_1, \widehat{C_2})/(2 + e(C_1, \widehat{C_2})))$.

3.3 Existence of the regular interposition tubes

Let us introduce a new notation: if Σ is a tube defined on some open interval I (see Remark 2.1.2), then we set

$$\mathrm{bd}(\Sigma) := \bigcup_{t \in I} \partial \overline{\Sigma}(t) . \tag{22}$$

The following result is the key point in the proof of the existence of the regular interposition tubes of Theorem 3.3 part (i).

Proposition 3.4 *Let* C_1, C_2, $(\bar{s}_1, \bar{y}_1) \in C_1$, $(\bar{s}_2, \bar{y}_2) \in C_2$ *be as in Theorem 3.3. There exist a* $\mathcal{C}^{1,1}$ *regular tube* Σ *defined on some interval* I *and some* $(t, x) \in](\bar{s}_1, \bar{y}_1), (\bar{s}_2, \bar{y}_2)[$ *such that*

$$t \in I, \quad x \in \partial \Sigma(t) \quad \text{and} \quad e(C_1\ ,\ \widehat{C_2}) = e(C_1\ ,\ \mathrm{bd}(\Sigma)) + e(\Sigma\ ,\ \widehat{C_2}) . \tag{23}$$

Above, $](\bar{s}_1, \bar{y}_1), (\bar{s}_2, \bar{y}_2)[$ denotes the open segment joining $(\bar{s}_1, \bar{y}_1)$ and $(\bar{s}_2, \bar{y}_2)$.

Proof of Proposition 3.4. Let us first fix some notation needed throughout the proof: we set

- $\bar{e} := e(C_1\ ,\ \widehat{C_2})$,
- $E := \{(t,x) \in I\!\!R^+ \times I\!\!R^N : (t,x) \in](s_1,y_1),(s_2,y_2)[$ where $(s_1,y_1) \in C_1$ and $(s_2,y_2) \in \widehat{C_2}$ satisfy $|(s_1,y_1)-(s_2,y_2)|_\sigma = \bar{e}\}$,
- $A_\rho := \{(t,x) \in I\!\!R^+ \times I\!\!R^N\ :\ d^\sigma_{C_1}(t,x) > \rho$ and $d^\sigma_{\widehat{C_2}}(t,x) > \rho\}$ for $\rho \in (0, \bar{e}/2)$, and, if I is an interval, then $A_\rho(I) = A_\rho \cap (I \times I\!\!R^N)$,
- $(t(s), x(s)) := s(\bar{s}_1, \bar{y}_1) + (1-s)(\bar{s}_2, \bar{y}_2)$ for all $s \in (0,1)$ and
$$(\bar{t}, \bar{x}) := (t(1/2), x(1/2)), \tag{24}$$
- $I_\tau := (\bar{t} - \tau, \bar{t} + \tau)$ for all $\tau > 0$.

For later use we note that

$$d^\sigma_{\widehat{C_2}}(t(s),x(s)) = s\bar{e} \quad \text{and} \quad d^\sigma_{C_1}(t(s),x(s)) = (1-s)\bar{e} \quad \text{for all } s \in (0,1)\ , \tag{25}$$

because of the definition of $(\bar{s}_1, \bar{y}_1)$ and $(\bar{s}_2, \bar{y}_2)$. Moreover, for a point $(t,x) \in I\!\!R^+ \times I\!\!R^N$, the equality $\bar{e} - d^\sigma_{\widehat{C_2}}(t,x) = d^\sigma_{C_1}(t,x)$ holds if and only if $(t,x) \in \overline{E}$.

We now reduce the construction of the tube Σ to the construction of a suitable function w:

Lemma 3.5 *Let I be a nonempty open interval of $I\!\!R^+$ and $w : A_\rho(I) \to I\!\!R$ be of class $\mathcal{C}^{1,1}$ (for some $\rho \in (0, \bar{e}/2)$) and such that*

$$\bar{e} - d^\sigma_{\widehat{C_2}}(s,y) \le w(s,y) \le d^\sigma_{C_1}(s,y) \qquad \forall (s,y) \in A_\rho(I)\ . \tag{26}$$

We also assume that there is some $\gamma \in (\rho, \bar{e} - \rho)$ and some $(t,x) \in E \cap A_\rho(I)$ with $w(t,x) = \gamma$ and such that

$$D_x w(s,y) \neq 0 \qquad \forall (s,y) \in A_\rho(I) \quad \text{with} \quad w(s,y) = \gamma\ . \tag{27}$$

Then the set $\Sigma = \{(s,y) \in A_\rho(I) \mid w(s,y) \le \gamma\}$ satisfies the requirements of Proposition 3.4.

Proof of Lemma 3.5. Let us first check that Σ is a tube of class $\mathcal{C}^{1,1}$ in the intervall I. Because of assumption (27) it suffices to show that $\mathrm{bd}(\Sigma) \subset A_\rho(I)$ where $\mathrm{bd}(\Sigma)$ is defined by (22). Using (26) and the fact that $\gamma \in (\rho, \bar{e} - \rho)$, we have, for all $(s, y) \in \mathrm{bd}(\Sigma)$,

$$d^\sigma_{\widehat{C_2}}(s, y) \geq \bar{e} - \gamma > \rho \quad \text{and} \quad d^\sigma_{C_1}(s, y) \geq \gamma > \rho \; .$$

Hence $(s, y) \in A_\rho(I)$.

Finally we show that (23) holds. If $w(s, y) = \gamma$, then $d^\sigma_{C_1}(s, y) \geq \gamma$ while $d^\sigma_{\widehat{C_2}}(s, y) \geq \bar{e} - \gamma$. Hence $e(C_1, \mathrm{bd}(\Sigma)) \geq \gamma$, $e(\widehat{C_2}, \Sigma) \geq \bar{e} - \gamma$ and

$$e(C_1, \mathrm{bd}(\Sigma)) + e(\widehat{C_2}, \Sigma) \geq e(C_1, \widehat{C_2}) \; .$$

For the reverse inequality let us first recall that $\bar{e} - d^\sigma_{\widehat{C_2}}(t, x) = d^\sigma_{C_1}(t, x)$ because $(t, x) \in E$. This implies that $\gamma = d^\sigma_{C_1}(t, x) \geq e(C_1, \mathrm{bd}(\Sigma))$ and $\bar{e} - \gamma = d^\sigma_{\widehat{C_2}}(t, x) \geq e(\widehat{C_2}, \Sigma)$. Hence (23) holds.

QED

Next we turn to the construction of a function w satisfying the assumptions of Lemma 3.5. We advice the reader to look at Figure 2 to follow the rest of the proof of Proposition 3.4.

The first step is the following result given in [9]: let us set

$$K_1 := \{(t, x) \in I\!R^+ \times I\!R^N : d^\sigma_{C_1}(t, x) \leq 15\bar{e}/16\} \; .$$

Then, we have

Lemma 3.6 *[9, Lemma 5.1] The function $\varphi(t, x) = d^\sigma_{\partial K_1}(t, x)$ is $\mathcal{C}^{1,1}$ in a bounded open neighborhood $\mathcal{O}_1$ of $E \cap (K_1 \backslash \partial K_1)$ and*

$$\bar{e} - d^\sigma_{\widehat{C_2}}(t, x) \leq \frac{15\bar{e}}{16} - \varphi(t, x) \leq d^\sigma_{C_1}(t, x) \quad \forall (t, x) \in I\!R^+ \times I\!R^N \; . \tag{28}$$

We now show that φ has a nonvanishing spatial gradient in a neighborhood of $E \cap \mathcal{O}_1$. Let $(t, x) \in E \cap \mathcal{O}_1$. Then there exists $(s_1, y_1) \in C_1$, $(s_2, y_2) \in \widehat{C_2}$ such that $|(s_1, y_1) - (s_2, y_2)|_\sigma = \bar{e}$ and $(t, x) \in](s_1, y_1), (s_2, y_2)[$. Therefore, $D\varphi(t, x) = ((s_2, y_2) - (s_1, y_1)))/\bar{e}$. From assumption (18), we know that $y_1 \neq y_2$. Thus $D_x\varphi(t, x) \neq 0$ for all $(t, x) \in E \cap \mathcal{O}_1$. By continuity of $D_x\varphi$ in $\mathcal{O}_1$, there is an open set $\mathcal{O}_2 \subset \mathcal{O}_1$ which contains $E \cap A_{\bar{e}/8}$ and such that

$$\eta := \min_{(t,x) \in \overline{\mathcal{O}_2}} |D_x\varphi(t, x)| > 0. \tag{29}$$

We are now going to modify φ far away from E. For this we need a technical lemma:

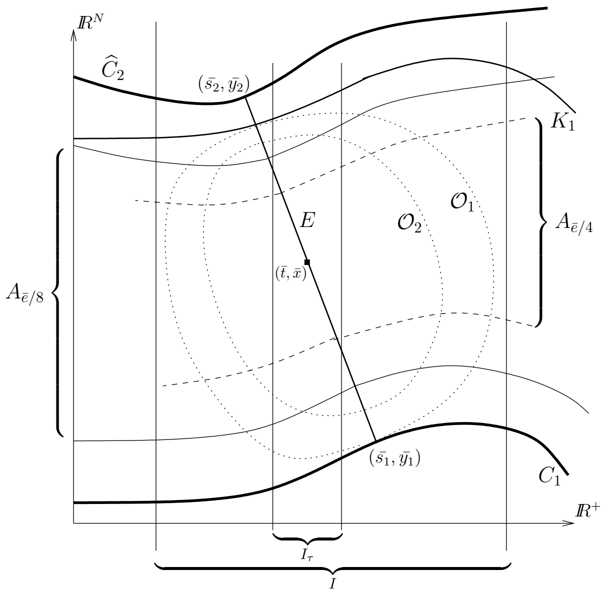

Figure 2: *Illustration of the proof of Proposition 3.4.*

Lemma 3.7 *Let f, g, h be continuous functions in $\mathbb{R}^k$ such that $g \leq f \leq h$ in $\mathbb{R}^k$. Suppose that $K := \{h = g\}$ is non empty and compact and that there is some open neighbourhood U of K such that f is $\mathcal{C}^{1,1}$ in U.*

Then, for any $\tilde{\eta} > 0$ and for any open subset U' such that $K \subset U' \subset\subset U$, there is a function $\psi : \mathbb{R}^k \to \mathbb{R}$ such that:

(i) ψ *is* $\mathcal{C}^{1,1}_{loc}$ *in* $\mathbb{R}^k$ *and* ψ *is* $\mathcal{C}^\infty$ *in* $\mathbb{R}^k \backslash U'$,

(ii) $g \leq \psi \leq h$ *in* $\mathbb{R}^k$ *and* $g < \psi < h$ *in* $\mathbb{R}^k \backslash U'$,

(iii) $|D(\psi - f)| \leq \tilde{\eta}$ *in* $\overline{U'}$.

Proof of Lemma 3.7. Let U_1 and U_2 be two open subsets of $\mathbb{R}^k$ such that $K \subset U_2 \subset\subset U_1 \subset\subset U'$ and fix some smooth map $\theta : \mathbb{R}^k \to [0, 1]$ such that $\theta = 1$ in $\overline{U_2}$ and $\theta = 0$ in $\mathbb{R}^k \backslash U_1$. Then we consider a smooth map $\xi : \mathbb{R}^k \to \mathbb{R}$ such that $g < \xi < h$ in $\mathbb{R}^k \backslash U_2$,

$$\|\xi - f\|_{L^\infty(U' \backslash U_2)} \leq \frac{\tilde{\eta}}{2\|D\theta\|_\infty} \quad \text{and} \quad \|D\xi - Df\|_{L^\infty(U' \backslash U_2)} \leq \frac{\tilde{\eta}}{2} \, .$$

The construction of such a function ξ is possible because $g < f < h$ outside of K and f is $\mathcal{C}^{1,1}$ in U. Then we set

$$\psi(x) = \theta(x) f(x) + (1 - \theta(x))\xi(x) \qquad \forall x \in \mathbb{R}^k \, .$$

Note first that (i) and (ii) obviously hold. As for (iii), it clearly holds in U_2 since $\psi = f$ in U_2. Moreover, for $x \in U' \backslash U_2$, we have

$$|D(\psi - f)(x)| \leq |D(\xi - f)(x)| + |f(x) - \xi(x)||D\theta(x)| \leq \frac{\tilde{\eta}}{2} + \frac{\tilde{\eta}|D\theta(x)|}{2\|D\theta\|_\infty} \leq \tilde{\eta} .$$

QED

Next, we apply Lemma 3.7 with $k := N+1$, $U := \mathcal{O}_1$, $U' := \mathcal{O}_2$, $\tilde{\eta} := \eta/2$ where η is given by (29),

$$g(t,x) := \bar{e} - d^\sigma_{\widehat{C_2}}(t,x) - d^\sigma_{A_{\bar{e}/8}}(t,x), \qquad h(t,x) := d^\sigma_{C_1}(t,x) + d^\sigma_{A_{\bar{e}/8}}(t,x)$$

and $f(t,x) := 15\bar{e}/16 - \varphi(t,x)$ for $(t,x) \in I\!\!R^+ \times I\!\!R^N$. We extend f, g and h for $t \leq 0$ by setting $f(t,x) = f(0,x)$, $g(t,x) = g(0,x)$ and $h(t,x) = h(0,x)$. From Lemma 3.6 we have $g \leq f \leq h$ in $I\!\!R^{N+1}$. Moreover, from assumption (18), the set $K := \{g = h\} = E \cap \overline{A_{\bar{e}/8}}$ is compact and contained in $(0,+\infty) \times I\!\!R^N$ and we know from Lemma 3.6 that f is $\mathcal{C}^{1,1}$ in the neighbourhood $\mathcal{O}_1$ of K. Lemma 3.7 states that there is a map $\psi : I\!\!R^{N+1} \to I\!\!R$ such that

$$\left\{ \begin{array}{l} \text{(i) } \psi \text{ is } \mathcal{C}^{1,1}_{loc} \text{ in } I\!\!R^{N+1} \text{ and } \mathcal{C}^\infty \text{ in } I\!\!R^{N+1} \backslash \mathcal{O}_2, \\ \text{(ii) } \bar{e} - d^\sigma_{\widehat{C_2}} \leq \psi \leq d^\sigma_{C_1} \text{ in } A_{\bar{e}/8}, \\ \text{(iii) } \bar{e} - d^\sigma_{\widehat{C_2}} < \psi < d^\sigma_{C_1} \text{ in } A_{\bar{e}/8} \backslash \mathcal{O}_2, \\ \text{(iv) } |D(\psi - f)| \leq \eta/2 \text{ in } \overline{\mathcal{O}_2}. \end{array} \right. \tag{30}$$

Putting together (29) and (iv) implies that

$$|D_x\psi(t,x)| \geq |D_x f(t,x)| - |D_x(\psi - f)(t,x)| \geq \frac{\eta}{2} \quad \forall (t,x) \in \overline{\mathcal{O}_2} .$$

We now choose two open subsets U_1 and U_2 of $I\!\!R^N$ and some $\tau > 0$ such that $\mathcal{O}_2(\bar{t}) \subset\subset U_2 \subset\subset U_1$ (recall that $\bar{t}$ is defined by (24)) and

$$|D_x\psi(t,x)| \geq \frac{\eta}{4} \qquad \forall (t,x) \in I_\tau \times \overline{U_1} . \tag{31}$$

We also fix some smooth function $\theta : I\!\!R^N \to [0,1]$ such that $\theta = 1$ in U_2, $\theta = 0$ in $I\!\!R^N \backslash U_1$ and we set

$$w(t,x) = \theta(x)\psi(t,x) + (1 - \theta(x))\psi(\bar{t},x) \qquad \forall (t,x) \in I\!\!R^{N+1} .$$

Note that w belongs to $\mathcal{C}^{1,1}_{loc}(I\!\!R^{N+1}) \cap \mathcal{C}^\infty(I\!\!R \times (I\!\!R^N \backslash U_1))$. We claim that we can choose $\tau > 0$ sufficiently small such that

$$\left\{ \begin{array}{l} \text{(i) } \bar{e} - d^\sigma_{\widehat{C_2}} \leq w \leq d^\sigma_{C_1} \text{ in } A_{\bar{e}/4}(I_\tau), \\ \text{(ii) } |D_x w| \geq \eta/8 \text{ in } I_\tau \times \overline{U_1}. \end{array} \right. \tag{32}$$

Let us prove the first assertion. Let $(t,x) \in A_{\bar{e}/4}(I_\tau)$. On the one hand, if $(t,x) \in A_{\bar{e}/4}(I_\tau) \cap (I_\tau \times U_2)$, then $\theta(x) = 1$ and $w(t,x) = \psi(t,x)$. Since $A_{\bar{e}/4}(I_\tau) \subset A_{\bar{e}/8}$, we conclude from (30(ii)). On the other hand, suppose $(t,x) \in A_{\bar{e}/4}(I_\tau) \backslash (I_\tau \times U_2)$. In particular, $(t,x) \notin \mathcal{O}_2(\bar{t})$ and $(\bar{t},x) \notin \mathcal{O}_2$. Therefore, from (30(iii)), we have

$$\bar{e} - d^{\sigma}_{\widehat{C_2}}(\bar{t},x) \ < \ \psi(\bar{t},x) \ < \ d^{\sigma}_{C_1}(\bar{t},x).$$

Using the uniform continuity of the above distance functions in the compact set $\overline{A_{\bar{e}/4}(I_\tau)}$, we obtain that, for τ small enough,

$$\bar{e} - d^{\sigma}_{\widehat{C_2}}(t,x) \ < \ \psi(\bar{t},x) \ < \ d^{\sigma}_{C_1}(t,x) \quad \forall (t,x) \in A_{\bar{e}/4}(I_\tau) \backslash (I_\tau \times U_2) \ .$$

Combining with (30(ii)), we conclude also in this case.

For the second assertion, we notice that, for $(t,x) \in I_\tau \times \overline{U_1}$, we have

$$|D_x(w - \psi)(t,x)| \leq |1 - \theta(x)||D_x(\psi(t,x) - \psi(\bar{t},x))| + |\psi(t,x) - \psi(\bar{t},x)||D\theta(x)|$$

with a right-handside smaller than $\eta/8$ provided τ is sufficiently small, because $\psi \in \mathcal{C}^{1,1}_{loc}$. Then, for $(t,x) \in I_\tau \times \overline{U_1}$, we have, from the choice of U_1 and τ in (31),

$$|D_x w(t,x)| \geq |D_x \psi(t,x)| - |D_x(w - \psi)(t,x)| \geq \eta/8 \ ,$$

which proves the second statement.

We now fix $\sigma \in (0, \bar{e}/4)$ such that

$$(t(s), x(s)) \in I_\tau \times \mathcal{O}_2(\bar{t}) \qquad \forall s \in (1/2 - \sigma, 1/2 + \sigma) \ .$$

This is possible because $(\bar{t},\bar{x}) = (t(1/2), x(1/2))$ belongs to $\mathcal{O}_2(\bar{t})$. From (25), an easy calculation gives $w(t(s), x(s)) = (1 - s)\bar{e}$. Since w is smooth in $\mathbb{R} \times (\mathbb{R}^N \backslash \overline{U_1})$, Sard Lemma states that we can find a level $\gamma \in ((1/2 - \sigma)\bar{e}, (1/2 + \sigma)\bar{e})$ such that γ is a non critical value of w in $\mathbb{R} \times (\mathbb{R}^N \backslash U_1)$. We claim that w and γ satisfy the requirements of Lemma 3.5. Note first that, for $s = (\bar{e} - \gamma)/\bar{e}$, the point $(t(s), x(s))$ belongs to $E \cap A_{\bar{e}/4}(I_\tau)$ and satisfies $w(t(s), x(s)) = \gamma$. Moreover, (26) holds from (32(i)). Finally we show that (27) holds. Indeed, if $w(t,x) = \gamma$ for some $(t,x) \in A_{\bar{e}/4}(I_\tau)$, then either $x \in \overline{U_1}$, in which case $D_x w(t,x) \neq 0$ thanks to (32(ii)), or $x \notin \overline{U_1}$ and $Dw(t,x) = (0, D_x w(,tx)) \neq 0$ because γ is a non critical value of w in $I_\tau \times (\mathbb{R}^N \backslash \overline{U_1})$. In each case we have $D_x w(t,x) \neq 0$. This completes the proof of Proposition 3.4.

QED

3.4 $\mathcal{C}^2$ regularization of a $\mathcal{C}^{1,1}$ tangent surface near contact points

The aim of this section is to show the following fact: if a $\mathcal{C}^{1,1}$ surface Σ is externally tangent to a set K at a point y, then it is possible to find a $\mathcal{C}^{1,1}$ surface $\tilde{\Sigma}$ which is close to Σ (in the $\mathcal{C}^{1,\mathrm{b}}$ sense) and is still externally tangent to K at a point $\tilde{y}$ close to y. Moreover, $\tilde{\Sigma}$ is more regular than Σ, namely is $\mathcal{C}^2$ in a neighborhood of $\tilde{y}$. In particular, we can use $\tilde{\Sigma}$ as a test set to estimate the curvature (see Remark 2.2).

Let us give the exact assumptions:

Let K be a subset of $\mathbb{R}^k$ for $k \geq 1$ and Σ be an open set with a $\mathcal{C}^{1,1}$ boundary $\partial\Sigma$, which is externally tangent to K at some point $y \in \partial K$. Let $x \notin K$ be such that y is the unique projection of x onto K and $p := Dd_K(x)$ is the outward normal to Σ at y. Suppose that there is a sequence of points $x_n \to x$, where d_K is twice differentiable with first and second derivative denoted respectively p_n and X_n, and finally assume that p_n converges to p while X_n converges to some X. (33)

Note that, by usual properties of the distance function at differentiability points, the projection of x_n onto K is unique and converges to y. We denote by y_n this projection.

Proposition 3.8 *Under Assumption (33) we can find a sequence of open sets Σ_n with $\mathcal{C}^{1,1}$ boundary such that*

(i) Σ_n is externally tangent to K at y_n,

(ii) Σ_n has a $\mathcal{C}^2$ boundary in a neighbourhood of y_n, with normal p_n and curvature equal to the restriction to $(p_n)^\perp$ of $-(X_n - \frac{1}{n}I_k)$,

(iii) Σ_n converges to Σ in the $\mathcal{C}^{1,\mathrm{b}}$ sense.

Before starting the proof of the proposition, we need two lemmas. The first one builds, from the derivatives of the distance function at a point a, a map ϕ which has a local maximum on K at the point b, projection of a onto K:

Lemma 3.9 *Suppose that $a \notin K$ and that d_K is twice differentiable at a. Let b be the projection of a onto K. Then, for any $\alpha > 0$, the (smooth) function*

$$\phi(z) = \langle Dd_K(a), z - b\rangle + \frac{1}{2}\langle (D^2 d_K(a) - \alpha I_k)(z - b), z - b\rangle$$

has a strict local maximum at b on K.

Proof of Lemma 3.9. For any $z \in K$, we have

$$\phi(z) = d_K(z+a-b) - \frac{\alpha}{2}|z-b|^2 - d_K(a) - |z-b|^2\epsilon(z-b)$$

because d_K is has a second order Taylor expansion at a. But, since $z \in K$, we have $d_K(z+a-b) \leq |(z+a-b)-z| = |a-b| = d_K(a)$. Therefore

$$\phi(z) \leq -\frac{\alpha}{2}|z-b|^2 - |z-b|^2\epsilon(z-b)$$

which is negative as soon as $z \in K$ is sufficiently close to b and $z \neq b$.

QED

From now on, we fix a smooth function $\theta : I\!R^+ \to [0,1]$ such that θ is nonincreasing, $\theta = 1$ on $[0,1/2]$ and $\theta = 0$ on $[1,+\infty)$.

We will use several times below the following interpolation. The proof relies on straightforward computations, so we skip it.

Lemma 3.10 *Let ϕ and ψ some $\mathcal{C}^{1,1}$ functions in some open set $\mathcal{O}$. Let $\bar{y} \in \mathcal{O}$ be such that $\phi(\bar{y}) = \psi(\bar{y})$, and let us set, for any $\rho > 0$,*

$$\xi_\rho(z) = \phi(z)\theta_\rho(z) + \psi(z)(1-\theta_\rho(z)) \qquad \text{where } \theta_\rho(z) = \theta\left(\frac{|z-\bar{y}|^2}{\rho^2}\right).$$

Then, for any $\rho > 0$ such that $B(\bar{y},\rho) \subset\subset \mathcal{O}$, we have

$$\|\xi_\rho - \psi\|_\infty \leq C(\eta\rho + (M_1+M_2)\rho^2)\ ,\ \|D\xi_\rho - D\psi\|_\infty \leq C(\eta + (M_1+M_2)\rho)$$

and

$$\|D^2\xi_\rho\|_\infty \leq \frac{C}{\rho}(\eta + (M_1+M_2)\rho)$$

for some constant $C = C(k) > 0$, where we have set $\eta = |D\phi(\bar{y}) - D\psi(\bar{y})|$, $M_1 = \|D^2\phi\|_\infty$ and $M_2 = \|D^2\psi\|_\infty$.

Remark 3.3
1. The norms $\|\cdot\|_\infty$ are of course taken on $B(\bar{y},\rho)$, since $\xi_\rho = \psi$ outside.
2. The key point is that ξ_ρ coincides with ϕ in a small neighbourhood of $\bar{y}$, but is not too far from ψ in the full set $\mathcal{O}$ provided ρ and η are small.

We are now ready to prove the proposition.

Proof of Proposition 3.8. From Lemma 3.9 the function ϕ_n defined by

$$\phi_n(z) = \langle Dd_K(x_n), z - y_n\rangle + \frac{1}{2}\langle (D^2 d_K(x_n) - \frac{1}{n} I_k)(z - y_n), z - y_n\rangle$$

has a strict local maximum at y_n on K. Since y_n is the unique projection of x_n onto K, the map $\psi_n(z) = d_K(x_n) - |z - x_n|$ has a global strict maximum on K at y_n. Hence

$$\zeta_n(z) = \phi_n(z)\theta_n(z) + \psi_n(z)(1 - \theta_n(z)) \qquad \text{where } \theta_n(z) = \theta\left(\frac{|z - y_n|^2}{\rho_n^2}\right),$$

has a global strict maximum at y_n on K, provided we choose $\rho_n > 0$, $\rho_n \to 0$ and n large enough. Since ϕ_n is globally smooth with uniformly bounded second order derivative, and since ψ_n is smooth with uniformly bounded second order derivative ouside of the ball $B(x, d_K(x)/2)$, and since finally $\psi_n(y_n) = \phi_n(y_n) = 0$ and $D\psi_n(y_n) = D\psi_n(y_n) = p_n$, Lemma 3.10 states that ζ_n and $D\zeta_n$ uniformly converge to the function $z \to d_K(x) - |z - x|$ and its derivative respectively, in the set $I\!R^k \backslash B(x, d_K(x)/2)$, and that $\|D^2\zeta_n\|_\infty$ is uniformly bounded in $I\!R^k \backslash B(x, d_K(x)/2)$.

Let us now denote by $\mathbf{d}_\Sigma$ the signed distance to Σ (see (5) for a definition). Since $\partial\Sigma$ is a $\mathcal{C}^{1,1}$ manifold, we can find some open neighbourhood $\mathcal{O}$ of $\partial\Sigma$ such that $\mathbf{d}_\Sigma$ is $\mathcal{C}^{1,1}$ in $\mathcal{O}$, with $\|D^2\mathbf{d}_\Sigma\|_\infty$ bounded in $\mathcal{O}$. For $z \in I\!R^k$, we define

$$d^n(z) = \mathbf{d}_\Sigma(z) - \beta_n |y - z|^2 \quad \text{where } \beta_n > 0,\ \beta_n \to 0.$$

Notice that d^n is $\mathcal{C}^{1,1}$ in $\mathcal{O}$, that d^n and its derivative converge locally uniformly to $\mathbf{d}_\Sigma$ whereas $\|D^2 d^n\|_\infty$ is bounded in $\mathcal{O}$. The advantage of introducing d^n is that $\{d^n \leq 0\}$ is still externally tangent to K at y with $\partial K \cap \partial\{d^n \leq 0\} = \{y\}$ (instead, Σ can touch K at many points). We claim that, if we choose $\beta_n = 2|y - y_n|^{1/3}$, then, at least for n large,

$$d^n(z) \leq d^n(y_n) \quad \text{for any } z \in K\backslash B(y_n, \beta_n/2). \tag{34}$$

Indeed, for $z \in K\backslash B(y_n, \beta_n/2)$,

$$\begin{aligned} d^n(z) - d^n(y_n) &\leq -\beta_n|y - z|^2 - \mathbf{d}_\Sigma(y_n) + \beta_n|y - y_n|^2 \\ &\leq -\beta_n(\beta_n/2 - |y - y_n|)^2 + |y - y_n| + \beta_n|y - y_n|^2 \\ &\leq -|y - y_n|(1 - 4|y - y_n|^{\frac{2}{3}}), \end{aligned}$$

which is nonpositive for large n since $y_n \to y$.

We introduce the maps

$$\xi_n(z) = \zeta_n(z)\tilde{\theta}_n(z) + [d^n(z) - d^n(y_n)](1 - \tilde{\theta}_n(z)) \quad \text{where } \tilde{\theta}_n(z) = \theta\left(\frac{|z-y_n|^2}{\sigma_n^2}\right)$$

and

$$\sigma_n = \max\{|p_n - Dd^n(y_n)|, \beta_n/\sqrt{2}\} .$$

Let us notice that $\sigma_n \to 0$ because $y_n \to y$ and p_n and $Dd^n(y_n) = D\mathbf{d}_\Sigma(y_n) - 2\beta_n(y_n - y)$ converge both to $p = D\mathbf{d}_\Sigma(y)$ since $D\mathbf{d}_\Sigma$ is continuous at y. We now use Lemma 3.10, with $\zeta_n(y_n) = [d^n(y_n) - d^n(y_n)] = 0$ and $\eta_n := |D\zeta_n(y_n) - Dd^n(y_n)| = |p_n - D\mathbf{d}_\Sigma(y_n) + 2\beta_n(y_n - y)| \to 0$. It states that ξ_n and $D\xi_n$ uniformly converge to $\mathbf{d}_\Sigma$ and $D\mathbf{d}_\Sigma$, respectively. Moreover, since $\eta_n \leq \sigma_n$, the second order derivative of ξ_n is uniformly bounded.

Let us finally prove that the set $\Sigma_n = \{\xi_n < 0\}$ satisfies our requirements. What we already proved on ξ_n shows that Σ_n converges to Σ (in the $\mathcal{C}^{1,\mathrm{b}}$ sense). From its construction, Σ_n is smooth in a neighborhood of y_n, with normal at the point y_n equal to p_n and curvature equal to the restriction to $(p_n)^\perp$ of $-(X_n - \frac{1}{n}I_k)$.

It remains to check that Σ_n is externally tangent to K at y_n. It suffices to prove that $\xi_n(z) \leq 0$ for any $z \in K$, because $\xi_n(y_n) = 0$. Let $z \in K$. If $|z - y_n| \leq \sigma_n/\sqrt{2}$, then $\xi_n(z) = \zeta_n(z) \leq 0$ from the construction of ζ_n. If $|z - y_n| > \sigma_n/\sqrt{2}$, then $|z - y_n| > \beta_n/2$ and thus, from (34), $d^n(z) \leq d^n(y_n)$. Since moreover ζ_n has a global maximum on K at y_n, we finally have

$$\begin{aligned} \xi_n(z) &= \zeta_n(z)\tilde{\theta}_n(z) + [d^n(z) - d^n(y_n)](1 - \tilde{\theta}_n(z)) \\ &\leq \zeta_n(y_n)\tilde{\theta}_n(z) + [d^n(y_n) - d^n(y_n)](1 - \tilde{\theta}_n(z)) = 0 . \end{aligned}$$

In conclusion we have proved that ξ_n has a global maximum on K at y_n, and the proof is complete.

QED

3.5 Proof of Theorem 3.3

The first part of the theorem is an immediate consequence of Proposition 3.4 : we set $\Sigma_1 = \Sigma - (t,x) + (\bar{s}_1, \bar{y}_1)$ and $\Sigma_2 = \Sigma - (t,x) + (\bar{s}_2, \bar{y}_2)$ where Σ and (t,x) are given by Proposition 3.4 and we check that Σ_1 and Σ_2 enjoy the desired properties.

Without loss of generality, we assume that $\delta \in (0, \bar{e}/(2+\bar{e}))$. Let us introduce, for all $(\tau_1, z_1, \tau_2, z_2) \in I\!R^+ \times I\!R^N \times I\!R^+ \times I\!R^N$,

$$f(\tau_1, z_1, \tau_2, z_2) = d^\sigma_{C_1}(\tau_1, z_1) + d^\sigma_{\widehat{C_2}}(\tau_2, z_2) + \frac{1}{2\delta}|(\tau_1, z_1) - (\tau_2, z_2)|^2_\sigma .$$

Then $d^\sigma_{C_1}$, $d^\sigma_{\widehat{C_2}}$ and f are semi-concave functions in $(I\!R^+ \times I\!R^N)^2 \backslash (C_1 \cup \widehat{C_2})^2$. We claim that f has a minimum at $(\bar\tau_1, \bar z_1, \bar\tau_2, \bar z_2)$, where

$$(\bar\tau_1, \bar z_1) = \frac{1}{2}(1+\delta)(\bar s_1, \bar y_1) + \frac{1}{2}(1-\delta)(\bar s_2, \bar y_2)$$

and

$$(\bar\tau_2, \bar z_2) = \frac{1}{2}(1-\delta)(\bar s_1, \bar y_1) + \frac{1}{2}(1+\delta)(\bar s_2, \bar y_2).$$

Indeed, on the one hand, an easy computation shows that $f(\bar\tau_1, \bar z_1, \bar\tau_2, \bar z_2) = \bar e - \delta/2$. On the other hand, for all $(\tau_1, z_1, \tau_2, z_2) \in I\!R^+ \times I\!R^N \times I\!R^+ \times I\!R^N$, we have

$$\begin{aligned} & f(\tau_1, z_1, \tau_2, z_2) \\ = \;& d^\sigma_{C_1}(\tau_1, z_1) + d^\sigma_{\widehat{C_2}}(\tau_2, z_2) + \frac{1}{2\delta}|(\tau_1, z_1) - (\tau_2, z_2)|^2_\sigma \\ = \;& |(\tau_1, z_1) - (t_1, x_1)|_\sigma + |(\tau_2, z_2) - (t_2, x_2)|_\sigma + \frac{1}{2\delta}|(\tau_1, z_1) - (\tau_2, z_2)|^2_\sigma \\ \geq \;& |(t_1, x_1) - (t_2, x_2)|_\sigma - |(\tau_1, z_1) - (\tau_2, z_2)|_\sigma + \frac{1}{2\delta}|(\tau_1, z_1) - (\tau_2, z_2)|^2_\sigma, \end{aligned}$$

where $(t_1, x_1) \in C_1$ and $(t_2, x_2) \in \widehat{C_2}$ are the points for which the distances $d^\sigma_{C_1}(\tau_1, z_1)$ and $d^\sigma_{\widehat{C_2}}(\tau_2, z_2)$ are achieved. It follows that $f(\tau_1, z_1, \tau_2, z_2) \geq \bar e - \delta/2$ since $|(t_1, x_1) - (t_2, x_2)|_\sigma \geq \bar e$ and since, for all $r \geq 0$, $-r + r^2/(2\delta) \geq -\delta/2$. Finally, $(\bar\tau_1, \bar z_1, \bar\tau_2, \bar z_2)$ is a minimum for f.

Since the semi-concave function f has a minimum at $(\bar\tau_1, \bar z_1, \bar\tau_2, \bar z_2)$, Jensen maximum principle [19] (see also [14]) states that one can find a sequence of points $(\bar\tau_{1,n}, \bar z_{1,n}, \bar\tau_{2,n}, \bar z_{2,n})$ which converges to $(\bar\tau_1, \bar z_1, \bar\tau_2, \bar z_2)$ and a non-negative symmetric matrix $\bar A \in \mathcal{S}_{2N+2}$ such that the functions $d^\sigma_{C_1}$, $d^\sigma_{\widehat{C_2}}$ and f are twice differentiable at $(\bar\tau_{1,n}, \bar z_{1,n})$, $(\bar\tau_{2,n}, \bar z_{2,n})$ and $(\bar\tau_{1,n}, \bar z_{1,n}, \bar\tau_{2,n}, \bar z_{2,n})$ respectively and such that

$$Df(\bar\tau_{1,n}, \bar z_{1,n}, \bar\tau_{2,n}, \bar z_{2,n}) \to 0 \quad \text{and} \quad D^2 f(\bar\tau_{1,n}, \bar z_{1,n}, \bar\tau_{2,n}, \bar z_{2,n}) \to \bar A \geq 0. \tag{35}$$

In particular, since $Dd^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n})$ and $Dd^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n})$ exist, the projections of $(\bar\tau_{1,n}, \bar z_{1,n})$ onto C_1 and $\widehat{C_2}$ respectively are unique, and equal to some $(\bar s_{1,n}, \bar y_{1,n})$ and $(\bar s_{2,n}, \bar y_{2,n})$. Note that $(\bar s_1, \bar y_1)$ is the unique projection onto C_1 of $(\bar t, \bar x)$, and therefore $(\bar s_{1,n}, \bar y_{1,n})$ converges to $(\bar s_1, \bar y_1)$. For the same reason, $(\bar s_{2,n}, \bar y_{2,n})$ converges to $(\bar s_2, \bar y_2)$. Since $d^\sigma_{C_1}$ and $d^\sigma_{\widehat{C_2}}$ are semi-concave functions, the matrices $D^2 d^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n})$ and $D^2 d^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n})$ are bounded from above: namely (see for instance [6, Proposition 22.2])

$$D^2_{xx} d^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n}) \le \frac{1}{d^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n})} I_N \tag{36}$$

and

$$D^2_{xx} d^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n}) \le \frac{1}{d^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n})} I_N \ . \tag{37}$$

Using (35), we get

$$\begin{pmatrix} D^2_{xx} d^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n}) + \frac{1}{\delta} I_N & -\frac{1}{\delta} I_N \\ -\frac{1}{\delta} I_N & D^2_{xx} d^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n}) + \frac{1}{\delta} I_N \end{pmatrix} \to A \ge 0, \tag{38}$$

where the matrix $A \in \mathcal{S}_{2N}$ is the restriction to $I\!R^N \times I\!R^N$ of $\bar A$. In particular, if we set $A = \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix}$, then

$$D^2_{xx} d^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n}) + D^2_{xx} d^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n}) + \frac{2}{\delta} I_N \to A_1 + A_2 \ge 0$$

and therefore $D^2_{xx} d^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n})$ and $D^2_{xx} d^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n})$ are in fact bounded. So, after relabelling all the sequences, we can assume that the restriction of $-D^2_{xx} d^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n})$ to $(D_x d^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n}))^\perp$ converges to some matrix X_1 while the restriction of $D^2_{xx} d^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n})$ to $(D_x d^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n}))^\perp$ converges to some X_2. Note that, from (38) we have

$$\begin{pmatrix} -X_1 + \frac{1}{\delta} I_{N-1} & -\frac{1}{\delta} I_{N-1} \\ -\frac{1}{\delta} I_{N-1} & X_2 + \frac{1}{\delta} I_{N-1} \end{pmatrix} \ge 0 \ .$$

Moreover, since

$$d^\sigma_{C_1}(\bar\tau_{1,n}, \bar z_{1,n}) \to d^\sigma_{C_1}(\bar\tau_1, \bar z_1) = \frac{\bar e(1-\delta)}{2}$$

and

$$d^\sigma_{\widehat{C_2}}(\bar\tau_{2,n}, \bar z_{2,n}) \to d^\sigma_{\widehat{C_2}}(\bar\tau_2, \bar z_2) = \frac{\bar e(1-\delta)}{2} \ ,$$

we get from (36, 37)

$$\begin{pmatrix} -X_1 & 0 \\ 0 & X_2 \end{pmatrix} \le \frac{2}{\bar e(1-\delta)} I_{2(N-1)} \le \frac{1}{\delta} I_{2(N-1)}$$

because $\delta < \bar e/(\bar e + 2)$. So we have proved (19).

We now apply Proposition 3.8 to the sets C_1 and Σ_1. Assumption (33) holds since the set Σ_1 is externally tangent to C_1 at $(\bar{s}_1, \bar{y}_1)$. Moreover, the point $(\bar{s}_1, \bar{y}_1)$ is the unique projection of the point $(\bar{\tau}_1, \bar{z}_1)$ onto C_1. The points $(\bar{\tau}_{1,n}, \bar{z}_{1,n})$ are points of twice differentiability of $d^\sigma_{C_1}$, converge to $(\bar{\tau}_1, \bar{z}_1)$ and have a unique projection $(\bar{s}_{1,n}, \bar{y}_{1,n})$ onto C_1. Therefore we can find a sequence of sets $\Sigma_{1,n}$ with $\mathcal{C}^{1,1}$ boundary, such that $\Sigma_{1,n}$ is externally tangent to C_1 at $(\bar{s}_{1,n}, \bar{y}_{1,n})$ and has a $\mathcal{C}^2$ boundary near $(\bar{s}_{1,n}, \bar{y}_{1,n})$. Note that, since Σ_1 is a $\mathcal{C}^{1,1}$ regular tube and since the sets $\Sigma_{1,n}$ converge to Σ_1 in the $\mathcal{C}^{1,\mathrm{b}}$ sense, $\Sigma_{1,n}$ are also $\mathcal{C}^{1,1}$ regular tubes provided n is sufficiently large.

From Proposition 3.8(ii), we also have that the curvature matrix of $\Sigma_{1,n}$ at $(\bar{s}_{1,n}, \bar{y}_{1,n})$ is equal to the restriction of $-(D^2 d^\sigma_{C_1}(\bar{\tau}_{1,n}, \bar{z}_{1,n}) + \frac{1}{n} I_{N+1})$ to the tangent space of $\Sigma_{1,n}$ at $(\bar{s}_{1,n}, \bar{y}_{1,n})$. Let us denote by $X_{1,n}$ the restriction of this curvature matrix to $\mathbb{R}^N$. We notice that $X_{1,n}$ converges to X_1.

In the same way, applying Proposition 3.8 to the complementary of the tube Σ_2 which is externally tangent to $\widehat{C_2}$ at $(\bar{s}_2, \bar{y}_2)$, we can find a sequence of $\mathcal{C}^{1,1}$ tubes $\Sigma_{2,n}$ which are externally tangent to $\widehat{C_2}$ at some points $(\bar{s}_{2,n}, \bar{y}_{2,n})$, such that $\Sigma_{2,n}$ are of class $\mathcal{C}^2$ near $(\bar{s}_{2,n}, \bar{y}_{2,n})$, and such that the curvature matrix $X_{2,n}$ to $\Sigma_{2,n}(\bar{s}_{2,n})$ at $\bar{y}_{2,n}$ converges to X_2.

QED

4 The inclusion principle

4.1 Statement of the main theorem. Existence, uniqueness and stability

Theorem 4.1 (Inclusion principle) *Let $0 \leq \lambda_1 < \lambda_2$ be fixed, $\mathcal{K}_1$ be a subsolution of the FPP (6) with speed h_{λ_1} on the time interval $[0, T)$ for some $T > 0$ and $\mathcal{K}_2$ be a supersolution on $[0, T)$ with speed h_{λ_2}. If*

$$\overline{\mathcal{K}_1}(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset ,$$

then, for all $t \in [0, T)$,

$$\overline{\mathcal{K}_1}(t) \cap \widehat{\mathcal{K}_2}(t) = \emptyset .$$

Before proving Theorem 4.1, we recall some applications of such inclusion principle, omiting the proofs which are easy adaptations of those of [7] and [9].

Concerning the existence of solutions, we have the following

Proposition 4.2 *For any initial position K_0, with $S \subset \mathrm{int}(K_0)$ and K_0 bounded, there is (at least) one solution to the FPP (6) for h_λ.*

Moreover, there is a largest solution and a smallest solution to this problem. The largest solution has a closed graph while the smallest solution has an open graph in $I\!R^+ \times I\!R^N$. The largest solution contains all the subsolutions of the FPP (6) with initial condition K_0, while the smallest solution is contained in any supersolution.

In general, one cannot expect to have a unique solution, i.e., the closure of the minimal solution is not necessarily equal to the maximal solution. However, uniqueness is generic:

Proposition 4.3 *Let $(K_\lambda)_{\lambda>0}$ be a strictly increasing family of initial positions (i.e., $K_{\lambda'} \subset\subset K_\lambda$ for $0 < \lambda' < \lambda$) such that $K_\lambda \in \mathcal{D}$ for all $\lambda > 0$. Then the solution of the FPP (6) for h_λ with initial position K_λ is unique but for a countable subset of the λ's.*

Stability of solutions is expressed by means of Kuratowski upperlimit of sets. Let us recall that, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets in $I\!R^M$, then the Kuratowski upperlimit $A^* = \mathrm{Lim\,sup}_n A_n$ is the subset of all accumulation points of somes sequences of points in $(A_n)_{n\in\mathbb{N}}$, namely

$$\begin{aligned} A^* \quad := \quad & \{z \in I\!R^M : \exists (n_k)_{k\in\mathbb{N}} \text{ increasing sequence of integers}, \exists (z_k)_{k\in\mathbb{N}}, \\ & \qquad z_k \in A_{n_k} \text{ and } z = \lim_k y_k\}. \end{aligned} \tag{39}$$

We define A_* as the complementary of $\mathrm{Lim\,sup}_n \widehat{A_n}$.

Proposition 4.4 *If $\mathcal{K}_n$ is a sequence of subsolutions for h_λ, locally uniformly bounded with respect to t, then the Kuratowski upperlimit $\mathcal{K}^* = \mathrm{Lim\,sup}_n \mathcal{K}_n$ is also a subsolution for h_λ.*

In a similar way, if $\mathcal{K}_n$ is a sequence of supersolutions for h_λ, locally uniformly bounded with respect to t, then $\mathcal{K}_*$ is also a supersolution for h_λ.

4.2 Proof of Theorem 4.1

Without loss of generality, we assume that $\mathcal{K}_1$ has a closed graph. We argue by contradiction, assuming there exists $0 \le T^* < T$ such that

$$\mathcal{K}_1(T^*) \cap \widehat{\mathcal{K}_2}(T^*) \neq \emptyset. \tag{40}$$

For $\sigma > 0$ and $\epsilon > 0$, we consider

$$\rho_\sigma(t) = \min_{x\in\mathcal{K}_1(t)} d^\sigma_{\widehat{\mathcal{K}_2}}(t,x)$$

and we set

$$T^{\epsilon,\sigma} = \inf\{t \geq 0 \ : \ \rho_\sigma(t) \leq \epsilon e^{-t} \ \} \ .$$

Recall that the notations $|\cdot|_\sigma$ and $d^\sigma_{\widehat{\mathcal{K}_2}}$ were introduced at the beginning of subsection 3.2. Let $r > 0$ sufficiently small such that $S + rB$ has a $\mathcal{C}^2$ boundary and $S + rB \subset\subset \mathcal{K}_1(t)$ and $S + rB \subset\subset \mathcal{K}_2(t)$ for all $t \in [0, T]$. We also fix $R > 0$ sufficiently large such that

$$\sup_{(t,x)\in\mathcal{K}_1,\ t\leq T} |x| + \sup_{(t,x)\in\mathcal{K}_2,\ t\leq T} |x| \leq R - r \ .$$

We denote by θ the constant defined in Proposition 2.4 for R and r. Let us recall that $\theta > 1/r$ and that, for any compact set K with $\mathcal{C}^{1,1}$ boundary such that $S_r \subset \mathrm{int}(K)$ and $K \subset B(0, R-r)$, for any $v \in I\!\!R^N$ with $|v| < 1/\theta$ and any $x \in \partial K$, we have

$$\bar{h}(x + v, K + v) \geq (1 - \theta|v|)^2 \bar{h}(x, K) \ . \tag{41}$$

We refer the reader to Figure 3 for an illustration of the proof.

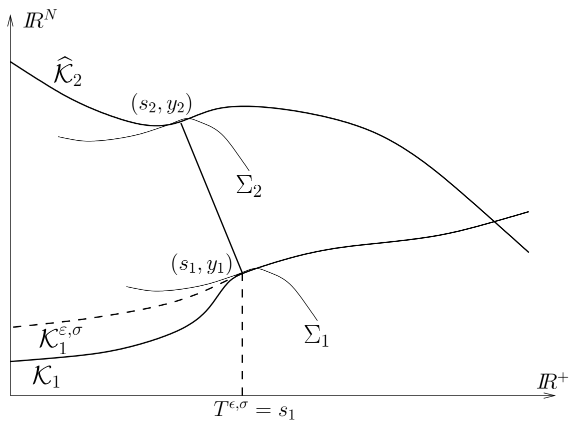

Figure 3: *Illustration of the proof of Theorem 4.1.*

Lemma 4.5 *We can choose $\epsilon > 0$ and $\sigma > 0$ sufficiently small so that*
(i) $\lambda_2(1 - \theta\epsilon)^2 > \lambda_1$ *and* $\epsilon\sigma < 1$,
(ii) $T^{\epsilon,\sigma} > \epsilon$,

(iii) $\rho_\sigma(T^{\epsilon,\sigma}) = \epsilon e^{-T^{\epsilon,\sigma}}$,
(iv) for any $y_1 \in \mathcal{K}_1(T^{\epsilon,\sigma})$ *and any* $(s_2, y_2) \in \widehat{\mathcal{K}_2}$ *such that*

$$\rho_\sigma(T^{\epsilon,\sigma}) = \min_{y \in \mathcal{K}_1(T^{\epsilon,\sigma})} d^\sigma_{\widehat{\mathcal{K}_2}}(T^{\epsilon,\sigma}, y) = |(T^{\epsilon,\sigma}, y_1) - (s_2, y_2)|_\sigma \,, \tag{42}$$

we have $y_1 \neq y_2$ *and* $s_2 > 0$.

The proof of the lemma is postponed to Section 4.3.

From now on we fix ϵ and σ as in Lemma 4.5 and we set $s_1 = T^{\epsilon,\sigma}$. Let us also set

$$\begin{array}{rcl} \mathcal{K}_1^{\epsilon,\sigma} & = & \{(s,y) \in I\!\!R^+ \times I\!\!R^N : \\ & & \quad \exists (\tau, z) \in \mathcal{K}_1 \text{ with } \tau \leq s_1 \text{ and } |(s,y) - (\tau, z)|_\sigma \leq \epsilon(e^{-\tau} - e^{-s_1})\}. \end{array}$$

Recall that, for any two subsets A_1 and A_2 of $I\!\!R^{N+1}$, we define the minimal distance between A_1 and A_2 by

$$e(A_1, A_2) = \inf_{(t_1,x_1)\in A_1,\ (t_2,x_2)\in A_2} |(t_2, x_2) - (t_1, x_1)|_\sigma \ .$$

Lemma 4.6 *The set* $\mathcal{K}_1^{\epsilon,\sigma}$ *is closed, with* $\mathcal{K}_1^{\epsilon,\sigma}(s_1) = \mathcal{K}_1(s_1)$ *and*

$$e(\mathcal{K}_1^{\epsilon,\sigma}, \widehat{\mathcal{K}_2}) = \epsilon e^{-s_1}.$$

Moreover there exist $y_1 \in \mathcal{K}_1^{\epsilon,\sigma}(s_1)$ *and* $(s_2, y_2) \in \widehat{\mathcal{K}_2}$*, such that*

$$|(s_1, y_1) - (s_2, y_2)|_\sigma = e(\mathcal{K}_1^{\epsilon,\sigma}, \widehat{\mathcal{K}_2}), \tag{43}$$

and, for such points y_1 *and* (s_2, y_2)*, we have* $y_1 \neq y_2$.

The proof is postponed. Lemma 4.6 is a kind of refinement of Lemma 4.5.(iv). Note that the proof of Lemma 4.5 and Lemma 4.6 only use the fact that $\overline{\mathcal{K}_1}(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset$ and that $\overline{\mathcal{K}_1}$ and $\widehat{\mathcal{K}_2}$ are left lower semicontinuous.

Next we give an estimate of the normal velocity of the tube $\mathcal{K}_1$ in terms of the normal velocity of $\mathcal{K}_1^{\epsilon,\sigma}$:

Lemma 4.7 *Assume that a* $\mathcal{C}^{1,1}$ *tube* Σ *is externally tangent to* $\mathcal{K}_1^{\epsilon,\sigma}$ *at some point* $(\bar{s}, \bar{y}) \in \partial\mathcal{K}_1^{\epsilon,\sigma}$*. Let* $(\bar{\tau}, \bar{z}) \in \mathcal{K}_1$ *be such that*

$$\bar{\tau} \leq s_1 \quad \text{and} \quad |(\bar{s}, \bar{y}) - (\bar{\tau}, \bar{z})|_\sigma \leq \epsilon(e^{-\bar{\tau}} - e^{-s_1}).$$

Then there is a $\mathcal{C}^{1,1}$ tube $\widetilde{\Sigma}$ which is externally tangent to the tube $\mathcal{K}_1 \cap ([0,\bar{\tau}] \times \mathbb{R}^N)$ at $(\bar{\tau}, \bar{z})$ and such that

$$V^{\widetilde{\Sigma}}_{(\bar{\tau},\bar{z})} \geq V^{\Sigma}_{(\bar{s},\bar{y})} + \epsilon e^{-s_1} \quad \text{and} \quad \widetilde{\Sigma}(\bar{\tau}) = \Sigma(\bar{s}) + (\bar{z} - \bar{y}).$$

If furthermore Σ is of class $\mathcal{C}^2$ in a neighbourhood of $(\bar{s}, \bar{y})$, then $\widetilde{\Sigma}$ is also $\mathcal{C}^2$ in a neighbourhood of $(\bar{\tau}, \bar{z})$.

We postpone the proof. We are now ready to use the interposition Theorem 3.3 that we apply to $C_1 := \mathcal{K}_1^{\epsilon,\sigma}$ and $C_2 := \widehat{\mathcal{K}_2}$. Note that condition (18) holds thanks to Lemma 4.6. Let us fix $(s_1, y_1) \in \mathcal{K}_1^{\epsilon,\sigma}$ and $(s_2, y_2) \in \widehat{\mathcal{K}_2}$ with

$$|(s_1, y_1) - (s_2, y_2)| = e(\mathcal{K}_1^{\epsilon,\sigma}, \widehat{\mathcal{K}_2}) \; .$$

From Theorem 3.3 we know that there exists a regular tube Σ_1 with $\mathcal{C}^{1,1}$ boundary such that Σ_1 is externally tangent to $\mathcal{K}_1^{\epsilon,\sigma}$ at (s_1, y_1) and $\Sigma_2 := \Sigma_1 + (s_2, y_2) - (s_1, y_1)$ is internally tangent to $\widehat{\mathcal{K}_2}$ at (s_2, y_2) (see Figure 3).

Futhermore, there exist $\mathcal{C}^{1,1}$ regular tubes $\Sigma_{1,n}$ and $\Sigma_{2,n}$ converging respectively to Σ_1 and Σ_2 in the $\mathcal{C}^{1,\mathrm{b}}$ sense, there exist $(s_{1,n}, y_{1,n}) \in \mathcal{K}_1^{\epsilon,\sigma}$ and $(s_{2,n}, y_{2,n}) \in \widehat{\mathcal{K}_2}$ converging respectively to (s_1, y_1) and (s_2, y_2), and there exist $(N-1) \times (N-1)$ matrices X_1, X_2, such that

(i) $\Sigma_{1,n}$ is externally tangent to $\mathcal{K}_1^{\epsilon,\sigma}$ at $(s_{1,n}, y_{1,n})$ and $\Sigma_{2,n}$ is internally tangent to $\widehat{\mathcal{K}_2}$ at $(s_{2,n}, y_{2,n})$,

(ii) For $i = 1, 2$, $\Sigma_{i,n}$ is of class $\mathcal{C}^2$ in a neighbourhood of $(s_{i,n}, y_{i,n})$ with

$$\lim_n \nu_{y_{1,n}}^{\Sigma_{1,n}(s_{1,n})} = \lim_n \nu_{y_{2,n}}^{\Sigma_{2,n}(s_{2,n})} = \nu_{y_1}^{\Sigma_1(s_1)} = \nu_{y_2}^{\Sigma_2(s_2)}, \tag{44}$$

$$\lim_n V^{\Sigma_{1,n}}_{(s_{1,n},y_{1,n})} = \lim_n V^{\Sigma_{2,n}}_{(s_{2,n},y_{2,n})} = V^{\Sigma_1}_{(s_1,y_1)} = V^{\Sigma_2}_{(s_2,y_2)}, \tag{45}$$

and, for $i = 1, 2$,

$$\lim_n H^{\Sigma_{i,n}}_{s_{i,n},y_{i,n}} \to X_i, \text{ with } X_1 - X_2 \leq 0 \; . \tag{46}$$

Since $\mathcal{K}_2$ is a supersolution for h_{λ_2} and $\Sigma_{2,n}$ is internally tangent to $\mathcal{K}_2$ at $(s_{2,n}, y_{2,n})$, we have

$$V^{\Sigma_{2,n}}_{(s_{2,n},y_{2,n})} \geq F(\nu_{y_{2,n}}^{\Sigma_{2,n}(s_{2,n})}, H_{y_{2,n}}^{\Sigma_{2,n}(s_{2,n})}) + \lambda_2 \bar{h}(y_{2,n}, \Sigma_{2,n}(s_{2,n})).$$

Letting $n \to +\infty$ gives, from the continuity property of $\bar{h}$, from (44), (45) and (46)

$$V^{\Sigma_2}_{(s_2,y_2)} \geq F(\nu^{\Sigma_2(s_2)}_{y_2}, X_2) + \lambda_2 \bar{h}(y_2, \Sigma_2(s_2)) \tag{47}$$

We now establish a symmetric inequality for Σ_1. For any n, let $(\tau_{1,n}, z_{1,n}) \in \mathcal{K}_1$ be such that

$$\tau_{1,n} \leq s_1 \quad \text{and} \quad |(s_{1,n}, y_{1,n}) - (\tau_{1,n}, z_{1,n})|_\sigma \leq \epsilon(e^{-\tau_{1,n}} - e^{-s_1}).$$

Note that $(\tau_{1,n}, z_{1,n}) \to (s_1, y_1)$ because $(s_{1,n}, y_{1,n}) \to (s_1, y_1)$ and $\mathcal{K}^{\epsilon,\sigma}_1$ has a closed graph with $\mathcal{K}^{\epsilon,\sigma}_1(s_1) = \mathcal{K}(s_1)$ (see Lemma 4.6). In particular, $\tau_{1,n} > 0$ for large n. Since $\Sigma_{1,n}$ is externally tangent to $\mathcal{K}^{\epsilon,\sigma}_1$ at $(s_{1,n}, y_{1,n})$, Lemma 4.7 states that one can find a $\mathcal{C}^{1,1}$ tube $\widetilde{\Sigma}_{1,n}$ externally tangent to $\mathcal{K}_1 \cap ([0, \tau_{1,n}] \times I\!\!R^N)$ at $(\tau_{1,n}, z_{1,n})$ with

$$V^{\widetilde{\Sigma}_{1,n}}_{(\tau_{1,n},z_{1,n})} \geq V^{\Sigma_{1,n}}_{(s_{1,n},y_{1,n})} + \epsilon e^{-s_1} \quad \text{and} \quad \widetilde{\Sigma}(\tau_{1,n}) = \Sigma(s_{1,n}) + (z_{1,n} - y_{1,n}).$$

Moreover $\widetilde{\Sigma}_{1,n}$ is also $\mathcal{C}^2$ in a neighbourhood of $(\tau_{1,n}, z_{1,n})$. Lemma 4.8, given below, states that the tube $\mathcal{K}_1 \cap ([0, \tau_{1,n}] \times I\!\!R^N)$ is still a subsolution to the evolution equation for h_{λ_1}. Therefore,

$$V^{\widetilde{\Sigma}_{1,n}}_{(\tau_{1,n},z_{1,n})} \leq F(\nu^{\widetilde{\Sigma}_{1,n}(\tau_{1,n})}_{z_{1,n}}, H^{\widetilde{\Sigma}_{1,n}(\tau_{1,n})}_{z_{1,n}}) + \lambda_1 \bar{h}(z_{1,n}, \widetilde{\Sigma}_{1,n}(\tau_{1,n})) .$$

Since $\Sigma_{1,n}(s_{1,n}) = \widetilde{\Sigma}_{1,n}(\tau_{1,n}) - (z_{1,n} - y_{1,n})$, we have $\nu^{\Sigma_{1,n}(s_{1,n})}_{y_{1,n}} = \nu^{\widetilde{\Sigma}_{1,n}(\tau_{1,n})}_{z_{1,n}}$ and $H^{\Sigma_{1,n}(s_{1,n})}_{y_{1,n}} = H^{\widetilde{\Sigma}_{1,n}(\tau_{1,n})}_{z_{1,n}}$. Therefore

$$V^{\Sigma_{1,n}}_{(s_{1,n},y_{1,n})} + \epsilon e^{-s_1} \leq F(\nu^{\Sigma_{1,n}(s_{1,n})}_{y_{1,n}}, H^{\Sigma_{1,n}(s_{1,n})}_{y_{1,n}}) + \lambda_1 \bar{h}(y_{1,n}, \Sigma_{1,n}(s_{1,n}) - (z_{1,n} - y_{1,n})) .$$

Letting $n \to +\infty$ we get

$$V^{\Sigma_1}_{(s_1,y_1)} + \epsilon e^{-s_1} \leq F(\nu^{\Sigma_1(s_1)}_{y_1}, X_1) + \lambda_1 \bar{h}(y_1, \Sigma_1(s_1)) . \tag{48}$$

The difference between (47) and (48) gives

$$\begin{aligned} 0 \;\; &\geq \;\; F(\nu^{\Sigma_2(s_2)}_{y_2}, X_2) + \lambda_2 \bar{h}(y_2, \Sigma_2(s_2)) \\ &\qquad - F(\nu^{\Sigma_1(s_1)}_{y_1}, X_1) - \lambda_1 \bar{h}(y_1, \Sigma_1(s_1)) + \epsilon e^{-s_1} \\ &\geq \;\; (\lambda_2(1-\theta\epsilon)^2 - \lambda_1)\bar{h}(y_1, \Sigma(\bar{t}) - e_1\bar{\nu}_x) + \epsilon e^{-s_1}, \end{aligned} \tag{49}$$

because, on the one hand, $V^{\Sigma_1}_{(s_1,y_1)} = V^{\Sigma_2}_{(s_2,y_2)}$, $\nu^{\Sigma_1(s_1)}_{y_1} = \nu^{\Sigma_2(s_2)}_{y_2}$ and $X_1 \leq X_2$ (from (44, 45,46)) and F is elliptic and because, on the another hand, from (41) and the definition of Σ_2,

$$\bar{h}(y_2, \Sigma_2(s_2)) = \bar{h}(y_2, \Sigma_1(s_1) + y_2 - y_1) \geq (1 - \theta|y_2 - y_1|)^2\, \bar{h}(y_1, \Sigma_1(s_1)),$$

where $|y_2 - y_1| \le \epsilon e^{-s_1} \le \epsilon$. Since $\bar{h} \ge 0$ and $\lambda_2(1-\theta\epsilon)^2 \ge \lambda_1$ from Lemma 4.5, there is a contradiction in (49) and the proof is complete.

QED

4.3 Proofs of Lemmas 4.5, 4.6, 4.7 and 4.8

Proof of Lemma 4.5. The proof of this lemma is close to the one of [7, Proposition 4.2]. We provide it for reader's convenience.

The assertion (i) is obvious.

To prove (ii), we first note that, since $\mathcal{K}_1(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset$ and $\mathcal{K}_1$ and $\widehat{\mathcal{K}_2}$ are closed, there exists $\tau > 0$ such that $\mathcal{K}_1(t) \cap \widehat{\mathcal{K}_2}(s) = \emptyset$ for all $0 \le s, t \le \tau$. Therefore

$$\mu := \inf\{|y - x| : 0 \le s, t \le \tau,\ x \in \mathcal{K}_1(t),\ y \in \widehat{\mathcal{K}_2}(s)\} > 0.$$

Let $\epsilon < \mu$ and $\sigma < \tau/(2\epsilon)$. Then, for $s \in [0, \tau/2]$ and $y \in \mathcal{K}_1(s)$ and for any $(t, x) \in \widehat{\mathcal{K}_2}$, we have

$$|(s, y) - (t, x)|_\sigma \ge \begin{cases} \mu & \text{if } t \le \tau, \\ \frac{\tau}{2\sigma} & \text{if } t \ge \tau. \end{cases}$$

Hence $d_{\widehat{\mathcal{K}_2}}(s, y) \ge \epsilon$, which proves that $T^{\epsilon,\sigma} \ge \tau/2 \ge \epsilon$.

We prove (iii). From the definition of $\rho_\sigma(T^{\epsilon,\sigma})$, there exists $t_n \downarrow T^{\epsilon,\sigma}$ with $\rho_\sigma(t_n) \le \epsilon e^{-t_n}$. Therefore there exists $y_n \in \mathcal{K}_1(t_n)$ such that $\rho_\sigma(t_n) = d^\sigma_{\widehat{\mathcal{K}_2}}(t_n, y_n)$ and, up to extract a subsequence, we can assume that $y_n \to y \in \mathcal{K}_1(T^{\epsilon,\sigma})$ (since $\mathcal{K}_1$ is closed). It follows

$$\rho_\sigma(t_n) = d^\sigma_{\widehat{\mathcal{K}_2}}(t_n, y_n) \to d^\sigma_{\widehat{\mathcal{K}_2}}(T^{\epsilon,\sigma}, y) \ge \rho_\sigma(T^{\epsilon,\sigma}).$$

Thus, we obtain the inequality $\rho_\sigma(T^{\epsilon,\sigma}) \le \epsilon e^{-T^{\epsilon,\sigma}}$ (note that we prove by the way that ρ_σ is a lower-semicontinuous function). It remains to prove that the equality holds. If not, there exists $y \in \mathcal{K}_1(T^{\epsilon,\sigma})$ such that $d^\sigma_{\widehat{\mathcal{K}_2}}(T^{\epsilon,\sigma}, y) < \epsilon e^{-T^{\epsilon,\sigma}}$. From the left lower-semicontinuity of the subsolution $\mathcal{K}_1$, for all sequence $t_n \uparrow (T^{\epsilon,\sigma})^-$, there exists $y_n \in \mathcal{K}_1(t_n)$ which converges to y. It follows that

$$\rho_\sigma(t_n) \le d^\sigma_{\widehat{\mathcal{K}_2}}(t_n, y_n) < \epsilon e^{-t_n} \text{ at least for } n \text{ large.}$$

We get a contradiction with the definition of $T^{\epsilon,\sigma}$ and conclude for the proof of (iii).

We turn to the proof of (iv). For this we fix $\epsilon > 0$ as in the proof of (ii) and we note that $T^{\epsilon,\sigma}$ is noncreasing with respect to σ. Since $T^{\epsilon,\sigma} \le T^*$, $\lim_{\sigma\to 0^+} T^{\epsilon,\sigma}$ exists and we denote it by $\bar{s}$. Note that $\bar{s} \ge \epsilon$.

We now argue by contradiction, assuming that there is a sequence $\sigma_n \to 0^+$, $y_{1,n} \in \mathcal{K}_1(T^{\epsilon,\sigma_n})$, $(s_{2,n}, y_{2,n}) \in \widehat{\mathcal{K}_2}$ such that

$$\rho_{\sigma_n}(T^{\epsilon,\sigma_n}) = |(T^{\epsilon,\sigma_n}, y_{1,n}) - (s_{2,n}, y_{2,n})|_{\sigma_n} = \epsilon e^{-T^{\epsilon,\sigma_n}} ,$$

with either $s_{2,n} = 0$ or $y_{1,n} = y_{2,n}$. Since $\mathcal{K}_1$ and $\widehat{\mathcal{K}_2}$ have closed graphs, up to extract subsequences, there exist $y_1 \in K_1(\bar{s})$ and $(s_2, y_2) \in \widehat{\mathcal{K}_2}$ such that $y_{1,n} \to y_1$ and $(s_{2,n}, y_{2,n}) \to (s_2, y_2)$. From the inequality

$$\epsilon \ge \left(\frac{1}{\sigma_n}(T^{\epsilon,\sigma_n} - s_{2,n})^2 + |y_{1,n} - y_{2,n}|^2\right)^{\frac{1}{2}} ,$$

we deduce that $T^{\epsilon,\sigma_n} - s_{2,n} \to 0$. But $T^{\epsilon,\sigma_n} \to \bar{s} > \epsilon$, and so $s_{2,n} > 0$ for n sufficiently large, which implies that $y_{1,n} = y_{2,n}$ and thus $y_1 = y_2$.

We now use the left lower-semicontinuity of $\mathcal{K}_1$ and $\widehat{\mathcal{K}_2}$: Let $t_n \to \bar{s}^-$. Since $y_1 = y_2 \in \mathcal{K}_1(\bar{s}) \cap \widehat{\mathcal{K}_2}(\bar{s})$, there exist $x_{1,n} \in \mathcal{K}_1(t_n)$ and $x_{2,n} \in \widehat{\mathcal{K}_2}(t_n)$ which converge to $y_1 = y_2$. Then

$$\rho_{\sigma_n}(t_n) \le d^{\sigma_n}_{\widehat{\mathcal{K}_2}}(t_n, x_{1,n}) \le |(t_n, x_{1,n}) - (t_n, x_{2,n})|_{\sigma} = |x_{1,n} - x_{2,n}| < \epsilon e^{-t_n}$$

as soon as n is sufficiently large. This is in contradiction with the definition of T^{ϵ,σ_n}.

QED

Proof of Lemma 4.6. The set $\mathcal{K}_1^{\epsilon,\sigma}$ is closed because so is $\mathcal{K}_1$. Let $y \in \mathcal{K}_1^{\epsilon,\sigma}(s_1)$. There is some $(\tau, z) \in \mathcal{K}_1$ such that $\tau \le s_1$ and $|(\tau, z) - (s_1, y)|_\sigma \le \epsilon(e^{-\tau} - e^{-s_1})$. Then

$$\frac{1}{\sigma^2}(\tau - s_1)^2 \le |(\tau, z) - (s_1, y)|^2_\sigma \le \epsilon^2(e^{-\tau} - e^{-s_1})^2 \le \epsilon^2(\tau - s_1)^2 .$$

Since $\epsilon\sigma < 1$ (from Lemma 4.5), we have $\tau = s_1$ and thus $y = z$. So we have proved that $\mathcal{K}_1^{\epsilon,\sigma}(s_1) \subset \mathcal{K}_1(s_1)$. The other inclusion being obvious, the equality holds.

Let $(s, y) \in \mathcal{K}_1^{\epsilon,\sigma}$. Then there exists $(\tau, z) \in \mathcal{K}_1$ with $\tau \le s_1$ and $|(s, y) - (\tau, z)|_\sigma \le \epsilon(e^{-\tau} - e^{-s_1})$. From the definition of ρ_σ and s_1, we have $d^\sigma_{\widehat{\mathcal{K}_2}}(\tau, z) \ge \rho_\sigma(\tau) \ge \epsilon e^{-\tau}$. It follows

$$d^\sigma_{\widehat{\mathcal{K}_2}}(s, y) \ge d^\sigma_{\widehat{\mathcal{K}_2}}(\tau, z) - |(s, y) - (\tau, z)|_\sigma \ge \epsilon e^{-s_1}. \tag{50}$$

Taking the infimum over $(s,y) \in \mathcal{K}_1^{\epsilon,\sigma}$, we get $e(\mathcal{K}_1^{\epsilon,\sigma}, \widehat{\mathcal{K}_2}) \geq \epsilon e^{-s_1}$.

Let us prove the opposite inequality. From Lemma 4.5, we can choose $y \in \mathcal{K}_1(s_1)$ such that $d^\sigma_{\widehat{\mathcal{K}_2}}(s_1, y) = \epsilon e^{-s_1}$. But $\mathcal{K}_1(s_1) \subset \mathcal{K}_1^{\epsilon,\sigma}(s_1)$. Therefore $e(\mathcal{K}_1^{\epsilon,\sigma}, \widehat{\mathcal{K}_2}) \leq d^\sigma_{\widehat{\mathcal{K}_2}}(s_1, y) = \epsilon e^{-s_1}$.

To prove the second assertion, let $y_1 \in \mathcal{K}_1(s_1)$ and $(s_2, y_2) \in \widehat{\mathcal{K}_2}$ be such that (42) in Lemma 4.5 holds. Then obviously (43) also holds. For such points, let $(\tau, z) \in \mathcal{K}_1$ be such that $\tau \leq s_1$ and

$$|(s_1, y_1) - (\tau, z)|_\sigma \leq \epsilon(e^{-\tau} - e^{-s_1}). \tag{51}$$

If $\tau < s_1$, then, by definition of s_1, we have $\rho_\sigma(\tau) > \epsilon e^{-\tau}$. From the computation (50) with $(s,y) := (s_1, y_1)$, it follows $d^\sigma_{\widehat{\mathcal{K}_2}}(s_1, y_1) > \epsilon e^{-\tau} > \epsilon e^{-s_1}$ which is a contradiction with (43). Hence $\tau = s_1$ and therefore (51) gives $(s_1, y_1) = (\tau, z) \in \mathcal{K}_1$. We conclude by Lemma 4.5.

QED

Proof of Lemma 4.7. Let $\mathbf{d}^\sigma_\Sigma$ be the signed distance to $\partial\Sigma$:

$$\mathbf{d}^\sigma_\Sigma(\tau, z) = \begin{cases} d^\sigma_\Sigma(\tau, z) & \text{if} \quad (\tau, z) \notin \Sigma, \\ -d^\sigma_{\partial\Sigma}(\tau, z) & \text{if} \quad (t, z) \in \Sigma. \end{cases} \tag{52}$$

Since Σ is of class $\mathcal{C}^{1,1}$, one can find $\eta > 0$ such that $\mathbf{d}^\sigma_\Sigma$ is of class $\mathcal{C}^{1,1}$ in $\{|\mathbf{d}^\sigma_\Sigma| < \eta\}$, with $D_x \mathbf{d}^\sigma_\Sigma \neq 0$ in this set.

Let us define $\widetilde{\Sigma}$ by

$$\widetilde{\Sigma} = \{(\tau, z) \in I\!\!R^+ \times I\!\!R^N \ : \ \mathbf{d}^\sigma_\Sigma((\tau, z) + (\bar{s}, \bar{y}) - (\bar{\tau}, \bar{z})) \leq -\epsilon(e^{-\tau} - e^{-\bar{\tau}})\}.$$

We first show that the tube $\widetilde{\Sigma}$ is externally tangent to $\mathcal{K}_1 \cap ([0, \bar{\tau}] \times I\!\!R^N)$ at $(\bar{\tau}, \bar{z})$. At first, if $(\tau, z) \in \mathcal{K}_1$ with $\tau \leq \bar{\tau}$, then by definition of $\mathcal{K}_1^{\epsilon,\sigma}$, we have $B_\sigma((\tau, z), \epsilon(e^{-\tau} - e^{-s_1})) \subset \mathcal{K}_1^{\epsilon,\sigma}$, where B_σ is the usual open ball related to the norm $|\cdot|_\sigma$. In particular, since $\mathcal{K}_1^{\epsilon,\sigma} \subset \Sigma$, we have $\mathbf{d}^\sigma_\Sigma(\tau, z) \leq -\epsilon(e^{-\tau} - e^{-s_1})$. Therefore

$$\begin{aligned} &\mathbf{d}^\sigma_\Sigma((\tau, z) + (\bar{s}, \bar{y}) - (\bar{\tau}, \bar{z})) \\ &\qquad \leq \mathbf{d}^\sigma_\Sigma(\tau, z) + |(\bar{s}, \bar{y}) - (\bar{\tau}, \bar{z})|_\sigma \\ &\qquad \leq -\epsilon(e^{-\tau} - e^{-s_1}) + \epsilon(e^{-\bar{\tau}} - e^{-s_1}) \\ &\qquad \leq -\epsilon(e^{-\tau} - e^{-\bar{\tau}}) \,. \end{aligned}$$

So we have proved that $\mathcal{K}_1 \cap ([0, \bar{\tau}] \times I\!\!R^N) \subset \widetilde{\Sigma}$. Moreover, we obviously have that $(\bar{\tau}, \bar{z}) \in \partial\widetilde{\Sigma}$ because $(\bar{s}, \bar{y}) \in \partial\Sigma$.

Let us show that $\widetilde{\Sigma}$ is a regular tube in an open interval containing $\bar{\tau}$. For this, recall that $\mathrm{bd}(\Sigma)$ is defined by (22). If (τ, z) belongs to $\mathrm{bd}(\Sigma)$ with $\epsilon|e^{-\tau} - e^{-\bar{\tau}}| < \eta$, then

$$|\mathbf{d}^{\sigma}_{\Sigma}((\tau, z) + (\bar{s}, \bar{y}) - (\bar{\tau}, \bar{z}))| \ \leq \epsilon|e^{-\tau} - e^{-\bar{\tau}}| < \eta$$

and so $\mathbf{d}^{\sigma}_{\Sigma}$ is differentiable in a neighbourhood of $(\tau, z) + (\bar{s}, \bar{y}) - (\bar{\tau}, \bar{z})$ with $D_x \mathbf{d}^{\sigma}_{\Sigma} \neq 0$. Thus $\widetilde{\Sigma}$ is as smooth as Σ. Finally,

$$V^{\widetilde{\Sigma}}_{(\bar{\tau}, \bar{z})} = -\frac{\frac{\partial \mathbf{d}^{\sigma}_{\Sigma}}{\partial t}(\bar{s}, \bar{y}) - \epsilon e^{-\bar{\tau}}}{|D_x \mathbf{d}^{\sigma}_{\Sigma}(\bar{s}, \bar{y})|} \geq V^{\Sigma}_{(\bar{s}, \bar{y})} + \epsilon e^{-s_1} \ ,$$

since $|D_x \mathbf{d}^{\sigma}_{\Sigma}(\bar{s}, \bar{y})| \leq |D \mathbf{d}^{\sigma}_{\Sigma}(\bar{s}, \bar{y})| = 1$.

QED

Lemma 4.8 *If $\mathcal{K}_1$ is a subsolution to the evolution equation for h_{λ_1}, then, for any $t > 0$, $\mathcal{K}_1 \cap ([0, t] \times I\!\!R^N)$ is also a subsolution for h_{λ_1}.*

Proof of Lemma 4.8. Let us set $\widetilde{\mathcal{K}_1} = \mathcal{K}_1 \cap ([0, t] \times I\!\!R^N)$. It is clear that $\widetilde{\mathcal{K}_1}$ is a left lower-semicontinuous tube because so is $\mathcal{K}_1$. Let Σ be a $\mathcal{C}^2$ tube defined on some open time-intervall I and which is external tangent to $\widetilde{\mathcal{K}_1}$ at some point (s, y) with $s > 0$. If $s < t$, then, assuming without loss of generality that $I \subset (0, t)$, Σ is also externally tangent to $\mathcal{K}_1$ and thus $V^{\Sigma}_{(s,y)} \leq h_{\lambda_1}(y, \Sigma(s))$.

We now suppose that $s = t$ and, without loss of generality, that $\partial\Sigma \cap \widetilde{\mathcal{K}_1} = \{(s, y)\}$. Let $\mathbf{d}^{\sigma}_{\Sigma}$ be the signed distance to $\partial\Sigma$ defined by (52). Note that, since $\partial\Sigma \cap \widetilde{\mathcal{K}_1} = \{(s, y)\}$, $\mathbf{d}^{\sigma}_{\Sigma}$ has a strict maximum on $\mathcal{K}_1$ at (s, y) (at least on the interval I). For $\gamma > 0$ we introduce the mapping

$$\varphi_\gamma(\tau, z) = \mathbf{d}^{\sigma}_{\Sigma}(\tau, z) + \gamma \log(t - \tau)$$

of class $\mathcal{C}^2$ for $\tau \in I \cap (0, t)$ and when $|\mathbf{d}^{\sigma}_{\Sigma}(\tau, z)|$ is small. From standard arguments (see for instance the proof of Lemma 4.2 of [8]), φ_γ has a maximum on $\mathcal{K}_1$ at some point $(s_\gamma, y_\gamma) \in \mathcal{K}_1$ which converge to (s, y) as $\gamma \to 0^+$ and such that $s_\gamma < t$. Moreover, the set

$$\Sigma_\gamma = \{(\tau, z) \in (I \cap (0, t)) \times I\!\!R^N : \varphi_\gamma(\tau, z) \leq \varphi_\gamma(s_\gamma, y_\gamma)\}$$

is a $\mathcal{C}^2$ tube on some open interval $I_\gamma \subset I \cap (0, t)$ with $s_\gamma \in I_\gamma$, and $\Sigma_\gamma(s_\gamma)$ converges in the $\mathcal{C}^2$ sense to $\Sigma(s)$. Now we note that Σ_γ is externally tangent to $\mathcal{K}_1$ at (s_γ, y_γ) and thus

$$V^{\Sigma_\gamma}_{(s_\gamma, y_\gamma)} \leq h_{\lambda_1}(y_\gamma, \Sigma_\gamma(s_\gamma)) \ ,$$

with

$$V^{\Sigma_\gamma}_{(s_\gamma,y_\gamma)} = -\frac{\frac{\partial\varphi}{\partial t}(s_\gamma, y_\gamma)}{|D\varphi(s_\gamma, y_\gamma)|} \ \geq \ V^{\Sigma}_{(s_\gamma,y_\gamma)} \ .$$

Hence

$$V^{\Sigma}_{(s_\gamma,y_\gamma)} \leq h_{\lambda_1}(y_\gamma, \Sigma_\gamma(s_\gamma)) \ ,$$

which gives the desired inequality as $\gamma \to 0^+$: $V^{\Sigma}_{(s,y)} \leq h_{\lambda_1}(y, \Sigma(s))$.

QED

5 Convergence to equilibria

In this section we investigate the asymptotic behavior of the solutions of our front propagation problem (6). More precisely, we show, under suitable assumptions on the source S and on F, that the solution converges as $t \to +\infty$ to the (weak) solution of the free boundary value problem:

$$\text{Find a set } K \in \mathcal{D} \text{ such that } h_\lambda(x, K) = 0 \text{ for all } x \in \partial K \qquad (53)$$

(recall that $\mathcal{D}$ is defined by (11)). This problem is a generalization of the Bernoulli exterior free boundary problem we recalled in the introduction.

Let us first introduce a notion of weak solution:

Definition 5.1 *We say that the set $K \in \mathcal{D}$ is a viscosity subsolution (respectively supersolution, solution) of the free boundary problem (in short FBP) (53) for h_λ if the constant tube $\mathcal{K}(t) = K$ for all $t \geq 0$ is a subsolution (respectively supersolution, solution) of the FPP (6) for h_λ.*

Remark 5.1 There are many other definitions of weak solutions for such FBP: see for instance the survey paper [15]. The one we introduce here is the more suitable to our purpose. The idea of using sub- and supersolution in this framework comes back to Beurling [5].

In order to ensure that the FBP (53) has a solution, we assume in the sequel the following:

$$\forall \lambda > 0, \ \exists R > 0 \text{ such that } \ \forall r \geq R, \forall x \in B(0,r), \ h_\lambda(x, B(0,r)) < 0. \quad (54)$$

This assumption states that $B(0,r)$ is a strict classical supersolution of the free boundary problem for h_λ for r sufficiently large. It is in particular fulfilled (i) when $F(\nu, A) = Tr(A) + F_1(\nu)$, where $F_1(\nu) \leq 0$, or (ii) when

$F = F(\nu) < 0$ for any ν with $|\nu| = 1$, because of the behavior of $\bar{h}$ for large balls (see Lemma 2.2). Note also that the assumption implies that, for any ball B,

$$F(\nu_x^B, H_x^B) < 0 \qquad \forall x \in \partial B \ ,$$

since F is elliptic and $\bar{h} \geq 0$.

Proposition 5.2 *We assume that (10) and (54) hold. Then, for any $\lambda > 0$, there is a largest and a smallest solution of the FBP (53) for h_λ, the largest being closed and containing any subsolution for h_λ, while the smallest is open and is contained in all the supersolutions.*

Proof of Proposition 5.2. The proof can be achieved by a direct application of Perron's method. Existence and bounds for sub- and supersolutions are ensured by assumption (54) and by Lemma 5.3 below.

QED

Lemma 5.3 *We assume that (10) and (54) hold. Let $\lambda > 0$ be fixed. Then there exist $\epsilon > 0$ and $R > 0$ such that, if K is a subsolution (respectively a supersolution) of the FBP (53) for h_λ, then $K \subset B(0, R)$ (respectively $S_\epsilon \subset K$ where S_ϵ is defined by (14)).*

Proof of Lemma 5.3. Let r be the smallest nonnegative real such that $K \subset B(0, r)$. Then there is some point $x \in \partial K \cap \partial(B(0, r))$. Using the constant tube $B(0, r) \times I\!\!R$ as a test-tube and the fact that K is a subsolution, we have $h_\lambda(x, B(0, r)) \geq 0$, which implies that $r < R$ where R is given by (54). Therefore we have $K \subset B(0, R)$.

The assertion for the supersolution can be proved in a similar way, by using assumption (10) and Lemma 2.3 saying that $\bar{h}(x, S_\epsilon)$ is large for $\epsilon > 0$ small and $x \in \partial S_\epsilon$.

QED

Next we address the uniqueness problem. The main assumption for this is that S is starshaped. We also suppose that $F = F(\nu, A)$ satisfies the subhomogeneity condition:

$$F(\nu, \gamma A) \geq \gamma F(\nu, A) \qquad \forall \gamma \geq 1, \tag{55}$$

and that the following compatibility condition between F and S holds:

$$F(\nu_x^S, H_x^S) < 0 \qquad \forall x \in \partial S \ . \tag{56}$$

Assumption (55) is fulfilled for instance if (i) $F(\nu, A) = Tr(A) + F_1(\nu)$, where $F_1(\nu) \leq 0$, or if (ii) $F = F(\nu) < 0$ for any ν with $|\nu| = 1$, while assumption (56) is always satisfied for F as in (ii), and is satisfied for sets with negative mean curvature if F is as in (i).

Theorem 5.4 *Let us assume that S is strictly starshaped at 0, with $0 \in \mathrm{int}(S)$, $g \equiv 1$ and that (10), (54), (55) and (56) hold. Then, for any $\lambda > 0$, the FBP (53) for h_λ has a unique solution denoted K_λ. Moreover, K_λ is starshaped at 0 for any $\lambda > 0$ and the map $\lambda \to \overline{K_\lambda}$ is continuous for the Hausdorff topology.*

Remark 5.2
1. Uniqueness of solution means that, if K_1 and K_2 are two solutions of the FBP for h_λ, then $\overline{K_1} = \overline{K_2}$ and $\mathrm{int}(K_1) = \mathrm{int}(K_2)$.
2. Such a uniqueness result is classical in the literature, see in particular Beurling [5], Tepper [27] and the survey paper [15].

In order to prove Theorem 5.4 we need three Lemmas. The first one explains that the homothetic of a subsolution is still a subsolution. The second one allows to compare sub and supersolutions of the FBP. The last one shows that subsolutions of the FBP for h_λ when λ is small are necessarily close to S.

Lemma 5.5 *Assume that S is starshaped at 0, $g \equiv 1$ and that (55) holds. If K is a subsolution of the FBP (53) for h_λ, then ρK is a subsolution of the FBP for $h_{\rho\lambda}$ for any $\rho \in (0, 1)$ such that $S \subset\subset \rho K$.*

Proof of Lemma 5.5. For sake of clarity, we do the proof in a formal way, by assuming that K is smooth. If not, it is enough to do the same computations for the test-surfaces. We first notice that

$$\bar{h}(x, \rho K) \geq \frac{1}{\rho^2}\bar{h}(\frac{x}{\rho}, K) \qquad \forall \rho \in (0, 1),\ \forall x \in \partial(\rho K)\ . \tag{57}$$

Indeed, if u is the solution to (9) with K instead of Ω, then $v(x) = u(x/\rho)$ is a subsolution of equation (9) with ρK instead of Ω (we use here the fact that S is starshaped, that $g \equiv 1$ and, thus, that $0 \leq u \leq 1$). Then

$$\bar{h}(x, \rho K) \geq |Dv(x)|^2 = \frac{1}{\rho^2}|Du(x/\rho)|^2 = \frac{1}{\rho^2}\bar{h}(x/\rho, K) \qquad \forall x \in \partial(\rho K)\ .$$

Next we also notice that $\nu_x^{\rho K} = \nu_{x/\rho}^K$ and $H_x^{\rho K} = \frac{1}{\rho} H_{x/\rho}^K$. Hence, for any $x \in \partial(\rho K)$, we have

$$\begin{aligned} h_{\rho\lambda}(x, \rho K) &= F(\nu_x^{\rho K}, H_x^{\rho K}) + \rho\lambda\bar{h}(x, \rho K) \\ &\geq F(\nu_{x/\rho}^K, \tfrac{1}{\rho} H_{x/\rho}^K) + \tfrac{\lambda}{\rho}\bar{h}(x/\rho, K) \\ &\geq (1/\rho) h_\lambda(x/\rho, K) \geq 0 \end{aligned}$$

because K is a subsolution for h_λ. Hence ρK is a subsolution for $h_{\rho\lambda}$.

QED

Lemma 5.6 *We assume that (54) holds. Let $0 < \lambda < \Lambda$, $R > 0$ and $\gamma > 0$ be fixed. Then, there is a constant $\kappa > 0$, such that for any $\lambda \leq \lambda_1 < \lambda_2 \leq \Lambda$, for any subsolution K_1 of the FBP (53) for h_{λ_1} and any supersolution K_2 for h_{λ_2} with*

$$S_\gamma \subset\subset K_1 \subset\subset K_2 \subset\subset B(0, R - \gamma) ,$$

we have

$$K_1 + \kappa(\lambda_2 - \lambda_1) B(0, 1) \subset K_2 ,$$

where the sum in the above inclusion denotes the Kuratowski sum between sets.

Proof of Lemma 5.6. Let $\theta > 0$ be the constant given by Lemma 2.4. From the assumption $K_1 \subset\subset K_2$, we have $\overline{K_1} \cap \overline{I\!R^N \setminus K_2} = \emptyset$ and we can find $y_1 \in \overline{K_1}$ and $y_2 \in \overline{I\!R^N \setminus K_2}$ such that

$$|y_1 - y_2| = \min_{z_1 \in \overline{K_1}, z_2 \in \overline{I\!R^N \setminus K_2}} |z_1 - z_2| .$$

Without loss of generality we can assume that $|y_1 - y_2| < 1/\theta$, since otherwise the result is obvious.

Using now the interposition and approximation results (see Proposition 3.1 and Theorem 3.2), the fact that K_1 is a subsolution for h_{λ_1} and K_2 a supersolution for h_{λ_2}, and proceeding as in the proof of Theorem 4.1, one can find an open set $\Sigma \subset I\!R^N$ with $\mathcal{C}^{1,1}$ boundary and $(N-1) \times (N-1)$ matrices $X_1 \leq X_2$ such that

$$0 \leq F(\nu_{y_1}^\Sigma, X_1) + \lambda_1 \bar{h}(y_1, \Sigma) \tag{58}$$

and

$$0 \geq F(\nu_{y_2}^{\Sigma + y_2 - y_1}, X_2) + \lambda_2 \bar{h}(y_2, \Sigma + y_2 - y_1) . \tag{59}$$

Since $\nu_{y_1}^{\Sigma} = \nu_{y_2}^{\Sigma+y_2-y_1}$ and $X_1 \leq X_2$, we get by subtracting (58) to (59) and using Lemma 2.4

$$0 \geq \left[\lambda_2 \left(1 - \theta|y_1 - y_2|\right)^2 - \lambda_1\right] \bar{h}(y_1, \Sigma) \, . \tag{60}$$

In order to complete the proof, we have now to check that $\bar{h}(y_1, \Sigma)$ is positive. By Hopf maximum principle, we just have to show that the connected component Σ' of $\overline{\Sigma}$ which contains y_1 has a non empty intersection with the source S. For this, we argue by contradiction, by assuming that $\Sigma' \cap S = \emptyset$ (see Figure 4 for an illustration). Let $K_1' := K_1 \cap \Sigma'$. Note that

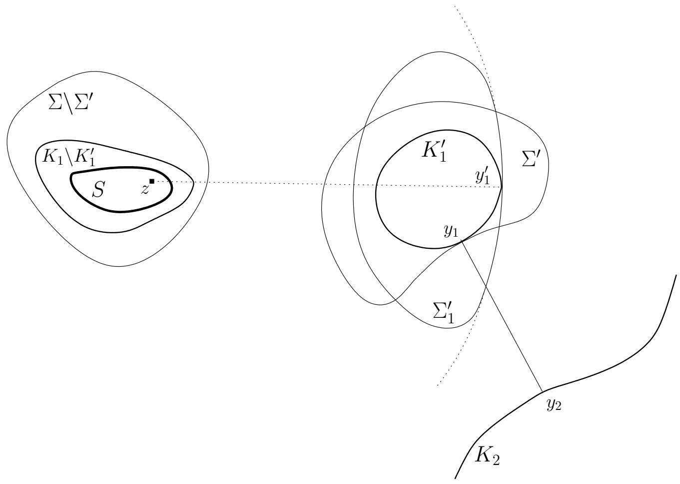

Figure 4: *Illustration of the proof of Lemma 5.6.*

$$e(K_1', \Sigma\backslash\Sigma') \geq e(\Sigma', \Sigma\backslash\Sigma') > 0 \tag{61}$$

since Σ is bounded with a smooth boundary. Let $z \in S$ and y_1' be a point of maximum of the euclidean norm $|\cdot - z|$ on K_1'. Note that $|y_1' - z| > 0$. The ball $B := B(z, |y_1' - z|)$ is externally tangent to K_1' at y_1'. Thanks to (61), one can build an open set Σ_1' with a $\mathcal{C}^2$ boundary, such that $K_1' \subset \overline{\Sigma_1'}$, $\overline{\Sigma_1'} \cap \overline{\Sigma\backslash\Sigma'} = \emptyset$, $y_1' \in \partial\Sigma_1'$, and for which there is a neighboourhood $\mathcal{O}$ of y_1' with $\Sigma_1' \cap \mathcal{O} = B \cap \mathcal{O}$. Note that $\overline{\Sigma_1'} \cap S = \emptyset$ since $S \subset\subset \Sigma\backslash\Sigma'$. Let us set $\Sigma_1 = \Sigma_1' \cup (\Sigma\backslash\Sigma')$. Note that Σ_1 is externally tangent to K_1 at y_1'. Moreover

$$h_\lambda(y_1', \Sigma_1) = F(\nu_{y_1'}^{\Sigma_1}, H_{y_1'}^{\Sigma_1}) + \lambda\bar{h}(y_1', \Sigma_1) = F(\nu_{y_1'}^{B}, H_{y_1'}^{B})$$

since $\bar{h}(y_1', \Sigma_1) = 0$. By (54), for all $r \geq R$ and all $x \in \partial B(0, r)$, we have $h_\lambda(x, B(0, r)) < 0$. Therefore $F(\nu_x^{B(0,r)}, H_x^{B(0,r)}) < 0$ since $\bar{h}(x, B(0, r)) \geq 0$.

By ellipticity, we have $F(\nu_{x'}^{B(0,r')}, H_{x'}^{B(0,r')}) < 0$ even for $r' \le R$ and $|x'| = r'$ since $\nu_{x'}^{B(0,r')} = \nu_{rx'/|x'|}^{B(0,r)}$. It follows

$$h_\lambda(y_1', \Sigma_1) = F(\nu_{y_1'}^B, H_{y_1'}^B) < 0$$

which is a contradiction since K_1 is a subsolution. So $\bar{h}(y_1, \Sigma) > 0$.

Then (60) leads to inequality

$$|y_1 - y_2| \ge \kappa(\lambda_2 - \lambda_1)$$

where $\kappa := 1/(2\theta\Lambda)$, which completes the proof.

QED

Lemma 5.7 *Under the assumptions of Theorem 5.4, for any $\epsilon > 0$, there is $\lambda_0 > 0$ such that, for any subsolution K of the FBP (53) for h_λ with $\lambda \in (0, \lambda_0)$, we have $K \subset S_\epsilon$ (see (14) for a definition of S_ϵ).*

Proof of Lemma 5.7. From assumption (56) and the regularity of the boundary of S, there is some $\alpha > 0$ such that

$$F(\nu_x^S, H_x^S) \le -\alpha \qquad \forall x \in \partial S \; .$$

Let us notice that a similar inequality also holds for ρS, for $\rho \ge 1$, because

$$F(\nu_x^{\rho S}, H_x^{\rho S}) = F(\nu_{x/\rho}^S, \frac{1}{\rho} H_{x/\rho}^S) \le \frac{1}{\rho} F(\nu_{x/\rho}^S, H_{x/\rho}^S) \le -\frac{\alpha}{\rho} \; ,$$

thanks to assumption (55).

Let us now fix $\epsilon > 0$ and $\rho_0 > 1$ such that $\rho_0 S \subset S_\epsilon$. Note that $S \subset\subset \rho_0 S$ since S is strictly starshaped. We set $\lambda_0 = \alpha/(\beta\rho_0)$, where $\beta = \sup_{x\in\partial(\rho_0 S)} \bar{h}(x, \rho_0 S)$. Let K be a subsolution for h_λ with $\lambda \in (0, \lambda_0)$. We denote by $\rho > 1$ the smallest real such that $K \subset \rho S$. In order to prove that $\rho \le \rho_0$, we argue by contradiction and assume that $\rho > \rho_0$. Since ρS is externally tangent to K at some point $x \in \partial K$ and K is a subsolution, we have

$$0 \le h_\lambda(x, \rho S) = F(\nu_x^{\rho S}, H_x^{\rho S}) + \lambda \bar{h}(x, \rho S) \le -\frac{\alpha}{\rho} + \lambda \bar{h}(x, \rho S)$$

where, from inequality (57),

$$\bar{h}(x, \rho S) \le \left(\frac{\rho_0}{\rho}\right)^2 \bar{h}(\frac{\rho_0 x}{\rho}, \rho_0 S) \le \left(\frac{\rho_0}{\rho}\right)^2 \beta \; .$$

Hence $0 \le -\frac{\alpha}{\rho} + \lambda \left(\frac{\rho_0}{\rho}\right)^2 \beta$, which implies that $\rho \le \lambda_0 \rho_0^2 \beta/\alpha = \rho_0$, a contradiction. So we have proved that $\rho \le \rho_0$. Therefore $K \subset \rho_0 S \subset S_\epsilon$.

QED

Proof of Theorem 5.4. Let us denote for any $\lambda > 0$ by K_λ the maximal solution of the BFP for h_λ. Note that $\lambda \to K_\lambda$ is nondecreasing since K_λ contains all the subsolutions for h_λ.

We first check that K_λ is starshaped at 0. Indeed, from Lemma 5.5, for any $\rho \in (0,1)$ sufficiently close to 1, the set ρK_λ is a subsolution for $h_{\rho\lambda}$, and thus for h_λ. Since K_λ contains all the subsolutions, we have $\rho K_\lambda \subset K_\lambda$ for any $\rho \in (0,1)$ sufficiently close to 1, which implies that K_λ is starshaped.

Next, we show that the map $\lambda \to K_\lambda$ is continuous for the Hausdorff topology. From the stability property of solutions (Proposition 4.4), the decreasing limit of the $K_{\lambda'}$ converges to K_λ when $\lambda' \to \lambda^+$. Hence we only have to show that $\mathrm{Lim}_{\lambda' \to \lambda^-} K_{\lambda'}$ equals K_λ, where Lim denotes the Kuratowski limit (see (39)).

Since, for any $\rho \in (0,1)$ sufficiently close to 1, the set ρK_λ is a subsolution for $h_{\rho\lambda}$, we have $\rho K_\lambda \subset K_{\rho\lambda}$. Therefore

$$K_\lambda = \operatorname*{Lim}_{\rho \to 1^-} \rho K_\lambda \subset \operatorname*{Lim}_{\lambda' \to \lambda^-} K_{\lambda'} \subset K_\lambda .$$

So we have checked that $\lambda \to K_\lambda$ is continuous.

Let us finally prove that, for any $\lambda > 0$, K_λ is the unique solution of for h_λ. Let K be another solution. Note that $K \subset K_\lambda$. From Lemma 5.7, we can find some $\overline{\lambda}_1 > 0$ such that $K_{\overline{\lambda}_1} \subset\subset K$ because $S \subset\subset K$. Let us set

$$\overline{\lambda} = \sup\{\lambda' \mid K_{\lambda'} \subset\subset K\} .$$

We now use Lemma 5.6 with $r > 0$ and R such that $S_r \subset K_{\overline{\lambda}_1}$ and $K_\lambda \subset B(0,R)$. There is a constant $\kappa > 0$ such that for any $\overline{\lambda}_1 < \lambda' < \overline{\lambda}$,

$$K_{\lambda'} + \kappa(\lambda - \lambda')B \subset K .$$

The continuity of the map $\lambda' \to K_{\lambda'}$ then implies that

$$K_{\overline{\lambda}} + \kappa(\lambda - \overline{\lambda})B \subset \overline{K} .$$

Therefore $\overline{\lambda} = \lambda$ since, otherwise, the continuity of $\lambda' \to K_{\lambda'}$ would also imply the existence of $\epsilon > 0$ such that $K_{\overline{\lambda}+\epsilon} \subset\subset \overline{K}$, a contradiction with the definition of $\overline{\lambda}$. Therefore $K_\lambda \subset \overline{K}$.

In order to prove that $\mathrm{int}(K) = \mathrm{int}(K_\lambda)$, we notice that $\mathrm{int}(K_\lambda) = \bigcup_{\lambda' < \lambda} K_{\lambda'}$, because $K_{\lambda'} \subset\subset K_\lambda$ for $\lambda' < \lambda$, and therefore the equality $\bar{\lambda} = \lambda$ implies that $\mathrm{int}(K_\lambda) \subset \mathrm{int}(K)$. Since the converse inequality is obvious, the proof of Theorem 5.4 is complete.

QED

Corollary 5.8 (Asymptotic behavior) *Under the assumptions of Theorem 5.4, if $\mathcal{K}$ is a solution of the FPP (6) for h_λ, then $\mathcal{K}(t)$ converges, for the Hausdorff metric as $t \to +\infty$, to the unique solution K_λ of the FBP (53) for h_λ while $\widehat{K}(t)$ converges to $\overline{\mathbb{R}^N \setminus K}$.*

Remark 5.3
1. Note that the above result holds for any solution $\mathcal{K}(t)$ of the FPP (6) with any initial position $\mathcal{K}(0) \in \mathcal{D}$.
2. The proof of the asymptotic behavior which follows relies strongly on the uniqueness of the solution of the limit problem (53).

Proof of Corollary 5.8. Let us fix $\lambda_1 < \lambda < \lambda_2$. From lemma 2.3 and (54), there are $r > 0$ and $R > 0$ such that S_r and $B(0, R)$ are respectively sub- and supersolution to the FBP for h_{λ_1} and h_{λ_2}. We can also choose $r > 0$ sufficiently small and $R > 0$ sufficiently large so that $S_r \subset\subset \mathcal{K}(0) \subset\subset B(0, R)$. The inclusion principle then states that

$$S_r \subset\subset \mathcal{K}(t) \subset\subset B(0,R) \qquad \forall t \geq 0\ .$$

Let K^* be the Kuratowski upperlimit of $\mathcal{K}(t)$ as $t \to +\infty$ (see (39)). Note that $S_r \subset K^* \subset B(0, R)$ and that the constant tube $\mathbb{R} \times K^*$ is actually the upperlimit of the solutions $\mathcal{K}(\cdot + \tau)$ as $\tau \to +\infty$. From the stability of solutions (see Proposition 4.4), the constant tube $\mathbb{R} \times K^*$ is a subsolution of the FPP for h_λ. Hence, K^* is a subsolution of the FBP (53) for h_λ and we have $K^* \subset K_\lambda$.

In the same way, if we set L^* to be the Kuratowski upperlimit of $\widehat{K}(t)$ as $t \to +\infty$, then $\mathbb{R}^N \setminus L^*$ is a supersolution to FBP (53) for h_λ. Since K_λ is the unique solution for h_λ, K_λ is also the smallest solution, which implies that $K_\lambda \subset \overline{\mathbb{R}^N \setminus L^*}$. Hence

$$K^* \subset K_\lambda \subset \overline{\mathbb{R}^N \setminus L^*}.$$

Since $\mathbb{R}^N \setminus L^* \subset K^*$, the proof is complete.

QED

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