Weak limit theorem in the Fourier tranform method for the estimation of multivariate volatility

Emmanuelle Clement, Arnaud Gloter

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Emmanuelle Clément* Arnaud Gloter†

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Abstract

In this paper, we prove some weak limit theorems for the Fourier estimator of multivariate volatility proposed by Malliavin and Mancino ([12], [13]). We first give a central limit theorem for the estimator of the integrated volatility assuming that we observe the whole path of the Ito process. Then we study the case of discrete time observations possibly non synchronous. In this framework we prove that the asymptotic variance of the estimator depends on the limit behavior of the ratio N/n where N is the number of Fourier coefficients and n the number of observations. We point out some optimal choices of N with respect to n to minimize this asymptotic variance.

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1 Introduction

A large literature is devoted to the computation of the volatility or the integrated volatility of financial asset returns. In practice, one observes discrete time realizations of some asset prices which time evolution is given by a multivariate Itô process X. In this typical framework of high frequency data, volatility can be estimated through parametric or non parametric methods. Many of these methods rely on the quadratic variation formula ([8], [3], [1]). Subsequently, modifications of the quadratic variation method have been introduced in order to cope with specific difficulties arising with financial data. Presence of jumps in data leads to the use of bi-power variation instead of the quadratic

*Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050, Université Paris-Est Marne-la-Vallée, 5 Bld Descartes, Champs-sur-marne, 77454 Marne-la-Vallée Cedex 2, France.

†Département de Mathématiques, Université d'Évry Val d'Essonne, 91025 Évry Cedex, France

variation ([4], [2]), microstructure noises are handled with pre-averaged variation or related methods ([18], [10], [17]). In the case of estimation of cross-volatility for assets observed at non synchronous instants, some alternative methods were introduced by Hayashi and Yoshida ([6], [7]). Indeed, in case of non synchronous data, the standard quadratic variation method yield to a biased estimation of the cross-volatility.

In [12], Malliavin and Mancino proposes an estimator of the volatility based on the computation of N Fourier coefficients of an Itô process X and study some of its properties in [13]. A main advantage of this method is that the estimator only relies on the Fourier coefficients of each components of X computed individually. A proxy for these Fourier coefficients seems easily available to the statistician and the method has direct applications in empirical studies ([15], [14]).

In this paper we focus on the asymptotic properties of the Fourier estimator and give new results. We first assume that we observe continuously the process X on a fixed time interval and we prove a central limit theorem for the estimator of the integrated volatility as N goes to infinity with the rate of convergence $\sqrt{N}$, where N denotes the highest Fourier frequency used in the estimation method. The asymptotic variance is half the so called 'quarticity' which is the asymptotic variance when the volatility is estimated by a discrete sampling with step $1/N$. Although the case of exact observation of the path is unfeasible in practice, this result serves as a benchmark for the case of discrete observations.

Then we give asymptotic results in the case of discrete time observations. In a first step we assume that the observations of X are synchronous and we note n the number of observations. In this situation, we show that the estimator of the integrated volatility is consistent as N and n go to infinity. In particular, this extend the result of [13] where the restriction that N/n goes to zero was imposed. Turning to the central limit theorem the situation is drastically different and the limit behavior of N/n is crucial. Indeed assuming that N/n goes to zero we are essentially in the previously studied situation of continuous time observations. Now if we assume that N/n converges to $a > 0$, we obtain a central limit theorem with rate of convergence $\sqrt{n}$ but the asymptotic variance depends on the parameter a. In particular we prove that there are some optimal choices of N with respect to n to minimize the asymptotic variance.

In a second step, we investigate the case of non synchronous data. Although the Fourier coefficients involved in the estimation method are computed individually for each components, results are rather different with the situation of synchronous sampling and much more surprising. We first show that the estimator may be biased, when the highest frequency used matches the number of data collected. We explicitly compute this bias in a general framework, and prove a central limit theorem associated to this convergence. To illustrate the situation we consider the specific example of an alternate sampling scheme (n equidistant data alternatively collected on different components of X). We establish that if N/n goes to zero the estimator of the integrated volatility is consistent. This is not surprising since the effect of non synchronous data disappears under this assumption. Now if N/n tends to $a > 0$ the estimator is not consistent. We are able to correct the explicit bias. Then, we find an explicit expression for the variance of the estimator. Moreover, numerical simulations shows that the corrected estimator performs well on simulated data.

The paper is organized as follows. Section 2 is devoted to the asymptotic results assuming that X is observed continuously. Section 3 contains the case of discrete synchronous observations. Finally in section 4 we consider non synchronous data.

2 Weak convergence in case of continuous time observations

Throughout this paper, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is the canonical space of a d-dimensional Brownian motion on the time interval $[0, 2\pi]$. We denote by $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq 2\pi}$ the usual augmentation of the natural filtration of W and we denote by $L^2_a([0, 2\pi])$ the space of real measurable adapted processes $X = (X(t))_{0 \leq t \leq 2\pi}$ such that $\mathbb{E} \int_0^{2\pi} |X(t)|^2 dt < +\infty$. We assume that we observe a J-dimensional continuous time process $X(t) = (X^1(t), \ldots, X^J(t))$ solution of the equations

$$dX^j(t) = b^j(t)dt + \sum_{r=1}^{d} \sigma^{j,r}(t) dW^r(t), \tag{1}$$

where $b^j \in L^2_a([0, 2\pi])$ and $\sigma^{j,r} \in L^2_a([0, 2\pi])$ for $1 \leq j \leq J$ and $1 \leq r \leq d$. In the following, we note σ^{j*} the transpose of the vector σ^j and more generally A^* the transpose of a matrix A. We define the volatility process as

$$\Sigma^{j,j'}(t) = \sum_{r=1}^{d} \sigma^{j,r}(t) \sigma^{j',r}(t) = (\sigma^{j*} \sigma^{j'})(t) \tag{2}$$

for $1 \leq j, j' \leq J$. Our aim is to study the asymptotic properties of the Fourier transform estimator of Σ proposed by Malliavin and Mancino [12]. We denote by $(c_k(\Sigma^{j,j'}))_{k \in \mathbb{Z}}$ the Fourier coefficients of $\Sigma^{j,j'}$, we have :

$$c_k(\Sigma^{j,j'}) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \Sigma^{j,j'}(t) dt. \tag{3}$$

Moreover for any measurable bounded function h, we note $c_k(h dX^j)$ the Fourier coefficient

$$c_k(h dX^j) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} h(t) dX^j(t). \tag{4}$$

With these notations we construct the estimator

$$\Gamma_N^{j,j'}(h) = \frac{2\pi}{2N+1} \sum_{|l| \le N} c_{-l}(dX^j) c_l(h dX^{j'}). \tag{5}$$

We remark that for $h_k(t) = e^{-ikt}$, $\Gamma_N^{j,j'}(h_k)$ is a consistent estimator of $c_k(\Sigma^{j,j'})$ (see [12]). In fact applying Ito's formula to the product of stochastic integrals $c_{-l}(dX^j)c_l(hdX^{j'})$ we have :

$$\Gamma_N^{j,j'}(h) = \frac{1}{2\pi} \int_0^{2\pi} h(t) \Sigma^{j,j'}(t) dt + R_N^{j,j'}(2\pi),$$

$$R_N^{j,j'}(t) = \frac{1}{2\pi} \left(\int_0^t (\int_0^s d_N(s-u) dX^j(u)) h(s) dX^{j'}(s) + \int_0^t (\int_0^s d_N(s-u) h(u) dX^{j'}(u)) dX^j(s) \right), \tag{6}$$

where d_N denotes the normalized Dirichlet kernel

$$d_N(u) = \frac{1}{2N+1} \sum_{k=-N}^{N} e^{iku} = \frac{1}{2N+1} \frac{\sin((2N+1)u/2)}{\sin(u/2)}. \tag{7}$$

In this section, we study the weak convergence of the process $(R_N^{j,j'}(t))_{t\in[0,2\pi]}$ normalized by $\sqrt{N}$. More precisely we prove that $\sqrt{N}R_N^{j,j'}$ converges stably in law. We refer to Jacod [9] for the definition of the stable convergence in law. In order to prove this result we need some regularity assumptions on the coefficients b and σ. In particular we assume that σ admits a Malliavin derivative. We denote by D the derivative operator and we refer the reader to Nualart [16] for the basic theory of Malliavin calculus. We make the following hypotheses.

H1. For $1 \le j \le J$ and $1 \le r \le d$ we assume that $\forall p \ge 1$:

$$\mathbb{E}(\sup_{t\in[0,2\pi]} |b^j(t)|^p) < +\infty, \quad \mathbb{E}(\sup_{t\in[0,2\pi]} |\sigma^{j,r}(t)|^p) < +\infty.$$

H2. We assume that almost surely the function $t \mapsto \sigma(t)$ is continuous on $[0, 2\pi]$.

H3. We assume that $\forall p \ge 1$ $\sigma \in \mathbb{D}^{1,p}$ and that for $1 \le j \le J$ and $1 \le r \le d$

$$\mathbb{E}(\sup_{s,t\in[0,2\pi]} |D_s \sigma^{j,r}(t)|^p) < +\infty.$$

This hypothesis is satisfied in particular for a diffusion process with regular coefficients.

Theorem 1 *Under H1, H2 and H3, the process $\sqrt{N}R_N^{j,j'}$ converges stably in law to $R^{j,j'}$ with*

$$R^{j,j'}(t) = \frac{1}{2\sqrt{\pi}} \int_0^t |h(s)| \sqrt{(\sigma^{j*}\sigma^j)(s)(\sigma^{j'*}\sigma^{j'})(s) + (\sigma^{j*}\sigma^{j'})^2(s)} d\tilde{W}(s),$$

where $\tilde{W}$ is a brownian motion independent of W.

Before proving Theorem 1, we recall some useful properties of the Dirichlet kernel d_N.

Lemma 1 *Let d_N the normalized Dirichlet kernel defined by (7), then the following properties are satisfied.*

i) $\int_0^{2\pi} |d_N(u)|^2 du = \frac{2\pi}{2N+1}$*;*

ii) $\forall p > 1$*, there exists a constant* C_p *such that* $\int_0^{2\pi} |d_N(u)|^p du \leq \frac{C_p}{2N+1}$ *;*

iii) $\forall 0 < \varepsilon < 2\pi$*,* $\lim_{N\to+\infty} N \int_0^{\varepsilon} |d_N(u)|^2 du = \frac{\pi}{2}$*.*

The proof of this Lemma is straightforward and we omit it.

Proof of Theorem 1.

First step. We first observe that the hypothesis $H1$ implies that for $p > 1$

$$R_N^{j,j'}(t) = M_N^{j,j'}(t) + \tilde{M}_N^{j,j'}(t) + o_P(\frac{1}{\sqrt{N}}), \tag{8}$$

with

$$M_N^{j,j'}(t) = \frac{1}{2\pi}\int_0^t \left(\int_0^s d_N(s-u)\sigma^{j*}(u)dW(u)\right) h(s)\sigma^{j'*}(s)dW(s), \tag{9}$$

$$\tilde{M}_N^{j,j'}(t) = \frac{1}{2\pi}\int_0^t \left(\int_0^s d_N(s-u)h(u)\sigma^{j'*}(u)dW(u)\right) \sigma^{j*}(s)dW(s). \tag{10}$$

Since X^j and $X^{j'}$ are solutions of equation (1), we can decompose $R_N^{j,j'}(t)$ as follows :

$$R_N^{j,j'}(t) = M_N^{j,j'}(t) + \tilde{M}_N^{j,j'}(t) + I_N^1(t) + I_N^2(t) + I_N^3(t) + \tilde{I}_N^1(t) + \tilde{I}_N^2(t) + \tilde{I}_N^3(t), \tag{11}$$

with

$$I_N^1(t) = \frac{1}{2\pi}\int_0^t \left(\int_0^s d_N(s-u)b^j(u)du\right) h(s)\sigma^{j'*}(s)dW(s), \tag{12}$$

$$I_N^2(t) = \frac{1}{2\pi}\int_0^t \left(\int_0^s d_N(s-u)\sigma^{j*}(u)dW(u)\right) h(s)b^{j'}(s)ds, \tag{13}$$

$$I_N^3(t) = \frac{1}{2\pi}\int_0^t \left(\int_0^s d_N(s-u)b^j(u)du\right) h(s)b^{j'}(s)ds. \tag{14}$$

The other terms are similar and we don't give them explicitly. We have from $H1$

$$\begin{aligned} \mathbb{E}I_N^1(t)^2 &= \frac{1}{(2\pi)^2}\int_0^t \mathbb{E}\left(\left(\int_0^s d_N(s-u)b^j(u)du\right)^2 h(s)^2(\sigma^{j'*}\sigma^{j'})(s)\right) ds, \\ &\leq C(\int_0^{2\pi} |d_N(u)|du)^2 \leq C\frac{1}{N^{2/p}}, \end{aligned} \tag{15}$$

where C is a constant whose value may change from a line to another and $p > 1$. The last inequality is a consequence of Hölder's inequality and property ii) of Lemma 1. We bound $\mathbb{E}I_N^3(t)^2$ on the same way. It remains to prove that $N\mathbb{E}I_N^2(t)^2$ tends to zero as N goes to infinity in order to obtain (8). This is a little bit more technical. To simplify the notation, we introduce the process

$$Y_N^j(t,s) = \int_0^s d_N(t-u)\sigma^{j*}(u)dW(u). \tag{16}$$

From Burkholder-Davis-Gundy inequality and $H1$, it is easy to check that :

$$(\mathbb{E}\sup_{s\leq t}|Y_N(t,s)|^4)^{1/4} \leq C(\int_0^t |d_N(t-u)|^2du)^{1/2} \leq \frac{C}{\sqrt{N}}, \tag{17}$$

With this notation we have

$$I_N^2(t)^2 = C\int_{[0,t]^2} Y_N(s,s)h(s)b^{j'}(s)Y_N(s',s')h(s')b^{j'}(s')dsds'.$$

By symmetry, it is enough to prove that

$$N\mathbb{E}\int_{[0,t]^2} Y_N(s,s)h(s)b^{j'}(s)Y_N(s',s')h(s')b^{j'}(s')1_{s'\leq s}dsds' \to 0.$$

Let $\varepsilon > 0$, for $s-\varepsilon \leq s' \leq s$ we have using $H1$ and (17)

$$N\mathbb{E}\int_{[0,t]^2} Y_N(s,s)h(s)b^{j'}(s)Y_N(s',s')h(s')b^{j'}(s')1_{s-\varepsilon\leq s'\leq s}dsds' \leq C\varepsilon.$$

Now if $s' < s-\varepsilon$, we observe that

$$\textstyle\int_{[0,t]^2} Y_N(s,s)h(s)b^{j'}(s)Y_N(s',s')h(s')b^{j'}(s')1_{s'<s-\varepsilon}dsds' = I_N^{2,1} + I_N^{2,2},$$

with

$$I_N^{2,1} = \int_{[0,t]^2} Y_N(s,s-\varepsilon)h(s)b^{j'}(s)Y_N(s',s')h(s')b^{j'}(s')1_{s'<s-\varepsilon}dsds',$$

$$I_N^{2,2} = \int_{[0,t]^2}(\int_{s-\varepsilon}^s d_N(s-u)\sigma^{j*}(u)dW(u))h(s)b^{j'}(s)Y_N(s',s')h(s')b^{j'}(s')1_{s'<s-\varepsilon}dsds'.$$

From Cauchy-Schwarz inequality, $H1$ and (17)

$$N\mathbb{E}I_N^{2,1} \leq C\sqrt{N\int_\varepsilon^t d_N^2(v)dv}$$

and from lemma 1, the right hand side term of the inequality goes to zero with N if $t < 2\pi$. At last we observe by conditioning on $\mathcal{F}_{s-\varepsilon}$ that

$$\mathbb{E}I_N^{2,2} = \mathbb{E}\int_{[0,t]^2}(\int_{s-\varepsilon}^s d_N(s-u)\sigma^{j*}(u)dW(u))h(s)(b^{j'}(s)-b^{j'}(s-\varepsilon))Y_N(s',s')h(s')b^{j'}(s')1_{s'<s-\varepsilon}dsds',$$

and consequently

$$N\mathbb{E}I_N^{2,2} \leq C(\mathbb{E}\int_0^t |b^{j'}(s) - b^{j'}(s-\varepsilon)|^4 ds)^{1/4}.$$

Finally letting ε go to zero we obtain the announced result.

Now to study the weak convergence of $\sqrt{N}R_N^{j,j'}$ we just have to study the limit behavior of the martingale $\sqrt{N}(M_N^{j,j'} + \tilde{M}_N^{j,j'})$. By symetry this can be reduced to the martingale $\sqrt{N}M_N^{j,j'}$. Following Jacod [9] and Jacod-Protter [11] we just have to determine the limit in probability of $\left\langle \sqrt{N}M_N^{j,j'}(t), W^r(t)\right\rangle$ for $1 \leq r \leq d$ and $\left\langle \sqrt{N}M_N^{j,j'}(t), \sqrt{N}M_N^{j,j'}(t)\right\rangle$, $\forall t \in [0, 2\pi]$.

Second step. We prove that for all $r \in \{1, \dots, d\}$, $\left\langle \sqrt{N}M_N^{j,j'}(t), W^r(t)\right\rangle$ tends to zero in $L^2(\Omega)$ as N goes to infinity. Using the notation (16) we can write

$$\mathbb{E}\left(\left\langle \sqrt{N}M_N^{j,j'}(t), W^r(t)\right\rangle^2\right) = \frac{N}{4\pi^2}\int_{[0,t]^2} \mathbb{E}\left(Y_N^j(s,s)Y_N^j(s',s')\sigma^{j',r}(s)\sigma^{j',r}(s')\right) h(s)h(s')dsds'$$

From $H3$, using the duality for stochastic integrals, we have

$$\mathbb{E}\left(Y_N^j(s,s)Y_N^j(s',s')\sigma^{j',r}(s)\sigma^{j',r}(s')\right) = \mathbb{E}\left(\int_0^s d_N(s-u)\sigma^{j*}(u)D_u(Y_N^j(s',s')\sigma^{j',r}(s)\sigma^{j',r}(s'))du\right),$$

and consequently

$$\mathbb{E}\left(Y_N^j(s,s)Y_N^j(s',s')\sigma^{j',r}(s)\sigma^{j',r}(s')\right) = E_N^1(s,s') + E_N^2(s,s') + E_N^3(s,s'),$$

with

$$\begin{aligned}
E_N^1(s,s') &= \mathbb{E}\left((\sigma^{j',r}(s)\sigma^{j',r}(s')\int_0^s d_N(s-u)d_N(s'-u)1_{\{u\leq s'\}}(\sigma^{j*}\sigma^j)(u)du\right),\\
E_N^2(s,s') &= \mathbb{E}\left(\sigma^{j',r}(s)\sigma^{j',r}(s')\int_0^s d_N(s-u)\sigma^{j*}(u)(\int_0^{s'} d_N(s'-v)D_u(\sigma^{j*}(v))dW(v))du\right),\\
E_N^3(s,s') &= \mathbb{E}\left(Y_N^j(s',s')\int_0^s d_N(s-u)\sigma^{j*}(u)D_u(\sigma^{j',r}(s)\sigma^{j',r}(s'))du\right).
\end{aligned}$$

This leads to the decomposition

$$\mathbb{E}\left\langle \sqrt{N}M_N^{j,j'}(t), W^r(t)\right\rangle^2 = \frac{N}{4\pi^2}\int_{[0,t]^2}(E_N^1(s,s') + E_N^2(s,s') + E_N^3(s,s'))h(s)h(s')dsds'. \tag{18}$$

Using successively $H1$ and Fubini's theorem, we obtain for the first term

$$\begin{aligned}
\frac{N}{4\pi^2}|\int_{[0,t]^2} E_N^1(s,s')h(s)h(s')dsds'| &\leq CN\int_{[0,t]^2}\int_0^s |d_N(s-u)d_N(s'-u)|1_{\{u\leq s'\}}dsds',\\
&= CN\int_0^t(\int_u^t |d_N(s-u)|ds\int_u^t |d_N(s'-u)|ds')du,\\
&\leq CN(\int_0^{2\pi} |d_N(s)|ds)^2 \leq \frac{C}{N^{2/p-1}}.
\end{aligned}$$

Choosing $p \in (0,1)$, we deduce that the first term in (18) goes to zero. For the second term, we have from $H1$, $H3$ and Cauchy-Schwarz inequality

$$\begin{aligned} |E_N^2(s,s')| &\leq \int_0^s |d_N(s-u)||\mathbb{E}\sigma^{j*}(u)\int_0^{s'} d_N(s'-v)D_u(\sigma^{j*}(v))dW(v)\sigma^{j',r}(s)\sigma^{j',r}(s')|du, \\ &\leq C\int_0^s |d_N(s-u)|du(\int_0^{s'} |d_N(s'-v)|^2 dv)^{1/2} \leq \frac{C}{N^{1/p+1/2}}, \end{aligned}$$

this yields

$$\frac{N}{4\pi^2}|\int_{[0,t]^2} E_N^2(s,s')h(s)h(s')dsds'| \leq \frac{C}{N^{1/p-1/2}},$$

We proceed similarly for the third term and this achieves the proof of the second step.

Third step. We prove in this section that $\forall t \in [0,2\pi]$ the following convergence holds in probability

$$\lim_N \left\langle \sqrt{N}M_N^{j,j'}(t), \sqrt{N}M_N^{j,j'}(t)\right\rangle = \frac{1}{8\pi}\int_0^t (\sigma^{j*}\sigma^j)(s)(\sigma^{j'*}\sigma^{j'})(s)h(s)^2 ds. \tag{19}$$

With the preceding notation we have

$$\left\langle \sqrt{N}M_N^{j,j'}(t), \sqrt{N}M_N^{j,j'}(t)\right\rangle = \frac{N}{4\pi^2}\int_0^t Y_N^j(s,s)^2(\sigma^{j'*}\sigma^{j'})(s)h(s)^2 ds$$

From Ito's formula we have

$$Y_N^j(s,s)^2 = \int_0^s d_N^2(s-u)(\sigma^{j*}\sigma^j)(u)du + 2\int_0^s Y_N^j(s,u)d_N(s-u)\sigma^{j*}(u)dW(u),$$

and consequently

$$\left\langle \sqrt{N}M_N^{j,j'}(t), \sqrt{N}M_N^{j,j'}(t)\right\rangle = T_N^1(t) + T_N^2(t),$$

with

$$T_N^1(t) = \frac{N}{4\pi^2}\int_0^t (\int_0^s d_N^2(s-u)(\sigma^{j*}\sigma^j)(u)du)(\sigma^{j'*}\sigma^{j'})(s)h(s)^2 ds, \tag{20}$$

$$T_N^2(t) = \frac{N}{2\pi^2}\int_0^t (\int_0^s Y_N^j(s,u)d_N(s-u)\sigma^{j*}(u)dW(u))(\sigma^{j'*}\sigma^{j'})(s)h(s)^2 ds. \tag{21}$$

a) We first prove that $T_N^2(t)$ tends to zero in $L^2(\Omega)$. We denote by $Z_N^j(t,s)$:

$$Z_N^j(t,s) = \int_0^s Y_N^j(t,u)d_N(t-u)\sigma^{j*}(u)dW(u) \tag{22}$$

and we use the notation :

$$\sigma^{j'}(s,s')^4 = (\sigma^{j'*}\sigma^{j'})(s)(\sigma^{j'*}\sigma^{j'})(s').$$

We proceed as in the second step, we have :

$$\mathbb{E}(T_N^2(t))^2 = \frac{N^2}{4\pi^4}\int_{[0,t]^2} h^2(s)h^2(s')\mathbb{E}\left(Z_N^j(s,s)Z_N^j(s',s')(\sigma^{j'*}\sigma^{j'})(s)(\sigma^{j'*}\sigma^{j'})(s')\right)dsds'.$$

Now from the duality formula

$$\mathbb{E}\left(Z_N^j(s,s)Z_N^j(s',s')\sigma^{j'}(s,s')^4\right) \quad = \quad F_N^1(s,s') + F_N^2(s,s') + F_N^3(s,s'),$$

with

$$\begin{aligned}
F_N^1(s,s') &= \mathbb{E}\left(\sigma^{j'}(s,s')^4 \int_0^s Y_N^j(s,u)d_N(s-u)(\sigma^{j*}\sigma^j)(u)Y_N^j(s',u)d_N(s'-u)1_{\{u\le s'\}}du\right), \\
F_N^2(s,s') &= \mathbb{E}\left(\sigma^{j'}(s,s')^4 \int_0^s Y_N^j(s,u)d_N(s-u)\sigma^{j*}(\int_0^{s'} d_N(s'-v)D_u(Y_N^j(s',v)\sigma^{j*}(v))dW(v))du\right), \\
F_N^3(s,s') &= \mathbb{E}\left(Z_N^j(s',s')\int_0^s Y_N^j(s,u)d_N(s-u)\sigma^{j*}D_u(\sigma^{j'}(s,s')^4)du\right).
\end{aligned}$$

From $H1$ and (17) we get :

$$(\mathbb{E}Z_N^j(s,s)^2)^{1/2} \le C/N.$$

Combining Cauchy-Schwarz inequality with the preceding inequality, this gives :

$$\begin{aligned}
\frac{N^2}{4\pi^4}\int_{[0,t]^2} h^2(s)h^2(s')F_N^1 ds ds' &\le CN\int_{[0,t]^2}\int_0^s |d_N(s-u)d_N(s'-u)|1_{\{u\le s'\}}du ds ds', \\
&= CN\int_0^t(\int_u^t |d_N(s-u)|ds)^2 du,
\end{aligned}$$

and finally from Hölder 's inequality and Lemma 1 ii) we obtain for $p>1$:

$$\frac{N^2}{4\pi^4}\int_{[0,t]^2} h^2(s)h^2(s')F_N^1 ds ds' \le \frac{C}{N^{2/p-1}}.$$

Turning to F_N^3 we have

$$\tfrac{N^2}{4\pi^4}\textstyle\int_{[0,t]^2} h^2(s)h^2(s')F_N^3 ds ds' \le CN^2\int_{[0,t]^2}\int_0^s |d_N(s-u)|\mathbb{E}|Z_N^j(s',s')Y_N^j(s,u)\sigma^{j*}D_u(\sigma^{j'}(s,s')^4)|du ds ds',$$

but

$$\mathbb{E}|Z_N^j(s',s')Y_N^j(s,u)\sigma^{j*}D_u(\sigma^{j'}(s,s')^4)| \le \frac{C}{\sqrt{N}}(\mathbb{E}(Z_N^j(s',s'))^2)^{1/2} \le \frac{C}{N\sqrt{N}},$$

consequently for any $p>1$

$$\frac{N^2}{4\pi^4}\int_{[0,t]^2} h^2(s)h^2(s')F_N^3 ds ds' \le \frac{C}{N^{1/p-1/2}}.$$

We bound the last term on a similar way observing that

$$\begin{aligned}
D_u(Y_N^j(s',v)\sigma^{j*}(v)) &= Y_N^j(s',v)D_u(\sigma^{j*}(v)) + d_n(s'-u)\sigma^j(u)1_{\{u\le v\}}\sigma^{j*}(v) \\
&\quad + \int_0^v d_N(s'-v')D_u(\sigma^j(v'))dW_{v'}\sigma^{j*}(v).
\end{aligned}$$

Finally we conclude that $\mathbb{E}(T_N^2(t))^2$ tends to zero as N goes to infinity.

b) Now we determine the limit in probability of $T_N^1(t)$ given by (20). For $0 \leq t < 2\pi$, one can easily check from the continuity assumption $H2$ and from Lemma 1 iii) that almost surely $\forall s \leq t$:

$$\lim_N N \int_0^s d_N^2(s-u)(\sigma^{j*}\sigma^j)(u)du = \lim_N N \int_0^s d_N^2(u)(\sigma^{j*}\sigma^j)(s-u)du = \frac{\pi}{2}(\sigma^{j*}\sigma^j)(s).$$

We deduce then from the dominated convergence Theorem that $\forall t < 2\pi$

$$\lim_N T_N^1 = \frac{1}{8\pi} \int_0^t (\sigma^{j*}\sigma^j)(s)(\sigma^{j'*}\sigma^{j'})(s)h^2(s)ds.$$

At last by a continuity argument the preceding result holds for $t \in [0, 2\pi]$.

Fourth step. We turn back to the decomposition of $R_N^{j,j'}(t)$ given in (8). We deduce from the second step that $\left\langle \sqrt{N}(M_N^{j,j'}(t) + \tilde{M}_N^{j,j'}(t)), W^r(t) \right\rangle$ tends to zero, $\forall r \in \{1, \ldots, d\}$. And from the third step we can prove that

$$\lim_N \left\langle \sqrt{N} N_N^{j,j'}(t), \sqrt{N} N_N^{j,j'}(t)) \right\rangle = \frac{1}{4\pi} \int_0^t h^2(s)((\sigma^{j*}\sigma^j)(s)(\sigma^{j'*}\sigma^{j'})(s) + (\sigma^{j*}\sigma^{j'})^2(s))ds,$$

where $N_N^{j,j'}(t) = M_N^{j,j'}(t) + \tilde{M}_N^{j,j'}(t)$. This achieves the proof of Theorem 1. $\diamond$

The estimator of the volatility studied in this section is constructed from the observation of $X(t)$ on the time interval $[0, 2\pi]$. However in practice we observe $X(t)$ at discrete time (t_k) and so we propose in the next section an estimator of $\Sigma^{j,j'}$ based on discrete time observations.

3 Weak convergence in case of discrete time observations

We assume in this section that we observe the process $X(t)$ solution of (1) at time $t_k = \frac{2\pi k}{n}$ for $k = 0, \ldots, n$. The case of non synchronous data will be treated in the next section. We denote by φ_n the function :

$$\varphi_n(t) = \frac{2\pi k}{n} \quad \text{if } \frac{2\pi k}{n} \leq t < \frac{2\pi(k+1)}{n}. \tag{23}$$

Our estimators are now based on the discrete Fourier coefficients

$$c_k^n(h_n dX^j) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi_n(t)} h_n(t) dX^j(t), \tag{24}$$

where $h_n(t) = h(\varphi_n(t))$. With these notations we define the discrete time estimators

$$\Gamma_{N,n}^{j,j'}(h) = \frac{2\pi}{2N+1} \sum_{|l| \leq N} c_{-l}^n(dX^j) c_l^n(h_n dX^{j'}). \tag{25}$$

Our aim is to study the asymptotic properties of these estimators as n and N go to infinity.

We first remark that for $h = 1$, $\Gamma^{j,j'}_{N,n}(1)$ is an estimator of the integrated volatility $\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{j,j'}(t)dt$. Moreover we can check that $\lim_N \Gamma^{j,j'}_{N,n}(1)$ is the discretized quadratic covariation of the process $(X^j, X^{j'})$ and consequently a central limit theorem holds as n goes to infinity with the classical rate of convergence $\sqrt{n}$ (see Genon-Catalot and Jacod [5]). In fact if we note $\Delta X^j_k = X^j(t_{k+1}) - X^j(t_k)$, we have

$$\begin{aligned} c^n_{-l}(dX^j)c^n_l(dX^{j'}) &= \frac{1}{4\pi^2}\sum_{k=0}^{n-1} e^{il\frac{2k\pi}{n}}\Delta X^j_k \sum_{k=0}^{n-1} e^{-il\frac{2k\pi}{n}}\Delta X^{j'}_k, \\ &= \frac{1}{4\pi^2}\sum_{k=0}^{n-1}\Delta X^j_k \Delta X^{j'}_k + \frac{1}{4\pi^2}\sum_{k\neq k'=0}^{n-1} e^{il\frac{2(k-k')\pi}{n}}\Delta X^j_k \Delta X^{j'}_{k'}. \end{aligned}$$

This gives

$$\Gamma^{j,j'}_{N,n}(1) = \frac{1}{2\pi}\sum_{k=0}^{n-1}\Delta X^j_k \Delta X^{j'}_k + \frac{1}{2\pi}\sum_{k\neq k'=0}^{n-1} d_N(\frac{2(k-k')\pi}{n})\Delta X^j_k \Delta X^{j'}_{k'}.$$

Assuming that n is fixed and letting N go to infinity, we have $\lim_N d_N(\frac{2(k-k')\pi}{n}) = 0$, for $k \neq k'$ and consequently

$$\lim_{N\to\infty}\Gamma^{j,j'}_{N,n}(1) = \frac{1}{2\pi}\sum_{k=0}^{n-1}\Delta X^j_k \Delta X^{j'}_k, \tag{26}$$

which is the expected result.

In this section we assume that N and n go to infinity simultaneously and that N/n tends to $a \in \mathbb{R}_+$. The case $a = 0$ has been studied essentially in the preceding section. In this case the process $(X(t))$ is observed continously. In the case $a = +\infty$ the estimator $\Gamma^{j,j'}_{\infty,n}$ corresponds to the discretized quadratic covariation of the process $(X(t))$ as we remark above.

Before stating our main result we give some useful properties of the discretized normalized Dirichlet kernel.

Lemma 2 *Let φ_n defined by (23) and d_N the normalized Dirichlet kernel then we have, assuming that $N/n \to a > 0$:*

i) $\forall p > 1$, $\exists C_{p,a}$ *such that* $\int_0^{2\pi} |d_N(\varphi_n(u))|^p du \le C_{p,a}/N$*;*

ii) $\forall 0 < \varepsilon < \pi$, $\lim_{N,n\to\infty} N\int_0^{\varepsilon} |d_N(\varphi_n(u))|^2 du = 2\pi a(\eta(2a) + 1)$, *with*

$$\eta(a) = \sum_{k=1}^{\infty}\frac{\sin^2(a\pi k)}{(a\pi k)^2} = \frac{1}{2a^2}r(a)(1 - r(a)), \quad r(a) = a - [a],$$

where $[x]$ denotes the integer part of x ;

iii) $\forall 0 < \varepsilon < \pi$, $\lim_{N,n\to\infty} N \int_{2\pi/n}^{\varepsilon} |d_N(\varphi_n(u))|^2 du = 2\pi a\eta(2a)$ *;*

iv) $\forall 0 < \varepsilon, \eta < \pi$, $\lim_{N,n\to\infty} N \int_{\varepsilon}^{2\pi-\eta} |d_N(\varphi_n(u))|^2 du = 0$.

We remark that the function $\eta(a) = 0$ if $a \in \mathbb{N}^*$. Moreover $a\eta(2a)$ tends to $1/4$ as a goes to zero and then ii) is coherent with Lemma 1 iii).

Proof We omit the proofs of i) and iv) which are straightforward.

ii) Let $0 < \varepsilon < \pi$. We note $k_n(\varepsilon) = n\varphi_n(\varepsilon)/2\pi$. We have

$$\int_0^{\varepsilon} |d_N(\varphi_n(u))|^2 du = \frac{2\pi}{n} \sum_{k=0}^{k_n(\varepsilon)-1} d_N^2(\frac{2k\pi}{n}) + \int_{\varphi_n(\varepsilon)}^{\varepsilon} d_N^2(\varphi_n(u))du.$$

Obviously $\lim_{N,n} N \int_{\varphi_n(\varepsilon)}^{\varepsilon} d_N^2(\varphi_n(u))du = 0$, so we just have to determine $\lim_{N,n} \sum_{k=0}^{k_n(\varepsilon)-1} d_N^2(\frac{2k\pi}{n})$. But

$$\sum_{k=0}^{k_n(\varepsilon)-1} d_N^2(\frac{2k\pi}{n}) = 1 + \sum_{k=1}^{\infty} \frac{1}{(2N+1)^2} \frac{\sin^2((2N+1)k\pi/n)}{\sin^2(k\pi/n)} 1_{\{k \le k_n(\varepsilon)\}}.$$

We have

$$\lim_{N,n} \frac{1}{(2N+1)^2} \frac{\sin^2((2N+1)k\pi/n)}{\sin^2(k\pi/n)} = \frac{\sin^2(2a\pi k)}{(2a\pi k)^2},$$

and since $k_n(\varepsilon)\pi/n \le \varepsilon/2 < \pi/2$, using $\sin x \ge 2x/\pi$ for $x \le \pi/2$,

$$\frac{1}{(2N+1)^2} \frac{\sin^2((2N+1)k\pi/n)}{\sin^2(k\pi/n)} 1_{\{k \le k_n(\varepsilon)\}} \le \frac{C}{(2ak)^2}.$$

We conclude then from the dominated convergence Theorem that

$$\lim_{N,n} \sum_{k=0}^{k_n(\varepsilon)-1} d_N^2(\frac{2k\pi}{n}) = 1 + \sum_{k=1}^{\infty} \frac{\sin^2(2a\pi k)}{(2a\pi k)^2},$$

and if we note

$$\eta(a) = \sum_{k=1}^{\infty} \frac{\sin^2(a\pi k)}{(a\pi k)^2},$$

we obtain

$$\lim_{N,n} N \int_0^{\varepsilon} |d_N(\varphi_n(u))|^2 du = 2\pi a(\eta(2a) + 1).$$

To achieve the proof of ii), it remains to explicit $\eta(a)$.

Defining $f(x) = x(1-x)$ on $[0,1]$, we have the Fourier Development

$$f(x) = \frac{1}{6} - \sum_{k\ge1} \frac{\cos(2\pi kx)}{\pi^2 k^2},$$

we deduce then that

$$\sum_{k\geq 1}\frac{\cos(2\pi ka)}{k^2}=\sum_{k\geq 1}\frac{\cos(2\pi k(a-[a]))}{k^2}=\pi^2(\frac{1}{6}-r(a)(1-r(a)),$$

with $r(a)=a-[a]$. Turning back to $\eta(a)$, we obtain

$$\begin{aligned}\eta(a) &= \sum_{k=1}^{\infty}\frac{1-\cos(2a\pi k)}{(2a^2\pi^2k^2)},\\ &= \frac{1}{2a^2}r(a)(1-r(a)).\end{aligned}$$

To prove iii), we just observe that

$$N\int_0^{2\pi/n}d_N(\varphi_n(u))du=d_N(0)2\pi N/n\to 2\pi a,$$

since $d_N(0)=1$. $\diamond$

We can now state the main result of this section. In the following, we note $d_{N,n}(s,u)=d_N(\varphi_n(s)-\varphi_n(u))$. From Ito's formula we have the decomposition

$$\Gamma^{j,j'}_{N,n}(h)=\frac{1}{2\pi}\int_0^{2\pi}h_n(t)\Sigma^{j,j'}(t)dt+R^{j,j'}_{N,n}(2\pi),\tag{27}$$

with

$$R^{j,j'}_{N,n}(t)=\frac{1}{2\pi}\left(\int_0^t(\int_0^s d_{N,n}(s,u)dX^j(u))h_n(s)dX^{j'}(s)ds+\int_0^t(\int_0^s d_{N,n}(s,u)h_n(u)dX^{j'}(u))dX^j(s)ds\right).\tag{28}$$

Theorem 2 *Let h a continuous bounded function with bounded derivative. We assume that N and n tend to infinity and that $\lim N/n=a>0$ then under $H1$, $H2$ and $H3$*

i) the process $\sqrt{N}R^{j,j'}_{N,n}$ converges stably in law to $R^{j,j'}$ with

$$R^{j,j'}(t)=\left(\frac{a(2\eta(2a)+1)}{2\pi}\right)^{1/2}\int_0^t|h(s)|\sqrt{(\sigma^{j*}\sigma^j)(s)(\sigma^{j'*}\sigma^{j'})(s)+(\sigma^{j*}\sigma^{j'})^2(s)}d\tilde{W}(s),$$

where $\tilde{W}$ is a brownian motion independent of W and $\eta(a)$ is defined in Lemma 2 ;

ii) $\sqrt{n}(\Gamma^{j,j'}_{N,n}(h)-\frac{1}{2\pi}\int_0^{2\pi}h(t)\Sigma^{j,j'}(t)dt)$ converges stably in law to $R^{j,j'}(2\pi)/\sqrt{a}$.

Recalling that $\eta(a)=0$ for $a\in\mathbb{N}^*$, we deduce from ii) that the optimal asymptotic variance is obtain for $2a\in\mathbb{N}^*$ and is equal to

$$\frac{1}{2\pi}\int_0^{2\pi}|h(s)|^2\left((\sigma^{j*}\sigma^j)(s)(\sigma^{j'*}\sigma^{j'})(s)+(\sigma^{j*}\sigma^{j'})^2(s)\right)ds.$$

In particular given n the number of observations, the choice of $N = n/2$ Fourier coefficients to estimate the integrated volatility is optimal and in this case, the variance is the same one as for the quadratic variation estimator. Remark that the choice $N = n/2$ was used in earlier empirical work [15] since it corresponds to the natural choice of the Nyquist frequency of the signal.

Remark 1 *Let us stress that the case $a = 0$ is excluded in theorem 2, and that the variance of the limit variable $R^{j,j'}(2\pi)/\sqrt{a}$ explodes as $a \to 0$. Actually it could be shown that for $a = 0$, $\sqrt{N}R^{j,j'}_{N,n}(t)$ converges to the limit term given in theorem 1. Thus, in this situation the effect of sampling disappears and the behaviour is analogous to the case of a continuous observation. However the rate of convergence $\sqrt{N}$ is slow compared to the number of observations $\sqrt{n}$. Hence, from the statistical point of view, this situation is not satisfactory.*

Proof ii) is a straightforward consequence of i) since from (27) we have

$$\Gamma^{j,j'}_{N,n}(h) = \frac{1}{2\pi}\int_0^{2\pi} h(t)\Sigma^{j,j'}(t)dt + O_P(\frac{1}{n}) + R^{j,j'}_{N,n}(2\pi).$$

i) We proceed as in the proof of Theorem 1.

First and second steps. As previously, from $H1$ and Lemma 2 i), iv), we have the decomposition:

$$R^{j,j'}_{N,n}(t) = M^{j,j'}_{N,n}(t) + \tilde{M}^{j,j'}_{N,n}(t) + o_P(\frac{1}{\sqrt{N}}), \tag{29}$$

with

$$M^{j,j'}_{N,n}(t) = \frac{1}{2\pi}\int_0^t \left(\int_0^s d_{N,n}(s,u)\sigma^{j*}(u)dW(u)\right) h_n(s)\sigma^{j'*}(s)dW(s), \tag{30}$$

$$\tilde{M}^{j,j'}_{N,n}(t) = \frac{1}{2\pi}\int_0^t \left(\int_0^s d_{N,n}(s,u)h_n(u)\sigma^{j'*}(u)dW(u)\right) \sigma^{j*}(s)dW(s). \tag{31}$$

Moreover from $H1$ and $H3$, $\left\langle \sqrt{N}(M^{j,j'}_{N,n}(t) + \tilde{M}^{j,j'}_{N,n}(t)), W^r(t)\right\rangle$ tends to zero in probability for $1 \le r \le d$, $\forall t \in [0, 2\pi]$.

Third step. We prove in this section that $\forall t \in [0, 2\pi]$ the following convergence holds in probability

$$\lim_N \left\langle \sqrt{N}M^{j,j'}_{N,n}(t), \sqrt{N}M^{j,j'}_{N,n}(t)\right\rangle = \frac{a(2\eta(a)+1)}{4\pi}\int_0^t (\sigma^{j*}\sigma^j)(s)(\sigma^{j'*}\sigma^{j'})(s)h(s)^2 ds. \tag{32}$$

We note

$$Y^j_{N,n}(t,s) = \int_0^s d_{N,n}(t,u)\sigma^{j*}(u)dW(u). \tag{33}$$

$$\tag{34}$$

We have

$$\left\langle \sqrt{N} M_{N,n}^{j,j'}(t), \sqrt{N} M_{N,n}^{j,j'}(t) \right\rangle = \frac{N}{4\pi^2} \int_0^t Y_{N,n}^j(s,s)^2 (\sigma^{j'*}\sigma^{j'})(s) h_n(s)^2 ds,$$

and from Ito's formula

$$\left\langle \sqrt{N} M_{N,n}^{j,j'}(t), \sqrt{N} M_{N,n}^{j,j'}(t) \right\rangle = T_{N,n}^1(t) + T_{N,n}^2(t),$$

with

$$T_{N,n}^1(t) = \frac{N}{4\pi^2} \int_0^t (\int_0^s d_{N,n}^2(s,u)(\sigma^{j*}\sigma^j)(u)du)(\sigma^{j'*}\sigma^{j'})(s) h_n(s)^2 ds, \tag{35}$$

$$T_{N,n}^2(t) = \frac{N}{2\pi^2} \int_0^t (\int_0^s Y_{N,n}^j(s,u) d_{N,n}(s,u)\sigma^{j*}(u)dW(u))(\sigma^{j'*}\sigma^{j'})(s) h_n(s)^2 ds. \tag{36}$$

a) Following the proof of Theorem 1, one can easily see that $T_{N,n}^2(t)$ tends to zero in $L^2(\Omega)$.

b) We just have to determine the limit in probability of $T_{N,n}^1(t)$ given by (35). For $0 \le t < 2\pi$ and $s \le t$ we have

$$\int_0^s d_{N,n}^2(s,u)(\sigma^{j*}\sigma^j)(u)du = V_{N,n}^1(s) + V_{N,n}^2(s),$$

with

$$V_{N,n}^1(s) = \int_{\varphi_n(s)}^s d_{N,n}^2(s,u)(\sigma^{j*}\sigma^j)(u)du, \tag{37}$$

$$V_{N,n}^2(s) = \int_0^{\varphi_n(s)} d_{N,n}^2(s,u)(\sigma^{j*}\sigma^j)(u)du. \tag{38}$$

It is easy to check that (since $D_N(0) = 1$)

$$V_{N,n}^1(s) = \int_{\varphi_n(s)}^s (\sigma^{j*}\sigma^j)(v)dv,$$

and consequently, assuming $H2$

$$\lim_{N,n} N \int_0^t V_{N,n}^1(s)(\sigma^{j'*}\sigma^{j'})(s) h_n(s)^2 ds = a\pi \int_0^t (\sigma^{j*}\sigma^j)(s)(\sigma^{j'*}\sigma^{j'})(s) h(s)^2 ds.$$

Now we set $v = \varphi_n(s) + 2\pi/n - u$ in $V_{N,n}^2(s)$ and we obtain for $0 < \varepsilon < \pi$:

$$NV_{N,n}^2(s) = N \int_{2\pi/n}^{\varepsilon} d_N^2(\varphi_n(v))(\sigma^{j*}\sigma^j)(\varphi_n(s) + 2\pi/n - v)dv \tag{39}$$

$$+N \int_{\varepsilon}^{\varphi_n(s)+2\pi/n} d_N^2(\varphi_n(v))(\sigma^{j*}\sigma^j)(\varphi_n(s) + 2\pi/n - v)dv. \tag{40}$$

From Lemma 2 iv), the second term tends to zero and from iii) we conclude that almost surely, $\forall s \le t$

$$\lim_N NV_{N,n}^2(s) = 2a\eta(2a)\pi(\sigma^{j*}\sigma^j)(s).$$

Finally, we obtain $\forall 0 \le t < 2\pi$:

$$\lim_{N,n} T^1_{N,n}(t) = \frac{a(2\eta(2a)+1)}{4\pi} \int_0^t (\sigma^{j*}\sigma^j)(s)(\sigma^{j'*}\sigma^{j'})(s)h(s)^2 ds,$$

and we end the proof as in section 2. $\diamond$

4 non synchronous data

4.1 General case

In this section we consider the case of non-synchronous data. To lighten the notations, we assume that $X = (X^1, X^2)$ is two-dimensional and respectively denote by $(t^1_k)_{k=0,\dots,M^1_n}$ and $(t^2_k)_{k=0,\dots,M^2_n}$ the instants when the data are collected for X^1 and X^2. There is no need that the number of data is the same for each component. For simplicity assume $t^1_0 = t^2_0 = 0$ and define for $j = 1, 2$ and $t \in [0, 2\pi]$:

$$\varphi^j_n(t) = \sup\{t^j_k \mid t^j_k \le t\}.$$

The estimator is based on the discrete Fourier coefficients:

$$c^{n,1}_k(dX^1) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi^1_n(t)} dX^1(t), \quad c^{n,2}_k(h^2_n dX^2) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi^2_n(t)} h(\varphi^2_n(t)) dX^2(t) \tag{41}$$

where $h^2_n(t) = h(\varphi^2_n(t))$. We shall focus on the covolatility estimator

$$\Gamma^{1,2}_{N,n}(h) = \frac{2\pi}{2N+1} \sum_{|l| \le N} c^{n,1}_{-l}(dX^1) c^{n,2}_l(h^2_n dX^2). \tag{42}$$

However, it is clear that by taking $X^2 = X^1$ with $\varphi^2_n(t) = \varphi^1_n(t)$, results for the covolatility estimator imply results for the volatility estimator of any components.

As suggested by the previous section, one need to calibrate the cut-off frequency $N = N_n$ with the sampling steps in order to get some result.

To maintain shorter notation, we define for $s, u \in [0, 2\pi]$:

$$d^{i,j}_{N,n}(s,u) = d_N(\varphi^i_n(s) - \varphi^j_n(u)), \quad \text{with } i,j \in \{1,2\} \tag{43}$$

We make the following assumptions.

A1 For $j = 1, 2$, $\sup_{k=0,\dots,M^j_n-1} \left| t^j_{k+1} - t^j_k \right| \xrightarrow{n\to\infty} 0$.

A2 The sequence of function $t \mapsto d^{1,2}_{N_n,n}(t,t)$ weakly converges to some integrable function $\gamma(t)$: for any continuous function f

$$\int_0^{2\pi} d^{1,2}_{N_n,n}(t,t) f(t) dt \xrightarrow{n\to\infty} \int_0^{2\pi} \gamma(t) f(t) dt.$$

Proposition 1 *Let h a continuous bounded function. Assume H1, H2, A1 and A2, then we have:*

$$\Gamma^{1,2}_{N_n,n}(h) \xrightarrow[\mathbf{P}]{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \gamma(t)h(t)\Sigma^{1,2}(t)dt.$$

Proof From Ito's formula we obtain a decomposition analogous to (27)–(28) in the case of non-synchronous data:

$$\Gamma^{1,2}_{N_n,n}(h) = \frac{1}{2\pi}\int_0^{2\pi} d^{1,2}_{N_n,n}(t,t)h^2_n(t)\Sigma^{1,2}(t)dt + R^{1,2}_{N_n,n}(2\pi), \tag{44}$$

$$R^{1,2}_{N_n,n}(t) = \frac{1}{2\pi}\left(\int_0^t (\int_0^s d^{1,2}_{N_n,n}(u,s)dX^1(u))h^2_n(s)dX^2(s) + \int_0^t (\int_0^s d^{2,1}_{N_n,n}(u,s)h^2_n(u)dX^2(u))dX^1(s)\right)$$

where we have used the notation (43). Remark that contrarily to the synchronous case, the kernels $d^{i,j}_{N,n}$ are non symmetric here.

Using the continuity of the functions h and $\Sigma^{1,2}$, the assumptions A1–A2 easily imply that the first term in the right hand side of (44) converges almost surely to $\frac{1}{2\pi}\int_0^{2\pi}\gamma(t)h(t)\Sigma^{1,2}(t)dt$.

It remains to check that $R^{1,2}_{N,n}(2\pi)$ converges to zero in $\mathbf{L^2}$ norm. Since X is solution of (1) we can find for $R^{1,2}_{N_n,n}(2\pi)$ a decomposition analogous to (11) and check that the two leading terms now are $M^{1,2}_{N_n,n}(2\pi)$ and $\tilde{M}^{1,2}_{N_n,n}(2\pi)$ with:

$$M^{1,2}_{N_n,n}(t) = \frac{1}{2\pi}\int_0^t \left(\int_0^s d^{1,2}_{N_n,n}(u,s)\sigma^{1*}(u)dW(u)\right) h^2_n(s)\sigma^{2*}(s)dW(s), \tag{45}$$

$$\tilde{M}^{1,2}_{N_n,n}(t) = \frac{1}{2\pi}\int_0^t \left(\int_0^s d^{2,1}_{N_n,n}(u,s)h^2_n(u)\sigma^{2*}(u)dW(u)\right) \sigma^{1*}(s)dW(s). \tag{46}$$

We finish the proof by showing that these terms converge to zero. Using assumption H1 with the Burkholder–Davis–Gundy inequality we get

$$E\left[(M^{1,2}_{N_n,n}(2\pi))^2\right] \le c\int_0^{2\pi}\int_0^s d^{1,2}_{N_n,n}(u,s)^2 duds, \tag{47}$$

for some constant c.

Assume $u \neq s$ are fixed, then by A1, for large enough n, we have $\left|\varphi^1_n(u) - \varphi^2_n(s)\right| > |u-s|/2 > 0$. Using that the Dirichlet kernel d_N uniformly converges to zero on compact subsets of $(0, 2\pi)$ we deduce that $d^{1,2}_{N_n,n}(u,s) \xrightarrow{n\to\infty} 0$. Since $d^{1,2}_{N_n,n}(u,s)$ is bounded by the constant 1, the dominated convergence theorem implies that the right hand side of (47) converges to zero. We treat $\tilde{M}^{1,2}_{N_n,n}(2\pi)$ on the same way and the proposition is shown. $\diamond$

Remark 2 *Let us stress that the conditions A1–A2 for convergence of the estimator are rather weak. The condition A2 relates the choice of the frequency N_n with the sampling scheme. In most situations, by choosing $1/N_n$ converging slowly to zero, it seems possible to get that $d^{1,2}_{N_n,n}(t,t)$ converges pointwise to 1. In this case the estimator is consistent for the co-volatility.*

If N_n is chosen too large, a bias may appear when $d^{1,2}_{N_n,n}(t,t)$ converges to a function not everywhere equal to one. We shall see that the weak convergence of $d^{1,2}_{N_n,n}(t,t)$ is the natural assumption in this circumstance.

The following assumptions are needed to obtain a central limit theorem related to proposition 1.

A3 $\forall p>1,\ \exists C_p,\ \forall n\geq 1,\ \forall i\in\{0,1\},\ \sup_{c\in[0,2\pi]}\int_0^{2\pi}\left|d_{N_n}(\varphi^i_n(s)-c)\right|^p ds\leq C_p/n.$

A4 There exists three integrable functions $\gamma^{1,2}$,$\tilde\gamma^{1,2}$, γ^c defined on $[0,2\pi]$ such that, for any continuous function $g:[0,2\pi]^2\mapsto\mathbb{R}$ the following convergences hold for all $t\in[0,2\pi)$,

$$n\int_0^t\int_0^s d^{1,2}_{N_n,n}(u,s)^2 g(u,s)duds \xrightarrow{n\to\infty} \int_0^t \gamma^{1,2}(u)g(u,u)du,$$

$$n\int_0^t\int_0^s d^{2,1}_{N_n,n}(u,s)^2 g(u,s)duds \xrightarrow{n\to\infty} \int_0^t \tilde\gamma^{1,2}(u)g(u,u)du,$$

$$n\int_0^t\int_0^s d^{1,2}_{N_n,n}(u,s)d^{2,1}_{N_n,n}(u,s) g(u,s)duds \xrightarrow{n\to\infty} \int_0^t \gamma^{c}(u)g(u,u)du.$$

In the statement of the assumption A4, we exclude the convergence in the case $t=2\pi$, as it will be convenient in the example given below. However under the assumption A3, the condition A4 is clearly equivalent if we include $t=2\pi$ in the statement.

Theorem 3 *Let h a continuous bounded function. Assume H1–H3, A1, A3–A4, then the sequence of random variables*

$$\sqrt{n}\left(\Gamma^{1,2}_{N_n,n}(h)-\frac{1}{2\pi}\int_0^{2\pi} d^{1,2}_{N_n,n}(t,t)h_n^2(t)\Sigma^{1,2}(t)dt\right)$$

converges stably in law to a variable with the representation,

$$\frac{1}{2\pi}\int_0^{2\pi}|h(t)|\sqrt{(\gamma^{1,2}(t)+\tilde\gamma^{1,2}(t))\Sigma^{1,1}(t)\Sigma^{2,2}(t)+2\gamma^c(t)(\Sigma^{1,2}(t))^2}d\tilde W_t.$$

Proof From (44) the theorem amount to show the convergence of $\sqrt{n}R^{1,2}_{N_n,n}(2\pi)$ as $n\to\infty$. We proceed as in the proof of theorem 1.

First and second steps. Again we use the decomposition, $R^{1,2}_{N_n,n}(t)=M^{1,2}_{N_n,n}(t)+\tilde M^{1,2}_{N_n,n}(t)+I^1_{N_n,n}(t)+I^2_{N_n,n}(t)+I^3_{N_n,n}(t)+\tilde I^1_{N_n,n}(t)+\tilde I^2_{N_n,n}(t)+\tilde I^3_{N_n,n}(t)$, where the main terms $M^{1,2}_{N_n,n}(t)$ and

$\tilde{M}^{1,2}_{N_n,n}(t)$ are explicitly given in (45)–(46) and the remainder terms are analogous to (12)–(14). We need to show that these remainder terms are negligible versus $1/\sqrt{n}$. For instance, by a computation analogous to (15) we find for any $p > 1$:

$$E[(I^1_{N_n,n}(t))^2] \leq C \int_0^t \left(\int_0^s d^{1,2}_{N_n,n}(u,s) du \right)^2 ds \leq \frac{C}{n^{2/p}}$$

where we have used the assumption A3. This gives the result for $I^1_{N_n,n}$. We omit the details for the other terms.

For the second step, using assumption A3, we show as in theorem 1 that $\left\langle \sqrt{n} M^{1,2}_{N_n,n}(t), W^r(t) \right\rangle$ $\left\langle \sqrt{n} \tilde{M}^{1,2}_{N_n,n}(t), W^r(t) \right\rangle$ tend to zero in $L^2(\Omega)$ for all $r \in \{1, \ldots, d\}$.

Third step. Here, we prove that the following three convergence properties hold in probability:

$$\lim_n \left\langle \sqrt{n} M^{1,2}_{N_n,n}(t), \sqrt{n} M^{1,2}_{N_n,n}(t) \right\rangle \xrightarrow{n\to\infty} \frac{1}{4\pi^2} \int_0^t \gamma^{1,2}(s) h(s)^2 \Sigma^{1,1}(s) \Sigma^{2,2}(s) ds,$$
$$\lim_n \left\langle \sqrt{n} \tilde{M}^{1,2}_{N_n,n}(t), \sqrt{n} \tilde{M}^{1,2}_{N_n,n}(t) \right\rangle \xrightarrow{n\to\infty} \frac{1}{4\pi^2} \int_0^t \tilde{\gamma}^{1,2}(s) h(s)^2 \Sigma^{1,1}(s) \Sigma^{2,2}(s) ds,$$
$$\lim_n \left\langle \sqrt{n} M^{1,2}_{N_n,n}(t), \sqrt{n} \tilde{M}^{1,2}_{N_n,n}(t) \right\rangle \xrightarrow{n\to\infty} \frac{1}{4\pi^2} \int_0^t \gamma^c(s) h(s)^2 \Sigma^{1,2}(s)^2 ds.$$

For the first property, we follow the proof of the part 3 in theorem 2 and easily get,

$$\left\langle \sqrt{n} M^{1,2}_{N_n,n}(t), \sqrt{n} M^{1,2}_{N_n,n}(t) \right\rangle = \frac{n}{4\pi^2} \int_0^t \int_0^s d^{1,2}_{N_n,n}(u,s)^2 \Sigma^{1,1}(u) du \Sigma^{2,2}(s) h^2_n(s)^2 ds + o_P(1).$$

Now the convergence of $\left\langle \sqrt{n} M^{1,2}_{N_n,n}(t), \sqrt{n} M^{1,2}_{N_n,n}(t) \right\rangle$ follows from assumptions A1, A4 and the continuity of h, $\Sigma^{1,1}$, $\Sigma^{2,2}$. The two other properties are shown in a similar way. $\diamond$

4.2 An example

We give in this section an example of a two dimensional process $(X^1(t), X^2(t))$ observed non synchronously. We assume that $(X^1(t))$ is observed at time $t^1_k = 2k\pi/n$ for $k = 0, \ldots, n$, and thus $\varphi^1_n(t) = 2k\pi/n$ if $2k\pi/n \leq t < 2(k+1)\pi/n$. The process $(X^2(t))$ is observed at time $t^2_k = 2\pi k/n + \pi/n$, for $k = 0, \ldots, n-1$, yielding to $\varphi^2_n(t) = 2k\pi/n + \pi/n$, for $2k\pi/n + \pi/n \leq t < 2(k+1)\pi/n + \pi/n$. We also assume that we observe $X^2(0)$ and $X^2(2\pi)$. This situation of alternate sampling is studied in [7] too. Although being a particular case of non synchronous observation, its main advantage is that all computation are explicitly tractable here. Our aim is to estimate the covolatility $\Sigma^{1,2}_t = (\sigma^{1*}\sigma^2)(t)$ and we consider the Fourier estimator $\Gamma^{1,2}_{N,n}(h)$ defined by the equations (41)–(42) of the previous section. We first establish that the estimator $\Gamma^{1,2}_{N,n}(h)$ is not consistent but we have an explicit bias.

Theorem 4 *Let h be a continuous function then, under $H1$–$H2$, $\Gamma^{1,2}_{N,n}(h)$ converges in probability to*

$$\frac{1}{2\pi}\int_0^{2\pi} h(t)\Sigma^{1,2}(t)dt\frac{\sin(a\pi)}{a\pi}$$

as N and n go to infinity and N/n goes to $a \in \mathbb{R}_+$.

If $a = 0$, which means that the number of observations n goes faster to infinity than the number of Fourier coefficients N, the effect of non synchronous data disappear and we have a consistent estimator of the integrated volatility $\frac{1}{2\pi}\int_0^{2\pi} h(t)\Sigma^{1,2}(t)dt$. If $a \in \mathbb{N}^*$, $\Gamma^{1,2}_{N,n}(h)$ goes to zero and the estimation method seems useless. Finally if $a \notin \mathbb{N}$, a consistent estimator is given by $\frac{a\pi}{\sin(a\pi)}\Gamma^{1,2}_{N,n}(h)$.

Proof Using the proposition 1 we just determine the limit of $\int_0^{2\pi} d^{1,2}_{N,n}(t,t)f(t)dt$ for any continuous function f. One can easily see that

$$\int_{2k\pi/n}^{2(k+1)\pi/n} d^{1,2}_{N,n}(t,t)f(t)dt \quad = \quad d_N(-\frac{\pi}{n})\int_{2k\pi/n}^{2k\pi/n+\pi/n} f(t)dt + d_N(\frac{\pi}{n})\int_{2k\pi/n+\pi/n}^{2(k+1)\pi/n} f(t)dt,$$

and consequently

$$\int_0^{2\pi} d^{1,2}_{N,n}(t,t)f(t)dt = d_N(\frac{\pi}{n})\int_0^{2\pi} f(t)dt.$$

We conclude the proof observing that $d_N(\frac{\pi}{n})$ converges to $\frac{\sin(a\pi)}{a\pi}$. ◇

Remark 3 *We could treat, in the same way, more general non synchronous sampling. Indeed if X^1 is sampled at times $t^1_k = 2k\pi/n$ and X^2 at times $t^2_k = (2k+s)\pi/n$, where s in any fixed shift in $)0,1($, we can show that the bias factor becomes $s\dfrac{\sin(2\pi(1-s)a)}{2\pi(1-s)a} + (1-s)\dfrac{\sin(2\pi sa)}{2\pi sa}$. However the associated central limit theorem is not explicit except if $s = 1/2$.*

We study now the rate of convergence of the estimator $\Gamma^{1,2}_{N,n}(h)$ and give some explicit limit.

Theorem 5 *We assume that $H1$, $H2$ and $H3$ hold true, that h is a function with a bounded derivative, and that N and n go to infinity such that $N/n = a + o(1/\sqrt{n})$ as $n \to \infty$, with $a > 0$. Then $\sqrt{n}(\Gamma^{1,2}_{N,n}(h) - \frac{1}{2\pi}\int_0^{2\pi} h(t)\Sigma^{1,2}(t)dt\frac{\sin(a\pi)}{a\pi})$ converges stably in law to*

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} |h(s)|\sqrt{\gamma_1(a)(\sigma^{1*}\sigma^1)(s)(\sigma^{2*}\sigma^2)(s) + \gamma_2(a)(\sigma^{1*}\sigma^2)^2(s)}d\tilde{W}(s),$$

where $\tilde{W}$ is a brownian motion independent of W and where γ_1 and γ_2 are defined by

$$\gamma_1(a) = 2\pi(\eta(a) - \eta(2a)), \tag{48}$$

$$\gamma_2(a) = \pi(\eta(a) - \eta(2a)) + \frac{\sin 2\pi a}{8a^2}(1_{]0,1/2[}(r(a)) - 1_{]1/2,1[}(r(a))). \tag{49}$$

Remark that for $a \in \mathbb{N}$ we have $\gamma_1(a) = \gamma_2(a) = 0$ and the estimator $\Gamma^{1,2}_{N,n}(h)$ converges (to zero) faster than the rate $\sqrt{n}$.

Proof We will use Theorem 3. First, observe that

$$\frac{1}{2\pi}\int_0^{2\pi} d^{1,2}_{N,n}(t,t)h_n^2(t)\Sigma^{1,2}(t)dt = d_N(\frac{\pi}{n})\frac{1}{2\pi}\int_0^{2\pi} h_n^2(t)\Sigma^{1,2}(t)dt,$$

consequently,

$$\frac{1}{2\pi}\int_0^{2\pi} d^{1,2}_{N,n}(t,t)h_n^2(t)\Sigma^{1,2}(t)dt - \frac{1}{2\pi}\int_0^{2\pi} h(t)\Sigma^{1,2}(t)dt\frac{\sin(\pi a)}{\pi a} = o(\frac{1}{\sqrt{n}}) \tag{50}$$

as soon as

$$d_N(\frac{\pi}{n}) - \frac{\sin(\pi a)}{\pi a} = o(\frac{1}{\sqrt{n}}),$$

which is equivalent to the condition given in the statement of the theorem: $N/n - a = o(\frac{1}{\sqrt{n}})$.

Now, the following lemma with theorem 3 and (50) give the result. $\diamond$

Lemma 3 *Let g a continuous bounded function on $\mathbb{R}^2$ and assume that N and n go to infinity with $N/n \to a > 0$ then we have $\forall t \in [0, 2\pi[$:*

i) $\lim_{N,n} n\int_0^t\int_0^s d^{1,2}_{N,n}(s,u)^2 g(s,u)duds = \gamma_1(a)\int_0^t g(s,s)ds$,

ii) $\lim_{N,n} n\int_0^t\int_0^s d^{2,1}_{N,n}(s,u)^2 g(s,u)duds = \gamma_1(a)\int_0^t g(s,s)ds$,

iii) $\lim_{N,n} n\int_0^t\int_0^s d^{1,2}_{N,n}(s,u)d^{2,1}_{N,n}(s,u)g(s,u)duds = \gamma_2(a)\int_0^t g(s,s)ds$,

where γ_1 and γ_2 are defined in Theorem 5.

iv) $\forall p > 1, \forall j \in \{0,1\}$, $\sup_{c\in[0,2\pi]}\int_0^{2\pi}\left|d_N(\varphi_n^j(t) - c)\right|^p dt \le C_p/n$.

Proof i) Let $s \in [0,t]$, we have

$$\begin{aligned} n\int_0^s d^{1,2}_{N,n}(s,u)^2 g(s,u)du &= n\int_{\varphi_n^1(s)}^s d^{1,2}_{N,n}(s,u)^2 g(s,u)du + n\int_0^{\varphi_n^1(s)} d^{1,2}_{N,n}(s,u)^2 g(s,u)du \\ &= V^1_{N,n}(s) + V^2_{N,n}(s). \end{aligned}$$

Now if $u \in [\varphi_n^1(s), s]$, $|\varphi_n^1(s) - \varphi_n^2(s)| = \pi/n$ and then $d^{1,2}_{N,n}(s,u) = d_N(\pi/n)$. This gives

$$V^1_{N,n}(s) = nd_N(\pi/n)^2\int_{\varphi_n^1(s)}^s g(s,u)du,$$

and finally

$$\lim_{N,n}\int_0^t V^1_{N,n}(s)ds = \pi\frac{\sin^2(a\pi)}{(a\pi)^2}\int_0^t g(s,s)ds.$$

To compute the limit of $V_{N,n}^2(s)$, we make the change of variable $v = \varphi_n^1(s) + 2\pi/n - u$. One can easily check that $\varphi_n^2(u + 2k\pi/n) = \varphi_n^2(u) + 2k\pi/n$ and that $\varphi_n^2(-u) = -\varphi_n^2(u) - 2\pi/n$, du a.e., if $|u| \geq \pi/n$. This leads to

$$V_{N,n}^2(s) = n \int_{2\pi/n}^{\varphi_n^1(s)+2\pi/n} d_N^2(\varphi_n^2(v)) g(s, \varphi_n^1(s) + 2\pi/n - v) dv.$$

Now since $\varphi_n^1(s) + 2\pi/n \leq t < 2\pi$, we have for all $\varepsilon > 0$:

$$\lim_{N,n} n \int_{\varepsilon}^{\varphi_n^1(s)+2\pi/n} d_N^2(\varphi_n^2(v)) g(s, \varphi_n^1(s) + 2\pi/n - v) dv = 0.$$

But we can establish for $0 < \varepsilon < \pi$

$$\lim_{N,n} n \int_{2\pi/n}^{\varepsilon} d_N^2(\varphi_n^2(v)) dv = \pi \left(2(\eta(a) - \eta(2a)) - \frac{\sin^2(a\pi)}{(a\pi)^2} \right), \tag{51}$$

it yields that

$$\lim_{N,n} V_{N,n}^2(s) = \pi \left(2(\eta(a) - \eta(2a)) - \frac{\sin^2(a\pi)}{(a\pi)^2} \right) g(s, s),$$

and finally

$$\lim_{N,n} \int_0^t (V_{N,n}^1(s) + V_{N,n}^2(s)) ds = 2\pi(\eta(a) - \eta(2a)) \int_0^t g(s, s) ds = \gamma_1(a) \int_0^t g(s, s) ds.$$

It remains to establish (51) to finish the proof of i). Let $\varphi_n^2(\varepsilon) = (2k_\varepsilon + 1)\pi/n$. We have

$$\lim_{N,n} n \int_{2\pi/n}^{\varepsilon} d_N^2(\varphi_n^2(v)) dv = \lim_{N,n} \int_{2\pi/n}^{\varphi_n^2(\varepsilon)-\pi/n} d_N^2(\varphi_n^2(v)) dv, \tag{52}$$

but

$$\begin{aligned} n \textstyle\int_{2\pi/n}^{\varphi_n^2(\varepsilon)-\pi/n} d_N^2(\varphi_n^2(v)) dv &= n \textstyle\sum_{k=1}^{k_\varepsilon - 1} \int_{2\pi k/n}^{2\pi(k+1)/n} d_N^2(\varphi_n^2(v)) dv \\ &= n\pi/n \textstyle\sum_{k=1}^{k_\varepsilon - 1} \left(d_N^2((2k-1)\pi/n) + d_N^2((2k+1)\pi/n) \right). \end{aligned}$$

By dominated convergence, we have

$$\lim_{N,n} \sum_{k=1}^{k_\varepsilon - 1} \left(d_N^2((2k-1)\pi/n) + d_N^2((2k+1)\pi/n) \right) = \sum_{k \geq 1} \left(\frac{\sin^2(a\pi(2k-1))}{(a\pi(2k-1))^2} + \frac{\sin^2(a\pi(2k+1))}{(a\pi(2k+1))^2} \right).$$

Now recalling that $\eta(a) = \sum_{k \geq 1} \frac{\sin^2(a\pi k)}{(a\pi k)^2}$ we deduce

$$\sum_{k \geq 1} \frac{\sin^2(a\pi(2k-1))}{(a\pi(2k-1))^2} = \eta(a) - \eta(2a),$$

and consequently (51) is proved.

We prove ii) on the same way by introducing the decomposition

$$\begin{aligned} n\int_0^s d_{N,n}^{2,1}(s,u)^2 g(s,u)du &= n\int_{\varphi_n^2(s)}^s d_{N,n}^{2,1}(s,u)^2 g(s,u)du + n\int_0^{\varphi_n^2(s)} d_{N,n}^{2,1}(s,u)^2 g(s,u)du \\ &= V_{N,n}^1(s) + V_{N,n}^2(s). \end{aligned}$$

As previously it is easy to see that

$$\lim_{N,n}\int_0^t V_{N,n}^1(s)ds = \pi\frac{\sin^2(a\pi)}{(a\pi)^2}\int_0^t g(s,s)ds.$$

To compute the limit of $V_{N,n}^2(s)$, we set $v = \varphi_n^2(s) + \pi/n - u$. We observe that $\varphi_n^1(u + 2k\pi/n) = \varphi_n^1(u) + 2k\pi/n$ and that $\varphi_n^1(-u) = -\varphi_n^1(u) - 2\pi/n$, du a.e. This gives

$$V_{N,n}^2(s) = n\int_{\pi/n}^{\varphi_n^2(s)+\pi/n} d_N^2(\varphi_n^1(v) + \pi/n) g(s, \varphi_n^2(s) + \pi/n - v)dv.$$

We conclude as in i) remarking that for $0 < \varepsilon < \pi$

$$\lim_{N,n} n\int_{\pi/n}^{\varepsilon} d_N^2(\varphi_n^1(v) + \pi/n)dv = \pi\left(2(\eta(a) - \eta(2a)) - \frac{\sin^2(a\pi)}{(a\pi)^2}\right).$$

We turn now to iii). We have

$$\begin{aligned} n\int_0^s d_{N,n}^{1,2}(s,u) d_N^{2,1}(s,u) g(s,u)du &= n\int_{\varphi_n^1(s)}^s d_{N,n}^{1,2}(s,u) d_N^{2,1}(s,u) g(s,u)du \\ &\quad + n\int_0^{\varphi_n^1(s)} d_{N,n}^{1,2}(s,u) d_N^{2,1}(s,u) g(s,u)du, \\ &= V_{N,n}^1(s) + V_{N,n}^2(s). \end{aligned}$$

As in i) and ii)

$$\lim_{N,n}\int_0^t V_{N,n}^1(s)ds = \pi\frac{\sin^2(a\pi)}{(a\pi)^2}\int_0^t g(s,s)ds.$$

It remains to identify $\lim_{N,n} V_{N,n}^2(s)$. Let $v = \varphi_n^1(s) + 2\pi/n - u$, we have

$$V_{N,n}^2(s) = n\int_{2\pi/n}^{\varphi_n^1(s)+2\pi/n} d_N(\varphi_n^1(v) + \varphi_n^2(s) - \varphi_n^1(s)) d_N(\varphi_n^2(v)) g(s, \varphi_n^1(s) + 2\pi/n - v)dv,$$

Now for $0 < \varepsilon < \pi$ and $\varphi_n^1(\varepsilon) = 2\pi k_\varepsilon/n$,

$$\begin{aligned} \lim_{N,n} n\int_{2\pi/n}^{\varepsilon} d_N(\varphi_n^1(v) + \varphi_n^2(s) - \varphi_n^1(s)) d_N(\varphi_n^2(v))dv &= \pi\lim_{N,n}\sum_{k=1}^{k_\varepsilon} d_N((2k + \delta_n(s))\pi/n)\left(d_N((2k-1)\pi/n)\right. \\ &\quad \left. + d_N((2k+1)\pi/n)\right), \end{aligned} \tag{53}$$

where $\delta_n(s) = 1$ if $2k\pi/n + \pi/n \le s < 2(k+1)\pi/n$ and $\delta_n(s) = -1$ otherwise. We deduce then that

$$\lim_{N,n}\int_0^t V_{N,n}^2(s)ds = \frac{\pi}{2}\sum_{k\ge 1}\left(\frac{\sin((2k+1)\pi a)}{(2k+1)\pi a} + \frac{\sin((2k-1)\pi a)}{(2k-1)\pi a}\right)^2\int_0^t g(s,s)ds. \tag{54}$$

A tedious calculation gives

$$\sum_{k\geq 1}\frac{\sin((2k+1)\pi a)}{(2k+1)\pi a}.\frac{\sin((2k-1)\pi a)}{(2k-1)\pi a}=-\frac{1}{2}\frac{\sin^2(\pi a)}{(\pi a)^2}+\frac{\sin(2\pi a)}{8\pi a^2}(1_{]0,1/2[}(r(a))-1_{]1/2,1[}(r(a))), \quad (55)$$

and putting this together we obtain

$$\lim_{N,n}\int_0^t V^2_{N,n}(s)ds=\pi\left((\eta(a)-\eta(2a))-\frac{\sin^2(\pi a)}{(\pi a)^2}+\frac{\sin(2\pi a)}{8\pi a^2}(1_{]0,1/2[}(r(a))-1_{]1/2,1[}(r(a)))\right),$$

and finally

$$\lim_{N,n}\int_0^t (V^1_{N,n}(s)+V^2_{N,n}(s))ds=\pi(\eta(a)-\eta(2a))+\frac{\sin(2\pi a)}{8a^2}(1_{]0,1/2[}(r(a))-1_{]1/2,1[}(r(a)))=\gamma_2(a).$$

iv) This property easily follows by simple computations with the condition $N/n\to a>0$. ⋄

4.3 Numerical results

We study the behaviour of the estimator on simulated data with a Monte Carlo method.

We first consider the framework of Section 3 and assume, for simplicity, that $X_t=B_t$. The data are collected at the instants $i2\pi/n$ with $i=0,\ldots,n-1$ and the estimator is $\Gamma^{1,1}_{N,n}:=\Gamma^{1,1}_{N,n}(h)$ given by (25) with $h\equiv 1$. Hence, we are estimating the integrated volatility $\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{1,1}_s ds$ which is equal, here, to the constant 1. The figure 1 shows the mean value and the standard deviation of the estimator in the case $n=100$ and for values of N ranging from $N=20$ to $N=1000$. We made 10000 replications. It is clear that the estimator has no bias whatever is the choice for N. According to theorem 2 the optimal choices of N, for minimizing the variance, are such that $2a\approx 2N/n\in\mathbb{N}_*$. We see that if $N<n/2=50$, the standard deviation increases while N decreases. This is natural since the variance in theorem 2 ii) explodes as $a\to 0$, actually we know that when N is too small ($a\approx 0$) the right rate of estimation is $1/\sqrt{N}$ (see remark 1). The choice $N=50$ corresponds to $a=1/2$ and is optimal, moreover, on the figure 1 the standard deviation seems almost constant for $N>n/2=50$. In graph 2 we plot the variance of the estimator multiplied by n for $N=50$ to $N=1000$. We, now, clearly see that $2N/n\in\mathbb{N}_*$ give minimal variances and the deviations are a bit higher otherwise. However, as N/n becomes very large, the variance of the estimator tends to the optimality, which is natural since the estimator becomes similar to the quadratic variation estimator (recall (26)).

We now consider the case of asynchronous data studied in section 4.2. We focus on the simple case, $X^1_t=B_t$ and $X^2_t=B_t$ for $t\in[0,2\pi]$, yielding to a constant integrated covolatility $\frac{1}{2\pi}\int_0^{2\pi}\Sigma^{1,2}_t dt=1$. The process X^1 (respectively X^2) is observed at the instants $2i\pi/n$ (resp. $(2i+1)\pi/n$). The estimator

$\Gamma^{1,2}_{N,n} := \Gamma^{1,2}_{N,n}(h)$ given in (41)–(42) with $h \equiv 1$ is biased by theorem 4, thus we define a corrected version:

$$\Gamma^{\text{unbiased}}_{N,n} = \left(\frac{\sin(\pi N/n)}{\pi N/n}\right)^{-1} \Gamma^{1,2}_{N,n}$$

when $\sin(\pi N/n) \neq 0$. If $N/n \to a > 0$ and $a \notin \mathbb{N}$ then, by theorem 4, $\Gamma^{\text{unbiased}}_{N,n}$ is a consistent estimator of the covolatility. Moreover the theorem 5 with simple computations implies that, $\sqrt{n}(\Gamma^{\text{unbiased}}_{N,n} - 1) \xrightarrow{n\to\infty} \mathcal{N}(0, v(a))$ where

$$v(a) = \frac{3\pi^2}{4\sin(\pi a)^2}\big(r(a)1_{\{r(a)\in(0,1/2]\}} + (1-r(a))1_{\{r(a)\in(1/2,1)\}}\big) + \frac{\pi}{8}\frac{\sin(2\pi a)}{\sin(\pi a)^2}\big(1_{\{r(a)\in(0,1/2)\}} - 1_{\{r(a)\in(1/2,1)\}}\big).$$

The variance of the estimator, $v(a)$, is a 1-periodic function of a and it is easy to check that $v(a) \sim_{a\to 0} 1/a$, which reflects that $\sqrt{n}$ is not the right rate of convergence for $a = 0$. As $a \to 1-$, we readily have $v(a) \sim 1/(1-a)$, which shows that the divergence of the factor $\frac{\pi a}{\sin(\pi a)}$ is partially compensated by the cancellation of $\gamma_1(a)$ and $\gamma_2(a)$ for $a = 1$ (recall (48)–(49)). In figure 3, we plot the graph of the function v for $a \in)0, 1($. Numerically, the variance is minimized for $a \simeq 0.42$ and $a \simeq 0.58$ where $v(a) \simeq 3.52$. This is very close to the theoretical variance, 7/2, of the 'non synchronous covariance estimator' introduced by Hayachi and Yoshida [6] [7]. Contrarily to the synchronous case, the choice $a = 1/2$ does not yield to the minimum variance, however $v(1/2) = 3\pi^2/8 \simeq 3.70$ remains close to the optimal choice.

In figure 4 we show the empirical mean and standard deviation of the estimator for $n = 100$ and with N ranging from 10 to 90. It is clear that the estimator is unbiased and the standard deviation remains low on all this range of choice for N. In figure 5, we plot the variance multiplied by n for $N = 10$ to $N = 90$. The graph shows a good concordance with the theoretical values shown in figure 3. On the whole, it appears that the estimator performs well for estimating the integrated co-volatility and is not very sensitive to the choice of $N \in \{30, \ldots 70\}$.

An other conclusion of this case study is that using the non-corrected version of the estimator (42) would give poor results. Indeed, keeping the bias low would yield to a choice of the ratio N/n close to zero, which in turn deteriorates the variance of estimation, since $\sqrt{n}$ is not any more the rate of estimation for $a = 0$.

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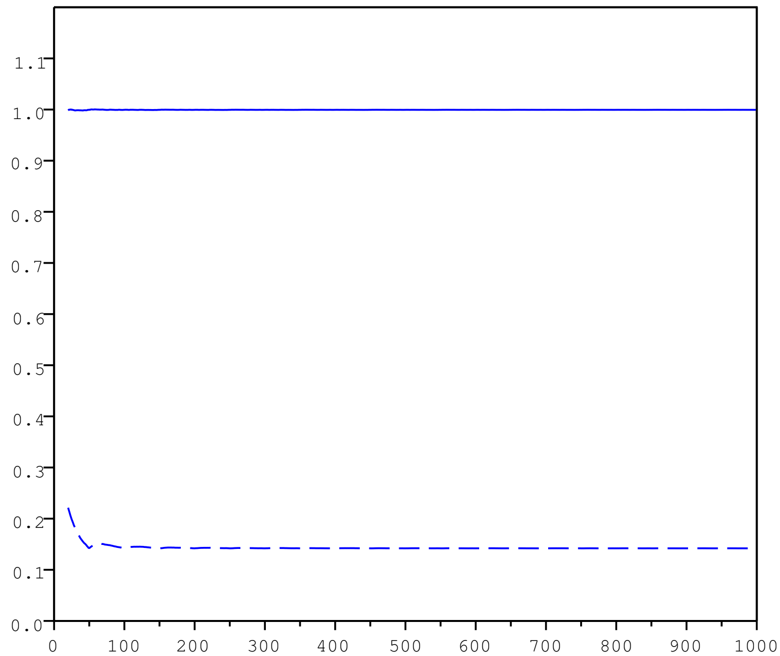

Figure 1: Mean (solid line) and standard deviation (dashed line) of $\Gamma^{1,1}_{N,n}$.

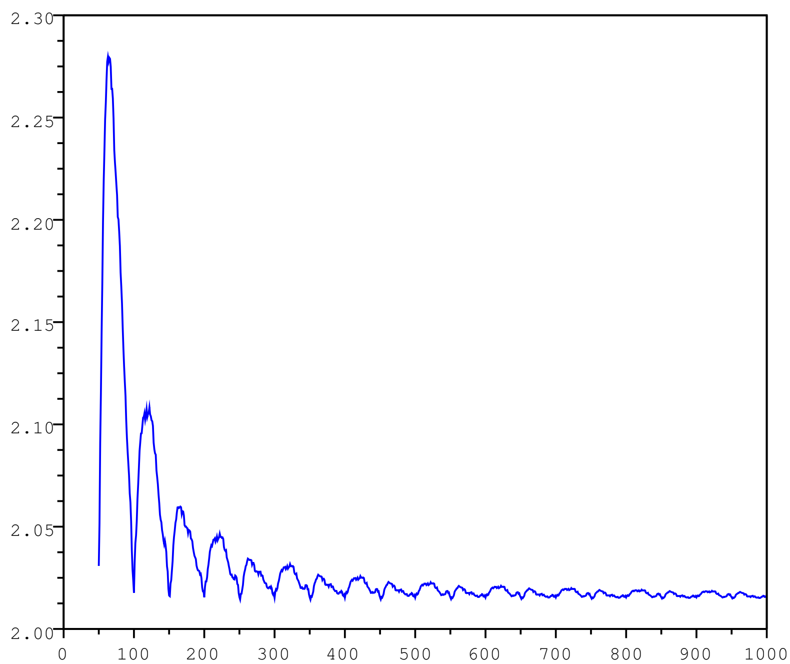

Figure 2: Variance of $\Gamma^{1,1}_{N,n}$ multiplied by n as a function of N.

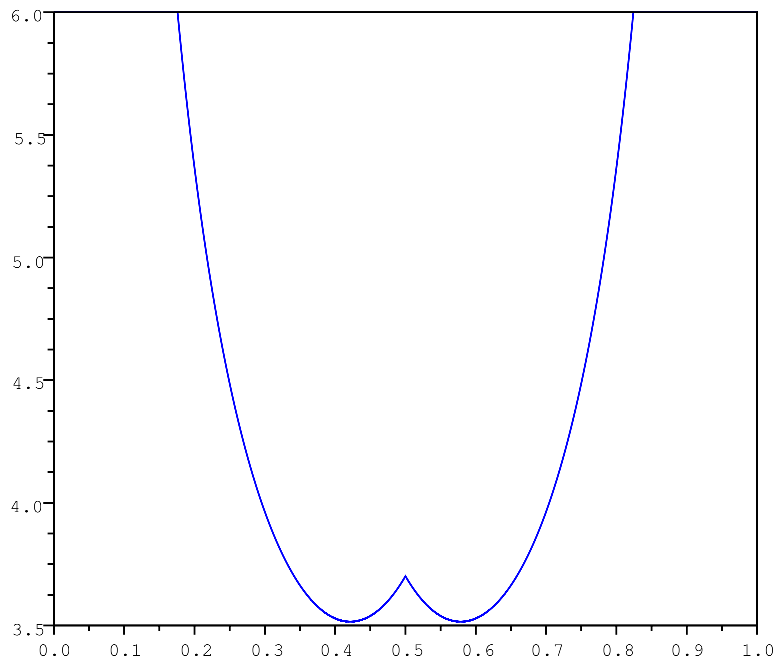

Figure 3: Plot of the function v

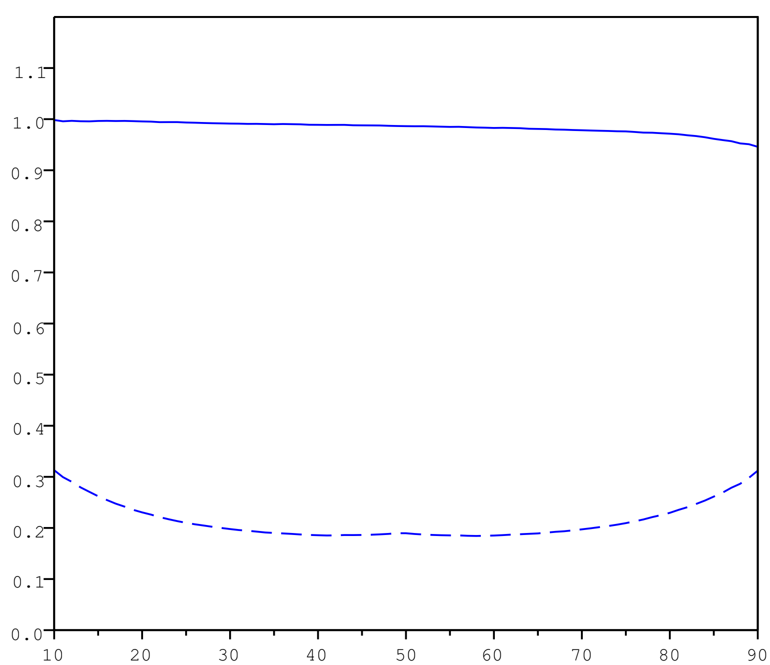

Figure 4: Mean (solid line) and standard deviation (dashed line) of $\Gamma^{\text{unbiased}}_{N,n}$ as a function of N. Non synchronous case.

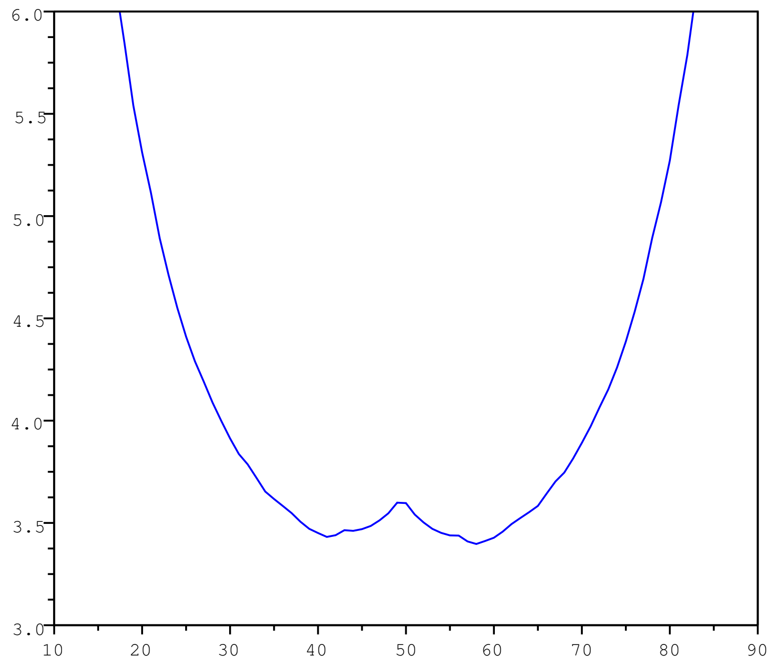

Figure 5: Variance of $\Gamma_{N,n}^{\text{unbiased}}$ multiplied by n as a function of N. Non synchronous case.