On sampling lattices with similarity scaling relationships
Steven Bergner, Dimitri Van de Ville, Thierry Blu, Torsten Möller

To cite this version:
Steven Bergner, Dimitri Van de Ville, Thierry Blu, Torsten Möller. On sampling lattices with similarity scaling relationships. Laurent Fesquet and Bruno Torrésani. SAMPTA’09, May 2009, Marseille, France. pp.General session, 2009. <hal-00453440>

HAL Id: hal-00453440
https://hal.archives-ouvertes.fr/hal-00453440
Submitted on 4 Feb 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
On sampling lattices with similarity scaling relationships

Steven Bergner (1), Dimitri Van De Ville(2), Thierry Blu(3), and Torsten Möller(1)

(1) GrUVi-Lab, Simon Fraser University, Burnaby, Canada.
(2) BIG, Ecole Polytechnique Fédérale de Lausanne, Switzerland.
(3) The Chinese University of Hong Kong, Hong Kong, China.

sbergner@cs.sfu.ca, thierry.blu@m4x.org, Dimitri.VanDeVille@epfl.ch, torsten@cs.sfu.ca

Abstract:
We provide a method for constructing regular sampling lattices in arbitrary dimensions together with an integer dilation matrix. Subsampling using this dilation matrix leads to a similarity-transformed version of the lattice with a chosen density reduction. These lattices are interesting candidates for multidimensional wavelet constructions with a limited number of subbands.

1. Primer on sampling lattices and related work

A sampling lattice is a set of points \( \{ Rk : k \in \mathbb{Z}^n \} \subset \mathbb{R}^n \) that is closed under addition and inversion. The non-singular generating matrix \( R \in \mathbb{R}^{n \times n} \) contains basis vectors in its columns. Lattice points are uniquely indexed by \( k \in \mathbb{Z}^n \) and the neighbourhood around each sampling point is identical. This makes them suitable sampling patterns for the reconstruction in shift-invariant spaces.

Subsampling schemes for lattices are expressed in terms of a dilation matrix \( K \in \mathbb{Z}^{n \times n} \) forming a new lattice with generating matrix \( RK \). The reduction rate in sampling density corresponds to

\[
|\det K| = \alpha^n = \delta \in \mathbb{Z}^+.
\]

Dyadic subsampling discards every second sample along each of the \( n \) dimensions resulting in a \( \delta = 2^n \) reduction rate. To allow for fine-grained scale progression we are particularly interested in low subsampling rates, such as \( \delta = 2 \) or 3.

As discussed by van de Ville et al. [8], the 2D quincunx subsampling is an interesting case permitting a two-channel relation. With the implicit assumption of only considering subsets of the Cartesian lattice it is shown that a similarity-two-channel dilation may not extend for \( n > 2 \).

Here, we show that by permitting more general basis vectors in \( \mathbb{R}^n \) the desired fixed-rate dilation becomes possible for any \( n \). Our construction produces a variety of lattices making it possible to include additional quality criteria into the search as they may be computed from the Voronoi cell of the lattice [9] including packing density and expected quadratic quantization error (second order moment). Agrell et al. [1] improve efficiency for the computation by extracting Voronoi relevant neighbours. Another possible sampling quality criterion appears in the

2. Lattice construction

We are looking for a non-singular lattice generating matrix \( R \) that, when sub-sampled by a dilation matrix \( K \) with reduction rate \( \delta = \alpha^n \), results in a similarity-transformed version of the same lattice, that is, it can be scaled and rotated by a matrix \( Q \) with \( Q^TQ = \alpha^2 I \). An illustration of a subsampling resulting in a rotation by \( \theta = \arccos \frac{1}{\sqrt{2}} \) in 2D is given in Figure 1. Formally, this kind of relationship can be expressed as

\[
QR = RK
\]

leading to the observation that subsampling \( K \) and scaled rotation \( Q \) are related by a similarity transform

\[
R^{-1}QR = K.
\]
Using a matrix $J_2 = \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}$ it is possible to diagonalize a 2D rotation matrix by the following similarity transform

$$
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} = J_2^{-1} \begin{bmatrix} e^{i\theta} & 0 \\
0 & e^{-i\theta}\end{bmatrix} J_2 = J_2^{-1} \Delta J_2.
$$

(4)

Using this observation to replace the scaled rotation matrix $Q$ in Equation 3 leads to

$$
K = R^{-1} Q R \\
K = \alpha R^{-1} J_n^{-1} S \Delta S^{-1} J_n R \\
K = \alpha P \Delta P^{-1}
$$

(5)

with

$$
R = J_n^{-1} S P^{-1} \\
Q = \alpha J_n^{-1} \Delta J_n.
$$

(6)

Thus, given a matrix $K$ that has an eigen-decomposition corresponding to that of a uniformly scaled rotation matrix, we can compute the lattice generating matrix $R$ as in Equation 6. The elements of the diagonal matrix $S$ inserted in the construction of $R$ scale the otherwise unit eigenvectors in the columns of $P$. Below, we will refer to this construction as function formRQ($K, S$) using $S = I$ by default.

### 2.2 Construction Algorithm

The steps for constructing lattices with the desired subsampling matrices are summarized in algorithm 1.

**Algorithm 1 genLattices($n, \alpha, C$)** is defined in the appendix. A possible implementation for the function compoly($n, \alpha, C$) is described in Section 2.1 and formRQ($K$) is defined below Equation 6.

### 2.1 Constructing suitable dilation matrices $K$

The eigenvalues of $K$, $\Delta$ and $Q$ impose restrictions on their shared characteristic polynomial $d(\lambda) = \det(K - \lambda I) = \sum_{k=0}^{n} c_k \lambda^k$ as discussed in the appendix. For the case $n$ is *even* with the only non-zero integer coefficients $c_0 = \delta$, $c_n^2 < 4 \delta$, $c_n = 1$ this leaves a finite number of different options for $c_n/2$. The case $n$ is *odd* permits a single possible polynomial with non-zero coefficients $c_0 = -\delta$, $c_n = 1$. For these monic polynomials it is possible to directly construct a candidate $K$ via the companion matrix ([16], p. 192)

$$
K = \begin{bmatrix}
0 & -c_0 \\
1 & -c_1 \\
& \ddots & \ddots & \ddots \\
& & 0 & -c_{n-2} \\
& & 1 & -c_{n-1}
\end{bmatrix}.
$$

(7)

This allows to construct a lattice fulfilling the self-similar subsampling condition for any dimensionality $n$, one for every possible characteristic polynomial.

With this starting point it is possible to construct additional suitable dilation matrices via a similarity transform with a unimodular matrix $T$

$$
K_T = T K T^{-1} = P_T \Delta P_T^{-1}.
$$

(8)

Using a unimodular rather than any non-singular $T$ guarantees that $T^{-1}$ is also unimodular following from the fact that $T^{-1}$ can be constructed from the adjugate (the transposed co-factor matrix) of $T$. Thus, $K_T$ remains an integer matrix by this transform. Possible generators for this unimodular group are discussed in ([5], pp. 23). Our implementation, referred to as function genUnimodular($n$), uses a construction of $T = LU$ from several random integer lower and upper triangular matrices having ones on their diagonal.

It is not guaranteed that all possible $K$ for a given characteristic polynomial can be generated through a similarity transform with some $T$. However, formRQ($K_T$) provides numerous non-equivalent $R_T$ lattice generators. Among them it is possible to apply further criteria to select the “best” lattice.

An alternative to transforming $K$ is the eigenvector scaling by diagonal matrix $S$ in Equation 6. Using non-unit scaling allows to produce further lattices for any given $K$ resulting in an $n$-dimensional continuous search space.

**Algorithm 2 genKompans($n, \delta$)**

1. $Ks = \{\}
2. if $n$ is even then
3. for all $C \in \mathbb{Z} : C^2 < 4 \delta$ do
4. $Ks \leftarrow Ks \cup \text{compoly}(n, \delta^\frac{1}{2}, C)$
5. end for
6. else if $n$ is odd
7. $Ks \leftarrow \{\text{compoly}(n, \delta^\frac{1}{2})\}$
8. end if
9. return $Ks$
5. Similarity transforms of one valid example is not guaran-

3. Constructions for different dimensions and subsampling ratios

For the 2D case we have created lattices permitting a re-
duction rate 2 in Figure 2 and rate 3 in Figure 3. In both
cases, familiar examples arise in the quincunx and the hex-
lattice for the respective ratios.
A search of 3D lattices enjoying the self-similar subsam-
pling property with rate 2 dilations resulted in 53 non-
equivalent cases. These lattices were compared in terms of
their dimensionless second order moments, corresponding to

The dimensionless second order moment for the Voronoi
Cell of this lattice is $G = 0.081904$. For comparison,
the Cartesian cube has $G_{cc} = 0.0833$ and the truncated octa-
hedron of the BCC lattice has $G_{bcc} = 0.0785$.

4. Discussion and potential applications

The current formation of candidate matrices $K$ based on
similarity transforms of one valid example is not guaran-

teed to produce all possible solutions. For 2D and 3D we
also employed an exhaustive search over a range of integer
matrices with values in $[-3, 3]$ resulting in the same num-
ber of non-equivalent 2D cases as the construction via $K_T$.
However, for dimensionality $n > 3$ the exhaustive search
had to be replaced by a random sampling of integer matrix-
escs ultimately rendering the method infeasible for $n > 5$.
In that light the current construction via scaled eigenvectors
of the companion matrix is a significant improvement
as it allows to produce a large number of non-equivalent
lattices for any dimensionality.

Our subsampling schemes may have applications for mul-
tidimensional wavelet transforms (7). Another direction
for possible investigation is the construction of sparse
grids that are employed in the context of high-dimensional
integration and approximation adapting to smoothness
conditions of the underlying function space (3).
Appendix: Characteristic polynomial of a scaled rotation matrix in $\mathbb{R}^n$

The similarity relationship between $K$ and $Q$ in Equation 2 implies that they share the same characteristic polynomial $d(\lambda) = \det(K - \lambda I) = \det(Q - \lambda I)$ leading to an agreement in eigenvalues $d(\lambda_k) = 0$ and determinant $d(0)$ ([6], p. 184). Further, since $K$ is an integer matrix the polynomial $d(\lambda) \in \mathbb{Z}[\lambda]$ has integer coefficients $c_k$. In order to find integer matrices $K$ with the eigenvalues of a scaled rotation matrix, it will be important to distinguish the two different forms of the diagonal matrix $\Delta$ in Equation 4 and 5 for the case $n = \text{even}$

$$\Delta = \text{diag}[e^{j\theta_1}, e^{-j\theta_2}, \ldots, e^{j\theta_{n/2}}, e^{-j\theta_{n/2}}]$$

and the case $n = \text{odd}$

$$\Delta = \text{diag}[1, e^{j\theta_1}, e^{-j\theta_2}, \ldots, e^{j\theta_{(n-1)/2}}, e^{-j\theta_{(n-1)/2}}]$$

with analogue block-wise constructions for $J_n$.

For dimensionality $n = \text{even}$ the characteristic polynomial of $K$ and $Q$ fulfills

$$d(\lambda) = \prod_{k=1}^{n/2} (\alpha e^{j\theta_k} - \lambda)(\alpha e^{-j\theta_k} - \lambda)$$

$$= \prod_{k=1}^{n/2} (\alpha^2 - 2\lambda \alpha \cos \theta_k + \lambda^2)$$

$$= \prod_{k=1}^{n/2} \left( \frac{\alpha^2}{\lambda^2} - 2 \frac{\alpha^2}{\lambda} \cos \theta_k + \frac{\alpha^4}{\lambda^4} \right)$$

$$= d\left( \frac{\alpha^2}{\lambda} \right) \left( \frac{\lambda}{\alpha} \right)^n$$

Thus, if

$$d(\lambda) = \sum_{k=0}^{n} c_k \lambda^k$$

$$= \sum_{k=0}^{n} c_k \left( \frac{\alpha^2}{\lambda} \right)^k \left( \frac{\lambda}{\alpha} \right)^n$$

$$= \sum_{k=0}^{n} c_{n-k} \alpha^{k-2} \lambda^k$$

$$\Leftrightarrow c_k = \alpha^{n-2k} c_{n-k} = \delta^{1-k/2} c_{n-k}. \tag{10}$$

If $c_k \neq 0$ and $c_k, \delta \in \mathbb{Z}$ then $\delta^{1-2k/2} \in Q$. This is impossible for $0 < 2k < n$, assuming small values of $\delta$, such as 2, 3 or any simple product of primes. This implies that $c_k = c_{n-k} = 0$ for $k = 1, 2, \ldots, \lfloor n/2 \rfloor - 1$. For $k = \lfloor n/2 \rfloor$ the $c_k$ can be non-zero leading to

$$d(\lambda) = \lambda^n + C \lambda^{n/2} + \alpha^n \tag{11}$$

with the requirement that $C^2 < 4\alpha^n$ so that the complex eigenvalues $d(\lambda_k) = 0$ are evenly distributed on the complex circle of radius $|\lambda_k| = \alpha$.

For dimensionality $n = \text{odd}$ the polynomial fulfills

$$d(\lambda) = (\alpha - \lambda) \prod_{k=1}^{(n-1)/2} (\alpha e^{j\theta_k} - \lambda)(\alpha e^{-j\theta_k} - \lambda)$$

$$\Rightarrow d(\lambda) = -\left( \frac{\lambda}{\alpha} \right)^n d\left( \frac{\alpha^2}{\lambda} \right)$$

Thus, if

$$d(\lambda) = \sum_{k=0}^{n} c_k \lambda^k$$

$$= -\sum_{k=0}^{n} c_k \left( \frac{\alpha^2}{\lambda} \right)^k \left( \frac{\lambda}{\alpha} \right)^n$$

$$= -\sum_{k=0}^{n} c_{n-k} \alpha^{n-2k} \lambda^k$$

$$\Leftrightarrow c_k = -\alpha^{n-2k} c_{n-k} = -\delta^{1-k} \delta^{2k} c_{n-k}. \tag{13}$$

By the same reasoning as for the even case, $c_k = 0$ for all $k = 1, 2, \ldots, \lfloor n/2 \rfloor$ resulting in only one possible characteristic polynomial

$$d(\lambda) = \lambda^n - \alpha^n. \tag{14}$$

To refer to the above procedure we will invoke a function $\text{comply}(n, \alpha, C)$ that returns a companion matrix (Equation 7) with a characteristic polynomial as in Equation 11 or 14.

References:


