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A non-conforming discontinuous Galerkin method for solving Maxwell’s equations

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Abstract — This paper reviews the main features of a high-order non-dissipative discontinuous Galerkin (DG) method recently investigated in [1]-[3] for solving Maxwell’s equations on non-conforming simplex meshes. The proposed method combines a centered approximation for the numerical fluxes at inter element boundaries, with either a second-order or a fourth-order leap-frog time integration scheme. Moreover, the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way resulting in arbitrary-level hanging nodes.

I. INTRODUCTION

In the recent years, several works have demonstrated the benefits of using DG methods for the simulation of time-domain electromagnetic wave propagation problems involving complex geometries and heterogeneous media. Being higher order versions of traditional finite volume method [1]-[5], Discontinuous Galerkin Time-Domain (DGTD) methods are flexible discretization methods that can handle complicated geometries, media and meshes, and achieve high-order accuracy by simply choosing suitable local basis functions. Whereas several conforming DGTD methods for the numerical resolution of the system of Maxwell equations have been developed so far [4], the design of non-conforming DGTD methods is still in its infancy. In practice, the non-conformity can result from a local refinement of the mesh (i.e. h-refinement), of the interpolation order (i.e. p-enrichment) or of both of them (i.e. hp-refinement).

II. DISCONTINUOUS GALERKIN METHOD

We consider the three-dimensional Maxwell equations on a bounded domain $\Omega$ of $\mathbb{R}^3$. The electric permittivity tensor $\epsilon(x)$ and the magnetic permeability tensor $\mu(x)$ are varying in space and both symmetric positive definite. The electric field $\vec{E}$ and the magnetic field $\vec{H}$ verify: $\partial_t \vec{E} = \nabla \times \vec{H}$ and $\partial_t \vec{H} = -\nabla \times \vec{E}$. We consider a partition $\Omega_h$ of $\Omega$ into a set of tetrahedra $\tau_i$. To each $\tau_i \in \Omega_h$ we assign an integer $p_i \geq 0$ (the local interpolation order) and we collect the $p_i$ in the vector $p = \{p_i : \tau_i \in \Omega_h\}$. On the domain boundary $\partial \Omega$, we use either a metallic boundary condition or a first-order Silver-Müller absorbing boundary condition. In the following, for a given partition $\Omega_h$ and vector $p$, we seek approximate solutions to Maxwell’s system in the finite dimensional subspace $V_p(\Omega_h) := \{v \in L^2(\Omega)^3 : v|_{\tau_i} \in \mathbb{P}_p(\tau_i), \forall \tau_i \in \Omega_h\}$, where $\mathbb{P}_p(\tau_i)$ denotes the space of nodal polynomials of total degree at most $p_i$ inside the element $\tau_i$. The DGTD method at the heart of this study is based on a totally centered numerical fluxes at the interface between elements of the mesh. We suppose that all electric (resp. magnetic) unknowns are gathered in a column vector $\vec{E}$ (resp. $\vec{H}$) of size $d$ (the total number of degrees of freedom), then the semi-discretized DGTD method can be rewritten as (see [1]-[2]-[3] for more details):

\[
\begin{align*}
M^e \partial_t \vec{E} &= K^e \vec{E} - A^e \vec{H} - B^e, \\
M^\mu \partial_t \vec{H} &= -K^\mu \vec{E} + A^\mu \vec{E} - B^\mu,
\end{align*}
\]

where we have the following definitions and properties:

- $M^e$, $M^\mu$ and $K$ are $d \times d$ block diagonal matrices with diagonal blocks equal to the local mass and stiffness matrices respectively. Therefore $M^e$ and $M^\mu$ are symmetric positive definite matrices, and $K$ is a symmetric matrix.
- $A$ is also a $d \times d$ block sparse matrix, whose non-zero blocks are equal to the internal interface matrix (corresponds to fluxes at internal interfaces of the mesh). Then $A$ is a symmetric matrix.
- $B$ is a $d \times d$ block diagonal matrix, whose non-zero blocks are equal to the metallic interface matrix (corresponds to fluxes at metallic boundary interfaces of the mesh). Then $B$ is a skew-symmetric matrix.

The DGTD-$P_p$ method using centered fluxes combined with $N$th order leap-frog ($LF_N$) time scheme [6] and arbitrary local accuracy and basis functions can be written, in function of the matrix $S = \kappa - A - B$, in the general form:

\[
\begin{align*}
M^e \frac{\vec{E}^{n+1} - \vec{E}^n}{\Delta t} &= S_N H^{n+\frac{1}{2}}, \\
M^\mu \frac{H^{n+\frac{1}{2}} - H^{n-\frac{1}{2}}}{\Delta t} &= -t_N E^{n+1},
\end{align*}
\]

where the matrix $S_N$ verifies:

\[
S_N = \begin{cases} 
S & \text{if } N = 2, \\
S(I - \frac{\Delta t^2}{24} M^{-\mu} tN M^{-e} \cdot S) & \text{if } N = 4.
\end{cases}
\]

III. STABILITY AND CONVERGENCE ANALYSIS

We define the following discrete version of the electromagnetic energy:

\[
E^n = \frac{1}{2} (\epsilon^e \cdot \vec{E}^n + \epsilon^m \cdot (\epsilon^\mu)^{\frac{1}{2}} \vec{M}^e \vec{E}^n + \mu^\mu \cdot \vec{H}^{n+\frac{1}{2}}).
\]

Then we have the following Lemmas [2]:

Lemma 1 Using the DGTD-$P_p$ method (2)-(3), the total discrete electromagnetic energy $E^n$ (4) is a positive definite quadratic form of all unknowns if:

\[
\Delta t \leq \frac{2}{d_N}, \text{ with } d_N = ||M^{-\frac{1}{2}} S_N M^{-\frac{1}{2}}||,
\]

where $||\cdot||$ denote the canonical norm of a matrix $(\forall X, ||AX|| \leq ||A|| \cdot ||X||)$, and the matrix $M^{-\frac{1}{2}}$ ($\sigma$ stands for $e$ or $\mu$) is the inverse square root of $M^\sigma$. Also, for a given mesh, the stability limit of the $LF_4$ scheme roughly 2.85 times larger than that of the $LF_2$ scheme.

Lemma 2 The convergence order in space and time of the DGTD-$P_p$ method (2)-(3) is:

\[
O(T h_{\min}^{\min(s,p)}) + O(\Delta t^N),
\]

where $\Delta t \in [0, T]$, $h$ is the mesh size and the solution belongs to $H^s(\Omega)$ with $s > 1/2$ a regularity parameter.
IV. NUMERICAL EXPERIMENTS

In the following, for a given non-conforming mesh, we assign to coarse (i.e. non refined) elements a high polynomial degree \( p_1 \) and to the refined region a low polynomial degree \( p_2 \). The resulting scheme is referred to as DGTD-\( P_p^{(p_1,p_2)} \). If \( p_1 = p_2 = p \), the scheme is simply called DGTD-\( P_p \).

A. eigenmode in a PEC cavity

The first test case that we consider is the propagation of an eigenmode in a unitary PEC cavity with \( \epsilon = \mu = 1 \).

The 2D case: We compare the LF\(_2\) and LF\(_4\) time schemes using the DGTD-\( P_p \) method. Numerical simulations make use of a non-conforming triangular mesh which consists of 782 triangles and 442 nodes (36 of them are hanging nodes) as shown on Fig. 1 right. We plot on Fig. 1 left the time evolution of the overall \( L^2 \) error of the DGTD-\( P_p \) method using the LF\(_2\) and LF\(_4\) schemes. Tab. I gives the \( L^2 \) error, the CPU time in minutes to reach 105 periods, and the convergence order “O”. It can be observed from Fig. 1 and Tab. I that the LF\(_2\) is more accurate and requires less CPU time than the LF\(_4\) scheme. Moreover, the convergence order is bounded by \( N \) which confirms the result of Lemma 2.

![Fig. 1. Error of the DGTD-\( P_p \) method (left) on the non-conforming triangular mesh (right).](image)

**Table I. Comparison between LF\(_2\) and LF\(_4\) DGTD-\( P_p \) method**

<table>
<thead>
<tr>
<th>( n )</th>
<th>LF(_2) scheme</th>
<th>LF(_4) scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L^2 ) error</td>
<td>CPU</td>
<td>( L^2 ) error</td>
</tr>
<tr>
<td>2</td>
<td>1.8E-03</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>3.1E-04</td>
<td>98</td>
</tr>
</tbody>
</table>

The 3D case: Here we compare the DGTD-\( P_p^{(p_1,p_2)} \) with several DGTD-\( P_p \) methods using the LF\(_2\) time scheme. Numerical simulations make use of an unstructured mesh which consists of 4406 tetrahedra and 962 nodes. Tab. II gives the \( L^2 \) error, the \# DOF and the CPU time in minutes to reach 30 periods. Note that, the low polynomial degree \( p_2 \) is used in only 5% of the tetrahedra of the mesh. We can deduce from Tab. II that to achieve a given accuracy, the DGTD-\( P_p^{(p_1,p_2)} \) requires less CPU time than the DGTD-\( P_p \). Moreover, for a given CPU time, the DGTD-\( P_p^{(p_1,p_2)} \) method is roughly 8 times more efficient than the DGTD-\( P_p \) method.

B. Scattering of a plane wave by a dielectric cylinder

In this problem, a plane wave impinges on a dielectric cylinder, experiencing reflection and refraction at the material interface. The material is non-magnetic, and the material exterior to the cylinder is assumed to be vacuum. The cylinder has a radius \( r_0 = 0.6 \) and bounds a material of relative permittivity \( \epsilon_r = 2.25 \). The computational domain is chosen as a cylinder of radius 1.5 centered at \( (0,0) \). At the artificial boundary, we apply a first-order Silver-Müller absorbing boundary condition. Here, we compare the DGTD-\( P_p \) method using a conforming mesh with the DGTD-\( P_p^{(p_1,p_2)} \) using a non-conforming mesh. To this end, we construct a conforming mesh consisting of 11920 triangles and 6001 nodes. Then, a non-conforming mesh is obtained by locally refining (two refinement levels) the cylindrical zone \( 0.55 \leq r \leq 0.65 \) of a coarse conforming mesh. The resulting non-conforming mesh consists of 5950 triangles and 3151 nodes (300 of them are hanging nodes). Tab. III shows the relative \( L^2 \) error, the \# DOF and the CPU time in minutes to reach \( t = 5 \). As expected, the gain in CPU time between the DGTD-\( P_p^{(p_1,p_2)} \) and the conforming DGTD-\( P_p \) methods is considerable. For instance, to achieve an error level of 5%, the DGTD-\( P_p^{(2,0)} \) scheme requires 3 times less DOF and 21 times less CPU time than the DGTD-\( P_p \) scheme.

**Table II. \# DOF, \( L^2 \) errors and CPU time in minutes**

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \epsilon )</th>
<th>( \mu )</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L^2 ) error</td>
<td>CPU</td>
<td># DOF</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_p )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>7.2E-01</td>
<td>2.0E-01</td>
<td>1.4E-02</td>
<td>8.0E-04</td>
</tr>
<tr>
<td>CPU</td>
<td>4</td>
<td>40</td>
<td>213</td>
<td>859</td>
</tr>
<tr>
<td># DOF</td>
<td>4406</td>
<td>17624</td>
<td>44060</td>
<td>88120</td>
</tr>
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</table>

**Table III. \# DOF, \( L^2 \) errors and CPU time in minutes**

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \epsilon )</th>
<th>( \mu )</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L^2 ) error</td>
<td>CPU</td>
<td># DOF</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_p^{(p_1,p_2)} )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>13.6%</td>
<td>7.15%</td>
<td>5.20%</td>
<td>5.22%</td>
</tr>
<tr>
<td>CPU</td>
<td>20</td>
<td>178</td>
<td>542</td>
<td>1817</td>
</tr>
<tr>
<td># DOF</td>
<td>11920</td>
<td>35760</td>
<td>71520</td>
<td>119200</td>
</tr>
</tbody>
</table>

**REFERENCES**


