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A non-conforming discontinuous Galerkin method for solving Maxwell’s equations

H. Fahs
INRIA, 2004 Route des Lucioles, BP 93, F-06902 Sophia Antipolis Cedex, France
E-mail: hassan.fahs@inria.fr

Abstract — This paper reviews the main features of a high-order non-dissipative discontinuous Galerkin (DG) method recently investigated in [1]-[3] for solving Maxwell’s equations on non-conforming simplex meshes. The proposed method combines a centered approximation for the numerical fluxes at inter element boundaries, with either a second-order or a fourth-order leap-frog time integration scheme. Moreover, the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way resulting in arbitrary-level hanging nodes.

I. INTRODUCTION

In the recent years, several works have demonstrated the benefits of using DG methods for the simulation of time-domain electromagnetic wave propagation problems involving complex geometries and heterogeneous media. Being higher order versions of traditional finite volume method [1]-[5], Discontinuous Galerkin Time-Domain (DGTD) methods are flexible discretization methods that can handle complicated geometries, media and meshes, and achieve high-order accuracy by simply choosing suitable local basis functions. Whereas several conforming DGTD methods for the numerical resolution of the system of Maxwell equations have been developed so far [4], the design of non-conforming DGTD methods is still in its infancy. In practice, the non-conformity can result from a local refinement of the mesh (i.e. h-refinement), of the interpolation order (i.e. p-enrichment) or of both of them (i.e. hp-refinement).

II. DISCONTINUOUS GALERKIN METHOD

We consider the three-dimensional Maxwell equations on a bounded domain \( \Omega \) of \( \mathbb{R}^3 \). The electric permittivity tensor \( \varepsilon \) and the magnetic permeability tensor \( \mu \) are varying in space and both symmetric positive definite. The electric field \( \vec{E} \) and the magnetic field \( \vec{H} \) verify: \( \varepsilon \partial_t \vec{E} = \mu \partial_t \vec{H} \) and \( \mu \partial_t \vec{H} = -\varepsilon \partial_t \vec{E} \). We consider a partition \( \Omega \) of \( \Omega \) into a set of tetrahedra \( \tau_i \). To each \( \tau_i \in \Omega \), we assign an integer \( p_i \geq 0 \) (the local interpolation order) and we collect the \( p_i \) in the vector \( p = \{ p_i : \tau_i \in \Omega \} \). On the domain boundary \( \partial \Omega \), we use either a metallic boundary condition or a first-order Silver-Müller absorbing boundary condition. In the following, for a given partition \( \Omega \) and vector \( p \), we seek approximate solutions to Maxwell’s system in the finite dimensional subspace \( \mathcal{V}_p(\Omega) := \{ v \in L^2(\Omega)^3 : v_{\tau_i} \in \mathbb{P}_{p_i}(\tau_i), \forall \tau_i \in \Omega \} \), where \( \mathbb{P}_{p_i}(\tau_i) \) denotes the space of nodal polynomials of total degree at most \( p_i \) inside the element \( \tau_i \). The DGTD method at the heart of this study is based on a locally centered numerical fluxes at the interface between elements of the mesh. We suppose that all electric (resp. magnetic) unknowns are gathered in a column vector \( \mathbf{E} \) (resp. \( \mathbf{H} \)) of size \( d \) (the total number of degrees of freedom), then the semi-discretized DGTD method can be rewritten as (see [1]-[2]-[3] for more details):

\[
\begin{align*}
\mathbf{M}^e \partial_t \mathbf{E} &= \mathbf{K} \mathbf{H} - \mathbf{A} \mathbf{H} - \mathbf{B} \mathbf{H}, \\
\mathbf{M}^e \partial_t \mathbf{H} &= -\mathbf{K} \mathbf{E} + \mathbf{A} \mathbf{E} - \mathbf{B} \mathbf{E},
\end{align*}
\]

where we have the following definitions and properties:

\begin{itemize}
  \item \( \mathbf{M}^e \), \( \mathbf{M}^\mu \) and \( \mathbf{K} \) are \( d \times d \) block diagonal matrices with diagonal blocks equal to the local mass and stiffness matrices respectively. Therefore \( \mathbf{M}^e \) and \( \mathbf{M}^\mu \) are symmetric positive definite matrices, and \( \mathbf{K} \) is a symmetric matrix.
  \item \( \mathbf{A} \) is also a \( d \times d \) block sparse matrix, whose non-zero blocks are equal to the internal interface matrix (corresponds to fluxes at internal interfaces of the mesh). Then \( \mathbf{A} \) is a symmetric matrix.
  \item \( \mathbf{B} \) is a \( d \times d \) block diagonal matrix, whose non-zero blocks are equal to the metallic interface matrix (corresponds to fluxes at metallic boundary interfaces of the mesh). Then \( \mathbf{B} \) is a skew-symmetric matrix.
\end{itemize}

The DGTD-\( \mathbb{IP}_{p_i} \) method using centered fluxes combined with \( N \)th order leap-frog (LF\( _N \)) time scheme [6] and arbitrary local accuracy and basis functions can be written, in function of the matrix \( \mathbf{S} = \mathbf{K} - \mathbf{A} - \mathbf{B} \), in the general form:

\[
\begin{align*}
\mathbf{M}^e \frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\Delta t} &= \mathbf{S}_N \mathbf{H}^{n+\frac{1}{2}}, \\
\mathbf{M}^\mu \frac{\mathbf{H}^{n+\frac{1}{2}} - \mathbf{H}^{n+\frac{1}{2}}}{\Delta t} &= -\mathbf{S}_N \mathbf{E}^{n+1},
\end{align*}
\]

where the matrix \( \mathbf{S}_N \) verifies:

\[
\mathbf{S}_N = \begin{cases} 
\mathbf{S} & \text{if } N = 2, \\
\mathbf{S} \left( \mathbf{I} - \frac{\Delta t^2 \mathbf{M}^{-\mu} \mathbf{S} \mathbf{M}^{-\frac{1}{2}}}{24} \right) & \text{if } N = 4.
\end{cases}
\]

III. STABILITY AND CONVERGENCE ANALYSIS

We define the following discrete version of the electromagnetic energy:

\[
\mathcal{E}^n = \frac{1}{2} \left( \mathbf{E}^n \mathbf{M}^e \mathbf{E}^n + \mathbf{H}^{n+\frac{1}{2}} \mathbf{M}^\mu \mathbf{H}^{n+\frac{1}{2}} \right).
\]

Then we have the following Lemmas [2]:

**Lemma 1** Using the DGTD-\( \mathbb{IP}_{p_i} \) method (2)-(3), the total discrete electromagnetic energy \( \mathcal{E}^n \) (4) is a positive definite quadratic form of all unknowns if:

\[
\Delta t \leq \frac{2}{d_N}, \quad \text{with } d_N = \| \mathbf{M}^e \| \mathbf{S}_N \mathbf{M}^\frac{1}{2},
\]

where \( \| . \| \) denote the canonical norm of a matrix \( \langle X, \| AX \| \leq \| A \| \| X \| \rangle \), and the matrix \( \mathbf{M}^e \) (\( \sigma \) stands for \( \epsilon \) or \( \mu \)) is the inverse square root of \( \mathbf{M}^e \). Also, for a given mesh, the stability limit of the LF\( _4 \) scheme is roughly 2.85 times larger than that of the LF\( _2 \) scheme.

**Lemma 2** The convergence order in space and time of the DGTD-\( \mathbb{IP}_{p_i} \) method (2)-(3) is:

\[
\mathcal{O}(T h^{\min(s,p)}) + \mathcal{O}(\Delta t^s),
\]

where \( \Delta t \in [0, T] \), \( h \) is the mesh size and the solution belongs to \( H^s(\Omega) \) with \( s > 1/2 \) a regularity parameter.
IV. NUMERICAL EXPERIMENTS

In the following, for a given non-conforming mesh, we assign to coarse \((i.e. \text{ non refined})\) elements a high polynomial degree \(p_1\) and to the refined region a low polynomial degree \(p_2\). The resulting scheme is referred to as \(\text{DGTD-}\mathbb{P}_{(p_1,p_2)}\). If \(p_1 = p_2 = p\), the scheme is simply called \(\text{DGTD-}\mathbb{P}_p\).

A. eigenmode in a PEC cavity

The first test case that we consider is the propagation of an eigenmode in a unitary PEC cavity with \(\epsilon = \mu = 1\).

The 2D case: We compare the LF\(_2\) and LF\(_4\) time schemes using the \(\text{DGTD-}\mathbb{P}_p\) method. Numerical simulations make use of a non-conforming triangular mesh which consists of 782 triangles and 442 nodes (36 of them are hanging nodes) as shown on Fig. 1 right. We plot on Fig. 1 left the time evolution of the overall \(L^2\) error of the \(\text{DGTD-}\mathbb{P}_p\) method using the LF\(_2\) and LF\(_4\) schemes. Tab. I gives the \(L^2\) error, the CPU time in minutes to reach 105 periods, and the convergence order “\(O\)”. It can be observed from Fig. 1 and Tab. I that the LF\(_2\) is more accurate and requires less CPU time than the LF\(_4\) scheme. Moreover, the convergence order is bounded by \(N\) which confirms the result of Lemma 2.

![Fig. 1. Error of the DGTD-\(p\) method (left) on the non-conforming triangular mesh (right).](image)

<table>
<thead>
<tr>
<th>(p)</th>
<th>(L^2) error (\times 10^{-3})</th>
<th>CPU</th>
<th>(O)</th>
<th>(L^2) error (\times 10^{-3})</th>
<th>CPU</th>
<th>(O)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.8E-03</td>
<td>11</td>
<td>2.28</td>
<td>5.5E-04</td>
<td>8</td>
<td>2.32</td>
</tr>
<tr>
<td>3</td>
<td>3.1E-04</td>
<td>98</td>
<td>2.10</td>
<td>2.4E-05</td>
<td>28</td>
<td>2.97</td>
</tr>
<tr>
<td>4</td>
<td>1.9E-04</td>
<td>97</td>
<td>2.10</td>
<td>1.5E-05</td>
<td>70</td>
<td>3.99</td>
</tr>
</tbody>
</table>

The 3D case: Here we compare the \(\text{DGTD-}\mathbb{P}_{(p_1,p_2)}\) with several \(\text{DGTD-}\mathbb{P}_p\) methods using the LF\(_2\) time scheme. Numerical simulations make use of an unstructured mesh which consists of 4406 tetrahedra and 962 nodes. Tab. II gives the \(L^2\) error, the \# DOF and the CPU time in minutes to reach 30 periods. Note that, the low polynomial degree \(p_2\) is used in only 5% of the tetrahedra of the mesh. We can deduce from Tab. II that to achieve a given accuracy, the \(\text{DGTD-}\mathbb{P}_{(p_1,p_2)}\) requires less CPU time than the \(\text{DGTD-}\mathbb{P}_p\). Moreover, for a given CPU time, the \(\text{DGTD-}\mathbb{P}_{(p_1,p_2)}\) method is roughly 8 times more efficient than the \(\text{DGTD-}\mathbb{P}_p\) method.

B. Scattering of a plane wave by a dielectric cylinder

In this problem, a plane wave impinges on a dielectric cylinder, experiencing reflection and refraction at the material interface. The material is non-magnetic, and the material exterior to the cylinder is assumed to be vacuum. The cylinder has a radius \(r_0 = 0.6\) and bounds a material of relative permittivity \(\epsilon_r = 2.25\). The computational domain is chosen as a cylinder of radius 1.5 centered at \((0, 0)\). At the artificial boundary, we apply a first-order Silver-Müller absorbing boundary condition. Here, we compare the \(\text{DGTD-}\mathbb{P}_p\) method using a conforming mesh with the \(\text{DGTD-}\mathbb{P}_{(p_1,p_2)}\) using a non-conforming mesh. To this end, we construct a conforming mesh consisting of 11920 triangles and 6001 nodes. Then, a non-conforming mesh is obtained by locally refining (two refinement levels) the cylindrical zone \(0.55 < r < 0.65\) of a coarse conforming mesh. The resulting non-conforming mesh consists of 5950 triangles and 3151 nodes (300 of them are hanging nodes). Tab. III shows the relative \(L^2\) error, the \# CPU and the CPU time in minutes to reach \(t = 5\) As expected, the gain in CPU time between the \(\text{DGTD-}\mathbb{P}_{(p_1,p_2)}\) and the conforming \(\text{DGTD-}\mathbb{P}_p\) methods is considerable. For instance, to achieve an error level of 5%, the \(\text{DGTD-}\mathbb{P}_{(2,0)}\) scheme requires 3 times less DOF and 21 times less CPU time than the \(\text{DGTD-}\mathbb{P}_2\) scheme.

![Table II. # DOF, \(L^2\) errors and CPU time in minutes](image)

<table>
<thead>
<tr>
<th>(p)</th>
<th>(L^2) error (\times 10^{-3})</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.2E-01</td>
<td>4</td>
<td>40</td>
</tr>
<tr>
<td>1</td>
<td>2.0E-01</td>
<td>40</td>
<td>213</td>
</tr>
<tr>
<td>2</td>
<td>1.4E-02</td>
<td>4060</td>
<td>88120</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(p_1,p_2)</th>
<th>(L^2) error (\times 10^{-3})</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2,0))</td>
<td>3.6E-02</td>
<td>35</td>
<td>42908</td>
</tr>
<tr>
<td>((2,1))</td>
<td>1.3E-02</td>
<td>106</td>
<td>43676</td>
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<tr>
<td>((3,1))</td>
<td>1.0E-03</td>
<td>260</td>
<td>87096</td>
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<tr>
<td>((3,2))</td>
<td>8.8E-04</td>
<td>499</td>
<td>86030</td>
</tr>
</tbody>
</table>

![Table III. # DOF, \(L^2\) errors and CPU time in minutes](image)

<table>
<thead>
<tr>
<th>(p)</th>
<th>(L^2) error (\times 10^{-3})</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>13.6%</td>
<td>20</td>
<td>11920</td>
</tr>
<tr>
<td>1</td>
<td>7.15%</td>
<td>178</td>
<td>35760</td>
</tr>
<tr>
<td>2</td>
<td>5.20%</td>
<td>542</td>
<td>71520</td>
</tr>
<tr>
<td>3</td>
<td>5.22%</td>
<td>1817</td>
<td>119200</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(p_1,p_2)</th>
<th>(L^2) error (\times 10^{-3})</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1,0))</td>
<td>11.6%</td>
<td>9</td>
<td>11450</td>
</tr>
<tr>
<td>((2,0))</td>
<td>5.36%</td>
<td>25</td>
<td>19700</td>
</tr>
<tr>
<td>((2,1))</td>
<td>5.39%</td>
<td>33</td>
<td>26100</td>
</tr>
<tr>
<td>((3,2))</td>
<td>5.37%</td>
<td>179</td>
<td>46700</td>
</tr>
</tbody>
</table>

REFERENCES


