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A non-conforming discontinuous Galerkin method for solving Maxwell’s equations

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Abstract — This paper reviews the main features of a high-order non-dissipative discontinuous Galerkin (DG) method recently investigated in [1]-[3] for solving Maxwell’s equations on non-conforming simplex meshes. The proposed method combines a centered approximation for the numerical fluxes at inter element boundaries, with either a second-order or a fourth-order leap-frog time integration scheme. Moreover, the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way resulting in arbitrary-level hanging nodes.

I. INTRODUCTION

In the recent years, several works have demonstrated the benefits of using DG methods for the simulation of time-domain electromagnetic wave propagation problems involving complex geometries and heterogeneous media. Being higher order versions of traditional finite volume method [1]-[3], Discontinuous Galerkin Time-Domain (DGTD) methods are flexible discretization methods that can handle complicated geometries, media and meshes, and achieve high-order accuracy by simply choosing suitable local basis functions. Whereas several conforming DGTD methods for the numerical resolution of the system of Maxwell equations have been developed so far [4], the design of non-conforming DGTD methods is still in its infancy. In practice, the non-conformity can result from a local refinement of the mesh (i.e. h-refinement), of the interpolation order (i.e. p-enrichment) or of both of them (i.e. hp-refinement).

II. DISCONTINUOUS GALERKIN METHOD

We consider the three-dimensional Maxwell equations on a bounded domain \( \Omega \) of \( \mathbb{R}^3 \). The electric permittivity tensor \( \varepsilon(x) \) and the magnetic permeability tensor \( \mu(x) \) are varying in space and both symmetric positive definite. The electric field \( \vec{E} \) and the magnetic field \( \vec{H} \) verify:

\[
\varepsilon \partial_t \vec{E} = \text{curl} \, \vec{H} \quad \text{and} \quad \mu \partial_t \vec{H} = -\text{curl} \, \vec{E}.
\]

We consider a partition \( \Omega_h \) of \( \Omega \) into a set of tetrahedra \( \tau_i \). To each \( \tau_i \in \Omega_h \) we assign an integer \( p_i \geq 0 \) (the local interpolation order) and we collect the \( p_i \) in the vector \( p = \{ p_i : \tau_i \in \Omega_h \} \). On the domain boundary \( \partial \Omega \), we use either a metallic boundary condition or a first-order Silver-Müller absorbing boundary condition. In the following, for a given partition \( \Omega_h \) and vector \( p \), we seek approximate solutions to Maxwell’s system in the finite dimensional subspace \( V_p(\Omega_h) := \{ \vec{v} \in L^2(\Omega)^3 : v_{\tau_i} \in \mathbb{P}_{p_i}(\tau_i), \forall \tau_i \in \Omega_h \} \), where \( \mathbb{P}_{p_i}(\tau_i) \) denotes the space of nodal polynomials of total degree at most \( p_i \) inside the element \( \tau_i \). The DGTD method at the heart of this study is based on a totally centered numerical fluxes at the interface between elements of the mesh. We suppose that all electric (resp. magnetic) unknowns are gathered in a column vector \( \vec{E} \) (resp. \( \vec{H} \)) of size \( d \) (the total number of degrees of freedom), then the semi-discretized DGTD method can be rewritten as (see [1]-[2]-[3] for more details):

\[
\begin{align*}
M^\varepsilon \partial_t \vec{E} &= KH - AH - BH, \\
M^\mu \partial_t \vec{H} &= -KE + AE - BE,
\end{align*}
\]

where we have the following definitions and properties:

- \( M^\varepsilon, M^\mu \) and \( K \) are \( d \times d \) block diagonal matrices with diagonal blocks equal to the local mass and stiffness matrices respectively. Therefore \( M^\varepsilon \) and \( M^\mu \) are symmetric positive definite matrices, and \( K \) is a symmetric matrix.
- \( A \) is also a \( d \times d \) block sparse matrix, whose non-zero blocks are equal to the internal interface matrix (corresponds to fluxes at internal interfaces of the mesh). Then \( A \) is a symmetric matrix.
- \( B \) is a \( d \times d \) block diagonal matrix, whose non-zero blocks are equal to the metallic interface matrix (corresponds to fluxes at metallic boundary interfaces of the mesh). Then \( B \) is a skew-symmetric matrix.
- \( \Delta t \in [0, T] \), \( h \) is the mesh size and the solution belongs to \( H^s(\Omega) \) with \( s > 1/2 \) a regularity parameter.

III. STABILITY AND CONVERGENCE ANALYSIS

We define the following discrete version of the electromagnetic energy:

\[
\mathcal{E}^n = \frac{1}{2} \left( \varepsilon \partial_t \vec{E}^n + H^t \partial_t \vec{H}^n + \mu H^t \partial_t \vec{E}^n + \mu E^t \partial_t \vec{H}^n \right).
\]

Then we have the following Lemmas [2]:

**Lemma 1** Using the DGTD-\textsuperscript{IP}_p method (2)-(3), the total discrete electromagnetic energy \( \mathcal{E}^n \) (4) is a positive definite quadratic form of all unknowns if:

\[
\Delta t \leq \frac{2}{d_N}, \quad \text{with } d_N = \| M^\varepsilon^{-1} S_N M^\mu \|,
\]

where \( \| \cdot \| \) denote the canonical norm of a matrix (\( \forall X, \| AX \| \leq \| A \| \| X \| \)), and the matrix \( M^\mu \) (\( \sigma \) stands for \( \varepsilon \) or \( \mu \)) is the inverse square root of \( M^\varepsilon \). Also, for a given mesh, the stability limit of the LF\textsubscript{4} scheme is roughly 2.85 times larger than that of the LF\textsubscript{2} scheme.

**Lemma 2** The convergence order in space and time of the DGTD-\textsuperscript{IP}_p method (2)-(3) is:

\[
\mathcal{O}(\tau^{\min(s,p)}) + \mathcal{O}(\Delta t^s),
\]

where \( \Delta t \in [0, T] \), \( h \) is the mesh size and the solution belongs to \( H^s(\Omega) \) with \( s > 1/2 \) a regularity parameter.
IV. NUMERICAL EXPERIMENTS

In the following, for a given non-conforming mesh, we assign to coarse (i.e., non refined) elements a high polynomial degree \( p_1 \) and to the refined region a low polynomial degree \( p_2 \). The resulting scheme is referred to as DGTD-\( P_{(p_1,p_2)} \). If \( p_1 = p_2 = p \), the scheme is simply called DGTD-\( P_p \).

A. eigenmode in a PEC cavity

The first test case that we consider is the propagation of an eigenmode in a unitary PEC cavity with \( \epsilon = \mu = 1 \).

The 2D case: We compare the LF\(_2\) and LF\(_4\) time schemes using the DGTD-\( P_p \) method. Numerical simulations make use of a non-conforming triangular mesh which consists of 782 triangles and 442 nodes (30 of them are hanging nodes) as shown on Fig. 1 right. We plot on Fig. 1 left the time evolution of the overall \( L^2 \) error of the DGTD-\( P_p \) method using the LF\(_2\) and LF\(_4\) schemes. Tab. I gives the \( L^2 \) error, the CPU time in minutes to reach 105 periods, and the convergence order “O”.

![Fig. 1. Error of the DGTD-\( P_p \) method (left) on the non-conforming triangular mesh (right).](image)

### Table I. Comparison between LF\(_2\) and LF\(_4\) DGTD-\( P_p \) method

<table>
<thead>
<tr>
<th>( p )</th>
<th>LF(_2) scheme</th>
<th>LF(_4) scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( L^2 ) error</td>
<td>CPU</td>
</tr>
<tr>
<td>2</td>
<td>1.8E-03</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>3.1E-04</td>
<td>98</td>
</tr>
<tr>
<td>4</td>
<td>1.9E-04</td>
<td>2.10</td>
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</table>

The 3D case: Here we compare the DGTD-\( P_{(p_1,p_2)} \) with several DGTD-\( P_{(p_1,p_2)} \) methods using the LF\(_2\) time scheme. Numerical simulations make use of an unstructured mesh which consists of 4406 tetrahedra and 962 nodes. Tab. II gives the \( L^2 \) error, the \# DOF and the CPU time in minutes to reach 30 periods. Note that, the low polynomial degree \( p_2 \) is used in only 5% of the tetrahedra of the mesh. We can deduce from Tab. II that to achieve a given accuracy, the DGTD-\( P_{(p_1,p_2)} \) requires less CPU time than the DGTD-\( P_p \). Moreover, for a given CPU time, the DGTD-\( P_{(p_1,p_2)} \) method is roughly 8 times more efficient than the DGTD-\( P_p \) method.

B. Scattering of a plane wave by a dielectric cylinder

In this problem, a plane wave impinges on a dielectric cylinder, experiencing reflection and refraction at the material interface. The material is non-magnetic, and the material exterior to the cylinder is assumed to be vacuum. The cylinder has a radius \( r_0 = 0.6 \) and bounds a material of relative permittivity \( \epsilon_r = 2.25 \). The computational domain is chosen as a cylinder of radius 1.5 centered at \((0, 0)\). At the artificial boundary, we apply a first-order Silver-Müller absorbing boundary condition. Here, we compare the DGTD-\( P_p \) method using a conforming mesh with the DGTD-\( P_{(p_1,p_2)} \) using a non-conforming mesh. To this end, we construct a conforming mesh consisting of 11920 triangles and 6001 nodes. Then, a non-conforming mesh is obtained by locally refining (two refinement levels) the cylindrical zone \( 0.55 \leq r \leq 0.65 \) of a coarse conforming mesh.

### Table II. \# DOF, \( L^2 \) errors and CPU time in minutes

<table>
<thead>
<tr>
<th>( p )</th>
<th>( L^2 ) error</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.9E-03</td>
<td>11450</td>
<td>44060</td>
</tr>
<tr>
<td>1</td>
<td>2.0E-01</td>
<td>40</td>
<td>213</td>
</tr>
<tr>
<td>2</td>
<td>2.33</td>
<td>11450</td>
<td>44060</td>
</tr>
<tr>
<td>3</td>
<td>2.28</td>
<td>40</td>
<td>213</td>
</tr>
</tbody>
</table>

### Table III. \# DOF, \( L^2 \) errors and CPU time in minutes

<table>
<thead>
<tr>
<th>( p )</th>
<th>( L^2 ) error</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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REFERENCES


