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To cite this version:
Hassan Fahs. A non-conforming discontinuous Galerkin method for solving Maxwell’s equations. NUMELEC 2008, Dec 2008, Liège, Belgium. 2-p. hal-00452258

HAL Id: hal-00452258
https://hal.archives-ouvertes.fr/hal-00452258
Submitted on 1 Feb 2010

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A non-conforming discontinuous Galerkin method for solving Maxwell’s equations

H. Fahs
INRIA, 2004 Route des Lucioles, BP 93, F-06902 Sophia Antipolis Cedex, France
E-mail: hassan.fahs@inria.fr

Abstract — This paper reviews the main features of a high-order non-dissipative discontinuous Galerkin (DG) method recently investigated in [1]-[3] for solving Maxwell’s equations on non-conforming simplex meshes. The proposed method combines a centered approximation for the numerical fluxes at inter element boundaries, with either a second-order or a fourth-order leap-frog time integration scheme. Moreover, the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way resulting in arbitrary-level hanging nodes.

I. INTRODUCTION

In the recent years, several works have demonstrated the benefits of using DG methods for the simulation of time-domain electromagnetic wave propagation problems involving complex geometries and heterogeneous media. Being higher order versions of traditional finite volume method [1]-[5], Discontinuous Galerkin Time-Domain (DGTD) methods are flexible discretization methods that can handle complicated geometries, media and meshes, and achieve high-order accuracy by simply choosing suitable local basis functions. Whereas several conforming DGTD methods for the numerical resolution of the system of Maxwell equations have been developed so far [4], the design of non-conforming DGTD methods is still in its infancy. In practice, the non-conformity can result from a local refinement of the mesh (i.e. \( h \)-refinement), of the interpolation order (i.e. \( p \)-enrichment) or of both of them (i.e. \( h/p \)-refinement).

II. DISCONTINUOUS GALERKIN METHOD

We consider the three-dimensional Maxwell equations on a bounded domain \( \Omega \subset \mathbb{R}^3 \). The electric permittivity tensor \( \epsilon(x) \) and the magnetic permeability tensor \( \mu(x) \) are varying in space and both symmetric positive definite. The electric permittivity tensor \( \epsilon \) and the magnetic permeability tensor \( \mu \) are varying in space and both symmetric positive definite. The electric field \( \vec{E} \) and the magnetic field \( \vec{H} \) verify: \( \vec{E} \times \vec{B} = \epsilon \vec{H} \) and \( \vec{E} \times \vec{B} = -\mu \vec{E} \). We consider a partition \( \Omega_h \) of \( \Omega \) into a set of tetrahedra \( T_i \). To each \( T_i \in \Omega_h \) we assign an integer \( p_i \geq 0 \) (the local interpolation order) and we collect the \( p_i \) in the vector \( p = \{ p_i : T_i \in \Omega_h \} \). On the domain boundary \( \partial \Omega \), we use either a metallic boundary condition or a first-order Silver-Müller absorbing boundary condition. In the following, for a given partition \( \Omega_h \) and vector \( p \), we seek approximate solutions to Maxwell’s system in the finite dimensional subspace \( V_p(T_i) := \{ v \in L^2(\Omega)^3 : v_{T_i} \in \mathbb{P}_p(T_i), \forall T_i \in \Omega_h \} \), where \( \mathbb{P}_p(T_i) \) denotes the space of nodal polynomials of total degree at most \( p_i \) inside the element \( T_i \). The DGTD method at the heart of this study is based on a totally centered numerical fluxes at the interface between elements of the mesh. We suppose that all electric (resp. magnetic) unknowns are gathered in a column vector \( \vec{E} \) (resp. \( \vec{H} \)) of size \( d \) (the total number of degrees of freedom), then the semi-discretized DGTD method can be rewritten as (see [1]-[2]-[3] for more details):

\[
\begin{align*}
\mathbb{M}_p^e \partial_t \vec{E} &= \mathbb{K}\vec{H} - \mathbb{A}\vec{H} - \mathbb{B}\vec{H}, \\
\mathbb{M}_p^h \partial_t \vec{H} &= -\mathbb{K}\vec{E} + \mathbb{A}\vec{E} - \mathbb{B}\vec{E},
\end{align*}
\]

where we have the following definitions and properties:

- \( \mathbb{M}_p^e, \mathbb{M}_p^h \) and \( \mathbb{K} \) are \( d \times d \) block diagonal matrices with diagonal blocks equal to the local mass and stiffness matrices respectively. Therefore \( \mathbb{M}_p^e \) and \( \mathbb{M}_p^h \) are symmetric positive definite matrices, and \( \mathbb{K} \) is a symmetric matrix.
- \( \mathbb{A} \) is also a \( d \times d \) block sparse matrix, whose non-zero blocks are equal to the internal interface matrix (corresponds to fluxes at internal interfaces of the mesh). Then \( \mathbb{A} \) is a symmetric matrix.
- \( \mathbb{B} \) is a \( d \times d \) block diagonal matrix, whose non-zero blocks are equal to the metallic interface matrix (corresponds to fluxes at metallic boundary interfaces of the mesh). Then \( \mathbb{B} \) is a skew-symmetric matrix.

The DGTD-\( \mathbb{P}_p \) method using centered fluxes combined with \( N \)th order leap-frog (LF) time scheme [6] and arbitrary local accuracy and basis functions can be written, in function of the matrix \( S = \mathbb{K} - \mathbb{A} - \mathbb{B} \), in the general form:

\[
\begin{align*}
\mathbb{M}_p^e \frac{\vec{E}^{n+1} - \vec{E}^n}{\Delta t} &= S_N \vec{H}^{n+\frac{1}{2}}, \\
\mathbb{M}_p^h \frac{\vec{H}^{n+\frac{1}{2}} - \vec{H}^{n-\frac{1}{2}}}{\Delta t} &= -t S_N \vec{E}^{n+1},
\end{align*}
\]

where the matrix \( S_N \) verifies:

\[
S_N = \begin{cases}
S & \text{if } N = 2, \\
S(I - \frac{\Delta t^2}{24} \mathbb{M}_{-\mu}^{-\frac{1}{2}} S \mathbb{M}_{-\epsilon}^{-\frac{1}{2}}) & \text{if } N = 4.
\end{cases}
\]

III. STABILITY AND CONVERGENCE ANALYSIS

We define the following discrete version of the electromagnetic energy.

\[
\mathcal{E}^n = \frac{1}{2} \left( \epsilon \vec{E}^n \cdot \vec{E}^n + \mu \vec{H}^n \cdot \vec{H}^n + \frac{1}{\beta} \mu \vec{H}^n \cdot \mathbb{M}^e \vec{E}^{n+\frac{1}{2}} \right).
\]

Then we have the following Lemmas [2]:

**Lemma 1** Using the DGTD-\( \mathbb{P}_p \) method (2)-(3), the total discrete electromagnetic energy \( \mathcal{E}^n \) (4) is a positive definite quadratic form of all unknowns if:

\[
\Delta t \leq \frac{2}{d_N}, \quad \text{with } d_N = ||\mathbb{M}_p^e + S_N \mathbb{M}_p^h||,
\]

where \( || \cdot || \) denote the canonical norm of a matrix (\( \forall X, ||AX|| \leq ||A|| ||X|| \)) and the matrix \( \mathbb{M}_p^e \) (\( \sigma \) stands for \( \epsilon \) or \( \mu \)) is the inverse square root of \( \mathbb{M}^e \). Also, for a given mesh, the stability limit of the LF_2 scheme is roughly 2.85 times larger than that of the LF_3 scheme.

**Lemma 2** The convergence order in space and time of the DGTD-\( \mathbb{P}_p \) method (2)-(3) is:

\[
O(T_h^{\min(p,s)}) + O(\Delta t^s),
\]

where \( \Delta t \in [0, T] \), \( h \) is the mesh size and the solution belongs to \( H^s(\Omega) \) with \( s > 1/2 \) a regularity parameter.
IV. NUMERICAL EXPERIMENTS

In the following, for a given non-conforming mesh, we assign to coarse (i.e., non refined) elements a high polynomial degree $p_1$ and to the refined region a low polynomial degree $p_2$. The resulting scheme is referred to as DGTD-$P_{(p_1,p_2)}$. If $p_1 = p_2 = p$, the scheme is simply called DGTD-$P_p$.

A. eigenmode in a PEC cavity

The first test case that we consider is the propagation of an eigenmode in a unitary PEC cavity with $\epsilon = \mu = 1$.

The 2D case: We compare the LF$_2$ and LF$_4$ time schemes using the DGTD-$P_p$ method. Numerical simulations make use of a non-conforming triangular mesh which consists of 782 triangles and 442 nodes (36 of them are hanging nodes) as shown on Fig. 1 right. We plot on Fig. 1 left the time evolution of the overall $L^2$ error of the DGTD-$P_p$ method using the LF$_2$ and LF$_4$ schemes. Tab. I gives the $L^2$ error, the CPU time in minutes to reach 105 periods, and the convergence order “C”. It can be observed from Fig. 1 and Tab. I that the LF$_2$ is more accurate and requires less CPU time than the LF$_4$ scheme. Moreover, the convergence order is bounded by $N$ which confirms the result of Lemma 2.

![Fig. 1. Error of the DGTD-$P_p$ method (left) on the non-conforming triangular mesh (right).](image)

![Table I. Comparison between LF$_2$ and LF$_4$ DGTD-$P_p$ method](table)

<table>
<thead>
<tr>
<th>$p$</th>
<th>LF$_2$ scheme</th>
<th>LF$_4$ scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>CPU</td>
</tr>
<tr>
<td>2</td>
<td>1.8E-03</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>3.1E-04</td>
<td>98</td>
</tr>
<tr>
<td></td>
<td>1.9E-04</td>
<td>2.10</td>
</tr>
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</table>

The 3D case: Here we compare the DGTD-$P_{(p_1,p_2)}$ with several DGTD-$P_p$ methods using the LF$_2$ time scheme. Numerical simulations make use of an unstructured mesh which consists of 4406 tetrahedra and 962 nodes. Tab. II gives the $L^2$ error, the # DOF and the CPU time in minutes to reach 30 periods. Note that, the low polynomial degree $p_2$ is used in only 5% of the tetrahedra of the mesh. We can deduce from Tab. II that to achieve a given accuracy, the DGTD-$P_{(p_1,p_2)}$ requires less CPU time than the DGTD-$P_p$. Moreover, for a given CPU time, the DGTD-$P_{(p_1,p_2)}$ method is roughly 8 times more efficient than the DGTD-$P_p$ method.

B. Scattering of a plane wave by a dielectric cylinder

In this problem, a plane wave impinges on a dielectric cylinder, experiencing reflection and refraction at the material interface. The material is non-magnetic, and the material exterior to the cylinder is assumed to be vacuum. The cylinder has a radius $r_0 = 0.6$ and bounds a material of relative permittivity $\epsilon_r = 2.25$. The computational domain is chosen as a cylinder of radius 1.5 centered at $(0, 0)$. At the artificial boundary, we apply a first-order Silver-Müller absorbing boundary condition. Here, we compare the DGTD-$P_p$ method using a conforming mesh with the DGTD-$P_{(p_1,p_2)}$ using a non-conforming mesh. To this end, we construct a conforming mesh consisting of 11920 triangles and 6001 nodes. Then, a non-conforming mesh is obtained by locally refining (two refinement levels) the cylindrical zone $0.55 \leq r \leq 0.65$ of a coarse conforming mesh. The resulting non-conforming mesh consists of 5950 triangles and 3151 nodes (390 of them are hanging nodes). Tab. III shows the relative $L^2$ error, the # DOF and the CPU time in minutes to reach $t = 5$. As expected, the gain in CPU time between the DGTD-$P_{(p_1,p_2)}$ and the conforming DGTD-$P_p$ methods is considerable. For instance, to achieve an error level of 5%, the DGTD-$P_{(2,0)}$ scheme requires 3 times less DOF and 21 times less CPU time than the DGTD-$P_2$ scheme.

![Table II. # DOF, $L^2$ errors and CPU time in minutes](table)

<table>
<thead>
<tr>
<th>$p$</th>
<th>$L^2$ error</th>
<th>CPU</th>
<th># DOF</th>
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<tbody>
<tr>
<td></td>
<td>7.2E-01</td>
<td>4</td>
<td>4406</td>
</tr>
<tr>
<td>1</td>
<td>2.0E-01</td>
<td>40</td>
<td>17624</td>
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<tr>
<td>2</td>
<td>1.4E-02</td>
<td>213</td>
<td>40600</td>
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<tr>
<td>3</td>
<td>8.0E-04</td>
<td>859</td>
<td>88120</td>
</tr>
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<table>
<thead>
<tr>
<th>$(p_1,p_2)$</th>
<th>$L^2$ error</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2,0)$</td>
<td>3.6E-02</td>
<td>35</td>
<td>42908</td>
</tr>
<tr>
<td>$(2,1)$</td>
<td>1.3E-02</td>
<td>106</td>
<td>43676</td>
</tr>
<tr>
<td>$(3,1)$</td>
<td>1.0E-03</td>
<td>260</td>
<td>87096</td>
</tr>
<tr>
<td>$(3,2)$</td>
<td>8.5E-04</td>
<td>499</td>
<td>86030</td>
</tr>
</tbody>
</table>

![Table III. # DOF, $L^2$ errors and CPU time in minutes](table)

<table>
<thead>
<tr>
<th>$p$</th>
<th>$L^2$ error</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>13.6%</td>
<td>20</td>
<td>11920</td>
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<td>1</td>
<td>7.15%</td>
<td>178</td>
<td>35760</td>
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<tr>
<td>2</td>
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<td>542</td>
<td>71520</td>
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<tr>
<td>3</td>
<td>5.22%</td>
<td>1817</td>
<td></td>
</tr>
<tr>
<td></td>
<td>119200</td>
<td></td>
<td>119200</td>
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</table>

<table>
<thead>
<tr>
<th>$(p_1,p_2)$</th>
<th>$L^2$ error</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1,0)$</td>
<td>11.6%</td>
<td>9</td>
<td>11450</td>
</tr>
<tr>
<td>$(2,0)$</td>
<td>5.36%</td>
<td>25</td>
<td>19700</td>
</tr>
<tr>
<td>$(2,1)$</td>
<td>5.39%</td>
<td>33</td>
<td>26100</td>
</tr>
<tr>
<td>$(3,2)$</td>
<td>5.37%</td>
<td>179</td>
<td></td>
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</tbody>
</table>

![REFERENCES](references)