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A non-conforming discontinuous Galerkin method for solving Maxwell’s equations

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Abstract — This paper reviews the main features of a high-order non-dissipative discontinuous Galerkin (DG) method recently investigated in [1]-[3] for solving Maxwell’s equations on non-conforming simplex meshes. The proposed method combines a centered approximation for the numerical fluxes at inter element boundaries, with either a second-order or a fourth-order leap-frog time integration scheme. Moreover, the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way resulting in arbitrary-level hanging nodes.

I. INTRODUCTION

In the recent years, several works have demonstrated the benefits of using DG methods for the simulation of time-domain electromagnetic wave propagation problems involving complex geometries and heterogeneous media. Being higher order versions of traditional finite volume method [1]-[3], Discontinuous Galerkin Time-Domain (DGTD) methods are flexible discretization methods that can handle complicated geometries, media and meshes, and achieve high-order accuracy by simply choosing suitable local basis functions. Whereas several conforming DGTD methods for the numerical resolution of the system of Maxwell equations have been developed so far [4], the design of non-conforming DGTD methods is still in its infancy. In practice, the non-conformity can result from a local refinement of the mesh (i.e. h-refinement), of the interpolation order (i.e. p-enrichment) or of both of them (i.e. hp-refinement).

II. DISCONTINUOUS GALERKIN METHOD

We consider the three-dimensional Maxwell equations on a bounded domain \( \Omega \subset \mathbb{R}^3 \). The electric permittivity tensor \( \varepsilon(x) \) and the magnetic permeability tensor \( \mu(x) \) are varying in space and both symmetric positive definite. The electric field \( \vec{E} \) and the magnetic field \( \vec{H} \) verify: \( \varepsilon \partial_t \vec{E} = \text{curl} \vec{H} \) and \( \mu \partial_t \vec{H} = -\varepsilon \text{curl} \vec{E} \). We consider a partition \( \Omega_h \) of \( \Omega \) into a set of tetrahedra \( \tau_i \). To each \( \tau_i \in \Omega_h \) we assign an integer \( p_i \geq 0 \) (the local interpolation order) and we collect the \( p_i \) in the vector \( p = \{p_i : \tau_i \in \Omega_h\} \). On the domain boundary \( \partial \Omega \), we use either a metallic boundary condition or a first-order Silver-Müller absorbing boundary condition. In the following, for a given partition \( \Omega_h \) and vector \( p \), we seek approximate solutions to Maxwell’s system in the finite dimensional subspace \( V_p(\Omega_h) := \{v \in L^2(\Omega)^3 : \forall \tau_i \in \Omega_h, \, \tau_i \in \Omega_h\} \), where \( \mathbb{P}_p(\tau_i) \) denotes the space of nodal polynomials of total degree at most \( p_i \) inside the element \( \tau_i \). The DGTD method at the heart of this study is based on a totally centered numerical fluxes at the interface between elements of the mesh. We suppose that all electric (resp. magnetic) unknowns are gathered in a column vector \( \vec{E} \) (resp. \( \vec{H} \)) of size \( d \) (the total number of degrees of freedom), then the semi-discretized DGTD method can be rewritten as (see [1]-[2]-[3] for more details):

\[
\begin{align*}
M^\epsilon \partial_t \vec{E} &= KH - AH - \vec{B}H, \\
M^\mu \partial_t \vec{H} &= -K\vec{E} + A\vec{E} - B\vec{E},
\end{align*}
\]

(1)

where we have the following definitions and properties:

- \( M^\epsilon, M^\mu \) and \( \mathbb{K} \) are \( d \times d \) block diagonal matrices with diagonal blocks equal to the local mass and stiffness matrices respectively. Therefore \( M^\epsilon \) and \( M^\mu \) are symmetric positive definite matrices, and \( \mathbb{K} \) is a symmetric matrix.
- \( \mathbb{A} \) is also a \( d \times d \) block sparse matrix, whose non-zero blocks are equal to the internal interface matrix (corresponds to fluxes at internal interfaces of the mesh). Then \( \mathbb{A} \) is a symmetric matrix.
- \( \mathbb{B} \) is a \( d \times d \) block diagonal matrix, whose non-zero blocks are equal to the metallic interface matrix (corresponds to fluxes at metallic boundary interfaces of the mesh). Then \( \mathbb{B} \) is a skew-symmetric matrix.

The DGTD-\( \mathbb{P}_p \) method using centered fluxes combined with \( N \)th order leap-frog (\( LF_N \)) time scheme [6] and arbitrary local accuracy and basis functions can be written, in function of the matrix \( S = \mathbb{K} - \mathbb{A} - \mathbb{B} \), in the general form:

\[
\begin{align*}
\begin{cases}
M^\epsilon \vec{E}^{n+1} - \vec{E}^n &= tS_N \mathbb{H}^{n+\frac{1}{2}}, \\
M^\mu \mathbb{H}^{n+\frac{1}{2}} - \mathbb{H}^{n+\frac{1}{2}} &= -tS_N \mathbb{E}^{n+1},
\end{cases}
\end{align*}
\]

(2)

where the matrix \( S_N \) verifies:

\[
S_N = \begin{cases} S & \text{if } N = 2, \\
S(I - \frac{t^2}{24}M^{-\mu}(SM^{-\epsilon - \mathbb{S}})) & \text{if } N = 4.
\end{cases}
\]

(3)

III. STABILITY AND CONVERGENCE ANALYSIS

We define the following discrete version of the electromagnetic energy:

\[
\mathcal{E}^n = \frac{1}{2} \left\{ (\mathbb{E}^n)^T M^\epsilon \mathbb{E}^n + (\mathbb{H}^{n+\frac{1}{2}})^T M^\mu (\mathbb{H}^{n+\frac{1}{2}}) \right\}. 
\]

(4)

Then we have the following Lemmas [2]:

**Lemma 1** Using the DGTD-\( \mathbb{P}_p \) method (2)-(3), the total discrete electromagnetic energy \( \mathcal{E}^n \) (4) is a positive definite quadratic form of all unknowns if:

\[
\Delta t \leq \frac{2}{d_N}, \quad \text{with } d_N = \|M^{\frac{1}{2}}M^{-\epsilon} S_N M^{\frac{1}{2}}H\|,
\]

where \( \| \| \) denote the canonical norm of a matrix (\( \forall X, \|AX\| \leq \|A\|\|X\| \)) and the matrix \( M^{\frac{1}{2}}(\sigma \text{ stands for } \epsilon \text{ or } \mu) \) is the inverse square root of \( M^\epsilon \). Also, for a given mesh, the stability limit of the LF scheme is roughly 2.85 times larger than that of the LF2 scheme.

**Lemma 2** The convergence order in space and time of the DGTD-\( \mathbb{P}_p \) method (2)-(3) is:

\[
O(T_h^{\min(s,p)}) = O(\Delta t^s),
\]

where \( \Delta t \in [0,T] \), \( h \) is the mesh size and the solution belongs to \( H^s(\Omega) \) with \( s > 1 \) a regularity parameter.
IV. NUMERICAL EXPERIMENTS

In the following, for a given non-conforming mesh, we assign to coarse (i.e., non refined) elements a high polynomial degree \( p_1 \) and to the refined region a low polynomial degree \( p_2 \). The resulting scheme is referred to as DGTD-P\((p_1, p_2)\). If \( p_1 = p_2 = p \), the scheme is simply called DGTD-P\(p\).

A. eigenmode in a PEC cavity

The first test case that we consider is the propagation of an eigenmode in a unitary PEC cavity with \( \epsilon = \mu = 1 \).

The 2D case: We compare the LF\(2\) and LF\(4\) time schemes using the DGTD-P\(p\) method. Numerical simulations make use of a non-conforming triangular mesh which consists of 782 triangles and 442 nodes (36 of them are hanging nodes) as shown on Fig. 1. We plot on Fig. 1 left the time evolution of the overall \( L^2 \) error of the DGTD-P\(p\) method using the LF\(2\) and LF\(4\) schemes. Tab. I gives the \( L^2 \) error, the CPU time in minutes to reach 105 periods, and the convergence order “\( C \)”. It can be observed from Fig. 1 and Tab. I that the LF\(2\) is more accurate and requires less CPU time than the LF\(4\) scheme. Moreover, the convergence order is bounded by \( N \) which confirms the result of Lemma 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\linewidth]{fig1.png}
\caption{Error of the DGTD-P\(p\) method (left) on the non-conforming triangular mesh (right).}
\end{figure}

\begin{table}[h]
\centering
\caption{Comparison between LF\(2\) and LF\(4\) DGTD-P\(p\) method}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( p \) & LF\(2\) scheme & & LF\(4\) scheme & & \\
\hline
 & \( L^2 \) error & CPU & \( \Delta t \) & \( L^2 \) error & CPU & \( \Delta t \) \\
\hline
2 & 1.8E-03 & 11 & 2.28 & 5.2E-04 & 8 & 2.32 \\
3 & 3.1E-04 & 98 & 2.10 & 2.4E-05 & 28 & 2.97 \\
\hline
\end{tabular}
\end{table}

The 3D case: Here we compare the DGTD-P\((p_1, p_2)\) with several DGTD-P\(p\) methods using the LF\(2\) time scheme. Numerical simulations make use of an unstructured mesh which consists of 4406 tetrahedra and 962 nodes. Tab. II gives the \( L^2 \) error, the # DOF and the CPU time in minutes to reach 30 periods. Note that, the low polynomial degree \( p_2 \) is used in only 5% of the tetrahedra of the mesh. We can deduce from Tab. II that to achieve a given accuracy, the DGTD-P\((p_1, p_2)\) requires less CPU time than the DGTD-P\(p\). Moreover, for a given CPU time, the DGTD-P\((p_1, p_2)\) method is roughly 8 times more efficient than the DGTD-P\(p\) method.

\begin{table}[h]
\centering
\caption{Comparison between LF\(2\) and LF\(4\) DGTD-P\(p\) method}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( p \) & LF\(2\) scheme & & LF\(4\) scheme & & \\
\hline
 & \( L^2 \) error & CPU & \( \Delta t \) & \( L^2 \) error & CPU & \( \Delta t \) \\
\hline
2 & 1.8E-03 & 11 & 2.28 & 5.2E-04 & 8 & 2.32 \\
3 & 3.1E-04 & 98 & 2.10 & 2.4E-05 & 28 & 2.97 \\
\hline
\end{tabular}
\end{table}

B. Scattering of a plane wave by a dielectric cylinder

In this problem, a plane wave impinges on a dielectric cylinder, experiencing reflection and refraction at the material interface. The material is non-magnetic, and the material exterior to the cylinder is assumed to be vacuum. The cylinder has a radius \( r_0 = 0.6 \) and bounds a material of relative permittivity \( \epsilon_r = 2.25 \). The computational domain is chosen as a cylinder of radius 1.5 centered at \((0,0)\). At the artificial boundary, we apply a first order Silver-Müller absorbing boundary condition. Here, we compare the DGTD-P\(p\) method using a conforming mesh with the DGTD-P\((p_1, p_2)\) using a non-conforming mesh. To this end, we construct a conforming mesh consisting of 11920 triangles and 6001 nodes. Then, a non-conforming mesh is obtained by locally refining (two refinement levels) the cylindrical zone \( 0.55 \leq r \leq 0.65 \) of a coarse conforming mesh. The resulting non-conforming mesh consists of 5960 triangles and 3151 nodes (360 of them are hanging nodes). Tab. III shows the relative \( L^2 \) error, the # DOF and the CPU time in minutes to reach \( t \geq 5 \). As expected, the gain in CPU time between the DGTD-P\((p_1, p_2)\) and the conforming DGTD-P\(p\) method is considerable. For instance, to achieve an error level of 5%, the DGTD-P\((2,0)\) scheme requires 3 times less DOF and 21 times less CPU time than the DGTD-P\(2\) scheme.

\begin{table}[h]
\centering
\caption{Comparison between LF\(2\) and LF\(4\) DGTD-P\(p\) method}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( p \) & LF\(2\) scheme & & LF\(4\) scheme & & \\
\hline
 & \( L^2 \) error & CPU & \( \Delta t \) & \( L^2 \) error & CPU & \( \Delta t \) \\
\hline
2 & 1.8E-03 & 11 & 2.28 & 5.2E-04 & 8 & 2.32 \\
3 & 3.1E-04 & 98 & 2.10 & 2.4E-05 & 28 & 2.97 \\
\hline
\end{tabular}
\end{table}

REFERENCES


