A non-conforming discontinuous Galerkin method for solving Maxwell’s equations
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Abstract — This paper reviews the main features of a high-order non-dissipative discontinuous Galerkin (DG) method recently investigated in [1]-[3] for solving Maxwell’s equations on non-conforming simplex meshes. The proposed method combines a centered approximation for the numerical fluxes at inter element boundaries, with either a second-order or a fourth-order leap-frog time integration scheme. Moreover, the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way resulting in arbitrary-level hanging nodes.

I. INTRODUCTION

In the recent years, several works have demonstrated the benefits of using DG methods for the simulation of time-domain electromagnetic wave propagation problems involving complex geometries and heterogeneous media. Being higher order versions of traditional finite volume method [1]-[3], Discontinuous Galerkin Time-Domain (DGTD) methods are flexible discretization methods that can handle complicated geometries, media and meshes, and achieve high-order accuracy by simply choosing suitable local basis functions. Whereas several conforming DGTD methods for the numerical resolution of the system of Maxwell equations have been developed so far [4], the design of non-conforming DGTD methods is still in its infancy. In practice, the non-conformity can result from a local refinement of the mesh (i.e. h-refinement), of the interpolation order (i.e. p-enrichment) or of both of them (i.e. hp-refinement).

II. DISCONTINUOUS GALERKIN METHOD

We consider the three-dimensional Maxwell equations on a bounded domain \( \Omega \) of \( \mathbb{R}^3 \). The electric permittivity tensor \( \varepsilon(x) \) and the magnetic permeability tensor \( \mu(x) \) are varying in space and both symmetric positive definite. The electric field \( \vec{E} \) and the magnetic field \( \vec{H} \) verify: \( \varepsilon \partial_t \vec{E} = \text{curl} \vec{H} \) and \( \mu \partial_t \vec{H} = -\text{curl} \vec{E} \). We consider a partition \( \Omega_h \) of \( \Omega \) into a set of tetrahedra \( \tau_i \). To each \( \tau_i \in \Omega_h \) we assign an integer \( p_i \geq 0 \) (the local interpolation order) and we collect the \( p_i \) in the vector \( p = \{ p_i : \tau_i \in \Omega_h \} \). On the domain boundary \( \partial \Omega \), we use either a metallic boundary condition or a first-order Silver-Müller absorbing boundary condition. In the following, for a given partition \( \Omega_h \) and vector \( p \), we seek approximate solutions to Maxwell’s system in the finite dimensional subspace \( V_p(\Omega_h) := \{ v \in L^2(\Omega)^3 : v_{\tau_i} \in \mathbb{P}_{p_i}(\tau_i), \forall \tau_i \in \Omega_h \} \), where \( \mathbb{P}_{p_i}(\tau_i) \) denotes the space of nodal polynomials of total degree at most \( p_i \) inside the element \( \tau_i \). The DGTD method at the heart of this study is based on a totally centered numerical fluxes at the interface between elements of the mesh. We suppose that all electric (resp. magnetic) unknowns are gathered in a column vector \( \vec{E} \) (resp. \( \vec{H} \)) of size \( d \) (the total number of degrees of freedom), then the semi-discretized DGTD method can be rewritten as (see [1]-[2]-[3] for more details):

\[
\begin{aligned}
M^e \partial_t \vec{E} &= KH - AH - BH, \\
M^\mu \partial_t \vec{H} &= -K\vec{E} + AE - BE,
\end{aligned}
\]

where we have the following definitions and properties:

- \( M^e, M^\mu \) and \( K \) are \( d \times d \) block diagonal matrices with diagonal blocks equal to the local mass and stiffness matrices respectively. Therefore \( M^e \) and \( M^\mu \) are symmetric positive definite matrices, and \( K \) is a symmetric matrix.
- \( A \) is also a \( d \times d \) block sparse matrix, whose non-zero blocks are equal to the internal interface matrix (corresponds to fluxes at internal interfaces of the mesh). Then \( A \) is a symmetric matrix.
- \( B \) is a \( d \times d \) block diagonal matrix, whose non-zero blocks are equal to the metallic interface matrix (corresponds to fluxes at metallic boundary interfaces of the mesh). Then \( B \) is a skew-symmetric matrix.

The DGTD-\( \mathbb{P}_p \) method using centered fluxes combined with \( N \)th order leap-frog (LF\textsuperscript{N}) time scheme [6] and arbitrary local accuracy and basis functions can be written, in function of the matrix \( S = K - A - B \), in the general form:

\[
\begin{aligned}
M^e \frac{\vec{E}_{n+1} - \vec{E}_n}{\Delta t} &= S_N \vec{H}_{n+\frac{1}{2}}, \\
M^\mu \frac{\vec{H}_{n+\frac{1}{2}} - \vec{H}_n}{\Delta t} &= -t_S N \vec{E}_{n+1},
\end{aligned}
\]

where the matrix \( S_N \) verifies:

\[
S_N = \begin{cases} 
S & \text{if } N = 2, \\
S(I - \frac{\Delta t^2}{24} M^{-\mu}(tSM^{-\epsilon})^2) & \text{if } N = 4.
\end{cases}
\]

III. STABILITY AND CONVERGENCE ANALYSIS

We define the following discrete version of the electromagnetic energy:

\[
\mathcal{E}^n = \frac{1}{2} \left( \| \vec{E}_n \|^2 \frac{M^e}{\mu} + \| \vec{H}_n \|^2 \frac{M^\mu}{\mu} \right) + \frac{1}{2}\| \vec{H}_n \|^2 \frac{M^e}{\mu} m^\epsilon H_{n+\frac{1}{2}}.
\]

Then we have the following Lemmas [2]:

**Lemma 1** Using the DGTD-\( \mathbb{P}_p \) method (2)-(3), the total discrete electromagnetic energy \( \mathcal{E}^n \) (4) is a positive definite quadratic form of all unknowns if:

\[
\Delta t \leq \frac{2}{d_N}, \quad \text{with } d_N = ||M^\frac{-\epsilon}{\mu} S_N M^\frac{-\epsilon}{\mu^2}||,
\]

where \( ||.|| \) denote the canonical norm of a matrix (\( \forall X, \| AX \| \leq \| A \| \| X \| \)), and the matrix \( M^\frac{-\epsilon}{\mu} \) (\( \sigma \) stands for \( \epsilon \) or \( \mu \)) is the inverse square root of \( M^\epsilon \). Also, for a given mesh, the stability limit of the LF\textsuperscript{4} scheme is roughly 2.85 times larger than that of the LF\textsuperscript{2} scheme.

**Lemma 2** The convergence order in space and time of the DGTD-\( \mathbb{P}_p \) method (2)-(3) is:

\[
O(T h^{\text{min}(s,p)}) + O(\Delta t^s),
\]

where \( \Delta t \in [0,T] \), \( h \) is the mesh size and the solution belongs to \( H^s(\Omega) \) with \( s > 1/2 \) a regularity parameter.
IV. NUMERICAL EXPERIMENTS

In the following, for a given non-conforming mesh, we assign to coarse (i.e. non refined) elements a high polynomial degree $p_1$ and to the refined region a low polynomial degree $p_2$. The resulting scheme is referred to as DGTD-P$_{p_1,p_2}$. If $p_1 = p_2 = p$, the scheme is simply called DGTD-P$_p$.

A. eigenmode in a PEC cavity

The first test case that we consider is the propagation of an eigenmode in a unitary PEC cavity with $\epsilon = \mu = 1$.

The 2D case: We compare the LF$_2$ and LF$_4$ time schemes using the DGTD-P$_p$ method. Numerical simulations make use of a non-conforming triangular mesh which consists of 782 triangles and 442 nodes (30 of them are hanging nodes) as shown on Fig. 1 right. We plot on Fig. 1 left the time evolution of the overall $L^2$ error of the DGTD-P$_p$ method using the LF$_2$ and LF$_4$ schemes. Tab. I gives the $L^2$ error, the CPU time in minutes to reach 105 periods, and the convergence order “$O$”. It can be observed from Fig. 1 and Tab. I that the LF$_2$ is more accurate and requires less CPU time than the LF$_4$ scheme. Moreover, the convergence order is bounded by $N$ which confirms the result of Lemma 2.

![Fig. 1. Error of the DGTD-P$_p$ method (left) on the non-conforming triangular mesh (right).](image)

**Table I. Comparison between LF$_2$ and LF$_4$ DGTD-P$_p$ method**

<table>
<thead>
<tr>
<th>$p$</th>
<th>LF$_2$ scheme</th>
<th>LF$_4$ scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>$L^2$ error</td>
</tr>
<tr>
<td></td>
<td>CPU</td>
<td>$O$</td>
</tr>
<tr>
<td>2</td>
<td>1.8E-03</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>3.1E-04</td>
<td>98</td>
</tr>
</tbody>
</table>

![The 3D case: Here we compare the DGTD-P$_{p_1,p_2}$ with several DGTD-P$_p$ methods using the LF$_2$ time scheme. Numerical simulations make use of an unstructured mesh which consists of 4406 tetrahedra and 962 nodes. Tab. II gives the $L^2$ error, the # DOF and the CPU time in minutes to reach 30 periods. Note that, the low polynomial degree $p_2$ is used in only 5% of the tetrahedra of the mesh. We can deduce from Tab. II that to achieve a given accuracy, the DGTD-P$_{p_1,p_2}$ requires less CPU time than the DGTD-P$_p$. Moreover, for a given CPU time, the DGTD-P$_{p_1,p_2}$ method is roughly 8 times more efficient than the DGTD-P$_p$ method.](image)

**B. Scattering of a plane wave by a dielectric cylinder**

In this problem, a plane wave impinging on a dielectric cylinder, experiencing reflection and refraction at the material interface. The material is non-magnetic, and the material exterior to the cylinder is assumed to be vacuum. The cylinder has a radius $r_0 = 0.6$ and bounds a material of relative permittivity $\epsilon_r = 2.25$. The computational domain is chosen as a cylinder of radius 1.5 centered at $(0, 0)$. At the artificial boundary, we apply a first-order Silver-Müller absorbing boundary condition. Here, we compare the DGTD-P$_p$ method using a conforming mesh with the DGTD-P$_{p_1,p_2}$ using a non-conforming mesh. To this end, we construct a conforming mesh consisting of 11920 triangles and 6001 nodes. Then, a non-conforming mesh is obtained by locally refining (two refinement levels) the cylindrical zone $0.55 < r < 0.65$ of a coarse conforming mesh. The resulting non-conforming mesh consists of 5950 triangles and 3151 nodes (300 of them are hanging nodes). Tab. III shows the relative $L^2$ error, the # DOF and the CPU time in minutes to reach $t = 5$. As expected, the gain in CPU time between the DGTD-P$_{p_1,p_2}$ and the conforming DGTD-P$_p$ methods is considerable. For instance, to achieve an error level of 5%, the DGTD-P$_{2,0}$ scheme requires 3 times less DOF and 21 times less CPU time than the DGTD-P$_2$ scheme.

![The first test case that we consider is the propagation of an eigenmode in a unitary PEC cavity with $\epsilon = \mu = 1$. The 2D case: We compare the LF$_2$ and LF$_4$ time schemes using the DGTD-P$_p$ method. Numerical simulations make use of a non-conforming triangular mesh which consists of 782 triangles and 442 nodes (30 of them are hanging nodes) as shown on Fig. 1 right. We plot on Fig. 1 left the time evolution of the overall $L^2$ error of the DGTD-P$_p$ method using the LF$_2$ and LF$_4$ schemes. Tab. I gives the $L^2$ error, the CPU time in minutes to reach 105 periods, and the convergence order “$O$”. It can be observed from Fig. 1 and Tab. I that the LF$_2$ is more accurate and requires less CPU time than the LF$_4$ scheme. Moreover, the convergence order is bounded by $N$ which confirms the result of Lemma 2.](image)

**Table II. # DOF, $L^2$ errors and CPU time in minutes**

<table>
<thead>
<tr>
<th>$p$</th>
<th>$L^2$ error</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.2E-01</td>
<td>4</td>
<td>4406</td>
</tr>
<tr>
<td>1</td>
<td>2.0E-01</td>
<td>40</td>
<td>17624</td>
</tr>
<tr>
<td>2</td>
<td>1.4E-02</td>
<td>213</td>
<td>40600</td>
</tr>
<tr>
<td>3</td>
<td>8.0E-04</td>
<td>859</td>
<td></td>
</tr>
</tbody>
</table>

**Table III. # DOF, $L^2$ errors and CPU time in minutes**

<table>
<thead>
<tr>
<th>$p$</th>
<th>$L^2$ error</th>
<th>CPU</th>
<th># DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>13.6%</td>
<td>20</td>
<td>11920</td>
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<tr>
<td>1</td>
<td>7.15%</td>
<td>178</td>
<td>35760</td>
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<tr>
<td>2</td>
<td>5.20%</td>
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<td>71520</td>
</tr>
<tr>
<td>3</td>
<td>5.22%</td>
<td>1817</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5.39%</td>
<td>25</td>
<td>33</td>
</tr>
<tr>
<td>3</td>
<td>5.37%</td>
<td>179</td>
<td></td>
</tr>
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</table>

**REFERENCES**


