The Generalized Annihilation Property:A Tool For Solving Finite Rate of Innovation Problems

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Abstract:

We describe a property satisfied by a class of nonlinear systems of equations that are of the form $\mathbf{F}(\Omega)\mathrm{X} = \mathrm{Y}$. Here $\mathbf{F}(\Omega)$ is a matrix that depends on an unknown K-dimensional vector Ω, X is an unknown K-dimensional vector and Y is a vector of $N \geq K$) given measurements. Such equations are encountered in superresolution or sparse signal recovery problems known as "Finite Rate of Innovation" signal reconstruction.

We show how this property allows to solve explicitly for the unknowns Ω and X by a direct, non-iterative, algorithm that involves the resolution of two linear systems of equations and the extraction of the roots of a polynomial and give examples of problems where this type of solutions has been found useful.

1. Introduction

We consider the signal resulting from the convolution between a window $\varphi(t)$ and the sum of K Diracs with amplitude x_k located at time t_k. Given the N uniform samples y_n ($T =$ sampling step)

$$y_n = \sum_{k=1}^{K} x_k \varphi(nT - t_k) \qquad \text{where } n = 1, 2, \ldots, N, \quad (1)$$

then FRI problems (see [1, 2]) consist in retrieving the parameters t_k and x_k. Solving such problems is conceptually interesting because it shows how to break the standard Nyquist-Shannon bandlimitation rule for the exact reconstruction of signals from their uniform samples [3].

The system of consistent equations (1) can be expressed under the generic form of a nonlinear problem as shown in Fig. 1 (see next page), where the parameters $\Omega = [\omega_1, \omega_2, \ldots \omega_K]$ are related unambiguously to the unknowns t_k's. Because of the variety of settings adapted to this general approach, it happens to be necessary to distinguish between the parameters ω_k—which we shall call "abstract parameters"—and the locations t_k: typically, the ω_k's will be the zeros of some polynomial and from these ω_k's, we will be able to retrieve the t_k's using a functional relation of the form $\omega_k = \lambda(t_k)$ for some invertible function $\lambda(t)$.

At first sight, solving such a nonlinear system of equations is a daunting task. Fortunately, if the matrix $\mathbf{F}(\Omega)$ satisfies a property that we shall call "Generalized Annihilation Property" (GAP), this reduces to solving two linear systems of equations sandwiching a nonlinear step that amounts to polynomial root extraction in practical cases. The filters $\varphi(t)$ that satisfy the GAP are thus especially interesting, since the related FRI problems enjoy a straight non-iterative solution.

2. The Generalized Annihilation Property (GAP)

We carry on with the previously identified general nonlinear problem, namely

$$\mathbf{F}(\Omega)\,\mathrm{X} = \mathrm{Y}, \quad (3)$$

where the unknowns are $\Omega = [\omega_1, \omega_2, \ldots \omega_K]$ and $\mathrm{X} = [x_1, x_2, \ldots x_K]$, and where the measurements are $\mathrm{Y} = [y_1, y_2, \ldots y_N]$.

This system is said to satisfy the Generalized Annihilation Property whenever there exist $K+1$ constant matrices, $\mathbf{A}_k$, and $K+1$ scalar functions of Ω, $h_k(\Omega)$, such that we have the identity

$$\sum_{k=0}^{K} h_k(\Omega)\,\mathbf{A}_k\,\mathbf{F}(\Omega) = \mathbf{0}. \quad (4)$$

for any vector of parameters Ω. By right multiplying with X, the above equation implies that any solution Ω of (3) is also a solution of the (generalized) annihilation equation

$$\sum_{k=0}^{K} h_k(\Omega)\mathbf{A}_k\mathrm{Y} = 0. \quad (5)$$

This equation can be expressed in a matrix form $\mathbf{A}\mathrm{H} = 0$ where the unknown is $\mathrm{H} = [\,h_0(\Omega), h_1(\Omega), \ldots, h_K(\Omega)\,]^{\mathrm{T}}$ and the matrix $\mathbf{A} = \big[\,\mathbf{A}_0\mathrm{Y}, \mathbf{A}_1\mathrm{Y}, \ldots, \mathbf{A}_K\mathrm{Y}\,\big]$. Thus, in order to solve (3) for Ω and X, the idea consists in finding the scalar coefficients $h_k(\Omega)$ that satisfy (5), then retrieving $\omega_1, \omega_2, \ldots, \omega_K$ from the knowledge of $h_k(\Omega)$, and finally finding X such that $\mathbf{F}(\Omega)\,\mathrm{X} = \mathrm{Y}$. Without elaborating on the conditions that make this solution unique, a

$$\underbrace{\begin{bmatrix} \varphi(T-t_1) & \varphi(T-t_2) & \cdots & \varphi(T-t_K) \\ \varphi(2T-t_1) & \varphi(2T-t_2) & \cdots & \varphi(2T-t_K) \\ \vdots & \vdots & & \vdots \\ \varphi(NT-t_1) & \varphi(NT-t_2) & \cdots & \varphi(NT-t_K) \end{bmatrix}}_{\mathbf{F}(\Omega)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{bmatrix}}_{\mathrm{X}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}}_{\mathrm{Y}} \tag{2}$$

Figure 1: Algebraic equivalent of the consistency equations (1).

minimal requirement is that the matrices $\mathbf{A}_k$ have at least K rows.

In the simple case where the $h_k(\Omega)$'s are related to the ω_k's through a polynomial relation

$$\sum_{k=0}^{K} h_k(\Omega) z^{-k} = \prod_{k=1}^{K} (1 - \omega_k z^{-1}), \tag{6}$$

solving (3) boils down to a three-step algorithm that can be summarized as follows:

1. Compute a solution $\mathrm{H} = [\, 1, h_1, \ldots, h_{K-1}, h_K \,]^{\mathrm{T}}$ of
$$\left[\, \mathbf{A}_0 \mathrm{Y}, \mathbf{A}_1 \mathrm{Y}, \ldots, \mathbf{A}_K \mathrm{Y} \,\right] \mathrm{H} = 0;$$
2. Compute the roots ω_k of the z-transform $H(z) = \sum_{k=0}^{K} h_k z^{-k}$;
3. Compute a solution X of $\mathbf{F}(\Omega)\,\mathrm{X} = \mathrm{Y}$.

Example—Spectral estimation problems boil down to a nonlinear problem of the form (3) involving the *Vandermonde* matrix:

$$\mathbf{F}(\Omega) = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_K \\ \omega_1^2 & \omega_2^2 & \cdots & \omega_K^2 \\ \vdots & \vdots & & \\ \omega_1^N & \omega_2^N & \cdots & \omega_K^N \end{bmatrix}$$

where the frequencies to retrieve, f_k, are related to ω_k through $\omega_k = e^{j2\pi f_k}$. This problem satisfies the GAP for band-diagonal matrices $\mathbf{A}_k$ which are more precisely given by:

$$\mathbf{A}_k = \left[\, \mathbf{0}_{N-K,k} \;\; \mathbf{I}_{N-K} \;\; \mathbf{0}_{N-K,K-k} \,\right],$$

where $\mathbf{0}_{m,n}$ is the $m \times n$ zero matrix and $\mathbf{I}_n$ is the $n \times n$ identity matrix. A minimal—yet not sufficient—condition for the unicity of the solution is $N \geq 2K$. Since the $\mathbf{A}_k$ can be seen as shifting operators by k samples, the annihilation equation is analogous to a filtering equation—with an annihilating filter. The annihilation algorithm is then equivalent to Prony's method [4]. Of course, spectral estimation in the presence of noise has been addressed by numerous researchers since the 1970's [5, 6, 7, 8, 9, 10, 11].

3. Some GAP Kernels

The GAP is actually shared by many interesting filters that can be used in sampling schemes, resulting in easily solvable FRI problems. Among them, the first ones to be identified were the periodized sinc, the infinite (i-e., not periodized) sinc and the Gaussian kernels [1]. Even more interestingly, recent research indicates that this property may somewhat be related to the Strang-Fix conditions which makes a very intriguing connection with approximation theory [12], and considerably broadens the class of FRI-admissible kernels. In all cases investigated so far, the scalar coefficients $h_k(\Omega)$ satisfy (6).

3.1 Periodized sinc (Dirichlet) filter

Solving the FRI problem in the case of a periodic stream of Diracs is equivalent to considering (1) where φ is a periodized sinc kernel, e.g., a Dirichlet kernel

$$\varphi(t) = \sum_{k' \in \mathbb{Z}} \mathrm{sinc}(B(t - k'\tau)) = \frac{\sin(\pi B t)}{B\tau \sin(\pi t/\tau)}$$

where τ is the period of the Dirac stream and B some bandwith (chosen so that $B\tau$ is an odd integer) [2]. This problem can be reformulated using the annihilation equation (4) by defining the following annihilation matrices

$$\mathbf{A}_k = \left[\, \mathbf{0}_{B\tau-K,k} \;\; \mathbf{I}_{B\tau-K} \;\; \mathbf{0}_{B\tau-K,B\tau-k} \,\right] \mathbf{W}$$

where $\mathbf{W} = [e^{-j2\pi mn/N}]$ for $|m| \leq \lfloor B\tau/2 \rfloor$ and $1 \leq n \leq N$, is the N-DFT submatrix of size $B\tau \times N$. Then, the abstract parameters ω_k are related to the locations t_k through $\omega_k = e^{-j2\pi t_k/\tau}$. This kernel has been found useful for the estimation of UWB channels [13] and for image superresolution [14].

3.2 Infinite sinc filter

The filter $\varphi(t)$ is given by $\varphi(t) = \mathrm{sinc}\, Bt$ with $B = 1/T$. When $\big(\varphi * x\big)(t)$ is sampled uniformly at frequency B, the nonlinear system of equations satisfies the GAP. The abstract parameters ω_k are related to the locations t_k through

$\omega_k = t_k$ and the annihilation matrices are given by

$$\mathbf{A}_k = \begin{bmatrix} \binom{K}{K} & \binom{K}{K-1} & \cdots & \binom{K}{0} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & \cdots & \binom{K}{K} & \binom{K}{K-1} & \cdots & \binom{K}{0} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 2^k & \ddots & & \vdots \\ \vdots & \ddots & 3^k & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & N^k \end{bmatrix}$$

3.3 Gaussian filter

The filter $\varphi(t)$ is given by $\varphi(t) = \exp(-t^2/\sigma^2)$. When $\big(\varphi * x\big)(t)$ is sampled uniformly at frequency T^{-1}, the nonlinear system of equations satisfies the GAP. The abstract parameters ω_k are related to the locations t_k through $\omega_k = \exp(2t_kT/\sigma^2)$ and the annihilation matrices are given by

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{0}_{N-K,k} & \mathbf{I}_{N-K} & \mathbf{0}_{N-K,K-k} \end{bmatrix} \times \begin{bmatrix} e^{\frac{T^2}{\sigma^2}} & 0 & \cdots & \cdots & 0 \\ 0 & e^{\frac{(2T)^2}{\sigma^2}} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & e^{\frac{(NT)^2}{\sigma^2}} \end{bmatrix}$$

A version of this solution (actually, for a Gabor kernel) was used in Optical Coherence Tomography, showing the possibility to resolve slices of a microscopic sample below the coherence length of the illuminating reference light [15].

3.4 Finite Support Strang-Fix filters

Through linear combinations of its shifts, the finite support filter $\varphi(t)$ is assumed to reconstruct polynomials up to some degree $L-1$ (standard Strang-Fix condition [16]) or exponentials $e^{a_l t}$ where $a_l - a_0$ is linear with $l = 0, 1, \ldots, L-1$. More precisely, in the standard Strang-Fix case, we denote by $c_{l,n}$ the coefficients such that

$$\sum_{n\in\mathbb{Z}} c_{l,n}\varphi(nT-t) = t^l \quad \text{where } l = 0, 1, \ldots, L-1,$$

by T the sampling step, and by $[0, S]$ the support of $\varphi(t)$. Then, the abstract parameters ω_k are related to the locations t_k through $\omega_k = t_k$ and the annihilation matrices are given by

$$\mathbf{A}_k = \begin{bmatrix} c_{k,1} & c_{k,2} & \cdots & c_{k,N} \\ c_{k-1,1} & c_{k-1,2} & \cdots & c_{k-1,N} \\ \vdots & \vdots & & \vdots \\ c_{k-L+1,1} & c_{k-L+1,2} & \cdots & c_{k-L+1,N} \end{bmatrix}.$$

Additionally, there is a constraint on the minimal number of samples N for the GAP to hold, which is that N be larger than $\lceil (S + \max_k\{t_k\})/T \rceil$.

4. Conclusion

We have shown how to unify the different techniques used in FRI signal reconstruction through an algebraic property that we call the Generalized Annihilation property. In essence, this property allows to solve nonlinear system of equations within two noniterative steps. We hope that this property can be used to solve other FRI problems (i.e, with new kernels) in particular in dimensions higher than 1 (for instance, like in [17]), and maybe to solve other types of problems not directly related to sampling.

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