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Abstract: In this paper, we formulate the problem of sampling sparse signals in fractional Fourier domain. The fractional Fourier transform (FrFT) can be seen as a generalization of the classical Fourier transform. Extension of Shannon's sampling theorem to the class of signals which are fractional bandlimited shows its association to a Nyquist-like bound. Thus proving that signals that have a non-bandlimited representation in FrFT domain cannot be sampled. We prove that under suitable conditions, it is possible to sample sparse (in time) signals by using the *Finite Rate of Innovation* (FRI) signal model. In particular, we propose a uniform sampling and reconstruction procedure for a periodic stream of Diracs, which have a non-bandlimited representation in FrFT domain. This generalizes the FRI sampling and reconstruction scheme in the Fourier domain to the FrFT domain.

1. Introduction

Shannon's sampling theorem [1] provides access to the digital world. Our understanding of this sampling theorem together with the reconstruction formula is solely based on the frequency content of the signal of interest. This is where the indispensable Fourier transform comes into the picture.

Almeida [2] introduced the fractional Fourier transform or the FrFT–a generalization of the Fourier transform–to the signal processing community in 1994. The generalization of the Fourier transform by FrFT has several interesting consequences from the signal processing perspective. For instance, non-bandlimited signals in the Fourier domain can still have a compactly supported representation in FrFT domain [3], when dealing with non stationary distortions, the FrFT based filters can perform better than Fourier domain based filters (in sense of mean square error) [4] etc. To give the reader an idea about the growing popularity of FrFT, it would be worth mentioning that on at least eight occasions including, [3, 5, 6, 7, 8, 9, 10, 11], Shannon's sampling theorem [1, 12] was independently extended to the class of fractional bandlimited signals. In [13], the FrFT of a signal or a function, say $x(t)$, is defined by

$$\widehat{x}_\theta\left(\omega\right) = \mathrm{FrFT}\{x(t)\} = \int x(t) K_\theta(t,\omega) dt \qquad (1)$$

where

$$K_\theta(t,\omega) \overset{\text{def}}{=} \begin{cases} \sqrt{\frac{1-j\cot\theta}{2\pi}} e^{j\frac{t^2+\omega^2}{2}\cot\theta - j\omega t \csc\theta}, & \theta \neq p\pi \\ \delta(t-\omega), & \theta = 2p\pi \\ \delta(t+\omega), & \theta + \pi = 2p\pi \end{cases} \qquad (2)$$

is the transformation kernel, parametrized by the fractional order $\theta \in \mathbb{R}$ and p is some integer.

The FrFT of a time-frequency representation e.g. Gabor Transform results in rotation of the plane by the fractional order of the FrFT [2]. Thus, we denote fractional order by θ and from now on, we will use fractional order and angle interchangeably. The inverse-FrFT with respect to angle θ is the FrFT at angle $-\theta$, given by,

$$x(t) = \int_{-\infty}^{\infty} \widehat{x}_\theta\left(\omega\right) K_{-\theta}\left(t,\omega\right) d\omega. \qquad (3)$$

Whenever $\theta = \pi/2$, (1) collapses to the classical Fourier transform definition. A direct consequence of the generalization of the Fourier transform by the FrFT results in a modification in the idea of bandlimitedness. Its impact is visible in the change that manifests in Shannon's sampling theorem for fractional bandlimited signals [11], which is stated in Theorem 1.

Theorem 1 (Shannon–FrFT). *Let $x(t)$ be a continuous-time signal. If the spectrum of $x(t)$, i.e. $\widehat{x}_\theta(\omega)$ is fractional bandlimited to ω_m which means, $\widehat{x}_\theta(\omega) = 0$, when $|\omega| > \omega_m$, then $x(t)$ is completely determined by giving its ordinates at a series of equidistant points spaced $T = \frac{\pi}{\omega_m}\sin\theta$ seconds apart.*

This theorem has an equivalence to the Shannon's sampling theorem for $\theta = \pi/2$. The reconstruction formula for fractional bandlimited signals is given in [11],

$$x(t) = \lambda_\theta^*\left(t\right) \sum_{n\in\mathbb{Z}} \lambda_\theta\left(nT\right) x(nT)\mathrm{sinc}\left((t-nT)\omega_m \csc\theta\right) \qquad (4)$$

where $\lambda_\theta\left(\cdot\right) \overset{\text{def}}{=} e^{j(\cdot)^2\frac{\cot\theta}{2}}$ is a domain independent chirp modulation function and the '*' in the superscript denotes complex conjugation. If $\widetilde{x}(t)$ is the approximation of $x(t)$, then $\|\widetilde{x}(t) - x(t)\|^2 = 0$ when $\omega_m \leqslant \frac{\omega_s}{2}\sin\theta$–the Nyquist rate for FrFT–where $\omega_s = 2\pi/T$ is the sampling frequency. Note that all the aforementioned results are equivalent to Shannon's sampling theorem with respect to

Fourier domain for $\theta = \pi/2$. Theorem 1 (for FrFT) has a striking similarity with the Shannon's sampling theorem (for FT), in that, sampling non-bandlimited signals is impossible. Consider Dirac's delta function or $\delta(t)$. Using (2), we have,

$$\widehat{\delta}_\theta(\omega) = \text{FrFT}\{\delta(t)\} = \sqrt{\tfrac{1-j\cot\theta}{2\pi}}\lambda_\theta(\omega) \qquad (5)$$

which is a non-bandlimited function (and least sparse when compared to the time-domain counterpart) and thus, Theorem 1 fails to answer the following question: If $x(t)$ is a fractional non-bandlimited signal, then, how can we sample and reconstruct such a signal? To make this statement clear, we introduce the fractional convolution operator, which is denoted by '$*_{\boldsymbol{\theta}}$'. Accordingly, filtering $x(t)$ by a filter, $h(t)$, in 'fractional sense' [1] is equivalent to [14],

$$x(t)*_\theta h(t) = \sqrt{\tfrac{1-j\cot\theta}{2\pi}}\lambda_\theta^*(t)\cdot([x(t)\lambda_\theta(t)] * [h(t)\lambda_\theta(t)]) \qquad (6)$$

where '$*$' denotes the usual convolution operator. In light of this definition, we wish to address the problem of recovering *parsimonious* $x(t)$ from the samples of its filtered version, i.e., $y(nT) = x(t) *_\theta h(t)|_{t=nT}, n \in \mathbb{Z}$. This problem has a natural/strong link with that of sparse sampling [15, 16, 17]. The Heisenberg-Gabor uncertainty principle for the FrFT [18] (a generalization of the Fourier duality) asserts that the product of spreads of $\widehat{x}_\theta(\omega)$ and $x(t)$ has a lower bound which is proportional to $\frac{\sin^2\theta}{4}$ (assuming that $\|x\| = 1$). This implies that sparsity in one domain will lead to loss of compact support in canonically conjugate domain.

Our contribution in this article is to propose a sampling and reconstruction scheme for signals which have a sparse representation in time domain and whose fractional spectrum is non-bandlimited. We model our sparse signal as a continuous periodic stream of Diracs which is being observed by an acquisition device which deploys a sinc-based filter.

The paper is organized as follows: We assume that the reader is familiar with basic ideas outlined in [12, 16, 17]. In Section II, we introduce our sparse signal model and the definition of the fractional Fourier series (FrFS). Using these as preliminaries, in Section III, we derive an equivalent representation of our signal in FrFT domain. In Section IV, we discuss the sampling theorem and its completeness and Section V is the conclusion.

2. Preliminaries

2.1 Sparse Signal Model

We model our sparse signal as a periodic stream of K Diracs, i.e.

$$x(t) = \sum_{k=0}^{K-1} c_k \sum_{n\in\mathbb{Z}} \delta(t - t_k - n\tau) \qquad (7)$$

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[1] We adhere to this modified definition of convolution operator as it inherits the fractional Fourier duality property, in that, $\text{FrFT}\{x(t) *_\theta h(t)\} = \lambda_\theta^*(\omega) \cdot \widehat{x}_\theta(\omega)\widehat{h}_\theta(\omega)$, which does not hold for the FrFT of $x(t) * h(t)$ unless $\theta = \frac{\pi}{2}$.

with period τ, weights $\{c_k\}_{k=0}^{K-1}$ and arbitrary shifts, $\{t_k\}_{k=0}^{K-1} \subset [0, \tau)$. In sense of [16], the signal has $2K$ degrees of freedom per period and the rate of innovation being $\rho = \frac{2K}{\tau}$. From now on, the signal $x(t)$ will denote the stream of Diracs.

2.2 Fractional Fourier Series (FrFS)

Periodic signals can be expanded in FrFT domain as a fractional Fourier series or FrFS [19]. The FrFS of a periodic signal, say $x(t)$, can be written as,

$$x(t) = \sum_{m\in\mathbb{Z}} \widehat{x}_\theta[m]\Phi_\theta(m, t) \qquad (8)$$

where,

$$\Phi_\theta^*(m, t) = \sqrt{\frac{\sin\theta - j\cos\theta}{\tau}} e^{j\frac{t^2 + (2\pi m \sin\theta/\tau)^2}{2}\cot\theta - j2\pi mt/\tau}$$

constitutes the basis for FrFS expansion for a τ-periodic $x(t)$. The FrFS coefficients are given by,

$$\widehat{x}_\theta[m] = \int_{\langle\tau\rangle} x(t)\Phi_\theta^*(m, t)dt = \langle x, \Phi_\theta(m, \cdot)\rangle \qquad (9)$$

where $\langle\tau\rangle$ denotes the integral width and $\langle a, b\rangle = \int a(t)b^*(t)dt$ denotes the inner product. The well-known Fourier series (FS) is just a special case of FrFS for $\theta = \frac{\pi}{2}$.

3. Stream of Diracs in Fractional Fourier Domain

In Fourier analysis, the Poisson summation formula (PSF) plays an important role. It is a well-known fact that a stream of Diracs (Dirac comb) in time-domain is another stream of Diracs in Fourier domain. In this subsection, we will derive the equivalent representation of Dirac comb in FrFT domain. This can be seen as a generalization of the Poisson summation formula for Dirac comb in FrFT domain.

Theorem 2. *Let $\sum_{n\in\mathbb{Z}}\delta(t - n\tau)$ be a Dirac comb, then*

$$\sum_{n\in\mathbb{Z}}\delta(t - n\tau) \xleftrightarrow{FrFT}$$

$$\frac{1}{\tau}\sqrt{\frac{2\pi}{1-j\cot\theta}}\sum_{k\in\mathbb{Z}}\widehat{\delta}_\theta[k\omega_0\sin\theta] e^{-j\left(t^2 + \frac{(k\omega_0\sin\theta)^2}{2}\right)\cot\theta + jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{\tau}$.

Proof. Let $s(t) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}}\delta(t - n\tau)$. The proof is done by expanding $s(t)$ in FrFS basis or,

$$s(t) = \sum_{k\in\mathbb{Z}} \underbrace{\langle s, \Phi_\theta\rangle}_{\widehat{s}_\theta[k]} \Phi_\theta(k, t). \qquad (10)$$

The coefficients of this expansion are given by,

$$\widehat{s}_\theta\,[k] \overset{(9)}{=} \langle s, \Phi_\theta\,(k,t)\rangle$$

$$= \frac{\kappa\,(\theta)}{\sqrt{\tau}} \int\limits_{t_0}^{t_0+\tau} s\,(t)\,\Phi_\theta^*\,(k,t)\,dt, \quad \forall t_0 \in \mathbb{R}$$

$$= \frac{\kappa\,(\theta)}{\sqrt{\tau}} \int\limits_{-\tau/2}^{\tau/2} \delta\,(t)\,e^{j\left(t^2+(k\omega_0\sin\theta)^2/2\right)\cot\theta - jk\omega_0 t}\,dt$$

$$\text{(since } s(t+\tau) = s(t) \text{ and } s(t) = \delta\,(t)\,, t \in \left[\tfrac{-\tau}{2}, \tfrac{\tau}{2}\right])$$

$$= \frac{\kappa\,(\theta)}{\sqrt{\tau}} e^{j\left((k\omega_0\sin\theta)^2/2\right)\cot\theta}$$

$$\overset{(5)}{=} \frac{\kappa\,(\theta)}{\sqrt{\tau}} \sqrt{\frac{2\pi}{1-j\cot\theta}} \widehat{\delta}_\theta\,[k\omega_0\sin\theta] \qquad (11)$$

where $\kappa\,(\theta) = \sqrt{\sin\theta - j\cos\theta}$. Back substitution of (11) in (10) results in,

$$s(t) = \frac{1}{\tau}\sqrt{\frac{2\pi}{1-j\cot\theta}}$$

$$\times \sum_{k\in\mathbb{Z}} \widehat{\delta}_\theta\,[k\omega_0\sin\theta]\,e^{-j\left(t^2+\frac{(k\omega_0\sin\theta)^2}{2}\right)\cot\theta+jk\omega_0 t}.$$

This concludes the proof. ∎

For sake of convenience, we will assume that the constant $\sqrt{\frac{1-j\cot\theta}{2\pi}}$ has been absorbed in τ. Note that at $\theta = \frac{\pi}{2}$, $s(t) = \frac{1}{\tau}\sum_{k\in\mathbb{Z}} e^{jk\omega_0 t}$ which is the result of applying the PSF on $s(t)$ in Fourier domain. Our immediate goal now is to derive the FrFS equivalent of $x(t)$ in (7). Since $x(t)$ is a linear combination of some $s(t)$ delayed by some time shift t_k, it will be useful to recall shift property of FrFT [2] which states that,

$$\text{FrFT}\,\{s\,(t-t_k)\}$$

$$= \widehat{s}_\theta\,(\omega - t_k\cos\theta)\,e^{j\frac{1}{2}t_k^2\sin\theta\cos\theta - j\omega t_k\sin\theta}. \qquad (12)$$

Therefore, call $x(t) = \sum_{k=0}^{K-1} c_k\cdot s_k(t)$ where $s_k(t)$ is the time-shifted version of $s(t)$ with shift parameter t_k. Using Theorem 2 and the shift-property of FrFT, we have,

$$s_k(t) = \sum_{n\in\mathbb{Z}} \delta(t-t_k-n\tau)$$

$$\overset{(8)}{=} \sum_{m\in\mathbb{Z}} \text{FrFT}\{\,\delta(t-t_k)\,\}\,|_{\omega=m\omega_0\sin\theta}\,\Phi_\theta\,(m,t)$$

$$\overset{(12)}{=} \underbrace{\frac{1}{\tau}\sum_{m\in\mathbb{Z}} e^{j\frac{\cot\theta}{2}\left(t_k^2-t^2\right)+jm\omega_0(t-t_k)}}_{\text{PSF for Dirac Comb in FrFT}}.$$

Having obtained the FrFT-version of $s_k(t)$, we can write,

$$x(t) = \sum_{k=0}^{K-1} c_k \cdot \sum_{n\in\mathbb{Z}} \delta(t-t_k-n\tau)$$

$$= \sum_{k=0}^{K-1} c_k \sum_{m\in\mathbb{Z}} e^{j\frac{\cot\theta}{2}\left(t_k^2-t^2\right)+jm\omega_0(t-t_k)}$$

$$= e^{-jt^2\frac{\cot\theta}{2}} \sum_{m\in\mathbb{Z}} \frac{1}{\tau}\underbrace{\left(\sum_{k=0}^{K-1} c_k e^{j\frac{\cot\theta}{2}\left(t_k^2\right)-jm\omega_0 t_k}\right)}_{p[m]} e^{j\frac{2\pi m}{\tau}t}.$$

Note that $x(t)$ is non-bandlimited, however, it can be completely described by the knowledge of $p[m]$ which in turn can be expanded as a linear combination of K complex exponentials.

4. Sampling and Reconstruction of Sparse Signals in Fractional Fourier Domain

We assume that a sinc–based kernel is used to pre-filter $x(t)$. In particular, we let the sampling kernel to be $\varphi_n(t) = e^{-j\frac{\cot\theta}{2}t^2}\,\text{sinc}(t-nT)$. Integer translates of $\varphi_n(t)$ form an orthonormal basis and the FrFT of $\varphi(t)(= \varphi_0(t))$ is given by $\widehat{\varphi}_\theta(\omega) = \sqrt{\frac{1-j\cot\theta}{2\pi}}\left(e^{-j\frac{\cot\theta}{2}\omega^2}\right)\text{rect}(\omega/2\pi)$. In light of the definition in (6), prefiltering the input signal $x(t)$ with the kernel/low-pass filter $\varphi(-t)$ and sampling can be written as, $y\,(nT) = x(t) *_\theta \varphi(-t)|_{t=nT}$. The main result is in the form of the following theorem.

Theorem 3. *Let $x(t)$ be a τ-periodic stream of Diracs weighted by coefficients $\{c_k\}_{k=0}^{K-1}$ and locations $\{t_k\}_{k=0}^{K-1}$ with finite rate of innovation $\rho = \frac{2K}{\tau}$. Let the sampling kernel/prefilter $\varphi(t)$ be an ideal low-pass filter which has fractional bandwidth $[-B\pi, B\pi]$, where B is chosen such that $B \geq \rho$. If the filtered version of $x(t)$, i.e. $y(t) = x(t) *_\theta \varphi(-t)$ is sampled uniformly at locations $t = nT$, $n = 0,\ldots,N-1$ then the samples,*

$$y\,(nT) = x(t) *_\theta \varphi\,(-t)|_{t=nT}\,, n = 0,\ldots,N-1,$$

are a sufficient characterization of $x(t)$, provided that $N \geq 2M_\theta + 1$ and $M_\theta = \left\lfloor\frac{B\tau\csc\theta}{2}\right\rfloor$.

Proof. Using the following FrFT pair,

$$\sqrt{\frac{1-j\cot\theta}{2\pi}}\lambda_\theta^*\,(\omega)\cdot\text{rect}(\tfrac{\omega}{2\pi B}) \overset{\text{FrFT}}{\longleftrightarrow}$$

$$(B\csc\theta)\,\lambda_\theta^*\,(t)\,\text{sinc}\,(Bt\csc\theta)$$

we define our sampling kernel as,

$$\varphi_B(t-nT) = \lambda_\theta^*\,(t)\,\varphi\,(B\csc\theta\,(t-nT))$$

which is compactly supported over $[-B\pi, B\pi]$. Prefiltering and sampling $x(t)$ results in,

$$y\,(nT) = x(t) *_\theta \varphi\,(-t)|_{t=nT}\,, n = 0,\ldots,N-1$$

$$= \frac{\lambda_\theta^*\,(nT)}{\tau} \sum_{m\in\mathbb{Z}} p[m]$$

$$\times \left\langle e^{j\frac{2\pi m}{\tau}t}, (B\csc\theta)\,\text{sinc}\,((B\csc\theta)\,(t-nT))\right\rangle.$$

The inner product in the above step is further simplified using the Fourier integral,

$$\left\langle e^{j\frac{2\pi m}{\tau}t}, (B\csc\theta)\,\text{sinc}\,((B\csc\theta)\,(t-nT))\right\rangle =$$

$$\text{rect}(\tfrac{m}{B\tau\csc\theta})e^{j\frac{2\pi m}{\tau}(nT)}.$$

We can therefore conclude that,

$$y\,(nT) = \frac{\lambda_\theta^*\,(nT)}{\tau} \sum_{m\in\mathbb{Z}} p[m]\,\text{rect}(\tfrac{m}{B\tau\csc\theta})e^{j\frac{2\pi m}{\tau}(nT)}$$

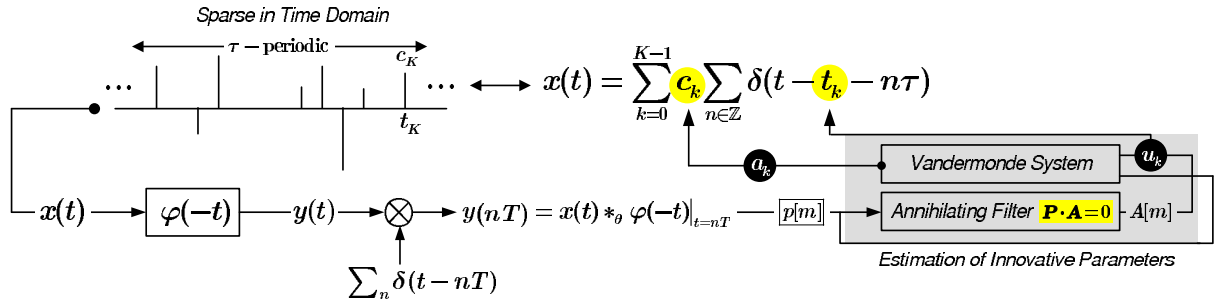$$= \frac{\lambda_\theta^*\,(nT)}{\tau} \sum_{m=-M_\theta}^{M_\theta} p[m]e^{j\frac{2\pi m}{\tau}(nT)}, n = 0,\ldots,N-1$$

Figure 1: Sampling and reconstruction of periodic stream of Diracs in FrFT domain.

where $M_\theta = \left\lfloor \frac{B\tau \csc\theta}{2} \right\rfloor$.

Signal reconstruction from its samples: Call $p[m] = \sum_{k=0}^{K-1} a_k u_k^m$ – a linear combination of K-complex exponentials, $u_k = \lambda_{\pi/2}^* \left(\sqrt{\omega_0 t_k}\right)$ with weights $a_k = c_k \cdot \lambda_\theta(t_k)$. The problem of calculating $\{a_k\}_{k=0}^{K-1}$ and $\{u_k\}_{k=0}^{K-1}$ is based on finding a suitable polynomial $A(z) = \prod_{k=0}^{K-1} \left(1 - u_k z^{-1}\right)$ whose inverse z-transform yields the annihilating filter coefficients, $A[m]$ which annihilate $p[m]$. In matrix notation, finding $A[m]$ is equivalent to finding a corresponding vector $\mathbf{A}$ that forms a null space of a suitable submatrix of $p[m]$ i.e. $\mathbf{P}^{(2M_\theta - K+1) \times (K+1)}$ – which is essentially the set $\text{Null}(\mathbf{P}) = \left\{\mathbf{A} \in \mathbb{R}^{K+1} : \mathbf{P} \cdot \mathbf{A} = \mathbf{0}\right\}$. For details of this computation, the reader is referred to (cf. Pg. 1427, [16]). Figure 1 shows the layout of this algorithm. ∎

5. Conclusion

We presented a scheme for sampling and reconstruction of sparse signals in fractional Fourier domain. A direct consequence of modeling our signal of interest as a *Finite Rate of Innovation* signal, is that, the outcome bears an acute resemblance with the results previously derived, for the Fourier domain case. This simplifies the problem to the extent that reconstruction strategy remains unchanged and as we have shown, one can obtain the precise locations and amplitudes of the stream of Diracs using the annihilating filter method. Since time and frequency domains are special cases of the FrFT domain, it turns out that the number of values (M_θ) required for exact reconstruction of time domain signal depends on the chirp rate of transformation, i.e. θ.

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