Convergence properties and numerical simulation by an adaptive FEM of the thermistor problem

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Abstract

In this paper, the convergence properties of the finite element approximation of the thermistor problem are investigated, both from theoretical and numerical point of view. From one hand, based on a duality argument, a theoretical convergence result is proved under low regularity assumption. From other hand, numerical experiments are performed based on a decoupled algorithm. Moreover, on a non convex domain, the convergence properties versus the mesh size are shown to be improved by using suitable mesh adaptation strategy and error estimator.

Key words: Thermistor problem ; L1 right-hand-side ; existence result ; convergence of the finite element method ; corner singularity ; mesh adaptation.

Introduction

Thermistor problem and similar problems coming from fluid mechanics have been the subject of many theoretical articles [6,9,5,1,17]. These article deal with existence and regularity results. In this paper, the thermistor problem is presented in a numerical point of view: existence of the approximate solution and its convergence to the exact solution under low regularity assumption. Moreover, we successfully compare some theoretical regularity results to numerical experiments.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, $d = 1, 2$ or 3, whose boundary $\partial\Omega$ is divided into two disjoint subdomains Γ_1 and Γ_2. The thermistor problem

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may be written as follows :
find u and ϕ, defined from Ω to $\mathbb{R}$ such that

$$\begin{cases} -\mathrm{div}(\kappa(u)\nabla u) = \sigma(u)\ |\nabla\phi|^2 \quad \text{in } \Omega, \\ -\mathrm{div}(\sigma(u)\nabla\phi) = f \quad \text{in } \Omega, \\ u = u_0 \quad \text{on } \Gamma_2 \\ -\kappa(u)\dfrac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1, \\ \phi = \phi_0 \quad \text{on } \Gamma_1 \\ -\sigma(u)\dfrac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_2. \end{cases} \tag{1}$$

The data f is known, the Dirichlet boundary conditions u_0 and ϕ_0 are piecewise linear functions defined respectively on Γ_2 and Γ_1. Let assume that κ and σ are positive and bounded. If $\phi \in H^1_{\phi_0} = \{v \in H^1(\Omega),\ v = \phi_0 \text{ on } \Gamma_1\}$, then the right-hand-side $|\nabla\phi|^2$ of the first equation in (1) belongs only to $L^1(\Omega)$ and the corresponding solution u does not belong to $H^1_{u_0}(\Omega) = \{v \in H^1(\Omega),\ v = u_0 \text{ on } \Gamma_2\}$ (see e.g. [5]).

Under certain assumptions (on the regularity of f and on the domain Ω) the solution ϕ is more regular. Let us suppose that $\phi \in W_0^{1,2r}(\Omega)$ for some $r > 1$: then $|\nabla\phi|^2 \in L^r(\Omega)$. In [5] the authors showed that there exists a solution $u \in W^{1,q}(\Omega)$ with $q \in [1, d/(d-1)[$ such that:

$$\|u - u_h\|_{W^{1,q}(\Omega)} \leq C h^{\min(1,\,2(1-1/r))} \left\| |\nabla\phi|^2 \right\|_{L^r(\Omega)}$$

where u_h is an usual piecewise linear finite element approximation of u. Moreover a classical result has been shown in [6], under several conditions on σ and κ, and the boundary $\partial\Omega$.

When $f = 0$, problem (1) models the electric heating of a conducting body with u being the temperature, ϕ the electrical potential. σ and k denotes respectively the electrical and the thermal conductivity which may depend on temperature (see Wiedemann-Frantz law bellow [6]). (1) is heavily used for modelling the current density $\sigma(u)\,\nabla\phi$ distribution which is a key issue in the developpement of high magnetic field magnets providing up to 34 Tesla [15,16].

The present paper is organized as follow: section 1 presents an existence result of the exact solution. Section 2 shows an existence and convergence result of the approximate solution with a low regularity hypothesis on the exact solution. Section 3 introduces a numerical algorithm and presents some numerical investigations of the solution and its regularity, by means of an adaptive FEM strategy.

1 Regularity results for the exact problem

Let assume that σ and κ are functions defined from $\Omega \mapsto \mathbb{R}$, and let $\overline{u} \in L^2(\Omega)$ be a given temperature, we want to solve:

(P) : *Find* $u \in H^1_{u_0}$ *and* $\phi \in H^1_{\phi_0}$ *such that:*

$$\begin{cases} -div\,(\kappa(\overline{u})\nabla u) &= \sigma(\overline{u})|\nabla\phi|^2 \ \text{ in } \Omega, \\ -div\,(\sigma(\overline{u})\nabla\phi) &= \quad f \ \text{ in } \Omega, \\ -\kappa(\overline{u})\dfrac{\partial u}{\partial n} &= \quad 0 \ \text{ on } \Gamma_1, \\ -\sigma(\overline{u})\dfrac{\partial\phi}{\partial n} &= \quad 0 \ \text{ on } \Gamma_2. \end{cases} \tag{2}$$

The regularity of u and ϕ depends on two factors: right-hand-side f, and convexity and regularity of Ω.

THEOREM **1** *Let* σ *and* κ *be functions defined on* $\Omega \mapsto \mathbb{R}$ *with regularity* $C^m(\overline{\Omega})$. *If* f *is* $H^m(\Omega)$ *and* Ω *is of class* C^2 *then* $\phi \in H^{m+2}(\Omega)$ *and* $u \in H^2(\Omega)$.

Proof: Under the assumptions of the theorem, $\phi \in H^{m+2}(\Omega)$ (see [10,11]). Sobolev embedding leads to the fact that $|\nabla\phi|^2 \in L^2(\Omega)$, since there is a continuous injection of $W^{1,1}(\Omega)$ in $L^2(\Omega)$ in dimension 2. As a conclusion, $u \in H^2(\Omega)$.

$\square$

The following theorem use the Sobolev embedding property:

If Ω *is polygonal and piecewise Lipschitz-continuous, and if the following conditions are satisfied:*

$$s \geq t, \quad 1 \leq \alpha \leq \beta, \quad \text{and } s - \frac{2}{\alpha} = t - \frac{2}{\beta}$$

then $\qquad W^{s,\alpha}(\Omega) \subset W^{t,\beta}(\Omega) \qquad$ *with continuous injection.*

THEOREM **2** *Let* σ *and* κ *be functions defined on* $\Omega \mapsto \mathbb{R}$ *with regularity* $C^m(\overline{\Omega})$. *Assume that* Ω *is polygonal and piecewise Lipschitz-continuous, and* $f \in L^s(\Omega)$, $s \in]1, +\infty[$. *Then* u *and* ϕ *belong to* $H^{1+2/q}(\Omega)$, *with* $q > \max\{q^*, 2\}$, *and* q^* *depending on the largest internal angle of* Ω.

Proof: Let ω_i the interior angle between two edges of polygon Ω, then the regularity depends on ω_i^* defined by:

$\omega_i^* = \omega_i$ if the two edges of angle i have the same boundary condition.
$\omega_i^* = 2\omega_i$ if each edge of angle i has a different boundary condition.

Let $\omega^* = \max\{\omega_1^*, \ldots, \omega_N^*\}$, $q^* = \dfrac{2}{\pi}\,\omega^*$ and p^* its conjugate exponent, then it can be shown that the solution of a Poisson equation with right-hand-side

$f \in L^s(\Omega)$, $s \in]1, +\infty[$ has the following regularity:

$$\phi \in W^{2,p}(\Omega) \text{ for } 1 < p < p^*.$$

Sobolev embedding leads to the conclusion that $\phi \in H^{1+2/q}(\Omega)$ with $q > max\{q^*, 2\}$. This ensures that $|\nabla\phi|^2 \in L^2(\Omega)$ and one can therefore conclude that u is also in $\phi \in H^{1+2/q}(\Omega)$ with $q > max\{q^*, 2\}$.

$\square$

If Ω is convex with only Dirichlet conditions then $q^* < 2$, and u and ϕ belong to $H^2(\Omega)$.

Remark **1** *The minimal condition required on ϕ in order to have $|\nabla\phi|^2 \in L^2(\Omega)$ is $\phi \in H^{3/2}(\Omega)$. Transposed to a condition on the geometry Ω, it leads to a maximum interior angle equals to $\omega^* = 2\pi$, which corresponds to a fissure on Ω. Since $\omega^* = 2\pi$ is the maximum possible interior angle, the solution always satisfies $|\nabla\phi|^2 \in L^2(\Omega)$ under the hypothesis of theorem 2.*

2 Convergence of the approximate problem

The convergence of the coupled problem has already been treated in literature: for instance in [9], despite the authors bypassed the L^1 regularity problem of the right-hand-side.

In this part, we adapt the duality argument used in [3] to get the convergence of the coupled approximated problem. This will be done under minimum regularity conditions on u and ϕ.

2.1 The variational formulation

The first step is to show in which sense the variational formulation is well-posed, to take the L^1-right-hand-side into account. Boundary condition on u is simplified, by taking homogeneous Dirichlet condition on $\partial\Omega$, without loss of generality [10].

2.1.1 The difficulties

Let $H^1_1 = \{v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_1\}$, and $(\mathcal{T}_h)_{h>0}$ a family of triangulation of Ω and $k \geq 1$. We introduce:

$$\begin{aligned}
V_h &= \{v_h \in H^1_0(\Omega); v_{h|K} \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\},\\
W_h &= \{v_h \in H^1_{\phi_0}(\Omega); v_{h|K} \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\},\\
X_h &= \{v_h \in H^1_1(\Omega); v_{h|K} \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\}.
\end{aligned}$$

The approximate variational formulation of (1) writes:
(FVP_h) : *find* $u_h \in V_h$ *and* $\phi_h \in W_h$ *such that*

$$\int_\Omega \kappa(u_h)\, \nabla u_h.\nabla v_h \,\mathrm{d}x = \int_\Omega \sigma(u_h)\; |\nabla \phi_h|^2\; v_h \,\mathrm{d}x, \quad \forall v \in V_h, \tag{3}$$

$$\int_\Omega \sigma(u_h)\nabla \phi_h.\nabla \psi_h \,\mathrm{d}x = \int_\Omega f\, \psi_h \,\mathrm{d}x, \quad \forall \psi_h \in X_h. \tag{4}$$

A first variational formulation of (1) has been proposed in [9]:
$(\widetilde{FVP})$: *find* $u \in H^1_0(\Omega)$ *and* $\phi \in H^1_{\phi_0}(\Omega)$ *such that*

$$\int_\Omega \kappa(u)\, \nabla u.\nabla v \,\mathrm{d}x = \int_\Omega \sigma(u)\; |\nabla \phi|^2\; v \,\mathrm{d}x, \quad \forall v \in H^1_0(\Omega),$$

$$\int_\Omega \sigma(u)\nabla \phi.\nabla \psi \,\mathrm{d}x = \int_\Omega f\, \psi \,\mathrm{d}x, \quad \forall \psi \in H^1_D(\Omega).$$

Nevertheless, the integral term in the right-hand-side of the first equation has **no sense** since $\sigma(u)|\nabla\phi|^2 \in L^1(\Omega)$ while $v \notin L^\infty(\Omega)$: we only have $v \in H^1_0(\Omega)$.

A second and more recent approach is the variational formulation proposed in [5, p. 7] for the Laplace problem and based on the **renormalized solution**. This approach is suitable for the proof of existence of solutions with low regularity. But it is limited to approximation with low polynom degree $k = 1$.

A third approach is presented in [3]: it consists in a **transposition** of the first equation of (1), associated to the L^1 right-hand-side. The unknown u is first transformed in an equivalent unknown denoted by θ, in order to replace the nonlinear operator $\mathrm{div}(\kappa(u)\nabla u)$ operator by the Laplace operator. Then, the problem admits a variational formulation by transposition.

2.1.2 The change of unknown

Let us denote by K a primitive of κ, defined by:

$$K(v) = \int_0^v \kappa(w)\,\mathrm{d}w.$$

Then the unknown u is replaced by a new unknown $\theta = K(u)$ such that $\nabla\theta = \kappa(u)\nabla u$. The function K is differentiable with bounded derivative and also increasing and nonnegative on $\mathbb{R}^+$ so that it admits an inverse K^{-1} from $\mathbb{R}^+$ into $\mathbb{R}^+$. Let $\tau(\theta) = \sigma \circ K^{-1}(\theta)$. The function τ is continuous, bounded and $\tau(\theta) \geq \sigma_{\min} > 0$ for all $\theta \in \mathbb{R}$. The problem (1) is replaced by:

(Q) : *find* θ *and* ϕ*, defined from* Ω *to* $\mathbb{R}$ *such that*

$$\begin{cases} -\Delta\theta = \tau(\theta)\ |\nabla\phi|^2 \quad \text{in } \Omega, \\ -\mathrm{div}(\tau(\theta)\nabla\phi) = f \quad \text{in } \Omega, \\ \theta = 0 \quad \text{on } \partial\Omega, \\ \phi = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5}$$

The corresponding approximate variational formulation writes:
(FVQ_h) : *find* $u_h \in V_h$ *and* $\phi_h \in W_h$ *such that*

$$\int_\Omega \nabla\theta_h.\nabla\zeta_h \,\mathrm{d}x = \int_\Omega \tau(\theta_h)\ |\nabla\phi_h|^2\ \zeta_h \,\mathrm{d}x, \quad \forall\zeta_h \in V_h, \tag{6}$$

$$\int_\Omega \tau(\theta_h)\nabla\phi_h.\nabla\psi_h \,\mathrm{d}x = \int_\Omega f\,\psi_h \,\mathrm{d}x, \quad \forall\psi_h \in X_h. \tag{7}$$

2.1.3 The variational formulation by transposition

Then, we multiply the first equation of (5) by a test-function ζ and integrate over Ω. Using two times the Green formulae, the left-hand-side becomes:

$$\begin{aligned} -\int_\Omega (\Delta\theta)\ \zeta \,\mathrm{d}x &= \int_\Omega \nabla\theta.\nabla\zeta \,\mathrm{d} - \int_{\partial\Omega} \frac{\partial\theta}{\partial n}\,\zeta \,\mathrm{d}s \\ &= -\int_\Omega \theta\ (\Delta\zeta)\ \mathrm{d}x + \int_{\partial\Omega} \theta\,\frac{\partial\zeta}{\partial n}\,\mathrm{d}s - \int_{\partial\Omega} \frac{\partial\theta}{\partial n}\,\zeta \,\mathrm{d}s. \end{aligned}$$

Let us denote Δ^{-1} the inverse of the Laplace operator associated to the homogeneous Dirichlet boundary condition. The operator Δ^{-1} associates to the data $\xi \in H^{-1}(\Omega)$ the solution $\zeta = \Delta^{-1}\xi \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta\zeta = \xi \text{ in } \Omega, \\ \zeta = 0 \text{ on } \partial\Omega. \end{cases}$$

When either Ω is convex or $\partial\Omega$ is $C^{1,1}$ then Δ^{-1} is an isomorphism from $L^2(\Omega)$ into $H_0^1(\Omega) \cap H^2(\Omega)$. When Ω is neither convex nor with a $C^{1,1}$ boundary, it is proved in [7] that Δ^{-1} is an isomorphism from $H^{-s}(\Omega)$ into $H_0^1(\Omega) \cap H^{2-s}(\Omega)$ where $s < 1 - \pi/\omega$ and $\omega \in]\pi, 2\pi[$ is the largest internal angle of $\partial\Omega$. We are assuming that Ω has no fissure, i.e. that $\omega < 2\pi$ and $s < 1/2$. Thus $s \in [0, 1/2[$.

A variational formulation by transposition of (5) writes:
(FVQ) : *find* $\theta \in L^2(\Omega)$ *and* $\phi \in H^1_{\phi_0}(\Omega)$ *such that*

$$\int_\Omega \theta\,\xi \,\mathrm{d}x = -\int_\Omega \tau(\theta)\ |\nabla\phi|^2\ \left(\Delta^{-1}\xi\right)\ \mathrm{d}x, \quad \forall\xi \in L^2(\Omega), \tag{8}$$

$$\int_\Omega \tau(\theta)\nabla\phi.\nabla\psi \,\mathrm{d}x = \int_\Omega f\,\psi \,\mathrm{d}x, \quad \forall\psi \in H_1^1(\Omega). \tag{9}$$

In [4], on a similar problem, it is proved that for small data, there exists at least one solution $(\theta, \phi) \in H^s(\Omega) \times H^1_1(\Omega)$, $\forall s < 1/2$.

Following [3], a direct numerical approximation of this formulation would need to explicitly know Δ^{-1} acting on elements of the discrete space, or some suitable approximation. The approximations (FVP_h) or (FVQ_h) provide regularizations of the data appearing in problem (2). Then, the standard Galerkin approximations (FVP_h) or (FVQ_h) of (2). makes sense and only internal approximation of $H^1_0(\Omega)$ are needed. Indeed, we are able to prove that the solution of (FVQ_h) approximates the solution by transposition of (FVQ).

2.2 A priori *estimates for decoupled subproblems*

We first prove the existence of the approximate solution of (FVP_h) by using the Brouwer's fixed point theorem.

We shall consider the two following partial problems:

$(FVR_{1,h})$: *given* $\overline{\theta}_h \in V_h$ *and* $\overline{\phi}_h \in W_h$, *find* $\theta_h \in V_h$ *such that*

$$\int_\Omega \nabla\theta_h.\nabla\zeta_h \,\mathrm{d}x = \int_\Omega \tau\left(\overline{\theta}_h\right) \left|\nabla\overline{\phi}_h\right|^2 \zeta_h \,\mathrm{d}x, \quad \forall \zeta_h \in V_h. \tag{10}$$

$(FVR_{2,h})$: *given* $\overline{\theta}_h \in V_h$, *find* $\phi_h \in W_h$ *such that*

$$\int_\Omega \tau\left(\overline{\theta}_h\right) \nabla\phi_h.\nabla\psi_h \,\mathrm{d}x = \int_\Omega f\, \psi_h \,\mathrm{d}x, \quad \forall \psi_h \in X_h. \tag{11}$$

Notice that both $(FVR_{2,h})$ and $(FVR_{1,h})$ are linear problems.

Theorem **3** (a priori *estimate for the first subproblem)*
Problem $(FVR_{1,h})$ *admits an unique solution, which satisfies the* a priori *estimate:*

$$\|\theta_h\|_{H^s(\Omega)} \leq C_1 \left(1 + h^{2-s-\frac{d}{2}} \left|\log\, h\right|^{1-\frac{1}{d}}\right) \left\|\tau\left(\overline{\theta}_h\right) \left|\nabla\overline{\phi}_h\right|^2\right\|_{L^1(\Omega)} \tag{12}$$

for a positive constant C_1.

Remark **2** *(on the bound)*
In [3] the authors proposed a comparable bound in the case $d = 2$ *and* $k = 1$*:*

$$\|\theta_h\|_{H^s(\Omega)} \leq C_1 \left(1 + h^{1-s}\right) \left\|\tau\left(\overline{\theta}_h\right) \left|\nabla\overline{\phi}_h\right|^2\right\|_{L^1(\Omega)}.$$

The bound (12) *is an extension of this bound to multidimensional systems* $d \geq 2$ *and higher polynomial degree* $k \geq 1$. *Our proof bases on the discrete inverse inequality proposed by [14] and involving the* log h *factor.*

Proof: The existence of the solution $\theta_h \in V_h$ of (10) follows from the Lax-Milgram theorem. As $\tau\left(\overline{\theta}_h\right)\left|\nabla\overline{\phi}_h\right|^2 \in L^\infty(\Omega)$, problem (10) admits an unique solution. The estimate (12) is more complex because of the H^s norm. Following [11], observe that $H_0^s(\Omega) = H^s(\Omega)$, $\forall s \in [0, \frac{1}{2}]$ and that $H_0^s(\Omega)$ is reflexif: $((H_0^s(\Omega)))'' = H^s(\Omega) = (H^{-s}(\Omega))' = H^s(\Omega) = H_0^s(\Omega)$. Thus the H^s norm could be computed by duality:

$$\begin{aligned}
\|\theta_h\|_{H^s(\Omega)} &= \sup_{\xi \in H^{-s}(\Omega)-\{0\}} \frac{\langle \theta_h,\, \xi\rangle_{H_0^s, H^{-s}}}{\|\xi\|_{H^{-s}(\Omega)}} \\
&= \sup_{\zeta \in H_0^1(\Omega)\cap H^{2-s}(\Omega)-\{0\}} \frac{\langle \theta_h,\, \Delta\zeta\rangle_{H_0^s, H^{-s}}}{\|\Delta\zeta\|_{H^{-s}(\Omega)}} \\
&\quad \text{since } H^{-s}(\Omega) \text{ and } H_0^1(\Omega)\cap H^{2-s}(\Omega) \text{ are isomorph.}
\end{aligned}$$

The linear application Δ^{-1} is continuous from H^{-s} into H^{2-s}: thus there exists a constant $C_2 > 0$ such that

$$\left\|\Delta^{-1}\xi\right\|_{H^{2-s}} \le C_2 \|\xi\|_{H^{-s}(\Omega)}, \quad \forall \xi \in H^{-s}(\Omega).$$

Using the isomorphism between $H^{-s}(\Omega)$ and $H_0^1(\Omega)\cap H^{2-s}(\Omega)$, the previous inequality writes:

$$\|\zeta\|_{H^{2-s}} \le C_2 \|\Delta\zeta\|_{H^{-s}(\Omega)}, \quad \forall \zeta \in H_0^1(\Omega)\cap H^{2-s}(\Omega).$$

Then

$$\|\theta_h\|_{H^s(\Omega)} = C_2^{-1} \sup_{\zeta \in H_0^1(\Omega)\cap H^{2-s}(\Omega)-\{0\}} \frac{\langle \theta_h,\, \Delta\zeta\rangle_{H_0^s, H^{-s}}}{\|\zeta\|_{H^{2-s}(\Omega)}}.$$

For all $\zeta \in H_0^1(\Omega)\cap H^{2-s}(\Omega)$, let $\tilde{\zeta} \in V_h$ its Lagrange interpolation. Then, from the Green formulae:

$$\begin{aligned}
\langle \theta_h,\, \Delta\zeta\rangle_{H_0^s, H^{-s}} &= \int_\Omega \nabla\theta_h.\nabla\zeta \,\mathrm{d}x - \int_{\partial\Omega} \theta_h \frac{\partial\zeta}{\partial n}\,\mathrm{d}s \\
&= \int_\Omega \nabla\theta_h.\nabla\tilde{\zeta}\,\mathrm{d}x + \int_\Omega \nabla\theta_h.\nabla\left(\zeta - \tilde{\zeta}\right)\mathrm{d}x \\
&= \int_\Omega \tau\left(\overline{\theta}_h\right)\left|\nabla\overline{\phi}_h\right|^2 \tilde{\zeta}\,\mathrm{d}x + \int_\Omega \nabla\theta_h.\nabla\left(\zeta - \tilde{\zeta}\right)\mathrm{d}x \\
&= A_1 + A_2.
\end{aligned}$$

We next estimate A_1 and A_2.

The A_1 estimate – As $\zeta \in H^{2-s}(\Omega) \subset L^\infty(\Omega)$ and since the Lagrange interpolation operator is continuous linear mapping from $L^\infty(\Omega)$ into $L^\infty(\Omega)$ there exists a constant $C_3 > 0$ such that $\left\|\tilde{\zeta}\right\|_{L^\infty(\Omega)} \le C_3 \|\zeta\|_{L^\infty(\Omega)}$, we have:

$$
\begin{aligned}
A_1 \leq & \left\| \tau\left(\overline{\theta}_h\right) \left|\nabla \overline{\phi}_h\right|^2 \right\|_{L^1(\Omega)} \left\|\tilde{\zeta}\right\|_{L^\infty(\Omega)} \\
\leq & C_3 \left\| \tau\left(\overline{\theta}_h\right) \left|\nabla \overline{\phi}_h\right|^2 \right\|_{L^1(\Omega)} \|\zeta\|_{L^\infty(\Omega)} .
\end{aligned}
$$

Since the injection from $H^{2-s}(\Omega)$ into $L^\infty(\Omega)$ is continuous, there exists a constant C_4 such that $\|\zeta\|_{L^\infty(\Omega)} \leq C_3 C_4 \|\zeta\|_{H^{2-s}(\Omega)}$ and then

$$
A_1 \leq C_4 \left\| \tau\left(\overline{\theta}_h\right) \left|\nabla \overline{\phi}_h\right|^2 \right\|_{L^1(\Omega)} \|\zeta\|_{H^{2-s}(\Omega)} .
$$

The A_2 estimate – A slight modification of the argument of Exercise 8.3 of [8] proves that there exists a constant depending only on Ω such that

$$
\left\|\nabla\left(\zeta - \tilde{\zeta}\right)\right\|_{L^2(\Omega)} \leq C_5 h^{1-s} \|\zeta\|_{H^{2-s}(\Omega)} .
$$

Then, from the Cauchy-Schwartz inequality:

$$
A_2 \leq C_5 h^{1-s} \|\nabla \theta_h\|_{L^2(\Omega)} \|\zeta\|_{H^{2-s}(\Omega)} .
$$

Since $\nabla \theta_{h|K} \in \mathbb{P}_{k-1}$ for all $K \in \mathcal{T}_h$, we have:

$$
\begin{aligned}
\|\nabla \theta_h\|_{L^2(\Omega)} = & \sup_{\zeta_h \in V_h} \frac{\displaystyle\int_\Omega \nabla \theta_h . \nabla \zeta_h \, \mathrm{d}x}{\|\nabla \zeta_h\|_{L^2(\Omega)}} \\
= & \sup_{\zeta_h \in V_h} \frac{\displaystyle\int_\Omega \tau\left(\overline{\theta}_h\right) \left|\nabla \overline{\phi}_h\right|^2 \zeta_h \, \mathrm{d}x}{\|\nabla \zeta_h\|_{L^2(\Omega)}} \quad \text{by using (10)} \\
\leq & \left\| \tau\left(\overline{\theta}_h\right) \left|\nabla \overline{\phi}_h\right|^2 \right\|_{L^1(\Omega)} \sup_{\zeta_h \in V_h} \frac{\|\zeta_h\|_{L^\infty(\Omega)}}{\|\nabla \zeta_h\|_{L^2(\Omega)}} .
\end{aligned}
$$

From [14] we have the following inverse inequality:

$$
\|\zeta_h\|_{L^\infty(\Omega)} \leq C_6 h^{1-\frac{d}{2}} |\log h|^{1-\frac{1}{d}} \|\zeta_h\|_{H^1(\Omega)} , \ \forall \zeta_h \in V_h .
$$

Then, using the Poincaré inequality:

$$
\|\nabla \theta_h\|_{L^2(\Omega)} \leq C_0^{-1} C_6 h^{1-\frac{d}{2}} |\log h|^{1-\frac{1}{d}} \left\| \tau\left(\overline{\theta}_h\right) \left|\nabla \overline{\phi}_h\right|^2 \right\|_{L^1(\Omega)}
$$

where C_0 is the constant of the Poincaré inequality. Grouping the previous inequality yields:

$$
A_2 \leq C_0^{-1} C_5 C_6 h^{2-s-\frac{d}{2}} |\log h|^{1-\frac{1}{d}} \left\| \tau\left(\overline{\theta}_h\right) \left|\nabla \overline{\phi}_h\right|^2 \right\|_{L^1(\Omega)} \|\zeta\|_{H^{2-s}(\Omega)} .
$$

Grouping the A_1 and A_2 estimates, we get:

$$\langle \theta_h,\, \Delta\zeta\rangle_{H_0^s,H^{-s}} \leq \left(C_3C_4 + C_0^{-1}C_5C_6h^{2-s-\frac{d}{2}}\left|\log\, h\right|^{1-\frac{1}{d}} \right) \left\| \tau\left(\overline{\theta}_h\right) \left|\nabla\overline{\phi}_h\right|^2 \right\|_{L^1(\Omega)} \|\zeta\|_{H^{2-s}(\Omega)}$$

and then

$$\|\theta_h\|_{H^s(\Omega)} \leq C_2^{-1}\left(C_3C_4 + C_0^{-1}C_5C_6h^{2-s-\frac{d}{2}}\left|\log\, h\right|^{1-\frac{1}{d}} \right) \left\| \tau\left(\overline{\theta}_h\right) \left|\nabla\overline{\phi}_h\right|^2 \right\|_{L^1(\Omega)}.$$

Finally, (12) is obtained with $C_1 = C_2^{-1}\max\left(C_3C_4, C_0^{-1}C_5C_6\right)$. $\square$

THEOREM **4** *(*a priori *estimate for the second subproblem)*
Problem $(FVR_{2,h})$ *admits an unique solution, which satisfies the* a priori *estimate:*

$$\|\phi_h\|_{H^1(\Omega)} \leq C_7\, \|f\|_{L^2(\Omega)} \tag{13}$$

for a positive constants C_7.

Proof: The existence of the solution $\phi_h \in W_h$ of (11) follows from the Lax-Milgram theorem. From one hand, choosing $\psi_h = \phi_h \in V_h$ in (11), we get:

$$\begin{aligned}\int_\Omega \tau\left(\overline{\theta}_h\right)\, |\nabla\phi_h|^2\, \mathrm{d}x &= \int_\Omega f\,\phi_h\,\mathrm{d}x\\ &\leq \|f\|_{L^2(\Omega)}\, \|\phi_h\|_{L^2(\Omega)}\\ &\leq \|f\|_{L^2(\Omega)}\, \|\phi_h\|_{H^1(\Omega)}\\ &\leq \frac{1}{2\beta}\, \|f\|^2_{L^2(\Omega)} + \frac{\beta}{2}\, \|\phi_h\|^2_{H^1(\Omega)},\ \forall\beta > 0.\end{aligned}$$

From other hand, from the hypothesis on σ and the Poincaré inequality:

$$\begin{aligned}\int_\Omega \tau\left(\overline{\theta}_h\right)\, |\nabla\phi_h|^2\, \mathrm{d}x &\geq \sigma_{\min}\int_\Omega |\nabla\phi_h|^2\, \mathrm{d}x\\ &\geq \sigma_{\min}C_0\, \|\phi_h\|^2_{H^1(\Omega)}\end{aligned}$$

where $C_0 > 0$ is the Poincaré inequality constant, depending only on Ω. Grouping the two previous inequalities, we get:

$$\sigma_{\min}C_0\, \|\phi_h\|^2_{H^1(\Omega)} \leq \frac{1}{2\beta}\, \|f\|^2_{L^2(\Omega)} + \frac{\beta}{2}\, \|\phi_h\|^2_{H^1(\Omega)},\ \forall\beta > 0.$$

Choosing $\beta = \sigma_{\min}$ we get:

$$\frac{\sigma_{\min}C_0}{2}\, \|\phi_h\|^2_{H^1(\Omega)} \leq \frac{1}{2\sigma_{\min}C_0}\, \|f\|^2_{L^2(\Omega)}$$

or equivalently:

$$\|\phi_h\|_{H^1(\Omega)} \leq \frac{1}{\sigma_{\min} C_0} \|f\|_{L^2(\Omega)} .$$

Thus, we get (13) with $C_7 = 1/(\sigma_{\min} C_0)$. $\square$

2.3 Existence result for the approximate problem

THEOREM **5** *(existence result)*
Assume that $h < h_0$. Then, the discrete problem (FVQ_h) admits always a solution which satisfies the estimates:

$$\|\theta_h\|_{H^s(\Omega)} \leq C_1 \left(1 + h^{2-s-\frac{d}{2}} |\log h|^{1-\frac{1}{d}}\right) \left\|\tau(\theta_h) |\nabla \phi_h|^2\right\|_{L^1(\Omega)} , \quad (14)$$
$$\|\phi_h\|_{H^1(\Omega)} \leq C_7 \|f\|_{L^2(\Omega)} . \quad (15)$$

Proof: Let us consider the continuous linear transformation $\mathcal{F}_i$, $i = 1, 2$ from $V_h \times W_h$ onto itself defined as follows: the image by $\mathcal{F}_i$ of $\left(\overline{\theta}_h, \overline{\phi}_h\right) \in V_h \times W_h$ is the element $(\theta_h, \phi_h) \in V_h \times W_h$ defined as the solution of $(FVR_{i,h})$. The transformations $\mathcal{F}_i$, $i = 1, 2$ are well defined, due to the uniqueness of solutions of subproblems $(FVR_{i,h})$. Let $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$ be the continuous linear transformation obtained by composition as follows: the image by $\mathcal{F}$ of $\left(\overline{\theta}_h, \overline{\phi}_h\right) \in V_h \times W_h$ is the element $(\theta_h, \phi_h) \in V_h \times W_h$ defined in two steps:

1. ϕ_h is the solution of $(FVR_{2,h})$ for a given $\overline{\theta}_h \in V_h$.
2. θ_h is the solution of $(FVR_{1,h})$ for a given $\left(\overline{\theta}_h, \phi_h\right) \in V_h \times W_h$;

From theorem 4 we have:

$$\|\phi_h\|_{H^1(\Omega)} \leq C_7 \|f\|_{L^2(\Omega)} ,$$

and from theorem 3:

$$
\begin{aligned}
\|\theta_h\|_{H^s(\Omega)} &\leq C_1\left(1+h^{2-s-\frac{d}{2}}\,|\log\,h|^{1-\frac{1}{d}}\right)\left\|\tau\left(\overline{\theta}_h\right)|\nabla\phi_h|^2\right\|_{L^1(\Omega)}\\
&\leq C_1\left(1+h_0^{2-s-\frac{d}{2}}\,|\log\,h_0|^{1-\frac{1}{d}}\right)\left\|\tau\left(\overline{\theta}_h\right)|\nabla\phi_h|^2\right\|_{L^1(\Omega)}\\
&\quad\text{for all } h<h_0 \text{ since } h\mapsto h^{\varepsilon}|\log\,h|^{\mu} \text{ is an increasing function, } \forall\varepsilon>0,\forall\mu\geq 0,\\
&\leq C_1\left(1+h_0^{2-s-\frac{d}{2}}\,|\log\,h_0|^{1-\frac{1}{d}}\right)\sigma_{\max}\,\|\nabla\phi_h\|_{L^2(\Omega)}\\
&\quad\text{since } \tau(.) \text{ is bounded,}\\
&\leq C_1\left(1+h_0^{2-s-\frac{d}{2}}\,|\log\,h_0|^{1-\frac{1}{d}}\right)\sigma_{\max}\,\|\phi_h\|_{H^1(\Omega)}\\
&\leq C_1C_7\left(1+h_0^{2-s-\frac{d}{2}}\,|\log\,h_0|^{1-\frac{1}{d}}\right)\sigma_{\max}\,\|f\|_{L^2(\Omega)}\\
&\quad\text{from theorem 4.}
\end{aligned}
$$

Thus, grouping the two previous inequalities:

$$
\left\|\mathcal{F}\left(\overline{\theta}_h,\,\overline{\phi}_h\right)\right\|_{H^s(\Omega)\times H^1(\Omega)} := \left(\|\theta_h\|^2_{H^s(\Omega)}+\|\phi_h\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}} \;\leq\; C_8\,\|f\|_{L^2(\Omega)}\,,
$$

where

$$
C_8 = C_7\left(1+\sigma^2_{\max}C_1^2\left(1+h_0^{2-s-\frac{d}{2}}\,|\log\,h_0|^{1-\frac{1}{d}}\right)^2\right)^{\frac{1}{2}}.
$$

By Brouwer's fixed point theorem, we conclude that problem (FVQ_h) always admits a solution. The estimates (14)-(15) are directly obtained from theorem 3 and 4 with the fixed point $\left(\overline{\theta}_h,\,\overline{\phi}_h\right)=(\theta_h,\,\phi_h)\in V_h\times W_h$. ∎

2.4 Convergence result of the approximate solution

Theorem **6** *(convergence to the continuous solution)*
Assume that $h<h_0$. Then, there exists a subsequence of the solutions $(\theta_h,\,\phi_h)_{h\in]0,h_0[}$ provided by (FVQ_h) that converge strongly in $H^s(\Omega)\times H^1(\Omega)$ to a solution of (FVQ).

Proof: We perform this proof in three steps.

step 1. *a priori* estimate – Let us start by finding a bound for θ_h. By theorem (5), the sequence $(\theta_h)_{h>0}$ is bounded in $H^s(\Omega)$. Also the sequence $(\phi_h)_{h>0}$ is bounded in $H^1(\Omega)$.

step 2. Limit for the potential equation – Let us recall that the embedding of $H^s(\Omega)$ in $L^2(\Omega)$ for any $s>0$ is compact. Then, the sequence $(\theta_h)_{h>0}$ contains a subsequence $(\theta_{h'})_{h'>0}$ which is strongly convergent in $L^2(\Omega)$ to a function $\theta\in L^2(\Omega)$.

From the estimate for the potentials, we may find a subsequence $(\phi_{h'})_{h'>0}$ of $(\phi_h)_{h>0}$ weakly convergent in $H^1(\Omega)$ to a function ϕ. In [4], in the similar context of two coupled turbulent fluids, the authors deduces that it contains a subsequence that converges strongly in $H^1(\Omega)$ to ϕ. Then we may pass to the limit in (7) and deduce that (θ, ϕ) satisfies (9).

step 3. Limit for the heat equation – Consider $\xi \in \mathcal{D}(\Omega)$ and denote $\zeta = \Delta^{-1}\xi$. As in the proof of theorem 3, let $\tilde{\zeta} \in V_h$ be the interpolation of ζ. We have:

$$\begin{aligned}\int_\Omega \theta_h \, \Delta\zeta \, \mathrm{d}x &= \lambda \int_\Omega \tau\left(\theta_h\right) \, |\nabla \phi_h|^2 \, \tilde{\zeta} \, \mathrm{d}x + \int_\Omega \nabla\theta_h . \nabla\left(\zeta - \tilde{\zeta}\right) \, \mathrm{d}x \\ &= A_1 + A_2.\end{aligned}$$

We next analyze the convergence of the two terms A_1 and A_2.

• **The A_1 limit** – As in Exercise 8.3 of [8]:

$$\left\|\zeta - \tilde{\zeta}\right\|_{L^\infty(\Omega)} \leq C_5 \, h^{\frac{1}{2}} \, \|\zeta\|_{W^{\frac{1}{2},\infty}(\Omega)} .$$

Then, since the embedding from $H^2(\Omega)$ into $W^{\frac{1}{2},\infty}(\Omega)$ is continuous, there exists a constant $C_9 > 0$ such that $\|\zeta\|_{W^{\frac{1}{2},\infty}(\Omega)} \leq C_9 \, \|\zeta\|_{H^2(\Omega)}$, $\forall \zeta \in H^2(\Omega)$. Thus, the previous inequality becomes:

$$\left\|\zeta - \tilde{\zeta}\right\|_{L^\infty(\Omega)} \leq C_5 C_9 \, h^{\frac{1}{2}} \, \|\zeta\|_{H^2(\Omega)} .$$

As the sequence $(\tau(\theta_h)|\nabla\phi_h|^2)_{h>0}$ converges strongly in $L^1(\Omega)$ to $\tau(\theta)|\nabla\phi|^2$, then

$$\lim_{h\to 0} A_1 = \lim_{h\to 0} \int_\Omega \tau(\theta_h)|\nabla\phi_h|^2 \, \tilde{\zeta} \, \mathrm{d}x = \int_\Omega \tau(\theta)|\nabla\phi|^2 \, \zeta \, \mathrm{d}x.$$

• **The A_2 limit** – As in the proof of theorem 3:

$$A_2 \leq C_0^{-1} C_5 C_6 h^{2-s-\frac{d}{2}} \, |\log h|^{1-\frac{1}{d}} \left\|\tau\left(\theta_h\right) \, |\nabla\phi_h|^2\right\|_{L^1(\Omega)} \|\zeta\|_{H^{2-s}(\Omega)} ,$$

and then

$$\lim_{h\to 0} A_2 = 0 \ .$$

Then, the limit satisfies

$$-\int_\Omega \theta \, \xi \, \mathrm{d}x = \int_\Omega \tau(\theta)|\nabla\phi|^2 \, \zeta \, \mathrm{d}x, \ \forall \xi = \Delta\zeta \in \mathcal{D}(\Omega).$$

Then (8) is satisfied since $\mathcal{D}(\Omega)$ is dense in $H^{-s}(\Omega)$. In conclusion, the limit (θ, ϕ) is a solution of (FVQ): this complete the proof of theorem 6. $\square$

3 Numerical experimentations

3.1 The algorithm

The system is not solved in the form given by (3), (4). Unlike studies on a similar case [17], we found that under certain conditions, a fixed point algorithm does not successfully converge towards a couple of discrete solutions (u_h, ϕ_h). An evolution problem is introduced and the steady state solution is computed by using a semi-implicit decoupled algorithm :

$$\frac{\partial u}{\partial t} \approx \frac{u^{n+1} - u^n}{\Delta t^n}.$$

The aim is to solve the semi-implicit system at iteration n :

Given $u^n \in H^1_{u_0}$, find $u^{n+1} \in H^1_{u_0}$ and $\phi^{n+1} \in H^1_{\phi_0}$, such that

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t^n} - \operatorname{div}(\kappa(u^n)\nabla u^{n+1}) = \sigma(u^n)\left|\nabla\phi^n\right|^2, \\ -\operatorname{div}(\sigma(u^n)\nabla\phi^{n+1}) = f, \\ -\kappa(u^n)\dfrac{\partial u^{n+1}}{\partial n} = 0 \quad \text{on } \Gamma_1, \\ -\sigma(u^n)\,\dfrac{\partial \phi^{n+1}}{\partial n} = 0 \quad \text{on } \Gamma_2. \end{cases}$$

Here Dirichlet condition for u is inhomogeneous, we define the proper finite element space $Y_h = \{v_h \in H^1_{u_0}(\Omega); v_{h|K} \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\}$, and Z_h whose functions are nul on Γ_2, e.g. $v_h = 0|_{\Gamma_2}$. The approximate variational formulation of (1) writes:

(FVP_h) : *Given $u_h^n \in Y_h$, find $u_h^{n+1} \in Y_h$ and $\phi_h^{n+1} \in W_h$ such that*

$$\begin{aligned} \int_\Omega u_h^{n+1}\, v_h \,\mathrm{d}x + \Delta t^n \int_\Omega \kappa(u_h^n)\, \nabla u_h^{n+1}.\nabla v_h \,\mathrm{d}x &= \Delta t^n \int_\Omega \sigma(u_h^n)\ \left|\nabla\phi_h^n\right|^2\ v_h \,\mathrm{d}x \\ &\quad + \int_\Omega u^n v_h \,\mathrm{d}x,\ \forall v_h \in Z_h, \\ \int_\Omega \sigma(u_h^n)\nabla\phi_h^{n+1}.\nabla w_h \,\mathrm{d}x &= \int_\Omega f\, w_h \,\mathrm{d}x, \quad \forall w_h \in X_h. \end{aligned}$$

The nonlinear coupled stationary problem is then solved by using a sequence of two linear decoupled subproblems of generalized Poisson type with non constant coefficients. The stopping criteria is related to the norm of the residual term of the stationary problem.

3.2 *Estimation of the order of convergence: first approach*

We expect a convergence behavior as:

$$\|u - u_h\|_{H^m} \approx h^{\gamma - m}. \tag{16}$$

A comparable convergence behavior is expected for ϕ with a possible different power index γ.

The power index γ associated to u and ϕ are investigated numerically by using three meshes h_n, h_{n-1} and h_{n-2} of a mesh family $(h_n)_{n>0}$, with $h_n < h_{n-1}$ for all $n > 0$. The power index is extracted from the following relation:

$$\frac{\|v_{h_{n-1}} - v_{h_{n-2}}\|_{L^2(\Omega)}}{\|v_{h_n} - v_{h_{n-1}}\|_{L^2(\Omega)}} = \frac{(h_{n-2}/h_{n-1})^\gamma - 1}{1 - (h_n/h_{n-1})^\gamma}. \tag{17}$$

Note that [3] used a comparable approach with a slightly different formulae. With the present, the γ power index is always defined: the right-hand-side as a function of γ is positive and increases with h_n, and (17) always admits a solution. Simulations of γ are presented in the following numerical experiment section.

3.3 *Estimation of the order of convergence: second approach*

The posteriori error estimator η introduced by Kelly et al. [12,13] is defined locally on each element K from the numerical solution u_h by :

$$(\eta_K^u)^2 = \frac{h}{24} \int_{\partial K} \left[\kappa(u_h) \, \frac{\partial u_h}{\partial \mathbf{n}} \right]^2 \, ds,$$

and globally on the whole domain Ω:

$$\eta^u = \sqrt{\sum_{K \in \mathcal{P}} (\eta_K^u)^2}.$$

A comparable estimator η^ϕ is introduced for ϕ_h with a variable coefficient $\sigma(u_h)$. Such estimators are able to detect high variations of the gradients of the approximate solutions ∇u_h, e.g. locally high value of the Hessian of u_h. For smooth solutions the gradient varies slowly, whereas at the vicinity of singularities the gradient and its derivatives increases dramatically. Since this estimator is expected to behave asymptotically as the H^1 error, we are looking for the convergence behavior of the error estimator η^u and η^ϕ with respect to the mesh parameter h:

$$\eta^u \approx h^{\lambda - 1}, \tag{18}$$

in the case of an uniform grid. Power index λ associated to ϕ could be different. Note that a recent a posteriori estimation for the thermistor problem [1] propose an error estimator for ϕ based on the jump gradient term across element boundaries, as for the Kelly estimate, plus some additional residual terms. Here, the contribution of these additional residual terms to the error estimator are not investigated numerically at the vicinity of a corner singularity ; we expect that the gradient jump term in the error estimator dominates the other residual terms.

3.4 Numerical investigations

Two test cases arising from the context of magnet modelling [15] are considered here. In the first case a disk made of copper alloy is studied. An explicit solution exists for (1) assuming constant coefficients $\sigma(u)$ and $\kappa(u)$ and no vanishing Dirichlet conditions u_0 and ϕ_0. For sake of symmetry we will restreint ourselves to one fourth of the domain.

In the second one we consider a non-convex geometry with a reentrant corner. Morevoer we assume that $\sigma(u)$ depends on the temperature, and that $\kappa(u)$ is given by the Wiedmann-Frantz law :

$$\kappa(u) = L\ \sigma(u)\ u, \tag{19}$$

where L is a constant, the so-called Lorentz number. See [6] for a theoretical proof of the existence and uniquess of (1) in that case.

3.4.1 Test case 1: convex geometry and constant coefficients

We assume here that $\sigma(u) = 50199996\ m/\Omega$ and that $\kappa(u) = 380\ W/(m.K)$ (i.e. electric and thermal conductivity of a copper alloy at room temperature). The domain Ω is shown on Fig. 1, with internal and external radius $r_1 = 0.0193\ m$ and $r_2 = 0.0242\ m$. The Dirichlet boundary condition is $\phi_0 = 0\ V$ on the blue border and $\phi_0 = U = 0.210\ V$ on the red one. The temperature is fixed to $u_0 = 293K$ on Γ_2 and $\frac{\partial u}{\partial n} = 0$ on Γ_1.

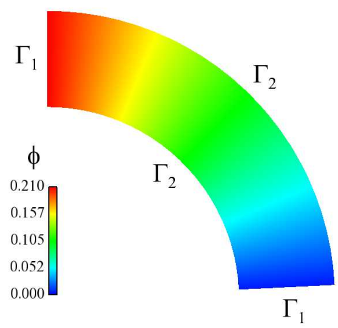

Figure 1. Test case 1: axisymmetric geometry and isovalues of the potential ϕ.

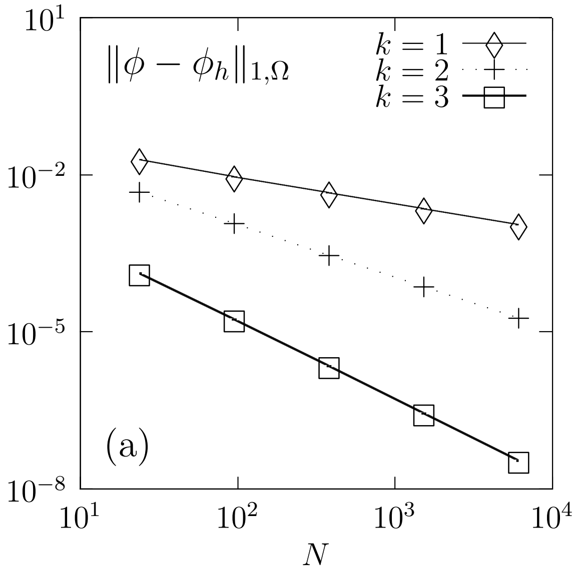

Figure 2. Test case 1: behavior of the error $\|\phi - \phi_h\|_{1,\Omega}$ versus the size of the finite element space N for $k = 1, 2, 3$.

Exact solution (u, ϕ) writes in terms of cylindrical coordinates:

$$u(r, \theta) = u_0 - \frac{\sigma U}{8\pi\kappa}\left(\ln^2(r) - \ln(r)\ln(r_1 r_2) + \ln(r_1)\ln(r_2)\right) \quad \text{and} \quad \phi(r, \theta) = \frac{U\theta}{2\pi}.$$

Note that both u and ϕ belong to $C^\infty(\overline{\Omega})$.

Fig. 2 shows the error $\|\phi - \phi_h\|_{1,\Omega}$ versus the size of the finite element space $N = \mathrm{cardV_h}$ for various values $k = 1, 2, 3$ of the polynom basis Q_k for a family of uniformly refined meshes. All computations are performed by using the *deal II* finite element library [2]. The error $\|\phi - \phi_h\|_{1,\Omega} \approx h^k$ as predicted by (16) with $\gamma = k + 1$ and $m = 1$. The convergence properties of the temperature u are comparable and are not represented here.

	Q_1		
	γ	λ	δ
ϕ	1.99	1.93	1.99
u	2.00	1.99	2.00

Table 1
Convergence order of the finite element method for test 1.

Table 1 estimates the order of convergence of the finite element method Q_1 by three methods and for (u_h, ϕ_h). The first method evaluates γ, the order of convergence by solving the non linear equation (17). The second method based on the Kelly error estimate, evaluates λ from (18) from a sequence of ten meshes by a least square procedure. The third method based on the knowledge of the exact solution (u, ϕ) computes directly the L^2-norm error and then estimates the convergence order denoted by δ by a least square

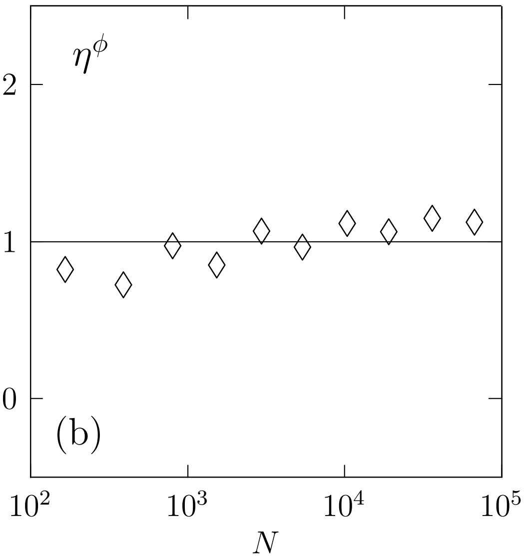

Figure 3. Test case 1: efficiency of the Kelly error estimate η^{ϕ} for $k = 1$.

procedure. Since the exact solution (u, ϕ) is regular for the test case 1, the expected values for all the orders of convergence γ, λ and δ is $k+1$. Values of γ and λ provide a rather good idea of the convergence order. Fig. 3 shows the error estimate efficiency $\eta^{\phi}/\|\phi - \phi_h\|_{1,\Omega}$: it stays globally around 1, with a slight increase when the number of degree of freedom N increases. Thus, the Kelly error estimate represents a very good estimation of the error for our problem.

3.4.2 Test case 2: non-convex geometry and non-constant coefficients

A non-convex geometry Ω with a reentrant corner is then considered (see Fig. 4). The interior angle equals to $\omega = 3\pi/2$.

Here $\sigma(u)$ depends on temperature u [15] :

$$\sigma(u) = \frac{\sigma_0}{1 + \beta(u - u_0)}$$

with $\sigma_0 = 50199996\ m/\Omega$, $\beta = 10^{-3}K^{-1}$, and $\kappa(u)$ is expressed by (19) with $L = 2.410^{-8}\Omega.W/K^2$.

The solution is represented on Fig. 4.

Both u and ϕ belong to the same Hilbert space H^s. Based on the interior angle $\omega^* = 3\pi/2$ of the domain Ω, theorem 2 provides a regularity result: both u and ϕ belongs to H^s for all $s < s^*$, where $s^* = 1 + 2/3 = 1.66\ldots$. The expected convergence order of the finite element method is $h^{\min(k+1,s^*)} = h^{s^*}$ that is independent of k. Table 2 confirms this slow convergence property, based on the estimations of the convergence order γ and λ introduced in the previous paragraph. Note that the estimation δ is no more available, since the exact

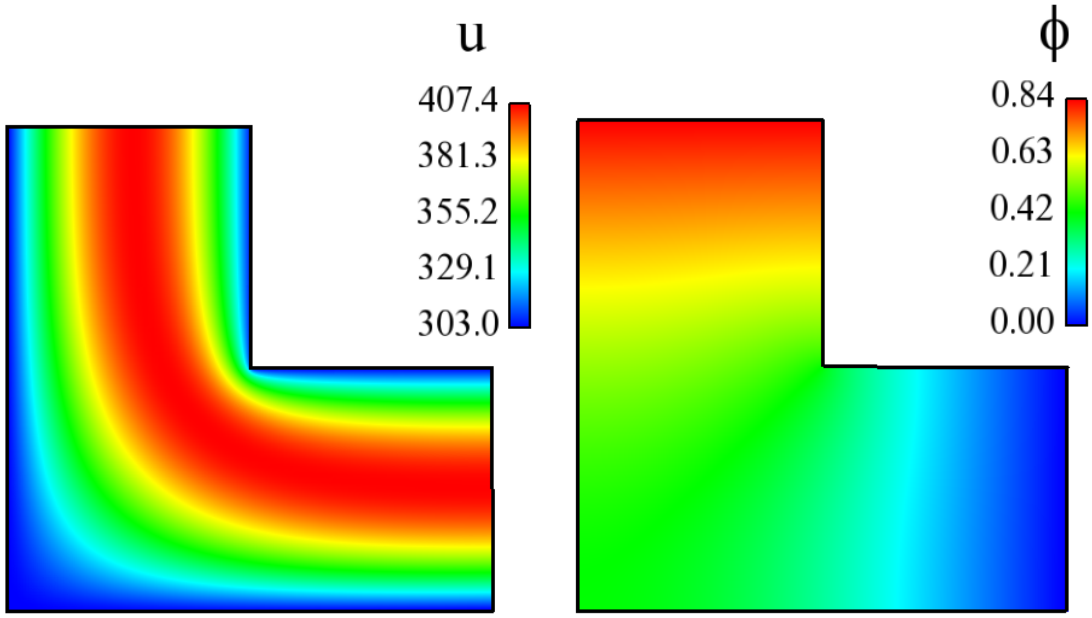

Figure 4. Isocontours of the temperature u and the potential ϕ for test 2.

Q_k	Q_1		Q_2		Q_3	
	γ	λ	γ	λ	γ	λ
ϕ	1.42	1.70	1.43	1.67	1.45	1.67
u	1.52	1.55	1.82	1.53	1.76	1.54

Table 2
Convergence order of the finite element method for test 2.

solution is unknown here.

In order to improve the poor convergence properties associated to singular solutions, let us turn to mesh adaptation procedure. The finite element degree is here fixed to $k = 1$. The refinement is done on a fixed fraction of elements associated to the highest local Kelly error estimate value. Fig. 5 shows adapted meshes obtained by this method. The refinement is concentrated at the vicinity of the singularity, located at the reentrant corner of the domain. Fig. 4 reveals that u and ϕ develops differently on Ω. Thus mesh adaptations based on η^u and η^ϕ look different despite both concentrate on the reentrant corner. Fig. 6 presents the Kelly error estimate for u and ϕ as a function of the number of degree of freedom of the finite element space N. Recall that the error estimator behaves as $h^{5/3}$ when using an uniform mesh refinement procedure. Since $N \approx h^{-2}$, the error estimator is expected to behave as $N^{-5/6} \approx N^{-0.83}$. When using the adaptive strategy, we observe that it is asymptotically improved roughly as $N^{-1.45}$.

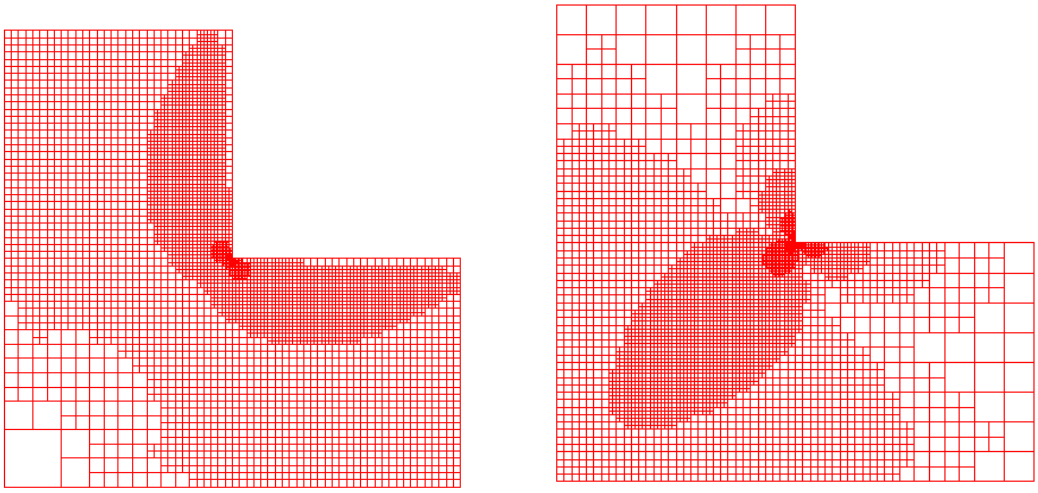

Figure 5. Adaptive meshes as obtained by a refinement criteria based on the Kelly error estimator: (left) for the temperature u ; (right) for the potential ϕ.

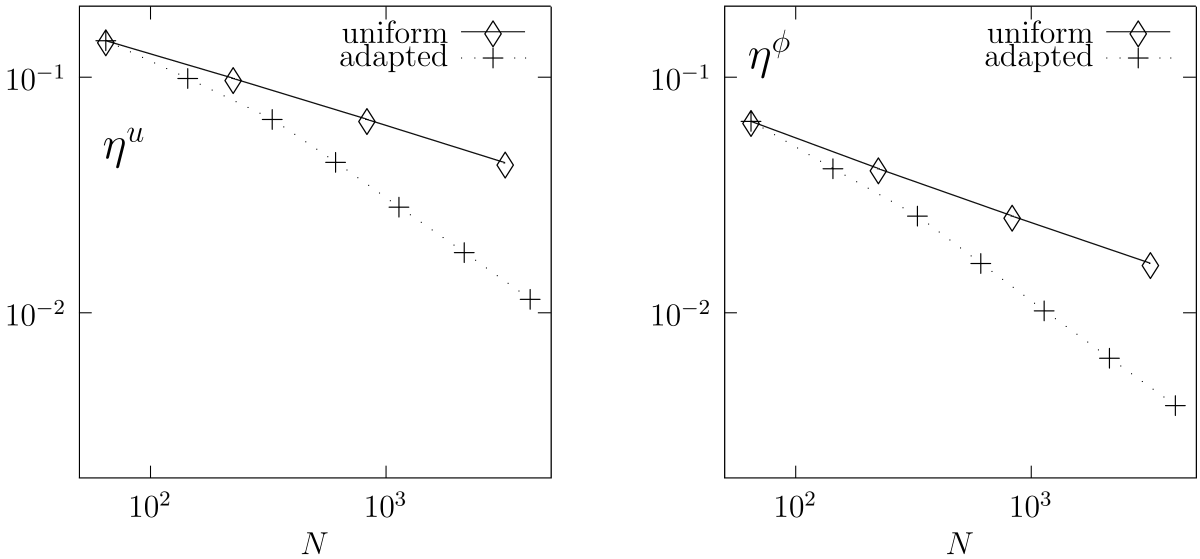

Figure 6. Acceleration of the convergence when using adapted meshes: (left) for the temperature u; (right) for the potential ϕ.

Conclusion

The convergence properties of the finite element approximation of the thermistor problem were investigated in this paper, both from theoretical and numerical point of view. Based on a duality argument, a theoretical convergence result was first proved under low regularity assumption. Next numerical experiments was performed based on a decoupled algorithm and the power index of the convergence versus the mesh size was extracted. Finally, a suitable mesh adaptation strategy was found to improve the convergence properties versus the mesh size on non convex domains.

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