Ultraparabolic H-measures and compensated compactness

E. Yu. Panov

Ultraparabolic H-measures and compensated compactness

E.Yu. Panov

Abstract

We present a generalization of compensated compactness theory to the case of variable and generally discontinuous coefficients, both in the quadratic form and in the linear, up to the second order, constraints. The main tool is the localization properties for ultra-parabolic H-measures corresponding to weakly convergent sequences.

1 Introduction

Recall the classical results of the compensated compactness theory (see [4, 9]). Suppose that Ω is an open subset of $\mathbb{R}^n$, and a sequence $u_r = (u_{1r}(x), \dots, u_{Nr}(x)) \in L^2(\Omega, \mathbb{R}^N)$, $r \in \mathbb{N}$, weakly converges to a vector-function $u(x)$ in $L^2(\Omega, \mathbb{R}^N)$. Assume that $a_{s\alpha k}$ are real constants for $s = 1, \dots, m$, $\alpha = 1, \dots, N$, $k = 1, \dots, n$, and the sequences of distributions

$$\sum_{\alpha=1}^{N} \sum_{k=1}^{n} a_{s\alpha k} \partial_{x_k} u_{\alpha r}, \quad s = 1, \dots, m, \quad r \in \mathbb{N}, \tag{1}$$

are strongly precompact in the space $H^{-1}_{loc}(\Omega) \doteq W^{-1}_{2,loc}(\Omega)$. Hereafter, we denote by $W^{-1}_{p,loc}(\Omega)$, $1 \le p \le \infty$ the locally convex space consisting of distributions $v \in \mathcal{D}'(\Omega)$ such that the distribution fv belongs to the Sobolev space $W^{-1}_p \doteq W^{-1}_p(\mathbb{R}^n)$ for all $f(x) \in C_0^\infty(\Omega)$. The topology in $W^{-1}_{p,loc}(\Omega)$ is generated by the family of semi-norms $u \to \|uf\|_{W^{-1}_p}$, $f(x) \in C_0^\infty(\Omega)$. Introduce the set

$$\Lambda = \Big\{ \lambda \in \mathbb{R}^N \mid \exists \xi \in \mathbb{R}^n, \xi \ne 0 : \sum_{\alpha=1}^{N} \sum_{k=1}^{n} a_{s\alpha k} \lambda_\alpha \xi_k = 0 \ \forall s = 1, \dots, m \ \Big\}.$$

Now, let $q(u) = \sum_{\alpha,\beta=1}^{N} q_{\alpha\beta} u_\alpha u_\beta$ be a quadratic functional on $\mathbb{R}^l$ such that
$q(\lambda) \ge 0$ for all $\lambda \in \Lambda$, and
$q(u_r) \to v$ weakly in the sense of distributions on Ω (in $\mathcal{D}'(\Omega)$).

Then, under the above assumptions,

$$q(u(x)) \le v \ \text{ in } \mathcal{D}'(\Omega)$$

(the weak low semicontinuity). In particular, if $q(\lambda) = 0$ on Λ then $v = q(u)$.

In this paper we generalize this result to the case when the differential constraints may contain second order terms, while all the coefficients are variable and may be discontinuous. Thus, assume that a sequence $u_r(x)$ is bounded in $L^p_{loc}(\Omega,\mathbb{R}^N)$, $2\le p\le\infty$ and converges weakly in $\mathcal{D}'(\Omega)$ to a vector-function $u(x)$ as $r\to\infty$. Let $d=p/(p-1)$ if $p<\infty$, and $d>1$ if $p=\infty$. Assume that the sequences

$$\sum_{\alpha=1}^N\sum_{k=1}^n \partial_{x_k}(a_{s\alpha k}u_{\alpha r})+\sum_{\alpha=1}^N\sum_{k,l=\nu+1}^n \partial_{x_kx_l}(b_{s\alpha kl}u_{\alpha r}),\quad s=1,\dots,m \tag{2}$$

are pre-compact in the anisotropic Sobolev space $W^{-1,-2}_{d,loc}(\Omega)$, which will be defined later in Section 2. Here ν is an integer number between 0 and n, and the coefficients $a_{s\alpha k}=a_{s\alpha k}(x)$, $b_{s\alpha kl}=b_{s\alpha kl}(x)$ belong to the space $L^{2q}_{loc}(\Omega)$, $q=p/(p-2)$ ($q=1$ in the case $p=\infty$), if $p>2$, and to the space $C(\Omega)$ if $p=2$. One example is given by $p=\infty$, $q=1$ and corresponds to the case when the functions $u_r(x)$ are uniformly locally bounded.

We introduce the set Λ (here $i=\sqrt{-1}$):

$$\Lambda=\Lambda(x)=\Big\{\lambda\in\mathbb{C}^N \mid \exists\xi\in\mathbb{R}^n,\xi\ne0:$$
$$\sum_{\alpha=1}^N\Big(i\sum_{k=1}^\nu a_{s\alpha k}(x)\xi_k-\sum_{k,l=\nu+1}^n b_{s\alpha kl}(x)\xi_k\xi_l\Big)\lambda_\alpha=0\ \forall s=1,\dots,m\ \Big\}. \tag{3}$$

Consider the quadratic form $q(x,u)=Q(x)u\cdot u$, where $Q(x)$ is a symmetric matrix with coefficients $q_{\alpha\beta}(x)$, $\alpha,\beta=1,\dots,N$ and $u\cdot v$ denotes the scalar multiplication on $\mathbb{R}^N$. The form $q(x,u)$ can be extended as Hermitian form on $\mathbb{C}^N$ by the standard relation

$$q(x,u)=\sum_{\alpha,\beta=1}^N q_{\alpha\beta}(x)u_\alpha\overline{u_\beta},$$

where we denote by $\overline{u}$ the complex conjugation of $u\in\mathbb{C}$. We suppose that the coefficients $q_{\alpha\beta}(x)\in L^q_{loc}(\Omega)$ if $p>2$, and $q_{\alpha\beta}(x)\in C(\Omega)$ if $p=2$.

Now, let the sequence $q(x,u_r)\to v$ as $r\to\infty$ weakly in $\mathcal{D}'(\Omega)$. Since for each $\alpha,\beta=1,\dots,N$ the sequences $u_{\alpha r}(x)u_{\beta r}(x)$ are bounded in $L^{p/2}_{loc}(\Omega)$ (here $p/2=\infty$ for $p=\infty$) then, passing to a subsequence if necessary, we may claim that

$$u_{\alpha r}(x)u_{\beta r}(x)\underset{r\to\infty}{\to}\zeta_{\alpha\beta}(x)$$

weakly in $L^{p/2}_{loc}(\Omega)$ if $p>2$ (hereafter, the weak convergence in $L^\infty_{loc}(\Omega)$ is understood in the sense of the weak-$*$ topology), and weakly in the space $M_{loc}(\Omega)$ of locally finite measures on Ω if $p = 2$. In view of the relation $\frac{1}{q} + \frac{2}{p} = 1$ this implies that

$$q(x, u_r) \underset{r\to\infty}{\to} \sum_{\alpha,\beta=1}^{N} q_{\alpha\beta}(x)\zeta_{\alpha\beta}(x)$$

weakly in $M_{loc}(\Omega)$ (weakly in $L^1_{loc}(\Omega)$ if $p > 2$) and therefore

$$v(x) = \sum_{\alpha,\beta=1}^{N} q_{\alpha\beta}(x)\zeta_{\alpha\beta}(x).$$

In particular, $v = v(x) \in L^1_{loc}(\Omega)$ for $p > 2$ and $v \in M_{loc}(\Omega)$ for $p = 2$.

Our main result is the following

Theorem 1. *Assume that $q(x, \lambda) \geq 0$ for all $\lambda \in \Lambda(x)$, $x \in \Omega$. Then $q(x, u(x)) \leq v$ (in the sense of measures).*

2 Main concepts

To prove Theorem 1 we will use the techniques of H-measures. Let

$$F(u)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot x} u(x)dx, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transformation extended as a unitary operator on the space $u(x) \in L^2(\mathbb{R}^n)$, let $S = S^{n-1} = \{ \ \xi \in \mathbb{R} \ \mid \ |\xi| = 1 \ \}$ be the unit sphere in $\mathbb{R}^n$.

The concept of an H-measure corresponding to some sequence of vector-valued functions bounded in $L^2(\Omega)$ was introduced by Tartar [10] and Gerárd [3] on the basis of the following result. For $r \in \mathbb{N}$ let $U_r(x) = \big(U_r^1(x), \dots, U_r^N(x)\big) \in L^2(\Omega, \mathbb{R}^N)$ be a sequence weakly convergent to the zero vector.

Proposition 1 (see [10, Theorem 1.1]). *There exists a family of complex Borel measures $\mu = \left\{\mu^{\alpha\beta}\right\}_{\alpha,\beta=1}^{N}$ in $\Omega\times S$ and a subsequence of $U_r(x)$ (still denoted U_r) such that*

$$\langle \mu^{\alpha\beta}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(U_r^\alpha \Phi_1)(\xi)\overline{F(U_r^\beta \Phi_2)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi \quad (4)$$

for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$.

The family $\mu = \left\{\mu^{\alpha\beta}\right\}_{\alpha,\beta=1}^{N}$ is called the H-measure corresponding to $U_r(x)$.

In [1] the new concept of parabolic H-measures was introduced. Here we need the more general variant of this concept recently developed in [5]. Suppose that $X \subset \mathbb{R}^n$ is a linear subspace, $X^\perp$ is its orthogonal complement, P_1, P_2 are orthogonal projections on X, $X^\perp$, respectively. We denote for $\xi \in \mathbb{R}^n$ $\tilde{\xi} = P_1\xi$, $\bar{\xi} = P_2\xi$, so that $\tilde{\xi} \in X$, $\bar{\xi} \in X^\perp$, $\xi = \tilde{\xi} + \bar{\xi}$. Let $S_X = \{ \xi \in \mathbb{R}^n \mid |\tilde{\xi}|^2 + |\bar{\xi}|^4 = 1 \}$. Then S_X is a compact smooth manifold of codimension 1; in the case when $X = \{0\}$ or $X = \mathbb{R}^n$, it coincides with the unit sphere $S = \{\xi \in \mathbb{R}^n \mid |\xi| = 1 \}$. Let us define a projection $\pi_X : \mathbb{R}^n \setminus \{0\} \to S_X$ by

$$\pi_X(\xi) = \frac{\tilde{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} + \frac{\bar{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}}.$$

Remark that in the case when $X = \{0\}$ or $X = \mathbb{R}^n$, $\pi_X(\xi) = \xi/|\xi|$ is the orthogonal projection on the sphere. We denote $p(\xi) = (|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}$. The following useful property of the projection π_X holds (see [5, Lemma 1]).

Lemma 1. *Let $\xi, \eta \in \mathbb{R}^n$, $\max(p(\xi), p(\eta)) \geq 1$. Then*

$$|\pi_X(\xi) - \pi_X(\eta)| \leq \frac{6|\xi - \eta|}{\max(p(\xi), p(\eta))}.$$

Proof. For $\xi \in \mathbb{R}^n$, $\alpha > 0$, we define $\xi_\alpha = \alpha^2\tilde{\xi} + \alpha\bar{\xi}$. Observe that for all $\alpha > 0$, $\pi_X(\xi_\alpha) = \pi_X(\xi)$. Without loss of generality we may suppose that $p(\xi) \geq p(\eta)$, and in particular $p(\xi) \geq 1$. Remark that $\pi_X(\xi) = \xi_\alpha$, $\pi_X(\eta) = \eta_\beta$, where $\alpha = 1/p(\xi)$, $\beta = 1/p(\eta)$. Therefore,

$$\begin{aligned} |\pi_X(\xi) - \pi_X(\eta)| = |\xi_\alpha - \eta_\beta| \leq |\xi_\alpha - \eta_\alpha| + |\eta_\alpha - \eta_\beta| = \\ \left(\alpha^4|\tilde{\xi} - \tilde{\eta}|^2 + \alpha^2|\bar{\xi} - \bar{\eta}|^2\right)^{1/2} + \left((\beta^2 - \alpha^2)^2|\tilde{\eta}|^2 + (\beta - \alpha)^2|\bar{\eta}|^2\right)^{1/2} \leq \\ \alpha|\xi - \eta| + (\beta - \alpha)\left((\beta + \alpha)^2|\tilde{\eta}|^2 + |\bar{\eta}|^2\right)^{1/2}. \end{aligned} \tag{5}$$

Here we take into account that $\alpha \leq 1$ and therefore $\alpha^4 \leq \alpha^2$. Since

$$(\beta + \alpha)^2 \leq 4\beta^2 = 4(|\tilde{\eta}|^2 + |\bar{\eta}|^4)^{-1/2} \leq 4/|\tilde{\eta}|,$$

we have the estimate

$$(\beta + \alpha)^2|\tilde{\eta}|^2 + |\bar{\eta}|^2 \leq 4(|\tilde{\eta}| + |\bar{\eta}|^2) \leq 4\left(2(|\tilde{\eta}|^2 + |\bar{\eta}|^4)\right)^{1/2} \leq 6(p(\eta))^2. \tag{6}$$

Concerning the term $\beta - \alpha$, we estimate it as follows

$$\begin{aligned} \beta - \alpha = \frac{p(\xi) - p(\eta)}{p(\xi)p(\eta)} = \frac{p(\xi)^4 - p(\eta)^4}{p(\xi)p(\eta)(p(\xi) + p(\eta))((p(\xi))^2 + (p(\eta))^2)} = \\ \frac{|\tilde{\xi}|^2 - |\tilde{\eta}|^2 + |\bar{\xi}|^4 - |\bar{\eta}|^4}{p(\xi)p(\eta)(p(\xi) + p(\eta))((p(\xi))^2 + (p(\eta))^2)} \leq \end{aligned}$$

$$\frac{(|\tilde{\xi}|+|\tilde{\eta}|)|\tilde{\xi}-\tilde{\eta}|+(|\bar{\xi}|+|\bar{\eta}|)(|\bar{\xi}|^2+|\bar{\eta}|^2)|\bar{\xi}-\bar{\eta}|}{p(\xi)p(\eta)(p(\xi)+p(\eta))((p(\xi))^2+(p(\eta))^2)} \leq$$
$$\frac{|\tilde{\xi}|+|\tilde{\eta}|+(|\bar{\xi}|+|\bar{\eta}|)(|\bar{\xi}|^2+|\bar{\eta}|^2)}{p(\xi)p(\eta)(p(\xi)+p(\eta))((p(\xi))^2+(p(\eta))^2)}|\xi-\eta| \leq$$
$$\frac{(p(\xi))^2+(p(\eta))^2+(p(\xi)+p(\eta))((p(\xi))^2+(p(\eta))^2)}{p(\xi)p(\eta)(p(\xi)+p(\eta))((p(\xi))^2+(p(\eta))^2)}|\xi-\eta| \leq$$
$$\frac{1+p(\xi)+p(\eta)}{p(\xi)+p(\eta)}\frac{|\xi-\eta|}{p(\xi)p(\eta)} \leq \frac{2|\xi-\eta|}{p(\xi)p(\eta)}. \tag{7}$$

Here we use that $\tilde{\xi} \leq (p(\xi))^2$, $\bar{\xi} \leq p(\xi)$, $\tilde{\eta} \leq (p(\eta))^2$, $\bar{\eta} \leq p(\eta)$, and that $p(\xi) + p(\eta) \geq 1$. Now it follows from (5), (6), (7) that

$$|\pi_X(\xi)-\pi_X(\eta)| \leq \frac{|\xi-\eta|}{p(\xi)} + \frac{2\sqrt{6}|\xi-\eta|}{p(\xi)} \leq \frac{6|\xi-\eta|}{p(\xi)} = \frac{6|\xi-\eta|}{\max(p(\xi),p(\eta))},$$

as was to be proved.□

Let $b(x) \in C_0(\mathbb{R}^n)$, $a(z) \in C(S_X)$. We introduce the pseudo-differential operators $\mathcal{B}, \mathcal{A}$ with symbols $b(x)$, $a(\pi_X(\xi))$, respectively. These operators are multiplication operators $\mathcal{B}u(x) = b(x)u(x)$, $F(\mathcal{A}u)(\xi) = a(\pi_X(\xi))F(u)(\xi)$. Obviously, the operators $\mathcal{B}, \mathcal{A}$ are well-defined and bounded in L^2. As was proved in [10], in the case when $S_X = S$, $\pi_X(\xi) = \xi/|\xi|$ the commutator $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ is a compact operator. In [5], using the assertion of Lemma 1, we extend this result for the general case (in the case $\dim X = 1$ this was done in [1]). For completeness we give the details below.

Lemma 2. *The operator* $[\mathcal{A}, \mathcal{B}]$ *is compact in* L^2.

Proof. We can find sequences $a_k(z) \in C^\infty(S_X)$, $b_k(x) \in C^\infty(\mathbb{R}^n)$, $k \in \mathbb{N}$ with the following properties: $F(b_k)(\xi) \in C_0^\infty(\mathbb{R}^n)$, and $a_k(z) \to a(z)$, $b_k(x) \to b(x)$ as $k \to \infty$ uniformly on S_X, $\mathbb{R}^n$, respectively. Then the sequences of the operators $\mathcal{A}_k$, $\mathcal{B}_k$ with symbols $a_k(\pi_X(\xi))$, $b_k(x)$ converge as $k \to \infty$ to the operators $\mathcal{A}$, $\mathcal{B}$, respectively (in the operator norm). Therefore, $[\mathcal{A}_k, \mathcal{B}_k] \underset{k\to\infty}{\to} [\mathcal{A}, \mathcal{B}]$ and it is sufficient to prove that the operators $[\mathcal{A}_k, \mathcal{B}_k]$ are compact for all $k \in \mathbb{N}$ (then $[\mathcal{A}, \mathcal{B}]$ is a compact operator as a limit of compact operators). Let $u = u(x) \in L^2(\mathbb{R}^n)$. Then by the known property $F(bu)(\xi) = F(b) * F(u)(\xi) = \int F(b)(\xi-\eta)F(u)(\eta)d\eta$,

$$F([\mathcal{A}_k, \mathcal{B}_k]u)(\xi) = F(\mathcal{A}_k\mathcal{B}_k u)(\xi) - F(\mathcal{B}_k\mathcal{A}_k u)(\xi) =$$
$$a_k(\pi_X(\xi))F(b_k u)(\xi) - F(b_k\mathcal{A}_k u)(\xi) =$$
$$\int_{\mathbb{R}^n} (a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))F(b_k)(\xi-\eta)F(u)(\eta)d\eta.$$

We have to prove that the integral operator $Kv(\xi) = \int_{\mathbb{R}^n} k(\xi,\eta)v(\eta)d\eta$ with the kernel $k(\xi,\eta) = (a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))F(b_k)(\xi-\eta)$ is compact on $L^2(\mathbb{R}^n)$.

Since $a_k \in C^\infty(S_X)$ then by Lemma 1

$$|a_k(\pi_X(\xi)) - a_k(\pi_X(\eta))| \le C\frac{|\xi-\eta|}{\max(p(\xi),p(\eta))}$$

for $\max(p(\xi),p(\eta)) \ge 1$, where $C = \text{const}$. Thus for all $\xi,\eta \in \mathbb{R}^n$ such that $\max(p(\xi),p(\eta)) > m > 1$

$$|a_k(\pi_X(\xi)) - a_k(\pi_X(\eta))| \le \frac{C}{m}|\xi-\eta|. \tag{8}$$

Let $\chi_m(\xi,\eta)$ be the indicator function of the set $\{\ (\xi,\eta)\in\mathbb{R}^{2n} \mid \max(p(\xi),p(\eta)) \le m\ \}$, and

$$k_m(\xi,\eta) = \chi_m(\xi,\eta)(a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))F(b_k)(\xi-\eta),$$
$$r_m(\xi,\eta) = (1-\chi_m(\xi,\eta))(a_k(\pi_X(\xi)) - a_k(\pi_X(\eta)))F(b_k)(\xi-\eta).$$

Then $k(\xi,\eta) = k_m(\xi,\eta) + r_m(\xi,\eta)$ and $K = K_m + R_m$, where K_m, R_m are integral operators with the kernels $k_m(\xi,\eta)$, $r_m(\xi,\eta)$, respectively. Since the function $k_m(\xi,\eta)$ is bounded and compactly supported then the operator K_m is a Hilbert-Schmidt operator, which is compact. On the other hand, in view of (8)

$$|R_m v(\xi)| \le \frac{C}{m}\int_{\mathbb{R}^n} |(\xi-\eta)F(b_k)(\xi-\eta)||v(\eta)|d\eta = [|\xi F(b_k)| * |v|](\xi)$$

and, by the Young inequality, for every $v \in L^2(\mathbb{R}^n)$

$$\|R_m v\|_2 \le \frac{C}{m}\|\xi F(b_k)\|_1 \|v\|_2.$$

Therefore, $\|R_m\| \le \text{const}/m$ and $R_m \to 0$ as $m \to \infty$. We conclude that $K_m \to K$ and therefore K is a compact operator, as a limit of compact operators. This completes the proof. $\square$

The ultra-parabolic H-measure $\mu^{\alpha\beta}$, $\alpha,\beta = 1,\dots,N$ corresponding to a subspace $X \subset \mathbb{R}^n$ and a sequence $U_r(x) \in L^2(\Omega,\mathbb{R}^N)$, weakly convergent to the zero vector, is defined on $\Omega\times S_X$ by the relation similar to (4): $\forall \Phi_1(x),\Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$

$$\langle \mu^{\alpha\beta}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\psi(\pi_X(\xi))d\xi. \tag{9}$$

The existence of the H-measure $\mu^{\alpha\beta}$ is proved exactly in the same way as in [10], using the statement of Lemma 2. For completeness we give the details below.

Proposition 2. *There exist a family of complex Borel measures* $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$ *in* $\Omega \times S_X$ *and a subsequence of* $U_r(x)$ *(still denoted by* U_r*) such that relation (9) holds for all* $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$. *Besides, the matrix-valued measure* μ *is Hermitian and positive definite, that is, for each* $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^n$ *the measure* $\mu\zeta \cdot \zeta = \sum_{\alpha,\beta=1}^N \mu^{\alpha\beta}\zeta_\alpha\overline{\zeta_\beta} \geq 0$.

Proof. Denote

$$I_r^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\psi(\pi_X(\xi))d\xi$$

and observe that, by the Buniakovskii inequality and the Plancherel identity,

$$|I_r^{\alpha\beta}| \leq \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty \cdot \|U_r^\alpha\|_{L^2(K)} \|U_r^\beta\|_{L^2(K)},$$

where $K \subset \Omega$ is a compact containing supports of Φ_1 and Φ_2. In view of the weak convergence of sequences U_r^α in $L^2(K)$ these sequences are bounded in $L^2(K)$. Therefore, for some constant C_K we have $\|U_r^\alpha\|_{L^2(K)}^2 \leq C_K$ for all $r \in \mathbb{N}$, $\alpha = 1, \dots, N$. Hence,

$$|I_r^{\alpha\beta}(\Phi_1, \Phi_2, \psi)| \leq C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty \tag{10}$$

and the sequences $I_r^{\alpha\beta}$ are bounded. Let D be a countable dense set in $(C_0(\Omega))^2 \times C(S_X)$. Using the standard diagonal process, we can extract a subsequence U_r (we keep the notation U_r for this subsequence) such that

$$I_r^{\alpha\beta}(\Phi_1, \Phi_2, \psi) \underset{r\to\infty}{\to} I^{\alpha\beta}(\Phi_1, \Phi_2, \psi) \tag{11}$$

for all triples $(\Phi_1, \Phi_2, \psi) \in D$. By estimate (10) we see that sequences $I_r^{\alpha\beta}(\Phi_1, \Phi_2, \psi)$ are uniformly continuous with respect to $(\Phi_1, \Phi_2, \psi) \in (C_0(\Omega))^2 \times C(S_X)$ and since D is dense in $(C_0(\Omega))^2 \times C(S_X)$, we conclude that limit relation (11) holds for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$. Passing in (10) to the limit as $r \to \infty$, we derive that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$

$$|I^{\alpha\beta}(\Phi_1, \Phi_2, \psi)| \leq C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty, \tag{12}$$

with $K = \operatorname{supp}\Phi_1 \cup \operatorname{supp}\Phi_2$. Now, we observe that

$$I_r^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = (\Phi_1 U_r^\alpha, \mathcal{A}(\Phi_2 U_r^\beta))_2, \tag{13}$$

where $\mathcal{A}$ is a pseudo-differential operator on $L^2 = L^2(\mathbb{R}^n)$ with symbol $\overline{\psi(\pi_X(\xi))}$, and $(\cdot,\cdot)_2$ is the scalar product in L^2. Let $\mathcal{B}$ be a pseudo-differential operator on L^2 with symbol $\Phi_2(x)$, and let $\omega(x) \in C_0(\mathbb{R}^n)$ be a function such that $\omega(x) \equiv 1$ on $\operatorname{supp}\Phi_2$. Then

$$\mathcal{A}(\Phi_2 U_r^\beta) = \mathcal{A}\mathcal{B}(\omega U_r^\beta) = \mathcal{B}\mathcal{A}(\omega U_r^\beta) + [\mathcal{A},\mathcal{B}](\omega U_r^\beta). \tag{14}$$

Since $\omega U_r^\beta \to 0$ as $r \to \infty$ weakly in L^2 while, by Lemma 2, the operator $[\mathcal{A},\mathcal{B}]$ is compact on L^2, we claim that $[\mathcal{A},\mathcal{B}](\omega U_r^\beta) \to 0$ as $r \to \infty$ strongly in L^2. Since the sequence $\Phi_1 U_r^\alpha$ is bounded in L^2, we conclude that $(\Phi_1 U_r^\alpha, [\mathcal{A},\mathcal{B}](\omega U_r^\beta))_2 \to 0$ as $r \to \infty$. It follows from this limit relation and (13), (14) that

$$\lim_{r\to\infty} (\Phi_1 U_r^\alpha, \mathcal{B}\mathcal{A}(\omega U_r^\beta))_2 = \lim_{r\to\infty} I_r^{\alpha\beta}(\Phi_1,\Phi_2,\psi) = I^{\alpha\beta}(\Phi_1,\Phi_2,\psi).$$

Taking into account that

$$(\Phi_1 U_r^\alpha, \mathcal{B}\mathcal{A}(\omega U_r^\beta))_2 = \int_{\mathbb{R}^n} \Phi_1(x)\overline{\Phi_2(x)} U_r^\alpha(x)\overline{\mathcal{A}(\omega U_r^\beta)(x)}dx,$$

we find that

$$I^{\alpha\beta}(\Phi_1,\Phi_2,\psi) = \tilde{I}^{\alpha\beta}(\Phi_1\overline{\Phi_2},\psi),$$

where $\tilde{I}^{\alpha\beta}(\Phi,\psi)$ is a bilinear functional on $C_0(\Omega)\times C(S_X)$ for each $\alpha,\beta = 1,\dots,N$. Taking in the above relation $\Phi_1 = \Phi(x)/\sqrt{|\Phi(x)|}$ (we set $\Phi_1(x) = 0$ if $\Phi(x) = 0$), $\Phi_2 = \sqrt{|\Phi(x)|}$, where $\Phi(x) \in C_0(\Omega)$, we find with the help of (12) that

$$\begin{aligned} |\tilde{I}^{\alpha\beta}(\Phi,\psi)| = |I^{\alpha\beta}(\Phi_1,\Phi_2,\psi)| &\le C_K\|\Phi_1\|_\infty\|\Phi_2\|_\infty\|\psi\|_\infty \\ &= C_K\|\Phi\|_\infty\|\psi\|_\infty,\ K = \operatorname{supp}\Phi. \end{aligned}$$

This estimate shows that the functionals $\tilde{I}^{\alpha\beta}(\Phi,\psi)$ are continuous on $C_0(\Omega)\times C(S_X)$. Now, we observe that for nonnegative $\Phi(x)$ and $\psi(\xi)$ the matrix $\tilde{I} \doteq \{\tilde{I}^{\alpha\beta}(\Phi,\psi)\}_{\alpha,\beta=1}^N$ is Hermitian and positive definite. Indeed, taking $\Phi_1(x) = \Phi_2(x) = \sqrt{\Phi(x)}$, we find

$$\begin{gathered} \tilde{I}^{\alpha\beta}(\Phi,\psi) = I^{\alpha\beta}(\Phi_1,\Phi_1,\psi) = \\ \lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_1 U_r^\beta)(\xi)}\psi(\pi_X(\xi))d\xi. \end{gathered} \tag{15}$$

For $\zeta = (\zeta_1,\dots,\zeta_N) \in \mathbb{C}^N$ we have, in view of (15),

$$\tilde{I}\zeta\cdot\zeta = \sum_{\alpha,\beta=1}^N \tilde{I}^{\alpha\beta}(\Phi,\psi)\zeta_\alpha\overline{\zeta_\beta} = \lim_{r\to\infty}\int_{\mathbb{R}^n} |F(\Phi_1 V_r)(\xi)|^2\psi(\pi_X(\xi))d\xi \ge 0,$$

where $V_r(x) = \sum\limits_{\alpha=1}^{N} U_r^\alpha \zeta_\alpha$. The above relation proves that the matrix $\tilde{I}$ is Hermitian and positive definite.

We see that for any $\zeta \in \mathbb{C}^n$ the bilinear functional $\tilde{I}(\Phi,\psi)\zeta\cdot\zeta$ is continuous on $C_0(\Omega)\times C(S_X)$ and nonnegative, that is, $\tilde{I}(\Phi,\psi)\zeta\cdot\zeta \geq 0$ whenever $\Phi(x)\geq 0$, $\psi(\xi)\geq 0$. It is rather well known (see for example [10, Lemma 1.10]), that such a functional is represented by integration over some unique locally finite non-negative Borel measure $\mu = \mu_\zeta(x,\xi) \in M_{loc}(\Omega\times S_X)$:

$$\tilde{I}(\Phi,\psi)\zeta\cdot\zeta = \int_{\Omega\times S_X} \Phi(x)\psi(\xi)d\mu_\zeta(x,\xi).$$

As a function of the vector ζ, μ_ζ is a measure valued Hermitian form. Therefore,

$$\mu_\zeta = \sum_{\alpha,\beta=1}^{N} \mu^{\alpha\beta}\zeta_\alpha\overline{\zeta_\beta} \tag{16}$$

with measure valued coefficients $\mu^{\alpha\beta}\in M_{loc}(\Omega\times S_X)$, which can be expressed as follows

$$\mu^{\alpha\beta} = [\mu_{e_\alpha+e_\beta} + i\mu_{e_\alpha+ie_\beta}]/2 - (1+i)(\mu_{e_\alpha}+\mu_{e_\beta})/2,$$

where $e_1,\dots,e_N$ is the standard basis in $\mathbb{C}^N$, and $i^2=-1$.

By (16)

$$\tilde{I}(\Phi,\psi)\zeta\cdot\zeta = \sum_{\alpha,\beta=1}^{l} \langle \mu^{\alpha\beta}, \Phi(x)\psi(\xi)\rangle \zeta_\alpha\overline{\zeta_\beta}$$

and since

$$\tilde{I}(\Phi,\psi)\zeta\cdot\zeta = \sum_{\alpha,\beta=1}^{l} \tilde{I}^{\alpha\beta}(\Phi,\psi)\zeta_\alpha\overline{\zeta_\beta},$$

then, comparing the coefficients, we find that

$$\langle \mu^{\alpha\beta}, \Phi(x)\psi(\xi)\rangle = \tilde{I}^{\alpha\beta}(\Phi,\psi). \tag{17}$$

In particular,

$$\langle \mu^{\alpha\beta}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = I^{\alpha\beta}(\Phi_1,\Phi_2,\psi) =$$
$$\lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\psi(\pi_X(\xi))d\xi.$$

To complete the proof, observe that for each $\zeta\in\mathbb{C}^N$ the measure

$$\sum_{\alpha,\beta=1}^{N} \mu^{\alpha\beta}\zeta_\alpha\overline{\zeta_\beta} = \mu_\zeta \geq 0.$$

Hence, μ is Hermitian and positive definite. $\square$

As it follows from the above Proposition, the matrix with component $\langle \mu^{\alpha\beta}, g(x,\xi)\rangle$ is Hermitian and positive definite for each real nonnegative $g(x,\xi) \in C_0(\Omega \times S_X)$.

Remark 1. We can replace the function $\psi(\pi_X(\xi))$ in relation (9) by a function $\tilde{\psi}(\xi) \in C(\mathbb{R}^n)$, which equals $\psi(\pi_X(\xi))$ for large $|\xi|$. Indeed, since $\Phi_2(x)$ is a function with compact support, $\Phi_2 U_r^\beta \underset{r\to\infty}{\to} 0$ weakly in $L^2(\mathbb{R}^n)$ as well as in $L^1(\mathbb{R}^n)$. Therefore, $F(\Phi_2 U_r^\beta)(\xi) \underset{r\to\infty}{\to} 0$ point-wise and in $L^2_{loc}(\mathbb{R}^n)$ (in view of the bound $|F(\Phi_2 U_r^\beta)(\xi)| \le \|\Phi_2 U_r^\beta\|_1 \le \text{const}$). Taking into account that the function $\chi(\xi) = \tilde{\psi}(\xi) - \psi(\pi_X(\xi))$ is bounded and has a compact support, we conclude that

$$\overline{F(\Phi_2 U_r^\beta)(\xi)}\chi(\xi) \underset{r\to\infty}{\to} 0 \quad \text{in } L^2(\mathbb{R}^n).$$

This implies that

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\chi(\xi) d\xi = 0.$$

Therefore,

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\tilde{\psi}(\xi) d\xi =$$
$$\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\psi(\pi_X(\xi)) d\xi = \langle \mu^{\alpha\beta}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle,$$

as required.

Let the sequence $U_r = \{U_r^\alpha\}_{\alpha=1}^N$ converges weakly as $r \to \infty$ to the zero vector, let it be bounded in $L^p_{loc}(\Omega, \mathbb{R}^N)$, $p \ge 2$, and let $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$ be an ultra-parabolic H-measure corresponding to this sequence. We define $\eta = \mathrm{Tr}\mu = \sum_{\alpha=1}^N \mu^{\alpha\alpha}$. As follows from Proposition 2, η is a locally finite non-negative measure on $\Omega \times S_X$. We assume that this measure is extended on σ-algebra of η-measurable sets, and in particular that this measure is complete. We denote by γ the projection of η on Ω, that is, $\gamma(A) = \eta(A \times S_X)$ if the set $A \times S_X$ is η-measurable. Obviously, γ is a complete locally finite measure on Ω, $\gamma \ge 0$. Under the above assumptions the following statements hold.

Proposition 3.
(i) As $r \to \infty$

$$|U_r|^2 = \sum_{\alpha=1}^N |U_r^\alpha(x)|^2 \to \gamma$$

weakly in $M_{loc}(\Omega)$; if $p > 2$ then $\gamma \in L^{p/2}_{loc}(\Omega)$ (here we identify γ and the corresponding density $\tilde{\gamma}$ of γ with respect to the Lebesgue measure dx, so that $\gamma = \tilde{\gamma}(x)dx$), and $|U_r|^2 \to \gamma(x)$ weakly in $L^{p/2}_{loc}(\Omega)$;

(ii) The H-measure μ is absolutely continuous with respect to η, more precisely, $\mu = H(x,\xi)\eta$, where $H(x,\xi) = \{h^{\alpha\beta}(x,\xi)\}_{\alpha,\beta=1}^N$ is a bounded η-measurable function taking values in the cone of positive definite Hermitian $N \times N$ matrices, besides $|h^{\alpha\beta}(x,\xi)| \le 1$.

Proof. By the Plancherel identity and relation (9) with $\psi \equiv 1$

$$\int_\Omega \Phi_1(x)\overline{\Phi_2(x)}|U_r|^2 dx = \sum_{\alpha=1}^N \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\alpha)(\xi)}d\xi$$
$$\underset{r\to\infty}{\to} \langle \eta(x,\xi), \Phi_1(x)\overline{\Phi_2(x)}\rangle = \langle \gamma, \Phi_1(x)\overline{\Phi_2(x)}\rangle$$

Since any function $\Phi(x) \in C_0(\Omega)$ can be represented in the form $\Phi(x) = \Phi_1(x)\overline{\Phi_2(x)}$ (for instance, one can take $\Phi_1(x) = \Phi(x)$, $\Phi_2(x)$ being arbitrary function in $C_0(\Omega)$ equal to 1 on $\operatorname{supp}\Phi_1(x)$), we conclude that $|U_r|^2 \to \gamma$ as $r \to \infty$ weakly in $M_{loc}(\Omega)$. In the case $p > 2$ (here $p/2 = \infty$ if $p = \infty$) the sequence $|U_r|^2$ is bounded in $L^{p/2}_{loc}(\Omega)$, and we conclude that $\gamma \in L^{p/2}_{loc}(\Omega)$. The first assertion is proved.

To prove (ii), remark firstly that $\mu^{\alpha\alpha} \le \eta$ for all $\alpha = 1, \dots, N$. Now, suppose that $\alpha, \beta \in \{1, \dots, N\}$, $\alpha \ne \beta$. By Proposition 2 for any compact set $B \subset \Omega \times S_X$ the matrix

$$\begin{pmatrix} \mu^{\alpha\alpha}(B) & \mu^{\alpha\beta}(B) \\ \overline{\mu^{\alpha\beta}(B)} & \mu^{\beta\beta}(B) \end{pmatrix}$$

is positive-definite; in particular,

$$|\mu^{\alpha\beta}(B)| \le \left(\mu^{\alpha\alpha}(B)\mu^{\beta\beta}(B)\right)^{1/2} \le \eta(B).$$

By regularity of measures $\mu^{\alpha\beta}$ and η this estimate is satisfied for all Borel sets B. This easily implies the inequality $\operatorname{Var}\mu^{\alpha\beta} \le \eta$. In particular, the measures $\mu^{\alpha\beta}$ are absolutely continuous with respect to η, and by the Radon-Nykodim theorem $\mu^{\alpha\beta} = h^{\alpha\beta}(x,\xi)\eta$, where the densities $h^{\alpha\beta}(x,\xi)$ are η-measurable and, as follows from the inequalities $\operatorname{Var}\mu^{\alpha\beta} \le \eta$, $|h^{\alpha\beta}(x,\xi)| \le 1$ η-a.e. on $\Omega \times S_X$. We denote by $H(x,\xi)$ the matrix with components $h^{\alpha\beta}(x,\xi)$. Recall that the H-measure μ is positive definite. This means that for all $\zeta \in \mathbb{C}^N$

$$\mu\zeta \cdot \zeta = H(x,\xi)\zeta \cdot \zeta\eta \ge 0. \tag{18}$$

Hence $H(x,\xi)\zeta \cdot \zeta \ge 0$ for η-a.e. $(x,\xi) \in \Omega \times S_X$. Choose a countable dense set $E \subset \mathbb{C}^N$. Since E is countable, then it follows from (18) that for a set

$(x,\xi)\in\Omega\times S_X$ of full η-measure $H(x,\xi)\zeta\cdot\zeta\ge 0\ \forall\zeta\in E$, and since E is dense we conclude that actually $H(x,\xi)\zeta\cdot\zeta\ge 0$ for all $\zeta\in\mathbb{C}^N$. Thus, the matrix $H(x,\xi)$ is Hermitian and positive definite for η-a.e. (x,ξ). After an appropriate correction on a set of null η-measure, we can assume that the above property is satisfied for all $(x,\xi)\in\Omega\times S_X$, and also $|h^{\alpha\beta}(x,\xi)|\le 1$ for all $(x,\xi)\in\Omega\times S_X$, $\alpha,\beta=1,\dots,N$. The proof is complete. $\square$

Corollary 1. *Suppose that the sequence $U_r=\{U_r^\alpha\}_{\alpha=1}^N$ is bounded in $L^p_{loc}(\Omega,\mathbb{R}^N)$, $p>2$. Let $q=p/(p-2)$ (as usual we set $q=1$ if $p=\infty$), and let $L^{2q}_0(\Omega)$ be the space of functions in $L^{2q}(\Omega)$ having compact supports. Then relation (9) still holds for all functions $\Phi_1(x),\Phi_2(x)\in L^{2q}_0(\Omega)$, $\psi(\xi)\in C(S_X)$.*

Proof. Let K be a compact subset of Ω and $\Phi_1(x),\Phi(x)\in L^{2q}(K)$. The functions from $L^{2q}(K)$ are supposed to be extended on Ω as zero functions outside of K. Using the Plancherel identity and the Hölder inequality (observe that $\frac{1}{2q}+\frac{1}{p}=\frac{1}{2}$), we get the following estimate

$$\left|\int_{\mathbb{R}^n}F(\Phi_1U_r^\alpha)(\xi)\overline{F(\Phi_2U_r^\beta)(\xi)}\psi(\pi_X(\xi))d\xi\right|$$
$$\le\|\psi\|_\infty\|\Phi_1U_r^\alpha\|_2\|\Phi_2U_r^\beta\|_2\le(C_K)^2\|\psi\|_\infty\cdot\|\Phi_1\|_{2q}\|\Phi_2\|_{2q},\tag{19}$$

where $C_K=\sup\limits_{r\in\mathbb{N}}\|U_r\|_{L^p(K)}$. On the other hand, by Proposition 3

$$|\langle\mu^{\alpha\beta},\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle|=|\langle\eta,h^{\alpha\beta}(x,\xi)\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle|$$
$$\le\|\psi\|_\infty\int_\Omega|\Phi_1(x)\Phi_2(x)|\gamma(x)dx\le\|\psi\|_\infty\|\gamma\|_{L^{p/2}(K)}\|\Phi_1\|_{2q}\|\Phi_2\|_{2q}\tag{20}$$

(in the last estimate we used again the Hölder inequality). Estimates (19), (20) show that both sides of relation (9) are continuous with respect to $(\Phi_1,\Phi_2)\in(L^{2q}(K))^2$. Since (9) holds for $\Phi_1,\Phi_2\in C_0(K)$ and the space $C_0(K)$ is dense in $L^{2q}(K)$, we claim that (9) holds for each $\Phi_1(x),\Phi_2(x)\in L^{2q}(K)$. To conclude the proof, it only remains to notice that K is an arbitrary compact subset of Ω. $\square$

We will need in the sequel some results about Fourier multipliers in spaces L^d, $d>1$. Recall that a function $a(\xi)\in L^\infty(\mathbb{R}^n)$ is a Fourier multiplier in L^d if the pseudo-differential operator $\mathcal{A}$ with the symbol $a(\xi)$, defined as $F(\mathcal{A}u)(\xi)=a(\xi)F(u)(\xi)$, $u=u(x)\in L^2(\mathbb{R}^n)\cap L^d(\mathbb{R}^n)$ can be extended as a bounded operator on $L^d(\mathbb{R}^n)$, that is

$$\|\mathcal{A}u\|_d\le C\|u\|_d\ \forall u\in L^2(\mathbb{R}^n)\cap L^d(\mathbb{R}^n),\quad C=\mathrm{const}.$$

We denote by M_d the space of Fourier multipliers in L^d. We also denote

$$\dot{\mathbb{R}}^n = (\mathbb{R} \setminus \{0\})^n = \{ \ \xi = (\xi_1, \ldots, \xi_n) \mid \prod_{k=1}^{n} \xi_k \neq 0 \ \}.$$

The following statement readily follows from the Marcinkiewicz multiplier theorem (see [8, Chapter 4]).

Theorem 2. *Suppose that $a(\xi) \in C^n(\dot{\mathbb{R}}^n)$ is a function such that for some constant C*

$$|\xi^\alpha D^\alpha a(\xi)| \leq C \quad \forall \xi \in \dot{\mathbb{R}}^n \tag{21}$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq n$. Then $a(\xi) \in M_d$ for all $d > 1$.

Here we use the standard notations $\xi^\alpha = \prod_{k=1}^n (\xi_k)^{\alpha_k}$, $D^\alpha = \displaystyle\prod_{k=1}^{n} \left(\frac{\partial}{\partial \xi_k} \right)^{\alpha_k}$. Actually (see [8]), it is sufficient to require that (21) is satisfied for multi-indexes α such that $\alpha_k \in \{0, 1\}$, $k = 1, \ldots, n$.

We also need the following simple lemma (see [5, Lemma 8]).

Lemma 3. *Let $h(y, z) \in C^n((\mathbb{R}^\nu \times \mathbb{R}^{n-\nu}) \setminus \{0\})$ be such that for some $k \in \mathbb{N}$, $\gamma \in \mathbb{R}$*

$$\forall t > 0 \quad h(t^k y, tz) = t^\gamma h(y, z). \tag{22}$$

Then there exists a constant $C > 0$ such that for each multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_\nu)$, $\beta = (\beta_1, \ldots, \beta_{n-\nu})$, $|\alpha| + |\beta| \leq n$ and all $y \in \mathbb{R}^\nu$, $z \in \mathbb{R}^{n-\nu}$, $y, z \neq 0$

$$|D_y^\alpha D_z^\beta h(y, z)| \leq C(|y|^2 + |z|^{2k})^{\frac{\gamma}{2k}} |y|^{-|\alpha|} |z|^{-|\beta|}.$$

Proof. In view of (22), for all $t > 0$ we have

$$D_y^\alpha D_z^\beta h(y, z) = t^{k|\alpha| + |\beta| - \gamma} (D_y^\alpha D_z^\beta h)(t^k y, tz).$$

Taking $t = (|y|^2 + |z|^{2k})^{-\frac{1}{2k}}$ in this relation, we find

$$D_y^\alpha D_z^\beta h(y, z) = (|y|^2 + |z|^{2k})^{\frac{\gamma - k|\alpha| - |\beta|}{2k}} (D_y^\alpha D_z^\beta h)(y', z'), \tag{23}$$

where $y' = t^k y$, $z' = tz$, so that $|y'|^2 + |z'|^{2k} = 1$. Since the set of such (y', z') is a compact subset of $\mathbb{R}^n \setminus \{0\}$ the derivatives $(D_y^\alpha D_z^\beta h)(y', z')$, $|\alpha| + |\beta| \leq n$, are bounded on this set, and relation (23) implies that for some constant $C > 0$

$$|D_y^\alpha D_z^\beta h(y, z)| \leq C(|y|^2 + |z|^{2k})^{\frac{\gamma}{2k}} (|y|^2 + |z|^{2k})^{-|\alpha|/2} (|y|^2 + |z|^{2k})^{-|\beta|/(2k)} \leq C(|y|^2 + |z|^{2k})^{\frac{\gamma}{2k}} |y|^{-|\alpha|} |z|^{-|\beta|}$$

for all $y, z \neq 0$. The proof is complete. $\square$

Now we can prove that some useful for us functions are Fourier multipliers. Namely, assume that X is a linear subspace of $\mathbb{R}^n$, and let $\pi_X : \mathbb{R}^n \to S_X$ be the projection defined in Section 2.

Proposition 4 (cf. [5, Proposition 6]). *The following functions are multipliers in spaces L^d for all $d > 1$:*

(i) $a_1(\xi) = \psi(\pi_X(\xi))$ *where* $\psi \in C^n(S_X)$;

(ii) $a_2(\xi) = \rho(\xi)(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2}$, *where* $\rho(\xi) \in C^\infty(\mathbb{R}^n)$ *is a function such that* $0 \leq \rho(\xi) \leq 1$, $\rho(\xi) = 0$ *for* $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \leq 1$, $\rho(\xi) = 1$ *for* $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \geq 2$;

(iii) $a_3(\xi) = (1 + |\xi|^2)^{1/2}(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2}$;

(iv) $a_4(\xi) = (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}(1 + |\xi|^2)^{-1}$.

Proof. Since the space M_d is invariant under non-degenerate linear transformations of the variables ξ (see [2, Chapter 6]) then we can assume that $X = \mathbb{R}^\nu = \{\xi \in \mathbb{R}^n \mid \xi = (y_1, \dots, y_\nu, 0, \dots, 0)\ \}$ while $X^\perp = \{\xi \in \mathbb{R}^n \mid \xi = (0, \dots, 0, z_1, \dots, z_{n-\nu})\ \}$. Since $\pi_X(t^2 y, tz) = \pi_X(y, z)$ for $t > 0$, $y \in X$, $z \in X^\perp$ then $h = a_1(\xi) = \psi(\pi_X(\xi))$ satisfies the assumptions of Lemma 3 with $k = 2$, $\gamma = 0$. By this Lemma for each multi-indexes α, β, $|\alpha| + |\beta| \leq n$

$$|y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta a_1(y, z)| \leq C = \text{const}.$$

This, in particular, implies that assumption (21) of Theorem 2 is satisfied. By this Theorem we conclude that $a_1(\xi) \in M_d$ for each $d > 1$.

To prove that $a_2(\xi) \in M_d$ we introduce the function $h_1(s, y, z) = (s^2 + |y|^2 + |z|^4)^{1/2}$, $s \in \mathbb{R}$. This function satisfies the assumptions of Lemma 3 with y replaced by $(s, y) \in \mathbb{R}^{\nu+1}$, and $k = \gamma = 2$. By this Lemma

$$|D_y^\alpha D_z^\beta h_1(s, y, z)| \leq C(s^2 + |y|^2 + |z|^4)^{1/2}|y|^{-|\alpha|}|z|^{-|\beta|},\ C = \text{const}.$$

Taking $s = 1$ in this relation, we arrive at the estimate

$$|D_y^\alpha D_z^\beta h_1(1, y, z)| \leq C(1 + |y|^2 + |z|^4)^{1/2}|y|^{-|\alpha|}|z|^{-|\beta|},$$

and by the Leibnitz formula we obtain that for each multi-indexes α, β such that $|\alpha| + |\beta| \leq n$

$$|D_y^\alpha D_z^\beta \rho(y, z) h_1(1, y, z)| \leq C_1(1 + |y|^2 + |z|^4)^{1/2}|y|^{-|\alpha|}|z|^{-|\beta|}, \tag{24}$$

$C_1 = \text{const}$ (we use that $\rho(y, z) = 1$ for $|y|^2 + |z|^4 \geq 2$). Let $h_2(y, z) = (|y|^2 + |z|^4)^{-1/2}$. This function satisfies (22) with $k = 2$, $\gamma = -2$. By Lemma 3 for some constant C_2 and every multi-indexes α, β such that $|\alpha| + |\beta| \leq n$

$$|D_y^\alpha D_z^\beta h_2(y, z)| \leq C_2(|y|^2 + |z|^4)^{-1/2}|y|^{-|\alpha|}|z|^{-|\beta|}. \tag{25}$$

By the Leibnitz formula we derive from (24), (25) the estimates

$$|D_y^\alpha D_z^\beta \rho(y,z)h_1(1,y,z)h_2(y,z)| \le$$
$$C_3(1+|y|^2+|z|^4)^{1/2}(|y|^2+|z|^4)^{-1/2}|y|^{-|\alpha|}|z|^{-|\beta|} \le 2C_3|y|^{-|\alpha|}|z|^{-|\beta|} \tag{26}$$

in the domain $|y|^2+|z|^4 \ge 1$, here $|\alpha|+|\beta| \le n$, $C_3 = \mathrm{const}$. In view of (26) we conclude that in this domain for each α, β, $|\alpha|+|\beta| \le n$

$$|y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta a_2(y,z)| \le \mathrm{const}.$$

Since $a_2(y,z) = 0$ for $|y|^2+|z|^4 < 1$ we see that the requirements of Theorem 2 are satisfied. Therefore, $a_2(\xi) \in M_d$ for all $d > 1$.

Now we introduce the functions $h_1(s,y,z) = (s^2+|y|^2+|z|^2)^{1/2}$, $h_2(s,y,z) = (s^2+|y|^2+|z|^2)^{-1}$, $h_3(s,y,z) = (s^2+|y|^2+|z|^4)^{-1/2}$, $h_4(s,y,z) = (s^2+|y|^2+|z|^4)^{1/2}$, $s \in \mathbb{R}$, $y \in X = \mathbb{R}^\nu$, $z \in X^\perp$. These functions satisfy (22) where y is replaced by $(s,y) \in \mathbb{R}^{l+1}$ with the parameters $k = \gamma = 1$; $k = 1, \gamma = -2$; $k = 2, \gamma = -2$; $k = \gamma = 2$, respectively. By Lemma 3 we find that for each α, β, $|\alpha|+|\beta| \le n$

$$|y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta h_1(1,y,z)| \le C(1+|y|^2+|z|^2)^{1/2},$$
$$|y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta h_2(1,y,z)| \le C(1+|y|^2+|z|^2)^{-1},$$
$$|y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta h_3(1,y,z)| \le C(1+|y|^2+|z|^4)^{-1/2},$$
$$|y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta h_4(1,y,z)| \le C(1+|y|^2+|z|^4)^{1/2},$$

where $C = \mathrm{const}$. Since $a_3(\xi) = h_1(1,y,z)h_3(1,y,z)$, $a_4(\xi) = h_2(1,y,z)h_4(1,y,z)$ where $y = \tilde{\xi}$, $z = \bar{\xi}$ then, using again the Leibnitz formula, we derive the estimates: for some constant C

$$|y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta a_3(y,z)| \le C(1+|y|^2+|z|^2)^{1/2}(1+|y|^2+|z|^4)^{-1/2} \le 2C,$$
$$|y|^{|\alpha|}|z|^{|\beta|}|D_y^\alpha D_z^\beta a_4(y,z)| \le C(1+|y|^2+|z|^2)^{-1}(1+|y|^2+|z|^4)^{1/2} \le 2C.$$

Here we take into account the following simple inequalities:

$$\frac{1+|y|^2+|z|^2}{1+|y|^2+|z|^4} = \frac{1+|y|^2}{1+|y|^2+|z|^4} + \frac{|z|^2}{1+|y|^2+|z|^4} \le 1+\min(|z|^2,|z|^{-2}) \le 2,$$
$$\frac{(1+|y|^2+|z|^4)^{1/2}}{1+|y|^2+|z|^2} \le \frac{(1+|y|^2)^{1/2}}{1+|y|^2+|z|^2} + \frac{|z|^2}{1+|y|^2+|z|^2} \le 2.$$

In view of Theorem 2, we conclude that $a_3(\xi), a_4(\xi) \in M_d$ for each $d > 1$. The proof is now complete.□

We define the anisotropic Sobolev space $W_d^{-1,-2}$ consisting of distributions $u(x)$ such that $(1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/2}F(u)(\xi) = F(v)(\xi)$, $v = v(x) \in L^d(\mathbb{R}^n)$. This is a Banach space with the norm $\|u\| = \|v\|_d$. The following proposition claims that this space lays between the spaces W_d^{-1} and W_d^{-2}.

Proposition 5 (cf. [5, Proposition 7]). *For each $d > 1$ $W_d^{-1} \subset W_d^{-1,-2} \subset W_d^{-2}$ and the both embeddings are continuous.*

Proof. Let $u \in W_d^{-1}$. This means that $(1+|\xi|^2)^{-1/2}F(u)(\xi) = F(w)(\xi)$, $w = w(x) \in L^d(\mathbb{R}^n)$. By Proposition 4(iii) $a_3(\xi) = (1+|\xi|^2)^{1/2}(1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{-1/2} \in M_d$. Therefore,

$$(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2}F(u)(\xi) = a_3(\xi)F(w)(\xi) = F(v)(\xi),\ v(x) \in L^d(\mathbb{R}^n),$$

that is, $u \in W_d^{-1,-2}$. We deduce that $W_d^{-1} \subset W_d^{-1,-2}$. Since $\|v\|_d \le C\|w\|_d$, $C = \mathrm{const}$ this embedding is continuous.

Now suppose that $u \in W_d^{-1,-2}$. Then $(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2}F(u)(\xi) = F(v)(\xi)$, $v = v(x) \in L^d(\mathbb{R}^n)$. By Proposition 4(iv) $a_4(\xi) = (1+|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}(1+|\xi|^2)^{-1} \in M_d$, and

$$(1 + |\xi|^2)^{-1}F(u)(\xi) = a_4(\xi)F(v)(\xi) = F(w)(\xi),\ w \in L^d(\mathbb{R}^n).$$

This means that $u \in W_d^{-2}$. We established that $W_d^{-1,-2} \subset W_d^{-2}$. The continuity of this embedding follows from the estimate $\|w\|_d \le C\|v\|_d$, $C = \mathrm{const}$. The proof is complete.□

We also introduce the local space $W_{d,loc}^{-1,-2}(\Omega)$ consisting of distributions $u(x)$ such that $uf(x)$ belongs to $W_d^{-1,-2}$ for all $f(x) \in C_0^\infty(\Omega)$. The space $W_{d,loc}^{-1,-2}(\Omega)$ is a locally convex space with the topology generated by the family of semi-norms $u \mapsto \|uf\|_{W_d^{-1,-2}}$, $f(x) \in C_0^\infty(\Omega)$. Analogously, we define the spaces $W_{d,loc}^{-1}(\Omega)$, $W_{d,loc}^{-2}(\Omega)$. As it readily follows from Proposition 5, $W_{d,loc}^{-1} \subset W_{d,loc}^{-1,-2} \subset W_{d,loc}^{-2}$ and these embeddings are continuous.

We will need also the following statement, which is rather well known (see, for example, [5, Lemma 6]).

Lemma 4. *Let $U_r(x)$ be a sequence bounded in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and weakly convergent to zero; let $a(\xi)$ be a bounded function on $\mathbb{R}^n$ such that $a(\xi) \to 0$ as $|\xi| \to \infty$. Then $a(\xi)F(U_r)(\xi) \underset{r\to\infty}{\to} 0$ in $L^2(\mathbb{R}^n)$.*

Proof. First, observe that by the assumption that $a(\xi) \to 0$ at infinity, for any $\varepsilon > 0$ we can choose $R > 0$ such that $|a(\xi)| < \varepsilon$ for $|\xi| > R$. Then

$$\int_{|\xi|>R} |a(\xi)|^2|F(U_r)(\xi)|^2 d\xi \le \varepsilon^2\|F(U_r)\|_2 = \varepsilon^2\|U_r\|_2 \le C\varepsilon^2, \tag{27}$$

where $C = \sup_{r\in\mathbb{N}} \|U_r\|_2$ is a constant independent of r.

Further, by our assumption $U_r \to 0$ as $r \to \infty$ weakly in L^1. This implies that $F(U_r)(\xi) \to 0$ point-wise as $r \to \infty$. Moreover, $|F(U_r)(\xi)| \le \|U_r\|_1 \le \text{const}$. Hence, using the Lebesgue dominated convergence theorem, we find that

$$\int_{|\xi|\le R} |a(\xi)|^2 |F(U_r)(\xi)|^2 d\xi \to 0 \tag{28}$$

as $r \to \infty$. It follows from (27), (28) that

$$\varlimsup_{r\to\infty} \int_{\mathbb{R}^n} |a(\xi)|^2 |F(U_r)(\xi)|^2 d\xi \le C\varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} |a(\xi)|^2 |F(U_r)(\xi)|^2 d\xi = 0,$$

that is, $a(\xi)F(U_r)(\xi) \underset{r\to\infty}{\to} 0$ in $L^2(\mathbb{R}^n)$. The proof is complete.□

3 Localization principle and proof of Theorem 1

Suppose that the sequence $u_r(x)$ converges weakly to $u(x)$ in $L^p_{loc}(\Omega, \mathbb{R}^N)$, and the sequences of distributions

$$\sum_{\alpha=1}^N \sum_{k=1}^n \partial_{x_k}(a_{s\alpha k} u_{\alpha r}) + \sum_{\alpha=1}^N \sum_{k,l=\nu+1}^n \partial_{x_k x_l}(b_{s\alpha kl} u_{\alpha r}), \ r \in \mathbb{N}, \quad s = 1, \dots, m,$$

are pre-compact in the anisotropic Sobolev space $W^{-1,-2}_{d,loc}(\Omega)$, where $d > 1$ is indicated in the Introduction. We will also assume that $d \le 2$. This assumption is not restrictive, because of the natural embeddings $W^{-1,-2}_{d,loc}(\Omega) \subset W^{-1,-2}_{d_1,loc}(\Omega)$ for each $d_1 < d$. Let $U_r = u_r(x) - u(x) = (U_r^1, \dots, U_r^N)$, $U_r^\alpha = u_{\alpha r}(x) - u_\alpha(x)$. Then $U_r \to 0$ as $r \to \infty$ weakly in $L^2_{loc}(\Omega, \mathbb{R}^N)$. Therefore, after extraction of a subsequence (still denoted U_r), we can assume that the parabolic H-measure $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$ corresponding to the subspace

$$X = \mathbb{R}^\nu = \{ \ \xi = (\xi_1, \dots, \xi_\nu, 0, \dots, 0) \in \mathbb{R}^n \ \}$$

is well defined.

Theorem 3 (localization principle). *For each $s = 1, \dots, m$; $\beta = 1, \dots, N$*

$$\sum_{\alpha=1}^N P_{s\alpha}(x, \xi)\mu^{\alpha\beta} = 0,$$

where

$$P_{s\alpha}(x,\xi) = 2\pi i \sum_{k=1}^{\nu} a_{s\alpha k}(x)\xi_k - 4\pi^2 \sum_{k,l=\nu+1}^{n} b_{s\alpha kl}(x)\xi_k\xi_l.$$

Proof.

Since the coefficients $a_{s\alpha k}(x), b_{s\alpha kl}(x)$ belong to $L^{2q}_{loc}(\Omega)$, and $\frac{1}{2q}+\frac{1}{p}=\frac{1}{2}$, the sequences $a_{s\alpha k}U_{\alpha r}$, $b_{s\alpha kl}U_{\alpha r}$ converge to zero as $r\to\infty$ weakly in $L^2_{loc}(\Omega,\mathbb{R}^N)$ and the sequences of distributions

$$\mathcal{L}_{sr} \doteq \sum_{\alpha=1}^{N}\sum_{k=1}^{n}\partial_{x_k}(a_{s\alpha k}U_r^\alpha) + \sum_{\alpha=1}^{N}\sum_{k,l=\nu+1}^{n}\partial_{x_kx_l}(b_{s\alpha kl}U_r^\alpha),\ r\in\mathbb{N},\quad s=1,\dots,m,$$

converge weakly to zero. Using the pre-compactness of these sequences in $W^{-1,-2}_{d,loc}(\Omega)$, we find that $\mathcal{L}_{sr}\to 0$ as $r\to\infty$ in $W^{-1,-2}_{d,loc}(\Omega)$. We choose $\Phi_1(x)\in C_0^\infty(\Omega)$ and consider the distributions

$$l_{sr} = \partial_{x_k}(a_{s\alpha k}\Phi_1U_r^\alpha - 2b_{s\alpha kl}U_r^\alpha\partial_{x_l}\Phi_1) + \partial_{x_kx_l}(b_{s\alpha kl}\Phi_1U_r^\alpha). \tag{29}$$

To simplify the notation, we use here and below the conventional rule of summation over repeated indexes, and suppose that the coefficients $b_{s\alpha kl}$ are defined for all $k,l=1,\dots,n$ with $b_{s\alpha kl}=0$ if $\min(k,l)\le\nu$. We can also assume that $b_{s\alpha kl}=b_{s\alpha lk}$ for $k,l=1,\dots,n$. Then, as it is easy to compute,

$$l_{sr} = \Phi_1\mathcal{L}_{sr} + a_{s\alpha k}U_r^\alpha\partial_{x_k}\Phi_1 - b_{s\alpha kl}U_r^\alpha\partial_{x_kx_l}\Phi_1. \tag{30}$$

Since the coefficients $a_{s\alpha k}(x), b_{s\alpha kl}(x)$ belong to $L^{2q}_{loc}(\Omega)$, and $\frac{1}{2q}+\frac{1}{p}=\frac{1}{2}$, the sequences $a_{s\alpha k}U_r^\alpha\partial_{x_k}\Phi_1$, $b_{s\alpha kl}U_r^\alpha\partial_{x_kx_l}\Phi_1$ are bounded in $L^2(\mathbb{R}^n)$. Noticing that the function $\Phi_1(x)$ has a compact support, we see that these sequences are bounded also in $L^d(\mathbb{R}^n)$ for all $s=1,\dots,m$, and they weakly converge to zero as $r\to\infty$. Therefore, they converge to zero strongly in $W_d^{-1}(\mathbb{R}^n)$ and, in view of Proposition 5, also in $W_d^{-1,-2}(\mathbb{R}^n)$. By our assumptions, $\Phi_1\mathcal{L}_{sr}\to 0$ as $r\to\infty$ in $W_d^{-1,-2}(\mathbb{R}^n)$. Hence, it follows from the above limit relations and (30) that $l_{sr}\to 0$ as $r\to\infty$ in $W_d^{-1,-2}(\mathbb{R}^n)$. Applying the Fourier transformation to this relation and then multiplying by $(1+|\tilde\xi|^2+|\bar\xi|^4)^{-1/2}$, we arrive at

$$(1+|\tilde\xi|^2+|\bar\xi|^4)^{-1/2}\Big(2\pi i\xi_kF(a_{s\alpha k}\Phi_1U_r^\alpha)(\xi) - \\ 4\pi i\xi_kF(b_{s\alpha kl}U_r^\alpha\partial_{x_l}\Phi_1)(\xi) - 4\pi^2\bar\xi_k\bar\xi_lF(b_{s\alpha kl}\Phi_1U_r^\alpha)(\xi)\Big) = F(v_{sr})(\xi), \tag{31}$$

where $v_{sr}\to 0$ as $r\to\infty$ in $L^d(\mathbb{R}^n)$. We take also into account that

$$\xi_k\xi_lF(b_{s\alpha kl}\Phi_1U_r^\alpha)(\xi) = \sum_{k,l=\nu+1}^{n}\xi_k\xi_lF(b_{s\alpha kl}\Phi_1U_r^\alpha)(\xi) = \bar\xi_k\bar\xi_lF(b_{s\alpha kl}\Phi_1U_r^\alpha)(\xi).$$

By Proposition 4(ii), we have

$$a_2(\xi) = \rho(\xi)(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2} \in M_d.$$

Therefore, it follows from (31) that

$$\rho(\xi)(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2}\Big(2\pi i\xi_k F(a_{s\alpha k}\Phi_1 U_r^\alpha)(\xi) - 4\pi i\xi_k F(b_{s\alpha kl}U_r^\alpha \partial_{x_l}\Phi_1)(\xi)$$
$$-4\pi^2\bar{\xi}_k\bar{\xi}_l F(b_{s\alpha kl}\Phi_1 U_r^\alpha)(\xi)\Big) = a_2(\xi)F(v_{sr})(\xi) = F(w_{sr})(\xi), \quad (32)$$

$w_{sr} \to 0$ as $r \to \infty$ in $L^d(\mathbb{R}^n)$ for all $s = 1, \dots, m$. Since

$$\frac{\rho(\xi)|\bar{\xi}|^2}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \le 1,$$

$$\frac{\rho(\xi)|\xi|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \le \rho(\xi)\frac{|\tilde{\xi}| + |\bar{\xi}|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \le 1 + \min(|\bar{\xi}|, |\bar{\xi}|^{-1}) \le 2$$

(recall that $0 \le \rho(\xi) \le 1$, and $\rho(\xi) = 0$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \le 1$), and $F(a_{s\alpha k}\Phi_1 U_r^\alpha)(\xi), F(b_{s\alpha kl}\Phi_1 U_r^\alpha)(\xi), F(b_{s\alpha kl}U_r^\alpha\partial_{x_l}\Phi_1)(\xi) \in L^2(\mathbb{R}^n)$, we see that $F(w_{sr})(\xi) \in L^2(\mathbb{R}^n)$, which implies that $w_{sr} \in L^2(\mathbb{R}^n)$ as well.

Since $b_{s\alpha kl} = 0$ for $k \le \nu$,

$$\tilde{\xi}_k F(b_{s\alpha kl}U_r^\alpha\partial_{x_l}\Phi_1)(\xi) = \sum_{k=1}^{\nu} \xi_k F(b_{s\alpha kl}U_r^\alpha\partial_{x_l}\Phi_1)(\xi) = 0. \quad (33)$$

Now, observe that for each k the function

$$a(\xi) = \frac{\rho(\xi)\bar{\xi}_k}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}},$$

satisfies the assumption of Lemma 4. Indeed, this follows from the estimate

$$|a(\xi)| \le \rho(\xi)(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/4}\frac{|\bar{\xi}|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}} \le \rho(\xi)(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/4}.$$

Since the sequences $a_{s\alpha k}\Phi_1 U_r^\alpha$, $b_{s\alpha kl}U_r^\alpha\partial_{x_l}\Phi_1$ are bounded in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and weakly converge to zero as $r \to \infty$, then by Lemma 4

$$\frac{\rho(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}}\bar{\xi}_k F(a_{s\alpha k}\Phi_1 U_r^\alpha)(\xi) \underset{r\to\infty}{\to} 0 \text{ in } L^2(\mathbb{R}^n), \quad (34)$$

$$\frac{\rho(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}}\bar{\xi}_k F(b_{s\alpha kl}U_r^\alpha\partial_{x_l}\Phi_1)(\xi) \underset{r\to\infty}{\to} 0 \text{ in } L^2(\mathbb{R}^n). \quad (35)$$

It follows from (33), (35) that

$$\frac{\rho(\xi)}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\xi_k F(b_{s\alpha kl}U_r^\alpha \partial_{x_l}\Phi_1)(\xi) \underset{r\to\infty}{\to} 0 \ \text{ in } L^2(\mathbb{R}^n). \tag{36}$$

Let $\Phi_2(x) \in C_0(\mathbb{R}^n)$, $\psi(\xi) \in C^n(S_X)$. Since the sequence $\Phi_2 U_r^\beta$ is bounded in $L^p(\Omega)$ and supported in the compact $\operatorname{supp}\Phi_2$, and $d' = d/(d-1) \le p$, this sequence is also bounded in $L^2(\mathbb{R}^n)\cap L^{d'}(\mathbb{R}^n)$, By Proposition 4(i) for a fixed $\beta = 1,\dots,N$ $\quad \overline{\psi(\pi_X(\xi))}F(\Phi_2 U_r^\beta)(\xi) = F(g_r)(\xi)$, where the sequence g_r is bounded in $L^2(\mathbb{R}^n)\cap L^{d'}(\mathbb{R}^n)$. We multiply (32) by $\psi(\pi_X(\xi))\overline{F(\Phi_2 U_r^\beta)(\xi)}$ and integrate the result over $\xi \in \mathbb{R}^n$. Passing then to the limit as $r\to\infty$ and taking into account relations (34), (36), we arrive at

$$\begin{aligned}
\lim_{r\to\infty}\int_{\mathbb{R}^n} & \frac{\rho(\xi)(2\pi i\tilde{\xi}_k F(a_{s\alpha k}\Phi_1 U_r^\alpha)(\xi) - 4\pi^2\bar{\xi}_k\bar{\xi}_l F(b_{s\alpha kl}\Phi_1 U_r^\alpha)(\xi))}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \\
& \times\overline{F(\Phi_2 U_r^\beta)}\psi(\pi_X(\xi))d\xi = \lim_{r\to\infty}\int_{\mathbb{R}^n} F(w_{sr})(\xi)\overline{F(g_r)(\xi)}d\xi \\
& = \lim_{r\to\infty}\int_{\mathbb{R}^n} w_{sr}(x)\overline{g_r(x)}dx = 0.
\end{aligned} \tag{37}$$

On the other hand, by relation (9), Remark 1 and Corollary 1 (in the case $p>2$), we see that

$$\begin{aligned}
\lim_{r\to\infty}\int_{\mathbb{R}^n} & \frac{\rho(\xi)(2\pi i\tilde{\xi}_k F(a_{s\alpha k}\Phi_1 U_r^\alpha)(\xi) - 4\pi^2\bar{\xi}_k\bar{\xi}_l F(b_{s\alpha kl}\Phi_1 U_r^\alpha)(\xi))}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}} \\
& \times\overline{F(\Phi_2 U_r^\beta)}\psi(\pi_X(\xi))d\xi = \\
& \langle\mu^{\alpha\beta}, (2\pi i a_{s\alpha k}(x)\tilde{\xi}_k - 4\pi^2 b_{s\alpha kl}(x)\bar{\xi}_k\bar{\xi}_l)\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle.
\end{aligned}$$

Then it follows from (37) that

$$\langle\mu^{\alpha\beta}, P_{s\alpha}(x,\xi)\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = 0, \tag{38}$$

where

$$\begin{aligned}
P_{s\alpha}(x,\xi) &= 2\pi i a_{s\alpha k}(x)\tilde{\xi}_k - 4\pi^2 b_{s\alpha kl}(x)\bar{\xi}_k\bar{\xi}_l = \\
& 2\pi i\sum_{k=1}^{\nu} a_{s\alpha k}(x)\xi_k - 4\pi^2\sum_{k,l=\nu+1}^{n} b_{s\alpha kl}(x)\xi_k\xi_l.
\end{aligned}$$

We underline that the functions $P_{s\alpha}(x,\xi)\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)$ are measurable and locally integrable with respect to the measure η. This is evident in the case $p=2$ (then $a_{s\alpha k}, b_{s\alpha kl} \in C(\Omega)$) while in the case $p>2$ this readily follows from

Proposition 3, from the assumptions $a_{s\alpha k}, b_{s\alpha kl} \in L^{2q}_{loc}(\Omega)$, and from the inequality $\frac{1}{2q} + \frac{2}{p} < \frac{1}{q} + \frac{2}{p} = 1$.

Since the functions $\Phi_1(x) \in C_0^\infty(\Omega)$, $\Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C^n(S_X)$ are arbitrary, we derive from (38) that $P_{s\alpha}(x,\xi)\mu^{\alpha\beta} = 0$ for each $s = 1,\dots,m$, $\beta = 1,\dots,N$. The proof is complete. $\square$

By Proposition 3 the H-measure μ admits the representation $\mu = H(x,\xi)\eta$, where $H(x,\xi) = \{h^{\alpha\beta}(x,\xi)\}_{\alpha,\beta=1}^N$ is an Hermitian matrix.

Corollary 2. *For η-a.e. $(x,\xi) \in \Omega \times S_X$ the image of $H(x,\xi)$ is contained in $\Lambda(x)$.*

Proof. By Theorem 3 $P_{s\alpha}(x,\xi)h^{\alpha\beta}(x,\xi)\eta = 0$. This can be written as $P(x,\xi)H(x,\xi) = 0$, where $P(x,\xi)$ is a $m \times N$ matrix with components $P_{s\alpha}$. Therefore, for η-a.e. $(x,\xi) \in \Omega \times S_X$ $\operatorname{Im} H(x,\xi) \subset \ker P(x,\xi)$. Now notice that if $\lambda = (\lambda_1,\dots,\lambda_N) \in \mathbb{C}^N$ belongs to $\ker P(x,\xi)$ then

$$\sum_{\alpha=1}^N \Big(i\sum_{k=1}^\nu a_{s\alpha k}(x)2\pi\xi_k - \sum_{k,l=\nu+1}^n b_{s\alpha kl}(x)2\pi\xi_k 2\pi\xi_l \Big)\lambda_\alpha = 0$$

for all $s = 1,\dots,m$. Remark that $2\pi\xi \neq 0$ because of the inclusion $\xi \in S_X$. Hence, $\lambda \in \Lambda(x)$. We conclude that $\ker P(x,\xi) \subset \Lambda(x)$, and $\operatorname{Im} H(x,\xi) \subset \ker P(x,\xi) \subset \Lambda(x)$, as was to be proved. $\square$

Now we are ready to prove our main Theorem 1.

Proof of Theorem 1. Since $H = H(x,\xi) \geq 0$ there exists a unique Hermitian matrix $R = R(x,\xi) = H^{1/2}$ such that $R \geq 0$ and $H = R^2$. By the known properties of Hermitian matrices $\ker R = \ker H$, which readily implies that $\operatorname{Im} R = \operatorname{Im} H$. By Corollary 2 we claim that $\operatorname{Im} R(x,\xi) \subset \Lambda(x)$ for η-a.e. $(x,\xi) \in \Omega \times S_X$. Now we represent the coefficients $q_{\alpha\beta}(x)$ of quadratic form $q(x,u)$ in the form $q_{\alpha\beta}(x) = q^{(1)}_{\alpha\beta}(x) q^{(2)}_{\alpha\beta}(x)$, where for $j = 1,2$ $q^{(j)}_{\alpha\beta}(x) \in L^{2q}_{loc}(\Omega)$ if $p > 2$, and $q^{(j)}_{\alpha\beta}(x) \in C(\Omega)$ if $p = 2$. For instance, we can set

$$q^{(1)}_{\alpha\beta}(x) = |q_{\alpha\beta}(x)|^{1/2} \operatorname{sign} q_{\alpha\beta}(x),\ q^{(2)}_{\alpha\beta}(x) = |q_{\alpha\beta}(x)|^{1/2}.$$

Taking into account Corollary 1, we find that for real $\Phi(x) \in C_0(\Omega)$

$$\int_\Omega (\Phi(x))^2 q(x, U_r(x))dx = \int_{\mathbb{R}^n} q^{(1)}_{\alpha\beta}(x)\Phi(x)U_r^\alpha(x) q^{(2)}_{\alpha\beta}(x)\Phi(x)U_r^\beta(x)dx =$$
$$\int_{\mathbb{R}^n} F(\Phi q^{(1)}_{\alpha\beta} U_r^\alpha)(\xi)\overline{F(\Phi q^{(2)}_{\alpha\beta} U_r^\beta)(\xi)}d\xi \underset{r\to\infty}{\to}$$
$$\langle \mu^{\alpha\beta}, (\Phi(x))^2 q_{\alpha\beta}(x)\rangle = \int_{\Omega\times S_X} (\Phi(x))^2 q_{\alpha\beta}(x) h^{\alpha\beta}(x,\xi)d\eta(x,\xi). \quad (39)$$

Since $H = R^2$ then $h^{\alpha\beta}(x,\xi) = r_{\alpha j}\overline{r_{\beta j}}$, where $r_{ij} = r_{ij}(x,\xi)$, $i,j = 1,\dots,N$ are components of matrix R. Therefore,

$$q_{\alpha\beta}(x)h^{\alpha\beta} = q_{\alpha\beta}(x)r_{\alpha j}\overline{r_{\beta j}} = \sum_{j=1}^{N} Q(x)Re_j \cdot Re_j, \tag{40}$$

where $\{e_j\}_{j=1}^N$ is the standard basis in $\mathbb{C}^N$. Since $Re_j \in \operatorname{Im} R \subset \Lambda(x)$ then it follows from the assumption of Theorem 1 that $Q(x)Re_j \cdot Re_j \geq 0$ for η-a.e. $(x,\xi) \in \Omega \times S_X$. In view of (40) we find that $q_{\alpha\beta}(x)h^{\alpha\beta}(x,\xi) \geq 0$ for η-a.e. $(x,\xi) \in \Omega \times S_X$. Now, it readily follows from (39) that

$$\lim_{r\to\infty} \int_\Omega (\Phi(x))^2 q(x, U_r(x))dx \geq 0 \tag{41}$$

for all real $\Phi(x) \in C_0(\Omega)$.

In view of the weak convergence $u_r \to u$, $q(x,u_r(x)) \to v$ as $r \to \infty$,

$$q(x,U_r(x)) = q(x,u_r(x)) + q(x,u(x)) - 2\operatorname{Re}(Q(x)u_r(x) \cdot u(x)) \to v - q(x,u(x))$$

weakly in $M_{loc}(\Omega)$, and we derive from (41) that

$$\langle v - q(x,u(x))dx, (\Phi(x))^2\rangle \geq 0$$

Since $(\Phi(x))^2$ is an arbitrary nonnegative function in $C_0(\Omega)$, this implies that $q(x,u(x)) \leq v$. The proof is complete. $\square$

Corollary 3. *Suppose that $q(x,\lambda) = 0$ for all $\lambda \in \Lambda(x)$, $x \in \Omega$. Then $v = q(x,u(x))$, that is, the functional $u \to q(x,u)$ is weakly continuous.*

Proof. Applying Theorem 1 to the quadratic forms $\pm q(x,u)$, we obtain the inequalities $\pm v \geq \pm q(x,u(x))$, which readily imply that $v = q(x,u(x))$. $\square$

Remark 2. In the particular case $\nu = n$ relations (2) are reduced to the requirement that the sequences of distributions

$$L_{sr} = \sum_{\alpha=1}^{N}\sum_{k=1}^{n} \partial_{x_k}(a_{s\alpha k}(x)u_{\alpha r}), \quad s = 1,\dots,m$$

are pre-compact in $W^{-1}_{d,loc}(\Omega)$. In applications to conservation laws, it usually happens that the sequences $u_{\alpha r}$ are bounded in $L^\infty_{loc}(\Omega)$ (so that $p = \infty$) while the sequences L_{sr} are bounded in $M_{loc}(\Omega)$. Since the space $M_{loc}(\Omega)$ is compactly embedded in $W^{-1}_{d,loc}(\Omega)$ for $d < n/(n-1)$ then condition (2) is satisfied.

In the case $\nu = 0$ the statement of Theorem 1 is a compensated compactness result under the second order constraints

$$L_{sr} = \sum_{\alpha=1}^{N} \sum_{k,l=1}^{n} \partial_{x_k x_l}(b_{s\alpha kl}(x)u_{\alpha r}), \quad s = 1, \dots, m,$$

which are required to be pre-compact in $W^{-2}_{d,loc}(\Omega)$. Observe also that in each of the cases $\nu = n, 0$ the set $\Lambda(x)$ may be defined as a subset of real space $\mathbb{R}^N$.

4 Some applications

We consider the parabolic operator

$$L(u) = \partial_t u - \sum_{k,l=1}^{n} \partial_{x_k x_l}(a_{kl}(t,x)g(t,x,u)), \quad u = u(t,x), \ (t,x) \in \Omega = (0,+\infty) \times V,$$

V being an open subset of $\mathbb{R}^n$. It is assumed that for $u = u(t,x)$

$$\begin{aligned} u, g(t,x,u) \in L^p_{loc}(\Omega), \ 2 \le p \le \infty, \text{ while} \\ a_{kl} = a_{kl}(t,x) \in L^{2q}_{loc}(\Omega), \quad \text{where } q = p/(p-2), \ p > 2, \\ \text{and } a_{kl} \in C(\Omega) \quad \text{if } p = 2. \end{aligned}$$

The matrix $A(t,x) = \{a_{kl}(t,x)\}_{k,l=1}^n$ is supposed to be symmetric and strictly positive: $A(t,x)\xi \cdot \xi > 0 \ \forall \xi \in \mathbb{R}^n$, $\xi \ne 0$. The function $g(t,x,u)$ is a Caratheodory function on $\Omega \times \mathbb{R}$, non-strictly increasing with respect to the variable u.

Assume that the sequences $u_r(t,x)$, $g(t,x,u_r(t,x))$, $r \in \mathbb{N}$ are bounded in $L^p_{loc}(\Omega)$, moreover, if $p = 2$ assume that the sequence $\rho(u_r(t,x)g(t,x,u_r(t,x)))$ is bounded in $L^1_{loc}(\Omega)$ for some positive super-linear function $\rho(u)$ (that is, $\rho(u)/|u| \to \infty$ as $|u| \to \infty$). Also suppose that $u_r \to u = u(t,x)$ as $r \to \infty$ weakly in $\mathcal{D}'(\Omega)$ while $f_r = L(u_r) \to f$ strongly in $W^{-1,-2}_{d,loc}(\Omega)$, where the latter space correspond to the subspace $X = \{ \ (\xi_0, 0, \dots, 0) \ \} \subset \mathbb{R}^{n+1}$, here $(\xi_0, \xi_1, \dots, \xi_n)$ are the dual variables (ξ_0 correspond to the time variable t), and $d = p/(p-1)$ ($d > 1$ in the case $p = \infty$).

Theorem 4. *Under the above assumptions, $L(u) = f$ in $\mathcal{D}'(\Omega)$. In addition, the sequence $g(t,x,u_r(t,x))$ converges to $g(t,x,u(t,x))$ as $r \to \infty$ strongly in $L^p_{loc}(\Omega)$.*

Proof. Let $u_{1r} = u_r(t,x)$, $u_{2r} = g(t,x,u_r(t,x))$. Passing to a subsequence if necessary, we can assume that $u_{2r}(t,x) \to u_2 = u_2(t,x)$ weakly as $r \to \infty$.

Then the sequence (u_{1r}, u_{2r}) converges weakly to $(u_1, u_2) \in L^p_{loc}(\Omega, \mathbb{R}^2)$ with $u_1 = u(t,x)$. Further, it satisfies the condition that the sequence of distributions

$$f_r = \partial_t u_{1r} - \sum_{k,l=1}^{n} \partial_{x_k x_l}(a_{kl}(t,x) u_{2r})$$

is pre-compact in $W^{-1,-2}_{d,loc}(\Omega)$. In accordance with (3), we define the set $\Lambda = \Lambda(t,x)$:

$$\Lambda = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \exists(\xi_0, \xi) \in (\mathbb{R} \times \mathbb{R}^n) \setminus \{0\} \quad i\xi_0\lambda_1 + (A(t,x)\xi \cdot \xi)\lambda_2 = 0 \ \}.$$

Since $(A(t,x)\xi \cdot \xi) > 0$ for $\xi \neq 0$ then $\Lambda = \{ \ (\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \operatorname{Re} \lambda_1 \overline{\lambda_2} = 0 \ \}$. Therefore, the quadratic functional $q = q(u) = (u_1\overline{u_2} + u_2\overline{u_1})/2$ is zero for $u = \lambda \in \Lambda$. By Corollary 3 (observe that all the assumptions of this Corollary are satisfied) we claim that

$$q(u_{1r}, u_{2r}) = u_{1r}u_{2r} \underset{r\to\infty}{\rightharpoonup} q(u_1, u_2) = u_1 u_2 \tag{42}$$

weakly in $L^1_{loc}(\Omega)$. Since the sequence u_r is bounded in $L^p_{loc}(\Omega)$, $p \geq 2$, then, extracting again a subsequence (still denoted by u_r), we may suppose that the Young measure $\nu_{t,x}$ corresponding to this subsequence is well defined. Recall that a Young measure $\nu_{t,x}$ on Ω is a weakly measurable map $(t,x) \to \nu_{t,x}$ of Ω into the space $\mathrm{Prob}(\mathbb{R})$ of probability measures on $\mathbb{R}$. The weak measurability means that for each bounded continuous function $p(\lambda)$ the function $(t,x) \to \int p(\lambda) d\nu_{t,x}(\lambda)$ is Lebesgue measurable on Ω. It is known (see, for example, [7]) that the Young measure corresponding to u_r satisfies the property that whenever the sequence $\psi(t,x,u_r(t,x))$ converges weakly in $L^1_{loc}(\Omega)$ for a Caratheodory function $\psi(x,\lambda)$, its weak limit is the function

$$\bar{\psi}(t,x) = \int \psi(t,x,\lambda) d\nu_{t,x}(\lambda).$$

Moreover, $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$, where $\delta(\lambda - u)$ is the Dirac mass at u, if and only if $u_r \to u$ in $L^1_{loc}(\Omega)$. Since $u_r \rightharpoonup u_1 = u(t,x)$, $g(t,x,u_r) \rightharpoonup u_2(t,x)$, $u_r g(t,x,u_r) = u_{1r}u_{2r} \rightharpoonup u_1u_2$ as $r \to \infty$ weakly in $L^1_{loc}(\Omega)$ then these limit functions admit the representations:

$$u_1 = \int \lambda d\nu_{t,x}(\lambda), \ u_2 = \int g(t,x,\lambda) d\nu_{t,x}(\lambda), \ u_1u_2 = \int \lambda g(t,x,\lambda) d\nu_{t,x}(\lambda).$$

It follows from these equalities that for a.e. $(t,x) \in \Omega$

$$u(t,x) \int g(t,x,\lambda) d\nu_{t,x}(\lambda) = \int \lambda g(t,x,\lambda) d\nu_{t,x}(\lambda).$$

It is reduced to the equality

$$\int (\lambda - u(t,x))g(t,x,\lambda)d\nu_{t,x}(\lambda) = 0,$$

and since $\int (\lambda - u(t,x))\nu_{t,x}(\lambda) = 0$, we arrive at the relation

$$\int (\lambda - u(t,x))(g(t,x,\lambda) - g(t,x,u(t,x)))d\nu_{t,x}(\lambda) = \\ \int (\lambda - u(t,x))g(t,x,\lambda)d\nu_{t,x}(\lambda) - \\ g(t,x,u(t,x)) \int (\lambda - u(t,x))\nu_{t,x}(\lambda) = 0 \tag{43}$$

for a.e. $(t,x) \in \Omega$. Taking into account the fact that the function $g(t,x,\lambda)$ is non-decreasing with respect to λ, we derive from (43) that for a.e. $(t,x) \in \Omega$ $g(t,x,\lambda) = g(t,x,u(t,x))$ on $\operatorname{supp} \nu_{t,x}$. Therefore,

$$u_2 = \int g(t,x,\lambda)d\nu_{t,x}(\lambda) = g(t,x,u(t,x))$$

almost everywhere in Ω. Hence, in the limit as $r \to \infty$

$$L(u_r) \to L(u) = \partial_t u - \sum_{k,l=1}^{n} \partial_{x_k x_l}(a_{kl}(t,x)g(t,x,u)) \text{ in } \mathcal{D}'(\Omega).$$

Since $L(u_r) = f_r \to f$ as $r \to \infty$ in $\mathcal{D}'(\Omega)$, we conclude that $L(u) = f$. Besides, the image of $\nu_{t,x}$ under the map $u \to g(t,x,u)$ coincides with the Dirac measure $\delta(\lambda - g(t,x,u(t,x)))$:

$$\tilde{\nu}_{t,x}(\lambda) \doteq (g(t,x,\cdot)^*\nu_{t,x})(\lambda) = \delta(\lambda - g(t,x,u(t,x))).$$

It is easy to see that $\tilde{\nu}_{t,x}(\lambda)$ is the Young measure corresponding to the sequence $g(t,x,u_r(t,x))$. Since this Young measure coincides with $\delta(\lambda - g(t,x,u(t,x)))$, we conclude that the sequence $g(t,x,u_r(t,x))$ converges to $g(t,x,u(t,x))$ strongly in $L^p_{loc}(\Omega)$. Finally, observe that the limit function does not depend on the prescribed above choice of a subsequence. Therefore, $g(t,x,u_r(t,x))$ also converges strongly to $g(t,x,u(t,x))$ for the original sequence u_r. The proof is complete. □

Remark 3. In the case when the function $g(t,x,u)$ is strictly monotone we deduce from Theorem 4 the strong pre-compactness property for weak solutions of the equation $L(u) = f = f(t,x) \in W^{-1,-2}_{loc}(\Omega)$, which satisfy the equation in $\mathcal{D}'(\Omega)$. Notice that for entropy solutions of this equation (with $f = f(t,x,u) \in$

$L^1_{loc}(\Omega, C(\mathbb{R})))$ the strong pre-compactness property follows from general results of [5, 6].

Acknowledgements. This work was carried out during the author stay at the University of Franche-Comté (Besançon, France). The authors thanks colleagues from the Laboratory of Mathematics and especially Boris Andreianov for hospitality and fruitful discussions on the subject of this paper.

The author also acknowledges the support of the Russian Foundation for Basic Research (grant No. 09-01-00490-a) and and Deutsche Forschungsgemeinschaft (DFG project No. 436 RUS 113/895/0-1).

References

[1] Antonić N., Lazar M: H-measures and variants applied to parabolic equations. *J. Math. Anal. Appl.* **343**, 207–225 (2008)

[2] Bergh, J., Löfström, J.: *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1980.

[3] Gerárd, P.: Microlocal defect measures. *Comm. Partial Diff. Equat.* **16**, 1761–1794 (1991)

[4] Murat,F.: Compacité par compensation. *Ann. Scuola Norm. sup. Pisa* **5**, 489–507 (1978)

[5] Panov, E.Yu.: Ultra-parabolic equations with rough coefficients. Entropy solutions and strong precompactness property. *Journal of Mathematical Sciences* **159**:2, 180–228 (2009)

[6] Panov, E.Yu.: On the strong pre-compactness property for entropy solutions of a degenerate elliptic equation with discontinuous flux. *J. Differential Equations* **247**:10, 2821-2870 (2009)

[7] Pedregal, P.: Parametrized measures and variational principles. *Progress in Nonlinear Differential Equations and their Applications* **30**. Birkhäuser Verlag, Basel, 1997.

[8] Stein E.M.: Singular integrals and differentiability properties of functions. *Princeton Mathematical Series*, No. 30, Princeton University Press, Princeton, N.J., 1970.

[9] Tartar, L.: Compensated compactness and applications to partial differential equations. *Nonlinear analysis and mechanics: Heriot. Watt Symposium, vol. 4 (Edinburgh 1979), Res. Notes Math.* **39**, 136–212 (1979)

[10] TARTAR, L.: H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations. *Proc. Roy. Soc. Edinburgh. Sect. A.* **115**:3-4, 193–230 (1990)

Novgorod State University,
41, B. St-Peterburgskaya,
173003 Veliky Novgorod, Russia.
email: Eugeny.Panov@novsu.ru