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Lasry-Lions Regularization and a Lemma of Ilmanen

Patrick Bernard (*)

Let $H$ be a Hilbert space. We define the following inf (sup) convolution operators acting on bounded functions $u : H \rightarrow \mathbb{R}$:

$$T_t u(x) := \inf_y \left( u(y) + \frac{1}{t} \| y - x \|^2 \right)$$

and

$$\hat{T}_t u(x) := \sup_y \left( u(y) - \frac{1}{t} \| y - x \|^2 \right).$$

We have the relation

$$T_t (-u) = -T_t (u).$$

Recall that these operators form semi-groups, in the sense that

$$T_t \circ T_s = T_{t+s} \quad \text{and} \quad \hat{T}_t \circ \hat{T}_s = \hat{T}_{t+s}$$

for all $t \geq 0$ and $s \geq 0$, as can be checked by direct calculation. Note also that

$$\inf u \leq T_t u(x) \leq u(x) \leq \hat{T}_t u(x) \leq \sup u$$

for each $t \geq 0$ and each $x \in H$. A function $u : H \rightarrow \mathbb{R}$ is called $k$-semi-concave, $k > 0$, if the function $x \mapsto u(x) - \|x\|^2/k$ is concave. We will occasionally consider semi-concave functions which take values in $[ -\infty, +\infty )$. The function $u$ is called $k$-semi-convex if $-u$ is $k$-semi-concave. A function $u$ is $t$-semi-concave and upper semi-continuous if and only if it belongs to the image of the operator $T_t$, this follows from Lemma 1 and Lemma 3 below. A function is called semi-concave if it is $k$-semi-concave for some $k > 0$. A function $u$ is said $C^{1,1}$ if it is Frechet differentiable and if the gradient of $u$ is Lipschitz. Note that a continuous function $u : H \rightarrow \mathbb{R}$ is $C^{1,1}$.

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if and only if it is semi-concave and semi-convex, see Lemma 5. Let us recall two important results in that language:

**Theorem 1** (Lasry-Lions, [6]). Let $u$ be a bounded function. For $0 < s < t$, the function $T_s \circ T_t u$ is $C^{1,1}$ and, if $u$ is uniformly continuous, then it converges uniformly to $u$ when $t \to 0$.

**Theorem 2** (Imanen, [5]). Let $u \geq v$ be two bounded functions on $H$ such that $u$ and $-v$ are semi-concave. Then there exists a $C^{1,1}$ function $w$ such that $u \geq w \geq v$.

Our goal in the present paper is to “generalize” simultaneously both of these results as follows:

**Theorem 3.** The operator $R_t := T_t \circ T_{2t} \circ T_t$ has the following properties:

- **Regularization**: For each function $f : H \to \mathbb{R}$ and each $t > 0$, the function $R_t(f)$ is $C^{1,1}$ provided it is locally bounded. This holds for all $t > 0$ if $f$ is bounded.
- **Approximation**: If $f : H \to \mathbb{R}$ is uniformly continuous, then $R_t(f)$ is $C^{1,1}$ and converges uniformly to $f$ as $t \to 0$.
- **Pinching**: If there exists a $k$-semi-concave continuous function $u$ and a $k$-semi-convex continuous function $v$ such that $v \leq f \leq u$, then, for all $t \in [0,k]$, we have $u \geq R_t(f) \geq v$, and $R_t(f)$ is $C^{1,1}$.

Theorem 3 does not, properly speaking, generalize Theorem 5. However, it offers a new (although similar) answer to the same problem: approximating uniformly continuous functions on Hilbert spaces by $C^{1,1}$ functions with a simple explicit formula.

Because of its symmetric form, the regularizing operator $R_t$ enjoys some nicer properties than the Lasry-Lions operators. For example, if $f$ is $C^{1,1}$, then it follows from the pinching property that $R_t f = f$ for $t$ small enough.

Theorem 2, can be proved using Theorem 3 by taking $w = R_t u$, for $t$ small enough. Note, in view of Lemma 3 bellow, that $R_t u = T_t \circ T_{2t} u$ when $t$ is small enough.

Theorem 3 can be somehow extended to the case of finite dimensional open sets or manifolds via partition of unity, at the price of losing the simplicity of explicit expressions. Let $M$ be a paracompact manifold of dimension $n$, equipped once and for all with an atlas $(\phi_i, i \in I)$ composed of
charts $\phi_i : B^n \to M$, where $B^n$ is the open unit ball of radius one centered at the origin in $\mathbb{R}^n$. We assume in addition that the image $\phi(B^n)$ is a relatively compact open set, and that the sets $\phi_i(B^n), i \in I$ form a locally finite open covering of the manifold $M$. Let us fix, once and for all, a partition of the unity $g_i$ subordinated to the open covering $(\phi_i(B^n), i \in I)$. It means that the function $g_i$ is non-negative with support inside $\phi_i(B^n)$, and that $\sum g_i = 1$ (note that this sum is finite at each point). Let us define the operator

$$G_\ell (u) := \sum_i \left[ R_{\ell n}((g_i u) \circ \phi_i) \right] \circ \phi_i^{-1},$$

where $a_i, i \in I$ are positive real numbers. In this expression, we consider each of the terms $[R_{\ell n}((g_i u) \circ \phi_i)] \circ \phi_i^{-1}$ as defined on the whole manifold $M$ with the value 0 outside of $\phi_i(B^n)$. The sum is then locally finite hence well-defined. We say that a function $u : M \to \mathbb{R}$ is locally semi-concave if, for each $i \in I$, there exists a constant $b_i$ such that the function $u \circ \phi_i - \| \cdot \|^2 / b_i$ is concave on $B^n$.

**Theorem 4.** Let $u \geq v$ be two continuous functions on $M$ such that $u$ and $-v$ are locally semi-concave. Then, the real numbers $a_i$ can be chosen such that, for each $t \in [0, 1]$ and each function $f$ satisfying $u \geq f \geq v$, we have:

- The function $G_t (f)$ is locally $C^{1,1}$.
- If $f$ is continuous, then $G_t (f)$ converges locally uniformly to $f$ as $t \to 0$.
- $u \geq G_t (f) \geq v$.

We will give some properties, most of which are well-known, of the operators $T_t$ and $T_t$ in Section 1, and derive the proof of the main results in Section 2.

**Notes and Acknowledgements.** Theorem 2 appears in Ilmanen’s paper [5] as Lemma 4G. Several proofs are sketch there but none is detailed. The proof we detail here follows lines similar to one of the sketches of Ilmanen. This statement also has a more geometric counterpart, Lemma 4E in [5]. A detailed proof of this geometric version is given in [2], Appendix. My attention was attracted to these statements and their relations with recent progresses on sub-solutions of the Hamilton-Jacobi equation (see [4, 1, 7]) by Pierre Cardialaguet, Albert Fathi and Maxime Zavidovique. Albert Fathi and Maxime Zavidovique also recently wrote a detailed proof of
Theorem 1, see [3]. This paper also proves how the geometric version follows from Theorem 2. There are many similarities between the tools used in the present paper and those used in [1]. Moreover, Maxime Zavidovique observed in [7] that the existence of $C^{1,1}$ subsolutions of the Hamilton-Jacobi equation in the discrete case can be deduced from Theorem 2. However, it seems that the main result of [1] (the existence of $C^{1,1}$ subsolutions in the continuous case) can’t be deduced easily from Theorem 2. Neither can Theorem 2 be deduced from it.

1. The operators $T_t$ and $\bar{T}_t$ on Hilbert spaces.

The proofs of the theorems follow from standard properties of the operators $T_t$ and $\bar{T}_t$ that we now recall in details.

**Lemma 1.** For each function $u : H \rightarrow \mathbb{R}$, the function $T_t u$ (which takes values in $[-\infty, +\infty]$), is $t$-semi-concave and upper semi-continuous. The function $\bar{T}_t u$ (which takes values in $(-\infty, +\infty]$), is $t$-semi-convex and lower semi-continuous. Moreover, if $u$ is $k$-semi-concave, then for each $t < k$ the function $\bar{T}_t u$ is $(k - t)$-semi-concave. Similarly, if $u$ is $k$-semi-convex, then for each $t < k$ the function $T_t u$ is $(k - t)$-semi-convex.

**Proof.** We shall prove the statements concerning $T_t$. We have

$$T_t u(x) - \|x\|^2/t = \inf_y (u(y) + \|y - x\|^2/t - \|x\|^2/t) = \inf_y (u(y) + \|y\|^2/t - 2x \cdot y/t),$$

this function is concave and upper semi-continuous as an infimum of continuous linear functions. On the other hand, we have

$$T_t u(x) + \|x\|^2/l = \inf_y (u(y) + \|y - x\|^2/t + \|x\|^2/l).$$

Setting $f(x, y) := u(y) + \|y - x\|^2/t + \|x\|^2/l$, the function $\inf_y f(x, y)$ is a convex function of $x$ if $f$ is a convex function of $(x, y)$. This is true if $u$ is $k$-semi-convex, $t < k$, and $l = k - t$ because we have the expression

$$f(x, y) = u(y) + \|y - x\|^2/t + \|x\|^2/l = (u(y) + \|y\|^2/k) + \left(\sqrt{\frac{t}{kt}} - \sqrt{\frac{k}{l}}\right)^2.$$

$\square$
Given a uniformly continuous function \( u : H \rightarrow \mathbb{R} \), we define its modulus of continuity \( \rho(r) : [0, \infty) \rightarrow [0, \infty) \) by the expression
\[
\rho(r) = \sup_{x \in B_1} u(x + re) - u(x),
\]
where the supremum is taken on all \( x \in H \) and all \( e \) in the unit ball of \( H \). The function \( \rho \) is non-decreasing, it satisfies 
\[
\rho(r + r') \leq \rho(r) + \rho(r'),
\]
and it converges to zero in zero (this last fact is equivalent to the uniform continuity of \( u \)). We say that a function \( \rho : [0, \infty) \rightarrow [0, \infty) \) is a modulus of continuity if it satisfies these properties. Given a modulus of continuity \( \rho(r) \), we say that a function \( u \) is \( \rho \)-continuous if 
\[
|u(y) - u(x)| \leq \rho(|y - x|)
\]
for all \( x \) and \( y \) in \( H \).

**Lemma 2.** If \( u : H \rightarrow \mathbb{R} \) is uniformly continuous, then the functions \( T_t u \) and \( T_t \bar{u} \) converge uniformly to \( u \) when \( t \rightarrow 0 \). Moreover, given a modulus of continuity \( \rho \), there exists a non-decreasing function \( \varepsilon(t) : [0, \infty) \rightarrow [0, \infty) \) satisfying \( \lim_{t \rightarrow 0} \varepsilon(t) = 0 \) and such that, for each \( \rho \)-continuous bounded function \( u \), we have:

- \( T_t u \) and \( T_t \bar{u} \) are \( \rho \)-continuous for each \( t \geq 0 \).
- \( u - \varepsilon(t) \leq T_t u(x) \leq u \) and \( u \leq T_t \bar{u} \leq u + \varepsilon(t) \) for each \( t \geq 0 \).

**Proof.** Let us fix \( y \in H \), and set \( v(x) = u(x + y) \). We have \( u(x) - \rho(|y|) \leq v(x) \leq u(x) + \rho(|y|) \). Applying the operator \( T_t \) gives \( T_t v(x) = u(x + y) + \rho(|y|)/t \). On the other hand, we have
\[
T_t v(x) = \inf_z \{ u(z + y) + ||z - x||^2/t \} = \inf_z \{ (u(z) + ||z - (x + y)||^2/t) = T_t u(x + y),
\]
so that
\[
T_t u(x) - \rho(|y|) \leq T_t u(x + y) \leq T_t u(x) + \rho(|y|).
\]
We have proved that \( T_t u \) is \( \rho \)-continuous if \( u \) is, the proof for \( T_t \bar{u} \) is the same.

In order to study the convergence, let us set \( \varepsilon(t) = \sup_{r > 0} (\rho(r) - r^2/t) \). We have
\[
\varepsilon(t) = \sup_{r > 0} (\rho(r(\sqrt{t}) - r^2) \leq \sup_{r > 0} (r(\sqrt{t})^2 - r^2) = \rho(\sqrt{t}) + \rho^2(\sqrt{t})/4.
\]
We conclude that \( \lim_{t \rightarrow 0} \varepsilon(t) = 0 \). We now come back to the operator \( T_t \), and observe that
\[
u(y) - ||y - x||^2/t \geq u(x) - \rho(|y - x|) + ||y - x||^2/t \geq u(x) - \varepsilon(t)
\]
for each \( x \) and \( y \), so that
\[
u - \varepsilon(t) \leq T_t u \leq u.
\]
**Lemma 3.** For each function $u : H \to (-\infty, +\infty]$, we have $T_t \circ T_t (u) \leq u$ and the equality $T_t \circ T_t (u) = u$ holds if and only if $u$ is t-semi-convex and lower semi-continuous. Similarly, given a function $v : H \to [-\infty, +\infty)$, we have $T_t \circ T_t (v) \geq v$, with equality if and only if $v$ is t-semi-concave and upper semi-continuous.

**Proof.** Let us write explicitly

$$T_t \circ T_t u(x) = \sup_y \inf_z (u(z) + ||z - y||^2/t - ||y - x||^2/t).$$

Taking $z = x$, we obtain the estimate $T_t \circ T_t u(x) \leq \sup_y u(z) = u(z)$. Let us now write

$$T_t \circ T_t u(x) + ||x||^2/t = \sup_y \inf_z (u(z) + ||z||^2/t + (2y/t) \cdot (x - z))$$

which by an obvious change of variable leads to

$$T_t \circ T_t u(x) + ||x||^2/t = \sup_y \inf_z (u(z) + ||z||^2/t + y \cdot (x - z)).$$

We recognize here that the function $T_t \circ T_t u(x) + ||x||^2/t$ is the Legendre bidual of the function $u(x) + ||x||^2/t$. It is well-know that a function is equal to its Legendre bidual if and only if it is convex and lower semi-continuous.

\[\square\]

2. **Proof of the main results.**

**Proof of Theorem 3.** For each function $f$ and each $t > 0$, the function $T_t \circ T_t \circ T_t f$ is both t-semi-concave and t-semi-convex. It is t-semi-convex by Lemma 1, and it is semi-concave because $T_{2t} (T_t f)$ is 2t-semi-concave by Lemma 1, which implies, still by Lemma 1, that $T_t \circ T_{2t} \circ T_t f$ is t-semi-concave. As a consequence, Lemma 5 below implies that the function $R_t f$ is $C^{1,1}$ provided it is locally bounded. The function $R_t (f)$ is bounded if $f$ is bounded, hence its $C^{1,1}$ in this case.

In the case where $f$ is uniformly continuous, Lemma 2 implies that

$$f - \varepsilon (2t) \leq R_t (f) \leq f + 2\varepsilon (t).$$

As a consequence, $R_t (f)$ is converging uniformly to $f$, and it is locally bounded hence $C^{1,1}$.

We now consider two continuous functions $u$ and $v$ such that $u$ and $-v$ are $k$ semi-concave, and such that $v \leq u$. We claim that

$$u \geq f \geq v \implies u \geq T_t \circ T_t f \geq v \text{ and } u \geq T_t \circ T_t f \geq v.$$
for $t \leq k$. This claim implies that $u \geq T_t \circ T_{2t} \circ T_t f \geq v$ when $u \geq f \geq v$ and $t \leq k$. Let us now prove the claim concerning $T_t \circ T_{t}$, the other part being similar. Since $v$ is $k$-semi-convex and continuous, we have $T_t \circ T_{t} v = v$ for $t \leq k$, by Lemma 3. Then,

$$u \geq f \geq T_t \circ T_{t} f \geq T_t \circ T_{t} v = v$$

where the second inequality follows from Lemma 3, and the third from the obvious fact that the operators $T_t$ and $T_t$ are order-preserving.

We have proved that $v \leq R_t(f) \leq u$ if $u \geq f \geq v$ and $t \leq k$. For $t \in [0, k]$, the function $R_t(f)$ is thus locally bounded hence $C^{1,1}$.

**Proof of Theorem 4.** Let $a_i$ be chosen such that the functions

$$(g_i u) \circ \hat{\phi}_i \text{ and } -(g_i v) \circ \hat{\phi}_i$$

are $a_i$-semi-concave on $\mathbb{R}^n$ (when extended by 0 outside of $B^n$). The existence of real numbers $a_i$ with this property follows from Lemma 4 below. Given $u \geq f \geq v$, we can apply Theorem 3 for each $i$ to the functions

$$(g_i u) \circ \hat{\phi}_i \geq (g_i f) \circ \hat{\phi}_i \geq (g_i v) \circ \hat{\phi}_i$$

extended by zero outside of $B^n$. We conclude that, for $t \in [0, 1]$, the function $R_{in}(g_i f) \circ \hat{\phi}_i$ is $C^{1,1}$ and satisfies

$$(g_i u) \circ \hat{\phi}_i \geq R_{in}(g_i f) \circ \hat{\phi}_i \geq (g_i v) \circ \hat{\phi}_i.$$ 

As a consequence, the function

$$[R_{in}(g_i f) \circ \hat{\phi}_i] \circ \hat{\phi}_i^{-1},$$

extended as a function on $M$ equal to 0 outside of $\hat{\phi}_i (B^n)$, is $C^{1,1}$. The function $G_t(f)$ is thus locally a finite sum of $C^{1,1}$ functions hence it is locally $C^{1,1}$. Moreover, we have

$$u = \sum_i g_i u \geq G_t(f) \geq \sum_i g_i v = v.$$ 

We have used:

**Lemma 4.** Let $u : B^n \to \mathbb{R}$ be a bounded function such that $u - \| \cdot \|^2 / a$ is concave, for some $a > 0$. For each compactly supported non-negative $C^2$ function $g : B^n \to \mathbb{R}$, the product $gu$ (extended by zero outside of $B^n$) is semi-concave on $\mathbb{R}^n$.

**Proof.** Since $u$ is bounded, we can assume that $u \geq 0$ on $B^n$. Let $K \subset B^n$ be a compact subset of the open ball $B^n$ which contains the support
of $g$ in its interior. Since the function $u - ||.||^2 / a$ is concave on $B_1$ it admits super-differentials at each point. As a consequence, for each $x \in B^n$, there exists a linear form $l_x$ such that

$$0 \leq u(y) \leq u(x) + l_x \cdot (y - x) + ||y - x||^2 / a$$

for each $y \in B^1$. Moreover, the linear form $l_x$ is bounded independently of $x \in K$. We also have

$$0 \leq g(y) \leq g(x) + dg_x \cdot (y - x) + C||y - x||^2$$

for some $C > 0$, for all $x, y$ in $R^n$. Taking the product, we get, for $x \in K$ and $y \in B^n$,

$$u(y)g(y) \leq u(x)g(x) + (g(x)l_x + u(x)dg_x) \cdot (y - x) + C||y - x||^2 + C||y - x||^3 + C||y - x||^4$$

where $C > 0$ is a constant independent of $x \in K$ and $y \in B^n$, which may change from line to line. As a consequence, setting $L_x = g(x)l_x + u(x)dg_x$, we obtain the inequality

$$(Lu)(y) \leq (Lu)(x) + L_x \cdot (y - x) + C||y - x||^2$$

for each $x \in K$ and $y \in B^n$. If we set $L_x = 0$ for $x \in R^n - K$, the relation (L) holds for each $x \in R^n$ and $y \in R^n$. For $x \in K$ and $y \in B^n$, we have already proved it. Since the linear forms $L_x, x \in K$ are uniformly bounded, we can assume that $L_x \cdot (y - x) + C||y - x||^2 \geq 0$ for all $x \in K$ and $y \in R^n - B^n$ by taking $C$ large enough. Then, (L) holds for all $x \in K$ and $y \in R^n$. For $x \in R^n - K$ and $y$ outside of the support $g$, the relation (L) holds in an obvious way, because $gu(x) = gu(y) = 0$, and $L_x = 0$. For $x \in R^n - K$ and $y$ in the support of $g$, the relation holds provided that $C \geq \max (gu)/d^2$, where $d$ is the distance between the complement of $K$ and the support of $g$. This is a positive number since $K$ is a compact set containing the support of $g$ in its interior. We conclude that the function $(gu)$ is semi-concave on $R^n$. $\square$

For completeness, we also prove, following Fathi:

**Lemma 5.** Let $u : H \rightarrow R$ be a locally bounded function which is both $k$-semi-concave and $k$-semi-convex. Then the function $u$ is $C^{1,1}$, and $6/k$ is a Lipschitz constant for the gradient of $u$.

**Proof.** It is well known that a locally bounded convex function is continuous. We conclude that $u$ is continuous. Let $u$ be a continuous function which is both $k$-semi-concave and $k$-semi-convex. Then, for each $x \in H$,
there exists a unique \( l_x \in H \) such that

\[
|u(x + y) - u(x) - l_x \cdot y| \leq \frac{\|y\|^2}{k}.
\]

We conclude that \( l_x \) is the gradient of \( u \) at \( x \), and we have to prove that the map \( x \rightarrow l_x \) is Lipschitz. We have, for each \( x, y \) and \( z \) in \( H \):

\[
l_x \cdot (y + z) - \|y + z\|^2 / k \leq u(x + y + z) - u(x) \leq l_x \cdot (y + z) + \|y + z\|^2 / k
\]

\[
l_{x+y} \cdot (-y) - \|y\|^2 / k \leq u(x) - u(x + y) \leq l_{x+y} \cdot (-y) + \|y\|^2 / k
\]

\[
l_{x+y} \cdot (-z) - \|z\|^2 / k \leq u(x + y) - u(x + y + z) \leq l_{x+y} \cdot (-z) + \|z\|^2 / k.
\]

Taking the sum, we obtain

\[
|l_{x+y} - l_x \cdot (y + z)| \leq \|y + z\|^2 / k + \|y\|^2 / k + \|z\|^2 / k.
\]

By a change of variables, we get

\[
|l_{x+y} - l_x \cdot (z)| \leq \|z\|^2 / k + \|y\|^2 / k + \|z - y\|^2 / k.
\]

Taking \( \|z\| = \|y\| \), we obtain

\[
|l_{x+y} - l_x \cdot (z)| \leq 6\|z\| \|y\| / k
\]

for each \( z \) such that \( \|z\| = \|y\| \), we conclude that

\[
\|l_{x+y} - l_x\| \leq 6\|y\| / k.
\]

\[
\square
\]

REFERENCES


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