Estimation for Lévy processes from high frequency data within a long time interval

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ESTIMATION FOR LÉVY PROCESSES FROM HIGH FREQUENCY DATA WITHIN A LONG TIME INTERVAL.

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Abstract. In this paper, we study nonparametric estimation of the Lévy density for Lévy processes, first without then with Brownian component. For this, we consider $2n$ (resp. $3n$) discrete time observations with step Δ. The asymptotic framework is: n tends to infinity, $\Delta = \Delta_n$ tends to zero while $n\Delta_n$ tends to infinity. We use a Fourier approach to construct an adaptive nonparametric estimator and to provide a bound for the global $\mathbb{L}^2$-risk. Estimators of the drift and of the variance of the Gaussian component are also studied. We discuss rates of convergence and give examples and simulation results for processes fitting in our framework.January 13, 2010

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1. Introduction

Let $(L_t, t \geq 0)$ be a real-valued Lévy process, *i.e.* a process with stationary independent increments and càdlàg sample paths. The distribution of the process $(L_t, t \geq 0)$ is completely specified by the characteristic function of the random variable L_t which has the form:

$$(1) \qquad \psi_t(u) = \mathbb{E}(\exp iuL_t) = \exp t(iu\tilde{b} - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}^*}(e^{iux} - 1 - iux1_{|x|\leq 1})N(dx)),$$

where $\tilde{b} \in \mathbb{R}$, $\sigma^2 \geq 0$ and $N(dx)$ is a positive measure on $\mathbb{R}^*$ satisfying

$$\int_{\mathbb{R}^*} x^2 \wedge 1 \; N(dx) < \infty$$

(see *e.g.* Bertoin (1996) or Sato (1997)). Thus, the statistical problem for Lévy processes is the joint estimation of its characteristic triple $(\tilde{b}, \sigma^2, N)$ where appears a finite-dimensional parameter $(\tilde{b}, \sigma^2)$ and an infinite dimensional parameter N, the Lévy measure. In most recent contributions, authors consider a discrete time observation of the sample path, with regular sampling interval Δ. Therefore, statistical procedures are based on the *i.i.d.* sample composed of the increments $(Z_k = Z_k^\Delta = L_{k\Delta} - L_{(k-1)\Delta}, k = 1, \ldots, n)$. In the general case, the distribution of the r.v. Z_k is not explicitly given as a function of $(\tilde{b}, \sigma^2, N)$. This is why authors rather use the relationship between the characteristic function ψ_Δ of Z_k and the characteristic triple. Assuming that $N(dx) = n(x)dx$ admits a density, several papers concentrate on the estimation of the Lévy density under various assumptions on the characteristic triple, including the case of $\tilde{b} = \sigma^2 = 0$ or assuming stronger integrability conditions on the Lévy density (see *e.g.* Watteel and Kulperger (2003), Jongbloed and van der Meulen (2006), van Es *et al.* (2007), Figueroa-López (2009) and the references therein, Comte and Genon-Catalot (2008, 2009a,b)). The joint estimation of $(\tilde{b}, \sigma^2, N)$ is investigated in Neumann and Reiss (2009) or Gugushvili (2009). The methods and results differ according to the asymptotic point of view. One may consider that the sampling interval Δ is fixed and that n tends to infinity (low frequency data). This approach, which is

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quite natural, raises mathematical difficulties and does not take into account the underlying continuous time model properties. One may consider that $\Delta = \Delta_n$ tends to 0 as n tends to infinity (high frequency data). Under the assumption that Δ_n tends to 0 within a fixed length time interval ($n\Delta_n = t$ fixed), the estimation of σ has been widely investigated for Lévy processes (see *e.g.* Woerner (2006), Barndorff-Nielsen *et al.* (2006), Jacod (2007)). However, the Lévy density cannot be identified from observations within a finite-length time interval. To identify all parameters in the high-frequency context, one has to assume both that Δ_n tends to 0 and $n\Delta_n$ tends to infinity. This is the point of view adopted in this paper. Our main focus is the nonparametric estimation of the Lévy density $n(.)$ by an adaptive deconvolution method which generalizes the study of Comte and Genon-Catalot (2009a). We also study estimators of the other parameters. More precisely, we assume that the Lévy density satisfies

(H1) $\int_{\mathbb{R}} x^2 n(x)dx < \infty.$

For statistical purposes, this assumption, which was proposed in Neumann and Reiss (2009), has several useful consequences. First, for all t, $\mathbb{E}L_t^2 < +\infty$ and as $\int_{\mathbb{R}}(e^{iux} - 1 - iux)n(x)dx$ is well defined, we get the following expression for (1):

$$\psi_t(u) = \mathbb{E}(\exp iuL_t) = \exp t(iub - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}}(e^{iux} - 1 - iux)n(x)dx), \tag{2}$$

where $b = \mathbb{E}L_1$ has a statistical meaning (contrary to $\tilde{b}$). Thus, the sample path can be expressed as:

$$L_t = bt + \sigma W_t + X_t, \tag{3}$$

where (W_t) and (X_t) are independent processes, (W_t) is a Brownian motion, (X_t) a square integrable pure-jump martingale:

$$X_t = \int_{]0,t]} \int_{\mathbb{R}^*} x(\hat{p}(du, dx) - dun(x)dx),$$

and

$$\hat{p}(du, dx) = \sum_{s \geq 0} \delta_{s,\Delta L_s}(du, dx)$$

is the random Poisson measure associated with the jumps of (L_t) (or (X_t)) with intensity $dun(x)dx$.

In Section 2, we present our main assumptions and some preliminary properties. In Section 3, we assume that $\sigma = 0$ and study the estimation of the function $h(x) = x^2 n(x)$. Using a sample of size $2n$, we build two collections of estimators $(\hat{h}_m, \bar{h}_m)_{m>0}$ indexed by a cut-off parameter m. The collections are obtained by Fourier inversion of two different estimators of the Fourier transform h^* of the function h. The estimators of h^* are built using empirical estimators of the characteristic function ψ_Δ and its first two derivatives. First, we give a bound for the $\mathbb{L}^2$-risk of $(\hat{h}_m, \bar{h}_m)$ for fixed m. Then, introducing an adequate penalty, we propose a data-driven choice of the cut-off parameter which yields an estimator $(\hat{h}_{\hat{m}}, \bar{h}_{\bar{m}})$ for each collection. The $\mathbb{L}^2$-risk of these estimators is studied. We prove that the optimal rate of convergence is automatically reached on Sobolev classes of regularity for the function h.

In Section 4, we consider the general case. To reach the Lévy density and get rid of the unknown σ^2, we must now use derivatives of ψ_Δ up to the order 3 and we estimate the function $p(x) = x^3 n(x)$ developing the Fourier inversion approach and adaptive choice of the cut-off parameter as for h. It is worth stressing that the point of view of small sampling interval is crucial to our study. Indeed, it helps obtaining simple estimators of ψ_Δ and its successive derivatives which are used to estimate the Fourier transform p^* pf p. Section 5 is devoted to the estimation of (b, σ). We study classical empirical means of the observations. This gives an estimator of b but cannot give estimators of σ. To estimate σ, we consider power variation estimators, introduced in Woerner (2006), Barndorff-Nielsen *et al.* (2006), Jacod (2007), Aït-Sahalia and Jacod (2007), under the asymptotic framework of high frequency data within a long time interval. Moreover, a new estimator of σ is obtained from the non parametric study. In Section 6, we give examples of Lévy models satisfying our set of assumptions. We provide numerical simulation results in Section 7. Section 9 contains the main proofs. In Section 10, two classical results, used in proofs, are recalled.

2. Assumptions and preliminary properties.

Let us consider the two functions

$$h(x) = x^2 n(x), \quad p(x) = x^3 n(x),$$

and the assumptions

(H2) (k) $\displaystyle\int_{\mathbb{R}} |x|^k n(x) dx < \infty$.
(H3) h belongs to $\mathbb{L}^2(\mathbb{R})$.
(H4) $\int x^8 n^2(x) dx = \int x^4 h^2(x) dx < \infty$
(H5) p belongs to $\mathbb{L}^2(\mathbb{R})$.
(H6) $\int x^{12} n^2(x) dx = \int x^6 p^2(x) dx < \infty$

Assumption (H2)(k) is a moment assumption. Indeed, according to Sato (1999, Section 5.25, Theorem 5.23), $\mathbb{E}|L_t|^k < \infty$ is equivalent to $\int_{|x|>1} |x|^k n(x) dx < \infty$. Below, for each stated result, the required value of k is given. Under (H1), the function h is integrable and Section 3 is devoted to the nonparametric estimation of h under the additional assumptions (H3)-(H4). Assumption (H4) is only required for the adaptive result. Under (H1)-(H2)(3), the function p is integrable and Section 4 concerns the estimation of p under (H5)-(H6).

Properties of the moments of $L_\Delta = Z_1^\Delta = Z_1$ for small Δ are used in the proofs below.

Lemma 2.1. *Let $p \geq 1$ be an integer and assume (H1)-(H2)(p) with $p \geq 3$. Then, $\mathbb{E}(|Z_1|^p) < +\infty$ and $\mathbb{E}(Z_1) = b\Delta$, $Var(Z_1) = \Delta(\sigma^2 + \int x^2 n(x) dx)$ and for $3 \leq \ell \leq p$, $\mathbb{E}(Z_1^\ell) = \Delta c_\ell + o(\Delta)$ where $c_\ell = \int x^\ell n(x) dx$.*

Proof. Under the assumption, ψ_Δ is p-times derivable and the result follows by computing the successive derivatives of ψ_Δ. □

Thus, under (H1), (H2)(p), $\mathbb{E}(Z_1^\ell/\Delta)$ is bounded for all $\ell \leq p$, for all Δ.

More precise results on the asymptotic behaviour of $(1/\Delta)\mathbb{E}f(Z_1)$ for unbounded functions f are given in Figueroa-López (2008) (see the Appendix). In the sequel, results on the behaviour of the characteristic function ψ_Δ (see (2)) for small Δ are needed.

Lemma 2.2. *Under (H1), $|\psi_\Delta(u) - 1| \leq \Delta|u|(c(u) + \sigma^2|u|)$ where $c(u) = |b| + |\int_0^u |h^*(v)| dv|$, $h^*(v) = \int e^{ivx} h(x) dx$ denotes the Fourier transform of h. If h^* is integrable on $\mathbb{R}$, then*

$$|\psi_\Delta(u) - 1| \leq \Delta|u|(|b| + |h^*|_1 + |u|\sigma^2).$$

Proof. By formula (2), under (H1), ψ_Δ is C^1 with:

$$\psi'_\Delta(u) = \Delta\psi_\Delta(u)(\phi(u) - \sigma^2 u), \tag{4}$$

where we have set, using that $e^{iux} - 1 = ix \int_0^u e^{ivx} dv$:

$$\phi(u) = ib - \int_0^u h^*(v)dv. \tag{5}$$

We have $|\phi(u)| \le |b| + |\int_0^u |h^*(v)|dv|$ and by the Taylor formula, $\psi_\Delta(u) - 1 = u\psi'_\Delta(c_u u)$ for some $c_u \in (0,1)$. The result follows. □

3. Case of no Gaussian component.

In this section, we assume that $\sigma^2 = 0$ and focus on the nonparametric estimation of h. For reasons that will appear below, we assume that we have at our disposal a $2n$-sample, $(Z_k)_{1\le k\le 2n}$, with $Z_k = Z_k^\Delta = L_{k\Delta} - L_{(k-1)\Delta}$. We assume that $\Delta = \Delta_n$ tends to 0 and $n\Delta_n$ tends to infinity. Hence, Δ and Z_k depend on n. However, to simplify notations, we omit the dependence on n and simply write Δ, Z_k.

3.1. **Definition of estimators depending on a cut-off parameter.** For a complex valued function f belonging to $\mathbb{L}^1(\mathbb{R})$, we denote its Fourier transform by $f^*(u) = \int e^{iux} f(x)dx$. For integrable and square integrable functions f, f_1, f_2, we use the following notations:

$$||f|| = \int |f(x)|^2 dx, \quad < f_1, f_2 >= \int f_1(x)\bar{f}_2(x)dx,$$

($\bar{z}$ denotes the conjugate of the complex number z). We have:

$$(f^*)^*(x) = 2\pi f(-x), \quad < f_1, f_2 >= \frac{1}{2\pi} < f_1^*, f_2^* > .$$

By formula (2), under (H1), ψ_Δ is C^2 and we have, as $\sigma^2 = 0$ (see (5)):

$$\frac{\psi'_\Delta(u)}{\psi_\Delta(u)} = i\Delta(b + \int \frac{e^{iux} - 1}{x} h(x)dx) = \Delta\phi(u).$$

Derivating again gives:

$$\frac{\psi"_\Delta(u)\psi_\Delta(u) - (\psi'_\Delta(u))^2}{\psi^2_\Delta(u)} = -\Delta \int e^{iux} h(x)dx = -\Delta h^*(u).$$

The construction of estimators is based on the formulae:

$$h^*(u) = -\frac{1}{\Delta}\left(\frac{\psi"_\Delta(u)\psi_\Delta(u) - (\psi'_\Delta(u))^2}{\psi^2_\Delta(u)}\right) = -\frac{1}{\Delta}\left(\frac{\psi"_\Delta(u)}{\psi_\Delta(u)} - \frac{(\psi'_\Delta(u))^2}{\psi^2_\Delta(u)}\right) \tag{6}$$

and the high frequency setting which yields that for all u, $\lim_{\Delta\to 0} \psi_\Delta(u) = 1$. By splitting the $2n$-sample into two independent subsamples of n observations, we introduce the following empirical unbiased estimators of $\psi_\Delta, \psi'_\Delta, \psi''_\Delta$:

$$\hat{\psi}^{(j)}_{\Delta,q}(u) = \frac{1}{n} \sum_{k=1+(q-1)n}^{qn} (iZ_k)^j e^{iuZ_k}, \ \ j = 0, 1, 2, \ q = 1, 2.$$

We also define, based on the full sample, the estimator of ψ''_Δ:

$$\hat{\psi}^{(2)}_\Delta(u) = \frac{1}{2n} \sum_{k=1}^{2n} (iZ_k)^2 e^{iuZ_k}.$$

We now build estimators of the Fourier transform h^* of h. Considering the first expression of h^* in (6), we replace $\psi_\Delta, \psi'_\Delta, \psi''_\Delta$ in the numerator by the empirical estimators built on the two independent subsamples of size n. In the denominator, ψ^2_Δ is simply replaced by 1. This yields:

$$\hat{h}^*(u) = \frac{1}{\Delta}\left(\hat{\psi}^{(1)}_{\Delta,1}(u)\hat{\psi}^{(1)}_{\Delta,2}(u) - \hat{\psi}^{(2)}_{\Delta,1}(u)\hat{\psi}^{(0)}_{\Delta,2}(u)\right). \tag{7}$$

Hence, using independence of the two subsamples,

$$\mathbb{E}\hat{h}^*(u) = \frac{1}{\Delta}\left((\psi'_\Delta(u))^2 - \psi"_\Delta(u)\psi_\Delta(u)\right) = h^*(u) + h^*(u)(\psi^2_\Delta(u) - 1).$$

Introducing a cut-off parameter m, we define an associated estimator of h, by:

$$\hat{h}_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-iux}\hat{h}^*(u)du.$$

This means that $\hat{h}^*_m(u) = \hat{h}^*(u)1_{[-\pi m,\pi m]}(u)$. By integration, noting that

$$\frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-iuz}du = \frac{\sin(\pi m z)}{\pi z},$$

the following expression is available

$$\hat{h}_m(x) = \frac{1}{n^2\Delta}\sum_{1\le j,k\le n}(Z_k^2 - Z_kZ_{n+j})\frac{\sin(\pi m(Z_k + Z_{j+n} - x))}{\pi(Z_k + Z_{j+n} - x)}.$$

We also define another estimator of h^* of h by setting:

$$\bar{h}^*(u) = -\frac{1}{\Delta}\hat{\psi}^{(2)}_\Delta(u). \tag{8}$$

Here, using (6), we get

$$\mathbb{E}\bar{h}^*(u) = -\frac{1}{\Delta}\psi"_\Delta(u) = h^*(u) + h^*(u)(\psi_\Delta(u) - 1) - \Delta\psi_\Delta(u)\phi^2(u). \tag{9}$$

Thus, $\bar{h}^*$ is simpler but has an additional bias term. We set:

$$\bar{h}_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-iux}\bar{h}^*(u)du = \frac{1}{2n\Delta}\sum_{k=1}^{2n} Z_k^2\,\frac{\sin(\pi m(Z_k - x))}{\pi(Z_k - x)}. \tag{10}$$

3.2. **Risk for a fixed cut-off parameter.** Next, let us define

$$h_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-iux}h^*(u)du.$$

Then we can prove the following result

Proposition 3.1. *Assume that (H1)-(H2)(4) and (H3) hold. Then we have*

$$\mathbb{E}(\|\hat{h}_m - h\|^2) \le \|h_m - h\|^2 + 72\mathbb{E}(Z_1^4/\Delta)\frac{m}{n\Delta} + \frac{4\Delta^2}{\pi}\int_{-\pi m}^{\pi m} u^2c^2(u)|h^*(u)|^2du. \tag{11}$$

And,

$$\mathbb{E}(\|\bar{h}_m - h\|^2) \le \|h_m - h\|^2 + \mathbb{E}(Z_1^4/\Delta)\frac{m}{n\Delta} + \frac{2\Delta^2}{\pi}\int_{-\pi m}^{\pi m} u^2c^2(u)|h^*(u)|^2du + C\Delta^2B_m, \tag{12}$$

with C a constant, $c(u)$ is defined in Lemma 2.2, B_m is defined in (16) and satisfies $B_m = O(m)$ if $h^* \in \mathbb{L}_1(\mathbb{R})$ and $B_m = O(m^5)$ otherwise.

Proof. First, the Parseval formula gives $\|\hat{h}_m - h\|^2 = (1/(2\pi))\|\hat{h}^*_m - h^*\|^2$ and we can note that $h^*(u) - h^*_m(u) = h^*(u)\mathbf{1}_{|u|\geq \pi m}$ is orthogonal to $\hat{h}^*_m - h^*_m$ which has its support in $[-\pi m, \pi m]$. Thus,

$$\|\hat{h}_m - h\|^2 = \frac{1}{2\pi}(\|h^* - h^*_m\|^2 + \|h^*_m - \hat{h}^*_m\|^2).$$

The first term $(1/(2\pi))\|h^* - h^*_m\|^2 = \|h - h_m\|^2$ is a classical squared bias term. Next,

$$\begin{aligned}\hat{h}^*_m(u) - h^*_m(u) &= [\hat{h}^*_m(u) - \mathbb{E}(\hat{h}^*_m(u))] + [\mathbb{E}(\hat{h}^*_m(u)) - h^*_m(u)] \\ &= [\hat{h}^*_m(u) - \mathbb{E}(\hat{h}^*_m(u))] + [\psi^2_\Delta(u) - 1]h^*(u)\mathbf{1}_{|u|\leq \pi m}.\end{aligned}$$

Bounding the norm of $\|\hat{h}^*_m - h^*_m\|^2$ by twice the sum of the norms of the two elements of the decomposition, we get

$$\begin{aligned}\mathbb{E}(\|\hat{h}_m - h_m\|^2) &\leq \frac{1}{\pi}\mathbb{E}\left(\int_{-\pi m}^{\pi m} |\hat{h}^*(u) - \mathbb{E}\hat{h}^*(u)|^2 du\right) + \frac{1}{\pi}\int_{-\pi m}^{\pi m} |\psi^2_\Delta(u) - 1|^2 |h^*(u)|^2 du \\ &\leq \frac{1}{\pi}\left(\int_{-\pi m}^{\pi m} Var(\hat{h}^*(u))du\right) + \frac{4\Delta^2}{\pi}\int_{-\pi m}^{\pi m} u^2 c^2(u)|h^*(u)|^2 du\end{aligned}$$

(see Lemma 2.2 for the upper bound of $|\psi_\Delta(u) - 1|$ and note that $|\psi_\Delta(u)| \leq 1$). Now, we use the decomposition:

$$\begin{aligned}\Delta(\hat{h}^*(u) - \mathbb{E}(\hat{h}^*(u))) &= (\hat{\psi}^{(1)}_{\Delta,1}(u) - \psi'_\Delta(u))(\hat{\psi}^{(1)}_{\Delta,2}(u) - \psi'_\Delta(u)) \\ &\quad + (\hat{\psi}^{(1)}_{\Delta,1}(u) - \psi'_\Delta(u))\psi'_\Delta(u) + (\hat{\psi}^{(1)}_{\Delta,2}(u) - \psi'_\Delta(u))\psi'_\Delta(u) \\ &\quad - (\hat{\psi}^{(2)}_{\Delta,1}(u) - \psi''_\Delta(u))(\hat{\psi}^{(0)}_{\Delta,2}(u) - \psi_\Delta(u)) \\ &\quad - (\hat{\psi}^{(2)}_{\Delta,1}(u) - \psi''_\Delta(u))\psi_\Delta(u) - (\hat{\psi}^{(0)}_{\Delta,2}(u) - \psi_\Delta(u))\psi''_\Delta(u).\end{aligned} \tag{13}$$

Considering each term consecutively and exploiting the independence of the samples, we obtain

$$\mathrm{Var}(\hat{h}^*(u)) \leq \frac{6}{\Delta^2}\left(\frac{\mathbb{E}^2(Z_1^2)}{n^2} + 2\frac{\mathbb{E}^2(Z_1^2)}{n} + \frac{\mathbb{E}(Z_1^4)}{n^2} + 2\frac{\mathbb{E}(Z_1^4)}{n}\right) \leq 36\frac{\mathbb{E}(Z_1^4/\Delta)}{n\Delta}. \tag{14}$$

Thus, the first risk bound is proved.

Analogously, we have

$$\mathbb{E}(\|\bar{h}_m - h\|^2) \leq \|h_m - h\|^2 + \frac{1}{\pi}\int_{-\pi m}^{\pi m} |\mathbb{E}\bar{h}^*(u) - h^*(u)|^2 du + \frac{1}{\pi}\int_{-\pi m}^{\pi m} Var(\bar{h}^*(u))du \tag{15}$$

For the variance of $\bar{h}^*(u)$, we use:

$$\bar{h}^*(u) - \mathbb{E}\bar{h}^*(u) = -\frac{1}{\Delta}(\hat{\psi}^{(2)}_\Delta(u) - \psi''_\Delta(u)).$$

Thus,

$$\mathrm{Var}(\bar{h}^*(u)) \leq \frac{1}{2n\Delta}\mathbb{E}(Z_1^4/\Delta).$$

Next, for the bias of $\bar{h}^*(u)$, we use (see (9)-(5)):

$$|\mathbb{E}\bar{h}^*(u) - h^*(u)|^2 \leq 2|h^*(u)|^2||\psi_\Delta(u) - 1|^2 + 2\Delta^2|\phi^4(u)|.$$

Hence, there is an additional term in the risk bound equal to

$$\frac{2}{\pi}\Delta^2\int_{-\pi m}^{\pi m} |\phi^4(u)|du = \Delta^2 B_m. \tag{16}$$

If h^* is integrable, $|\phi(u)| \leq C$, and $B_m = O(m)$. Otherwise, $|\phi^4(u)| \leq C|u|^4$, and $B_m = O(m^5)$. □

Remark 3.1. *We stress that the estimator $\hat{h}_m$ is more complicated to study, but $\bar{h}_m$ has an additional bias term.*

3.3. **Rates of convergence in Sobolev classes.** We consider classes of functions h belonging to the set:

$$\mathcal{C}(a,L)=\{f\in(\mathbb{L}^1\cap\mathbb{L}^2)(\mathbb{R}),\int(1+u^2)^a|f^*(u)|^2du\le L\}. \tag{17}$$

Then we can prove the following result:

Proposition 3.2. *Assume that (H1)-(H2)(4) and (H3) hold and that h belongs to $\mathcal{C}(a,L)$ with $a>1/2$. Consider the asymptotic setting where $n\to+\infty$, $\Delta\to 0$, $n\Delta\to+\infty$ and assume that $m\le n\Delta$. If $n\Delta^2\le 1$, then, for the optimal choice $m=O((n\Delta)^{1/(2a+1)})$, we have:*

$$\mathbb{E}(\|\hat{h}_m-h\|^2)\le O((n\Delta)^{-2a/(2a+1)}).$$

If $a\ge 1$, the condition $n\Delta^2\le 1$ can be replaced by $n\Delta^3\le 1$.

The same result holds for $\bar{h}_m$.

Proof. As $\|h-h_m\|^2=(1/\pi)\int_{|u|\ge\pi m}|h^*(u)|^2du$, the definition of $\mathcal{C}(a,L)$ implies clearly that $\|h-h_m\|^2\le(L/2\pi)(\pi m)^{-2a}$. The compromise between this term and the variance term of order $m/(n\Delta)$ is standard: it leads to choose $m=O((n\Delta)^{1/(2a+1)})$ and yields the order $O((n\Delta)^{-2a/(2a+1)})$.

For $a>1/2$, we have

$$\begin{aligned}|\int_0^u|h^*(v)|dv| &= |\int_0^u|h^*(v)|(1+v^2)^{a/2}(1+v^2)^{-a/2}dv|\\ &\le \left(\int|h^*(v)|^2(1+v^2)^a\int(1+v^2)^{-a}dv\right)^{1/2}\\ &\le \sqrt{L\int(1+v^2)^{-a}dv},\end{aligned}$$

where $\int(1+v^2)^{-a}dv<+\infty$. Therefore, h^* is integrable and $|\phi(u)|\le|b|+|h^*|_1$.

The last term in the risk bound (11) is less than

$$K\Delta^2\int_{-\pi m}^{\pi m}u^2|h^*(u)|^2du\le L\Delta^2(\pi m)^{2(1-a)_+}.$$

If $a\ge 1$ and $n\Delta^3\le 1$, we have $\Delta^2(\pi m)^{2(1-a)_+}=\Delta^2\le(n\Delta)^{-1}$.

If $a\in(1/2,1)$, the inequality $\Delta^2m^{2(1-a)}\le m^{-2a}$ is equivalent to $\Delta^2m^2\le 1$. As $m\le n\Delta$, $\Delta^2m^2\le 1$ holds if $n\Delta^2\le 1$.

For the additional bias term appearing in the risk bound of $\bar{h}_m$, we have $B_m=O(m)$. Thus, $m\Delta^2\le m^{-2a}$ holds, for $m=O((n\Delta)^{1/(2a+1)})$, if $m^{1+2a}\Delta^2=(n\Delta)\Delta^2\le 1$ which in turn holds if $n\Delta^3\le 1$. □

Remark 3.2. *We can also discuss the case where $a\in(0,1/2]$. If $a\le 1/2$, $|\int_0^u|h^*(v)|dv|=O(|u|^{1/2-a})$. Hence, the last term in (11) is of order Δ^2m^{3-4a} which is less than m^{-2a} if $\Delta^2m^{3-2a}\le 1$ and thus $\Delta^2m^3\le 1$. This requires $n\Delta^{5/3}\le 1$. The same constraint appears for $\bar{h}_m$ by looking at the exact formula for Δ^2B_m (see (16)).*

3.4. **Adaptation.** The estimators $\hat{h}_m, \bar{h}_m$ are deconvolution estimators that can also be described as minimum contrast estimators and projection estimators. For details, the reader is referred to Comte and Genon-Catalot (2009a,2009b). For $m > 0$, let

$$S_m = \{f \in \mathbb{L}^2(\mathbb{R}),\ \text{support}(f^*) \subset [-\pi m, \pi m]\}.$$

The space S_m is generated by an orthonormal basis, the sinus cardinal basis, defined by:

$$\varphi_{m,j}(x) = \sqrt{m}\varphi(mx - j), j \in \mathbb{Z}, \quad \varphi(x) = \frac{\sin \pi x}{\pi x} \quad (\varphi(0) = 1).$$

This is due to the fact that

$$\varphi^*_{m,j}(u) = \frac{e^{iuj/m}}{\sqrt{m}} 1_{[-\pi m, \pi m]}(u), j \in \mathbb{Z}.$$

For a function $f \in \mathbb{L}^2(\mathbb{R})$, $f_m(x) = 1/2\pi \int_{-\pi m}^{\pi m} e^{-iux} f^*(u)du$ is the orthogonal projection of f on S_m. Introducing, for a function $t \in S_m$,

$$\gamma_n(t) = ||t||^2 - \frac{1}{\pi} < \hat{h}^*, t^* >= ||t||^2 - 2 < \hat{h}_m, t >,$$

we get:

$$\hat{h}_m = \arg \min_{t \in S_m} \gamma_n(t),$$

and $\gamma_n(\hat{h}_m) = -\|\hat{h}_m\|^2$. We have

$$\hat{h}_m = \sum_{j \in \mathbb{Z}} \hat{a}_{m,j}\varphi_{m,j}, \quad \text{with} \quad \hat{a}_{m,j} = \frac{1}{2\pi} \int_{-\pi m}^{\pi m} \hat{h}^*(u)\varphi^*_{m,j}(-u)du.$$

and

$$\|\hat{h}_m\|^2 = \frac{1}{2\pi} \int_{-\pi m}^{\pi m} |\hat{h}^*(u)|^2 du$$

The coefficients $\hat{a}_{m,j}$ of the series as well as $\|\hat{h}_m\|^2$ can be explicitly computed by integration. In the same way, we set

$$\Gamma_n(t) = ||t||^2 - \frac{1}{\pi} < \bar{h}^*, t^* >= ||t||^2 - 2 < \bar{h}_m, t >,$$

and obtain:

$$\bar{h}_m = \arg \min_{t \in S_m} \Gamma_n(t),$$

Analogously, $\bar{h}_m$ has a series expansion on the sinus cardinal basis with explicit coefficients and $\|\bar{h}_m\|^2$ has a closed-form formula. We give the explicit expression of $\|\bar{h}_m\|^2$ which is less cumbersome than $\|\hat{h}_m\|^2$:

$$\|\bar{h}_m\|^2 = \frac{m}{4n^2\Delta^2} \sum_{1 \le k,l \le 2n} Z_k^2 Z_l^2 \varphi(m(Z_k - Z_l)). \tag{18}$$

Now, we need to select the best m as possible, in a set $\mathcal{M}_n = \{m \in \mathbb{N}, 1 \le m \le n\Delta\} = \{1, \dots, m_n\}$. For the estimators $\hat{h}_m$, we propose to take

$$\hat{m} = \arg \min_{m \in \mathcal{M}_n} \left(-\|\hat{h}_m\|^2 + \text{pen}(m)\right) \tag{19}$$

with

$$\text{pen}(m) = \kappa \frac{m}{n\Delta^2} \left((\frac{1}{n}\sum_{k=1}^{n} Z_k^2)(\frac{1}{n}\sum_{k=n+1}^{2n} Z_k^2) + \frac{1}{n}\sum_{k=1}^{n} Z_k^4 \right).$$

The intuition for this choice is the following. The expression of $\text{pen}(m)$ is an estimator of the variance term of the risk bound (11) as close as possible of the variance (see (14)). The term $-\|\hat{h}_m\|^2$ is an estimator of $-\|h_m\|^2 = \|h - h_m\|^2 - \|h\|^2$, which is up to a constant, the bias term of the bound (11). This is why $\hat{m}$ mimics the optimal bias-variance compromise.

For the estimators $\bar{h}_m$, we define

$$\bar{m} = \arg\min_{m \in \mathcal{M}_n} \left(-\|\bar{h}_m\|^2 + \kappa' \frac{m}{n\Delta^2} \left(\frac{1}{2n} \sum_{k=1}^{2n} Z_k^4 \right) \right). \tag{20}$$

The following result shows that the above data-driven choices of the cut-off parameter are indeed relevant.

Theorem 3.1. *Assume (H1)-(H2)(16)-(H3)-(H4). If, moreover, $h^* \in \mathbb{L}^1(\mathbb{R})$ and $n\Delta^3 \leq 1$, there exist a numerical constants κ, κ' such that*

$$\begin{aligned}
\mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2) &\leq C \inf_{m \in \mathcal{M}_n} \left(\|h - h_m\|^2 + \kappa(\Delta \mathbb{E}^2(\frac{Z_1^2}{\Delta}) + \mathbb{E}(\frac{Z_1^4}{\Delta})) \frac{m}{n\Delta} \right) \\
&\quad + \frac{\Delta^2}{\pi} \int_{-\pi m_n}^{\pi m_n} u^2 |h^*(u)|^2 du + C \frac{\ln^2(n\Delta)}{n\Delta}, \\
\mathbb{E}(\|\bar{h}_{\bar{m}} - h\|^2) &\leq C \inf_{m \in \mathcal{M}_n} \left(\|h - h_m\|^2 + \kappa' \mathbb{E}(\frac{Z_1^4}{\Delta}) \frac{m}{n\Delta} \right) \\
&\quad + \frac{\Delta^2}{\pi} \int_{-\pi m_n}^{\pi m_n} u^2 |h^*(u)|^2 du + \Delta^2 B_{m_n} + C \frac{\ln^2(n\Delta)}{n\Delta},
\end{aligned}$$

where $B_{m_n} = O(m_n)$ (B_{m_n} is defined in (16)).

The numerical constants κ, κ' have to be calibrated via simulations (see discussion in Comte and Genon-Catalot (2009a)).

By computations analogous to those in the proof of Proposition 3.2, we obtain the following Corollary.

Corollary 3.1. *Assume that the assumptions of Theorem 3.1 are fulfilled. If, for some positive L, $h \in \mathcal{C}(a, L)$ with $a > 1/2$, then $\mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2) = O((n\Delta)^{-2a/(2a+1)})$ provided that $n\Delta^2 \leq 1$. The same holds for $\mathbb{E}(\|\bar{h}_{\bar{m}} - h\|^2)$. If $a \geq 1$, the constraint $n\Delta^3 \leq 1$ is enough.*

4. Study of the general case ($\sigma^2 \neq 0$)

In this section, we assume (H1)-(H2)(3) and study the estimation of the function

$$p(x) = x^3 n(x).$$

We suppose that we have a sample of size $3n$, $(Z_k)_{1 \leq k \leq 3n}$, $Z_k = L_{k\Delta} - L_{(k-1)\Delta}$. As previously, we can build two estimators of p^*, one is simpler but has more bias than the other one.

4.1. Definition of the estimators.

We have to compute the three first derivatives of ψ_Δ to get rid of σ^2 (see (5):

$$\frac{\psi'_\Delta(u)}{\psi_\Delta(u)} = \Delta(ib - u\sigma^2 + i \int \frac{e^{iux} - 1}{x} h(x) dx) = \Delta(\phi(u) - u\sigma^2)$$

Derivating again gives:

$$\frac{\psi''_\Delta(u)\psi_\Delta(u) - (\psi'_\Delta(u))^2}{(\psi_\Delta(u))^2} = \Delta(\phi'(u) - \sigma^2) = -\Delta(\sigma^2 + \int e^{iux} x^2 n(x) dx), \tag{21}$$

and lastly

$$\frac{\psi_\Delta^{(3)}(u)\psi_\Delta^2(u) - 3\psi''_\Delta(u)\psi'_\Delta(u)\psi_\Delta(u) + 2[\psi'_\Delta(u)]^2}{\psi_\Delta^3(u)} = -\Delta i p^*(u).$$

Therefore, we get the formulae:

$$\begin{aligned} p^*(u) &= \frac{i}{\Delta}\left(\frac{\psi_\Delta^{(3)}(u)\psi_\Delta^2(u) - 3\psi''_\Delta(u)\psi'_\Delta(u)\psi_\Delta(u) + 2[\psi'_\Delta(u)]^3}{\psi_\Delta^3(u)}\right) \\ &= \frac{i}{\Delta}\left(\frac{\psi_\Delta^{(3)}(u)}{\psi_\Delta(u)} - 3\frac{\psi''_\Delta(u)\psi'_\Delta(u)}{\psi_\Delta^2(u)} + 2\frac{[\psi'_\Delta(u)]^3}{\psi_\Delta^3(u)}\right), \end{aligned}$$

where $\psi_\Delta(u)$ is close to one. We define empirical estimators of $\psi_\Delta(u), \psi'_\Delta(u), \psi''_\Delta(u), \psi_\Delta^{(3)}(u)$ using the independent subsamples:

$$\hat{\psi}_{\Delta,q}^{(j)}(u) = \frac{1}{n}\sum_{k=1+(q-1)n}^{qn} (iZ_k)^j e^{iuZ_k}, \quad j = 0,1,2,3, \quad q = 1,2,3.$$

The following estimator of p^* is obtained by replacing expressions in the numerator of p^* by their empirical counterpart (using the independent subsamples); moreover, the denominator is replaced by 1. We get

$$\hat{p}^*(u) = \frac{i}{\Delta}\left(\hat{\psi}_{\Delta,1}^{(3)}(u)\hat{\psi}_{\Delta,2}^{(0)}(u)\hat{\psi}_{\Delta,3}^{(0)}(u) - 3\hat{\psi}_{\Delta,1}^{(2)}(u)\hat{\psi}_{\Delta,2}^{(1)}(u)\hat{\psi}_{\Delta,3}^{(0)}(u) + 2\prod_{q=1}^{3}\hat{\psi}_{\Delta,q}^{(1)}(u)\right). \tag{22}$$

The bias of $\hat{p}^*(u)$ is given by

$$\mathbb{E}\hat{p}^*(u) - p^*(u) = (\psi_\Delta^3(u) - 1)p^*(u) \tag{23}$$

Note that, as $|\psi_\Delta(u)| \le 1$, $|\psi_\Delta^3(u) - 1| \le 3|\psi_\Delta(u) - 1|$. The estimator of p associated to $\hat{p}^*(u)$ with cut-off parameter m is:

$$\hat{p}_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-iux}\hat{p}^*(u)du,$$

which has a closed-form formula obtained by integration. We also define, based on all the observations,

$$\hat{\psi}_\Delta^{(3)}(u) = \frac{1}{3n}\sum_{k=1}^{3n}(iZ_k)^3 e^{iuZ_k}.$$

And

$$\bar{p}^*(u) = \frac{i}{\Delta}\hat{\psi}_\Delta^{(3)}(u) \quad \text{and} \quad \bar{p}_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-iux}\bar{p}^*(u)du.$$

Here,

$$\mathbb{E}\bar{p}^*(u) - p^*(u) = (\psi_\Delta(u) - 1)p^*(u) + 3i\frac{\psi''_\Delta(u)\psi'_\Delta(u)}{\Delta\psi_\Delta(u)} - 2i\frac{[\psi'_\Delta(u)]^3}{\Delta\psi_\Delta^2(u)} \tag{24}$$

Let us set:

$$\tilde{\phi}(u) = \phi(u) - u\sigma^2 = ib - \int_0^u h^*(v)dv - u\sigma^2. \tag{25}$$

Using $\psi'_\Delta(u) = \Delta\psi_\Delta(u)\tilde{\phi}(u)$ and some computations, we get:

$$\mathbb{E}\bar{p}^*(u) - p^*(u) = (\psi_\Delta(u) - 1)p^*(u) - 3i\Delta\psi_\Delta(u)\tilde{\phi}(u)(\sigma^2 + h^*(u)) + i\Delta^2\psi_\Delta(u)(\tilde{\phi}(u))^3, \tag{26}$$

which shows additional terms in comparison with the bias (23). The corresponding estimator has the expression:

$$\bar{p}_m(x) = \frac{1}{3n\Delta} \sum_{k=1}^{3n} Z_k^3 \, \frac{\sin(\pi m(Z_k - x))}{\pi(Z_k - x)}. \tag{27}$$

4.2. **Risk of the estimators.** Let

$$p_m(x) = \frac{1}{2\pi} \int_{-\pi m}^{\pi m} e^{-iux} p^*(u) du$$

denote the orthogonal projection of p on S_m. The risk of the estimators with fixed cut-off parameter is bounded as follows.

Proposition 4.1. *Under (H1)-(H2)*(6) *and (H5),*

$$\mathbb{E}(\|\hat{p}_m - p\|^2) \le \|p - p_m\|^2 + C_0 \mathbb{E}(Z_1^6/\Delta) \frac{m}{n\Delta} + C\frac{\Delta^2}{\pi} \int_{-\pi m}^{\pi m} u^2(1+u^2)|p^*(u)|^2 du \tag{28}$$

where C_0 in a numerical constant much larger than 1.
We also have

$$\begin{aligned} \mathbb{E}(\|\bar{p}_m - p\|^2) &\le \|p - p_m\|^2 + \mathbb{E}(Z_1^6/\Delta) \frac{m}{n\Delta} \\ &\quad + C(\Delta^2 \int_{-\pi m}^{\pi m} u^2(1+u^2)|p^*(u)|^2 du + \Delta^2 m^3 + \Delta^4 m^7). \end{aligned}$$

Proof. As previously,

$$\|\hat{p}_m - p\|^2 = \frac{1}{2\pi}(\|p^* - p_m^*\|^2 + \|p_m^* - \hat{p}_m^*\|^2).$$

The first term $(1/(2\pi))(\|p^* - p_m^*\|^2 = \|p - p_m\|^2$ is the projection bias on S_m. Next,

$$\begin{aligned} \hat{p}_m^*(u) - p_m^*(u) &= [\hat{p}_m^*(u) - \mathbb{E}(\hat{p}_m^*(u))] + [\mathbb{E}(\hat{p}_m^*(u)) - p_m^*(u)] \\ &= [\hat{p}_m^*(u) - \mathbb{E}(\hat{p}_m^*(u))] + [(\psi_\Delta(u))^3 - 1] p^*(u) \mathbf{I}_{|u| \le \pi m}. \end{aligned}$$

Using Lemma 2.2 and (23), we get

$$\begin{aligned} \mathbb{E}(\|\hat{p}_m - p_m\|^2) &\le \frac{1}{\pi} \int_{-\pi m}^{\pi m} C\Delta^2 u^2(1+u^2)|p^*(u)|^2 du + \frac{1}{\pi}\mathbb{E}\left(\int_{-\pi m}^{\pi m} |\hat{p}^*(u) - \mathbb{E}\hat{p}^*(u)|^2 du\right) \\ &\le C\frac{\Delta^2}{\pi} \int_{-\pi m}^{\pi m} u^2(1+u^2)|p^*(u)|^2 du + \frac{1}{\pi}\left(\int_{-\pi m}^{\pi m} Var(\hat{p}^*(u)) du\right). \end{aligned}$$

The estimator $\hat{p}^*$ is the sum of three terms, each term being the product of three independent variables. To bound the variance of $\hat{p}^*$, for each term, we subtract the corresponding expectation and use a decomposition with only centered terms. For instance, the first centered term of $\hat{p}^*$ is split as follows:

$$\begin{aligned} & \hat{\psi}_{\Delta,1}^{(3)} \hat{\psi}_{\Delta,2}^{(0)} \hat{\psi}_{\Delta,3}^{(0)} - \psi_\Delta^{(3)} [\psi_\Delta]^2 \\ = \; & (\hat{\psi}_{\Delta,1}^{(3)} - \psi_\Delta^{(3)})(\hat{\psi}_{\Delta,2}^{(0)} - \psi_\Delta)(\hat{\psi}_{\Delta,3}^{(0)} - \psi_\Delta) \\ & + (\hat{\psi}_{\Delta,1}^{(3)} - \psi_\Delta^{(3)})(\hat{\psi}_{\Delta,2}^{(0)} - \psi_\Delta)\psi_\Delta + (\hat{\psi}_{\Delta,1}^{(3)} - \psi_\Delta^{(3)})(\hat{\psi}_{\Delta,3}^{(0)} - \psi_\Delta)\psi_\Delta + \psi_\Delta^{(3)}(\hat{\psi}_{\Delta,2}^{(0)} - \psi_\Delta)(\hat{\psi}_{\Delta,3}^{(0)} - \psi_\Delta) \\ & + (\hat{\psi}_{\Delta,1}^{(3)} - \psi_\Delta^{(3)})[\psi_\Delta]^2 + \psi_\Delta^{(3)}(\hat{\psi}_{\Delta,2}^{(0)} - \psi_\Delta)\psi_\Delta + \psi_\Delta^{(3)}\psi_\Delta(\hat{\psi}_{\Delta,3}^{(0)} - \psi_\Delta), \end{aligned}$$

and analogously for the two other terms in the formula of the estimator.

Considering each term consecutively and exploiting the independence of the samples, we obtain

$$\text{(29)} \qquad \mathrm{Var}(\hat{p}^*(u)) \leq \frac{7\times 3}{\Delta^2}\left(\mathbb{E}(Z_1^6)+9\mathbb{E}(Z_1^4)\mathbb{E}(Z_1^2)+4[\mathbb{E}(Z_1^2)]^3\right)\left(\frac{1}{n^3}+\frac{3}{n^2}+\frac{3}{n}\right)$$

The main term is $\mathbb{E}(Z_1^6)/(n\Delta^2) = [\mathbb{E}(Z_1^6)/\Delta]/(n\Delta)$ coming from $\mathbb{E}(|\hat{\psi}^{(3)}_{\Delta,1}(u)-\psi^{(3)}_{\Delta}(u)|^2)|\psi_\Delta(u)|^4$. By the Hölder formula, we have $\mathbb{E}(Z_1^4)\leq(\mathbb{E}(Z_1^6))^{2/3}$ and $\mathbb{E}(Z_1^2)\leq(\mathbb{E}(Z_1^6))^{1/3}$. Thus,

$$\frac{1}{2\pi}\int_{-\pi m}^{\pi m} Var(\hat{p}^*(u))du \leq 2058\ [\mathbb{E}(Z_1^6)/\Delta]m/(n\Delta).$$

This ends the study of $\hat{h}_m$.

Now, let us study $\bar{h}_m$. Here, the variance satisfies:

$$\mathrm{Var}(\bar{p}^*(u)) \leq \frac{\mathbb{E}(Z_1^6)}{n\Delta^2} = \frac{\mathbb{E}(Z_1^6/\Delta)}{n\Delta}.$$

As previously,

$$\mathbb{E}(\|\bar{p}_m - p_m\|^2) = \frac{1}{2\pi}\mathbb{E}(\|\bar{p}^*_m - p^*_m\|^2) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m}\left(\mathrm{Var}(\bar{p}^*(u)) + |\mathbb{E}(\bar{p}^*(u)) - p^*(u)|^2\right)du,$$

where the first term is bounded by $\mathbb{E}(Z_1^6/\Delta)m/(n\Delta)$. It remains to study the bias term using (26). We have $|h^*(u)|\leq|h|_1$. By Lemma 2.2, $|\tilde{\phi}(u)|\leq|b|+|u|(|h|_1+\sigma^2)\leq C(1+|u|)$. Inserting these bounds in (26) implies

$$\text{(30)} \qquad |\mathbb{E}(\bar{p}^*(u)) - p^*(u)| \leq C\Delta|p^*(u)||u|(1+|u|) + C'\Delta(1+|u|) + C''\Delta^2(1+|u|)^3$$

Gathering the terms gives the announced bound for the risk of $\bar{p}_m$. This ends the proof of proposition 4.1. □

Remark 4.1. *Again here, both estimators have the same rate of variance with different constants. The simpler estimator has additional bias terms.*

We can state the result analogous to the one of Proposition 3.2.

Proposition 4.2. *Assume that (H1), (H2)(6), (H5) hold and that p belongs to $\mathcal{C}(a,L)$. Consider the asymptotic setting where $n\to+\infty$, $\Delta\to 0$ and $n\Delta\to+\infty$. If $n\Delta^{3/2}\leq 1$, then we have, for the optimal choice $m=O((n\Delta)^{1/(2a+1)})$,*

$$\mathbb{E}(\|\hat{p}_m - p\|^2) \leq O((n\Delta)^{-2a/(2a+1)}).$$

If $a\geq 1/2$, the condition $n\Delta^{3/2}\leq 1$ can be replaced by $n\Delta^2\leq 1$.
If $a\geq 3/2$, the condition $n\Delta^{3/2}\leq 1$ can be replaced by $n\Delta^3\leq 1$.
Moreover, if $n\Delta^{11/7}\leq 1$, then

$$\mathbb{E}(\|\bar{p}_m - p\|^2) \leq O((n\Delta)^{-2a/(2a+1)}).$$

If $a\geq 1/2$, the condition $n\Delta^{7/5}\leq 1$ can be replaced by $n\Delta^2\leq 1$.

Proof. Let us take $m=O((n\Delta)^{1/(2a+1)})$. When $p\in\mathcal{C}(a,L)$, the first two terms of (28) are of order $O((n\Delta)^{-2a/(2a+1)})$. The last term is $O(\Delta^2 m^{2(2-a)_+})$. If $a\geq 2$, its order is Δ^2 and is less than $1/(n\Delta)$ if $n\Delta^3\leq 1$.
If $a\in(0,2)$, $\Delta^2 m^{2(2-a)} = O(\Delta^2(n\Delta)^{2(2-a)/(1+2a)})$ which has lower rate than $O((n\Delta)^{-2a/(2a+1)})$ if $\Delta^2(n\Delta)^{4/(1+2a)}\leq O(1)$, that is $n\Delta^{1+(1+2a)/2} = n\Delta^{3/2+a}\leq O(1)$. This gives the results for $\hat{p}_m$.

For $\bar{p}_m$, we must consider in addition the terms $\Delta^2 m^3$ and $\Delta^4 m^7$. As previously, $\Delta^2 m^3 \leq (n\Delta)^{-2a/(2a+1)}$ if $n\Delta^{(6a+5)/(2a+3)} \leq O(1)$ that is $n\Delta^{5/3} \leq 1$ if $a > 0$ and $n\Delta^2$ if $a \geq 1/2$. Moreover, $\Delta^4 m^7 \leq (n\Delta)^{-2a/(2a+1)}$ if $n\Delta^{(10a+11)/(2a+7)} \leq 1$ that is $n\Delta^{11/7} \leq 1$ if $a > 0$ and $n\Delta^2 \leq 1$ if $a \geq 1/2$. □

4.3. **Adaptive strategy.** The adaptive selection of the best possible m imposes here a restricted collection of models. We choose $M_n = \{m \in \mathbb{N}/\{0\}, m \leq \sqrt{n\Delta} := \mu_n\}$.

From the bias point of view, the estimator to consider is $\hat{p}_{\tilde{m}}$ with:

$$\tilde{m} = \arg\min_{m \in M_n} \left(-\|\hat{p}_m\|^2 + \widetilde{\mathrm{pen}}(m)\right) \tag{31}$$

and

$$\widetilde{\mathrm{pen}}(m) = \kappa \frac{m}{n\Delta^2} \left(\frac{1}{n} \sum_{k=1}^{n} Z_k^6 + (\frac{1}{n} \sum_{k=1}^{n} Z_k^4)(\frac{1}{n} \sum_{k=n+1}^{2n} Z_k^2) \right),$$

where the penalty follows closely the variance term of the risk bound of $\hat{p}_m$. For simplicity and under additional restrictions, we can also consider the estimator $\bar{p}_{\bar{m}}$ where

$$\bar{m} = \arg\min_{m \in M_n} \left(-\|\bar{p}_m\|^2 + \overline{\mathrm{pen}}(m)\right) \text{ with } \overline{\mathrm{pen}}(m) = \kappa' \frac{m}{n\Delta^2} \left(\frac{1}{3n} \sum_{k=1}^{3n} Z_k^6 \right). \tag{32}$$

We can prove the following result:

Theorem 4.1. *Under assumption (H1), (H2)(24), (H5), (H6) and with $n\Delta^2 \leq 1$. There exist numerical constants κ, κ' such that*

$$\begin{aligned} \mathbb{E}(\|\hat{p}_{\hat{m}} - p\|^2) \quad &\leq \quad C \inf_{m \in M_n} \left(\|p - p_m\|^2 + \kappa[\mathbb{E}(\frac{Z_1^6}{\Delta}) + \Delta \mathbb{E}(\frac{Z_1^2}{\Delta})\mathbb{E}(\frac{Z_1^4}{\Delta})] \frac{m}{n\Delta} \right) \\ &\quad + \frac{\Delta^2}{\pi} \int_{-\pi\mu_n}^{\pi\mu_n} u^2(1+u^2)|p^*(u)|^2 du + C \frac{\ln^2(n\Delta)}{n\Delta}. \end{aligned}$$

$$\begin{aligned} \mathbb{E}(\|\bar{p}_{\bar{m}} - p\|^2) \quad &\leq \quad C \inf_{m \in M_n} \left(\|p - p_m\|^2 + \kappa' \mathbb{E}(\frac{Z_1^6}{\Delta}) \frac{m}{n\Delta} \right) \\ &\quad + C \left(\frac{\Delta^2}{\pi} \int_{-\pi\mu_n}^{\pi\mu_n} u^2(1+u^2)|p^*(u)|^2 du + \Delta^2 \mu_n^3 + \Delta^4 \mu_n^7 + \frac{\ln^2(n\Delta)}{n\Delta} \right). \end{aligned}$$

($\mu_n = \sqrt{n\Delta}$).

The result is given for $\hat{p}_{\hat{m}}$ but only proved for $\bar{p}_{\bar{m}}$ to avoid technicalities analogous to those studied in the case of estimation of h.

The consequence of Theorem 4.1 is that the adaptive estimators reach automatically the optimal rate of convergence when p belongs to a Sobolev class. This can be seen by computations analogous to those of Proposition 4.2.

5. Parameter estimation

Under (H1), the observed process may be written as $L_t = bt + \sigma W_t + X_t$ where (W_t) is a standard Brownian motion, (X_t) is a Lévy process, independent of (W_t), of the form

$$X_t = \int_{]0,t]} \int_{\mathbb{R}} x(\hat{p}(ds, dx) - dsn(x)dx),$$

where $\hat{p}(ds, dx)$ is the random jump measure of (L_t) (and (X_t)) which has compensator $dsn(x)dx$.

If moreover,

$$\int |x|n(x)dx < \infty.$$

then, the observed process may be written as $L_t = b_0 t + \sigma W_t + \Gamma_t$ where $b_0 = b - \int xn(x)dx$ and

$$\Gamma_t = \int_{]0,t]} \int_{\mathbb{R}} x\hat{p}(ds, dx) = X_t + t \int xn(x)dx = \sum_{s \le t} \Gamma_s - \Gamma_{s_-}$$

is of bounded variation on compact sets. We consider here a sample of size n. By using empirical means of the data Z_k^ℓ, it is possible to obtain consistent and asymptotically Gaussian estimators of b ($\ell = 1$) and, under suitable integrability assumptions on the Lévy density, of $\int x^\ell n(x)dx$ for $\ell \ge 3$. But this method fails to estimate σ for $\ell = 2$ (see below). For this, one has to use another approach either based on power variations or deduced for the nonparametric estimation.

5.1. **Some small time properties.** To study estimators of b and σ, small time properties of moments are required.

In Figueroa-López (2008), conditions under which $\frac{1}{\Delta}\mathbb{E}f(L_\Delta)$ converges, as Δ tends to 0, to $\int f(x)n(x)dx$ for unbounded functions f are investigated (with new results and an exhaustive review of previous ones). We sum up below some of these.

First, we recall that a non-negative locally bounded function g is said to be submultiplicative (resp. subadditive) if there exists a constant $K > 0$ such that, for all x, y, $g(x+y) \le Kg(x)g(y)$ (resp. $g(x+y) \le K(g(x) + g(y))$). Let

$$S(n) = \{g(x) := p(x)k(x), p \text{ subadditive}, k \text{ submultiplicative, and } \int_{|x|>1} g(x)n(x)dx < \infty\}.$$

Theorem 5.1. *Let f be locally bounded, $n(x)dx$-a.e. continuous and such that there exists a function $g \in S(n)$ such that* $\limsup_{|x|\to\infty} |f(x)|/g(x) < \infty$. *If moreover, $f(x) = o(x^2)$ as $x \to 0$, then, as $\Delta \to 0$,*

$$\frac{1}{\Delta}\mathbb{E}f(L_\Delta) \to \int f(x)n(x)dx.$$

If instead $f(x) \sim x^2$ as $x \to 0$, then

$$\frac{1}{\Delta}\mathbb{E}f(L_\Delta) \to \sigma^2 + \int f(x)n(x)dx.$$

This is Theorem 1.1 of Figueroa-López (2008). In particular, if for $k \ge 2$, $\int |x|^k n(x)dx < \infty$, the functions $f_u(x) = e^{iux}x^k$, $u \in \mathbb{R}$, satisfy the assumptions. Of course, the result can be obtained directly by derivating the characteristic function ψ_Δ as seen above. If for r real number, $r \ge 2$, $\int |x|^r n(x)dx < \infty$, $f(x) = |x|^r$ satisfies the assumptions.

For the case of $f(x) = |x|^r$ with $r < 2$, which is not included in the previous theorem, we state the two following special propositions.

Proposition 5.1. *(i) Let (Γ_t) be a Lévy process with no continuous component and Lévy measure $n(\gamma)d\gamma$. If $\int |\gamma|n(\gamma)d\gamma < \infty$, $b = \int \gamma n(\gamma)d\gamma$ and for $r \le 1$, $\int |\gamma|^r n(\gamma)d\gamma < \infty$. There exists a constant C such that, for all Δ,*

$$\mathbb{E}|\Gamma_\Delta|^r \le C\Delta.$$

(Under the assumption, (Γ_t) has finite mean and bounded variation on compact sets).
(ii) Let $X_t = B_{\Gamma_t}$ where (Γ_t) is a subordinator with Lévy density n_Γ satisfying $b = \int_0^{+\infty} \gamma\, n_\Gamma(\gamma)d\gamma <$

∞ and (B_t) is a Brownian motion independent of (Γ_t). The Lévy measure of (X_t) has a density given by

$$n_X(x) = \int_0^{+\infty} e^{-x^2/2\gamma} \frac{1}{\sqrt{2\pi\gamma}} n_\Gamma(\gamma) d\gamma. \tag{33}$$

Consequently, if $C = \int_0^{+\infty} \gamma^{r/2} n_\Gamma(\gamma) d\gamma < \infty$ with $r \le 2$, then

$$\mathbb{E}|X_\Delta|^r \le C\Delta.$$

(iii) Let (X_t) be a Lévy process with no Gaussian component. Then, $X_\Delta/\sqrt{\Delta}$ converges to 0 as Δ tends to 0 in probability and in $\mathbb{L}^r$ for all $r < 2$.

5.2. **Estimator of b.** Consider a Lévy process (L_t) satisfying (H1) and set $Z_k = L_{k\Delta} - L_{(k-1)\Delta}$ as above. Let us define the empirical means:

$$\hat{b} = \frac{1}{n\Delta} \sum_{k=1}^n Z_k, \quad \hat{c}_\ell = \frac{1}{n\Delta} \sum_{k=1}^n Z_k^\ell \ \text{ for } \ell \ge 2. \tag{34}$$

We prove now that $\hat{b}$, $\hat{c}_\ell, \ell \ge 2$ are consistent and asymptotically Gaussian estimators of the quantities b, $c_\ell, \ell \ge 2$ where

$$c_2 = \sigma^2 + \int x^2 n(x) dx, \quad c_\ell = \int x^\ell n(x) dx, \quad \text{for} \quad \ell \ge 3.$$

Proposition 5.2. *Assume (H1) and n tends to infinity, Δ tends to 0, $n\Delta$ tends to infinity.*

(i) Under (H2)$(2+\varepsilon)$ for some positive ε,

$$\sqrt{n\Delta}(\hat{b} - b) \textit{ converges in distribution to } \mathcal{N}(0, c_2).$$

(ii) Under (H2)$(2(\ell+\varepsilon))$ for some positive ε, and if $n\Delta^3$ tends to 0, $\sqrt{n\Delta}(\hat{c}_\ell - c_\ell)$ converges in distribution to $\mathcal{N}(0, c_{2\ell})$.

Proof. We have $\mathbb{E}(Z_k) = \Delta b$ and, for $\ell \ge 2$, $\mathbb{E}(Z_k^\ell) = \Delta c_\ell + o(\Delta)$. Therefore, $\hat{b}$ is an unbiased estimator of b and, for $\ell \ge 2$, $\sqrt{n\Delta}|\mathbb{E}\hat{c}_\ell - c_\ell| = \sqrt{n\Delta} O(\Delta)$. Hence, the additional condition $n\Delta^3 = o(1)$ to erase the bias.

Setting $c_1 = b$, $\hat{c}_1 = \hat{b}$, as $\mathrm{Var} Z_k^\ell = \Delta c_{2\ell} + o(\Delta)$ for $\ell \ge 1$, we have $n\Delta \mathrm{Var}\hat{c}_\ell = c_{2\ell} + O(\Delta)$. Writing

$$\sqrt{n\Delta}(\hat{c}_\ell - \mathbb{E}\hat{c}_\ell) = \frac{1}{\sqrt{n\Delta}} \sum_{k=1}^n (Z_k^\ell - \mathbb{E}Z_k^\ell) = \sum_{k=1}^n \chi_{k,n},$$

it is now enough to prove that $\sum_{k=1}^n \mathbb{E}|\chi_{k,n}|^{2+\varepsilon}$ tends to 0. Under the assumption, we have

$$\sum_{k=1}^n \mathbb{E}|\chi_{k,n}|^{2+\varepsilon} \le \frac{C}{n^{\varepsilon/2}\Delta^{1+\varepsilon/2}} \left(\mathbb{E}|Z_k|^{\ell(2+\varepsilon)} + |\mathbb{E}(Z_k^\ell)|^{2+\varepsilon} \right) \le \frac{C}{(n\Delta)^{\varepsilon/2}},$$

which gives the result. □

We stress that this method provides an estimator of b which is easy to compute and very good in practice (see Section 7), but cannot provide an estimator of σ^2.

5.3. **Estimation of σ^2.**

5.3.1. *Power variations.* Estimators of σ based on power variations of (L_t) have been proposed and mostly studied in the case where $n\Delta = 1$. They are studied for high frequency data within a long time interval in Aït-Sahalia and Jacod (2007). In the latter paper, the context is more general than ours, which implies that proofs are of high complexity. For Lévy processes fitting in our set of assumptions, we can derive the asymptotic properties of power variations estimators with a specific proof given in Section 9. Consider

$$\hat{\sigma}_n^{(r)} = \frac{1}{m_r n\Delta^{r/2}} \sum_{k=1}^{n} |Z_k|^r, \tag{35}$$

where $m_r = \mathbb{E}|X|^r$ for X a standard Gaussian variable (recall that $Z_k = L_{k\Delta} - L_{(k-1)\Delta}$).

Proposition 5.3. *As n tends to infinity, Δ tends to 0 and $n\Delta$ tends to infinity, if $n\Delta^{2-r} = o(1)$, $\sqrt{n}(\hat{\sigma}_n^{(r)} - \sigma^r)$ converges in distribution to a $\mathcal{N}(0, \sigma^{2r}(m_{2r}/m_r^2 - 1))$ for:*
(i) (L_t) a Lévy process satisfying (H1) and such that $\int |x|n(x)dx < \infty$ and $\int |x|^r n(x)dx < \infty$ for $r < 1$.
(ii) $(L_t = bt + \sigma W_t + X_t)$, with $X_t = B_{\Gamma_t}$, where W, B, Γ are independent processes, W, B are Brownian motions, Γ is a subordinator with Lévy measure n_Γ satisfying $b = \int_0^{+\infty} \gamma\, n_\Gamma(\gamma)d\gamma < \infty$ and $\int_0^{+\infty} \gamma^{r/2}\, n_\Gamma(\gamma)d\gamma < \infty$ for $r < 1$.

For other cases of Lévy processes, the result depends on the rate of convergence to 0 of $\mathbb{E}|X_\Delta|^r/\Delta^{r/2}$ (see Proposition 5.1 (iii)) and will still hold if $\sqrt{n\Delta}\mathbb{E}|X_\Delta|^r/\Delta^{r/2}$ tends to 0.

Remark 5.1.
- *It is worth noting that the rate of convergence of the estimators $\hat{\sigma}_n^{(r)}$ is $\sqrt{n}$. For $r = 1$, the estimator $\hat{\sigma}_n^{(1)}$ is consistent but not asymptotically Gaussian (because of its asymptotic bias). We have implemented these estimators for $r = 1/2$, $r = 1/4$ (see Section 7) for processes satisfying $\int |x|^r n(x)dx < +\infty$ for all positive r.*
- *Other estimators based on truncated power variations can be considered (see Aït-Sahalia and Jacod (2007)).*
- *We always give integrability conditions on $\mathbb{R}$ for the Lévy density. This simplifies the presentation but induces some redundancies. One should distinguish integrability conditions near 0 and near infinity to avoid these redundancies.*

5.3.2. *Another estimator of σ^2.* The previous power variation estimators depend on r and have rate $\sqrt{n}$ under the condition that $n\Delta^{2-r} = o(1)$ where the value $r < 1$ is such that $\int |x|^r n(x)dx < \infty$. This imposes a stronger integrability condition around 0 than (H1).

Under (H1) and (H2)(3), using an estimator of p^* and empirical mean estimators, we can build an estimator of σ^2 with rate $\sqrt{n\Delta}$. As seen above, estimators of $c_2 = \sigma^2 + \int x^2 h(x)dx = \sigma^2 + h^*(0)$ are available. Using a sample of size $3n$, we get an unbiased estimator of c_2 by setting (see Proposition 5.2):

$$\tilde{c}_2 = \frac{1}{3n\Delta} \sum_{k=1}^{3n} Z_k^2 - \Delta \left(\frac{1}{n\Delta} \sum_{k=1}^{n} Z_k \right) \left(\frac{1}{n\Delta} \sum_{\ell=n+1}^{2n} Z_\ell \right).$$

Following the same proof as above, we can prove that $\sqrt{n\Delta}(\tilde{c}_2 - c_2)$ converges in distribution to the centered Gaussian law with variance c_4 as Δ tends to 0, $n\Delta$ tends to infinity with no additional condition on Δ. We build an estimator of σ^2 by subtracting an estimator of $h^*(0)$. In addition to (H1)-(H2)(3), assume that $p^*(u)$ belongs to $\mathbb{L}_1(\mathbb{R})$.

Note that $h^*(u) = \int e^{iux} x^2 n(x)dx$ is such that $(d/du)(h^*(u)) = ip^*(u)$. Integrating from u to ∞, and using that $\lim_{|u|\to+\infty} h^*(u) = 0$ yields $h^*(u) = -i \int_u^{+\infty} p^*(v)dv$ and thus $h^*(0) =$

$-i\int_0^{+\infty} p^*(v)dv$. Since $h^*(0)$ is a real number, we must have:

$$h^*(0) = \int_0^{+\infty} \mathrm{Im} p^*(v)dv.$$

Therefore, for M to be chosen adequately later on, we propose to estimate σ^2, using the estimator $\bar{p}^*$ of p^*, by:

$$\hat{\sigma}^2_M = \tilde{c}_2 - \int_0^M \mathrm{Im}\bar{p}^*(v)dv. \tag{36}$$

Integration yields

$$\int_0^M \mathrm{Im}\bar{p}^*(v)dv = \frac{1}{3n\Delta}\sum_{k=1}^{3n} Z_k^2(1-\cos MZ_k). \tag{37}$$

Thus

$$\hat{\sigma}^2_M = \frac{1}{3n\Delta}\sum_{k=1}^{3n} Z_k^2 \cos MZ_k - \Delta\left(\frac{1}{n\Delta}\sum_{k=1}^{n} Z_k\right)\left(\frac{1}{n\Delta}\sum_{\ell=n+1}^{2n} Z_\ell\right). \tag{38}$$

Then we can prove the following result:

Proposition 5.4. *Assume (H1)-(H2)(4), (H5) and that p^* belongs to $\mathbb{L}_1(\mathbb{R})$. If, in addition, $p \in \mathcal{C}(a, L)$ with $a > 5/2$, $M = M(\Delta) = \Delta^{-1/4}$ and $n\Delta^2$ tends to 0, $\sqrt{n\Delta}(\hat{\sigma}^2_{M(\Delta)} - \sigma^2)$ converges in distribution to a centered Gaussian law with variance $(1/2)\int x^4 n(x)dx = c_4/2$.*

6. Examples

In this section, we give examples of models fitting in our framework.

Example 1. Drift + Brownian motion+ Compound poisson process.
Let

$$L_t = b_0 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \tag{39}$$

where N_t is a Poisson process with constant intensity c and Y_i is a sequence of i.i.d. random variables with density f, independent of the process (N_t). Then, $\sum_{i=1}^{N_t} Y_i$ is a compound Poisson process and (L_t) is a Lévy process with Lévy density $n(x) = cf(x)$. Note that $\mathbb{E}L_1 = b = b_0 + \int xn(x)dx$. For the estimation of p, the rates that can be obtained depend on the density f provided that f satisfies the assumptions of Theorem 4.1, which are essentially here moment assumptions for the r.v.'s Y_i. Any order can be obtained as shown in Table 1 where optimal rates are computed for f a standard Gaussian, an exponential with parameter 1 and a Beta distribution with parameters $(1, 3)$ (for p to be regular enough).
As $\int |x|^r n(x)dx < \infty$ for all $r < 1$ (actually, for all $r \le 2$), estimation of σ^r is possible using $\hat{\sigma}_n^{(r)}$ for any value of $0 < r < 1$ (provided that $n\Delta^{2-r} = o(1)$). The estimator $\hat{\sigma}^2_M$ is possible if p is regular enough ($p \in \mathcal{C}(a, L)$ with $a > 5/2$).

Example 2. Drift + Brownian motion + Lévy-Gamma process.
Consider $L_t = b_0 t + \sigma W_t + \Gamma_t$ where (Γ_t) is a Lévy gamma process with parameters (β, α), *i.e.* is a subordinator such that, for all $t > 0$, Γ_t has distribution Gamma with parameters $(\beta t, \alpha)$

$f(x)$	$\mathcal{N}(0,1)$	$\mathcal{E}(1)$	$\beta(1,3)$
$p(x) = cx^3 f(x)$	$\propto x^3 e^{-x^2}$	$\propto x^3 e^{-x}\mathbf{1}_{x>0}$	$\propto x^3(1-x)^2\mathbf{1}_{[0,1]}(x)$
$p^*(u)$	$\propto (u^3-3u)e^{-u^2/2}$	$\propto 1/(1-iu)^4$	$O(1/\|u\|^3)$ for large $\|u\|$.
$\int_{\|u\|\geq \pi m} \|p^*(u)\|^2 du$	$O((\pi m)^5 e^{-(\pi m)^2})$	$O((\pi m)^{-7})$	$O((\pi m)^{-5})$
$\int_{\|u\|\leq \pi\mu_n} u^4 \|p^*(u)\|^2 du$	$O(1)$	$O(1)$	$O(1)$
$\breve{m}$ (best choice of m)	$\sqrt{\log(n\Delta) - \frac{5}{2}\log\log(n\Delta)}/\pi$	$O((n\Delta)^{1/8})$	$O((n\Delta)^{1/6})$
Rate $\propto$	$\dfrac{\sqrt{\log(n\Delta)}}{n\Delta}$	$(n\Delta)^{-7/8}$	$(n\Delta)^{-5/6}$

TABLE 1. Rates for different "Drift+ Brownian motion +Compound Poisson process".

and density: $\alpha^{\beta t} x^{\beta t-1} e^{-\alpha x}/\Gamma(\beta t)\mathbf{1}_{x\geq 0}$. The Lévy density of (L_t) is $n(x) = \beta x^{-1} e^{-\alpha x}\mathbf{1}_{x>0}$. We have $\mathbb{E}L_1 = b = b_0 + \int xn(x)dx$ and $p(x) = \beta x^2 e^{-\alpha x}\mathbf{1}_{x>0}$.

We find $p^*(u) = 2\beta/(\alpha - iu)^3$, $\int_{|u|\geq \pi m} |p^*(u)|^2 du = O(m^{-5})$ and $\int_{-\pi\mu_n}^{\pi\mu_n} u^4 |p^*(u)|^2 du = O(1)$. Therefore the rate for estimating p is $O((n\Delta)^{-5/6})$ for a choice $\breve{m} = O((n\Delta)^{1/6})$.
As for all $r > 0$, $\int x^r n(x)dx < \infty$, $\hat{\sigma}_n^{(r)}$ is authorized, for any value of $0 < r < 1$, to estimate σ^r.

Example 2 (continued). Drift + Brownian motion + A specific class of subordinators.
Let $L_t = b_0 t + \sigma W_t + \Gamma_t$ where (Γ_t) is a subordinator of pure jump type with Lévy density of the form $n(x) = \beta x^{\delta-1/2} x^{-1} e^{-\alpha x}\mathbf{1}_{x>0}$ with $\delta > -1/2$ (thus $\int xn(x)dx < \infty$). This class of subordinators includes compound Poisson processes ($\delta > 1/2$) and Lévy Gamma processes ($\delta = 1/2$). When $\delta > 0$, the function $xn(x)$ is both integrable and square integrable. This case was discussed in Genon-Catalot and Comte (2009) where the estimation of $xn(x)$, when $b_0 = 0$, $\sigma = 0$, is studied. Here, we consider the case $-1/2 < \delta \leq 0$ which includes the Lévy Inverse Gaussian process ($\delta = 0$). Assumptions (H1)-(H6) are satisfied. The function $p(x) = x^3 n(x)$ can be estimated in presence (or not) of additional drift and Brownian component. We can compute

$$p^*(u) = \beta \frac{\Gamma(\delta + 5/2)}{(\alpha - iu)^{\delta+5/2}}.$$

Thus, $\int_{|u|\geq \pi m} |p^*(u)|^2 du = O(m^{-(2\delta+4)})$. As $2\delta + 1 \leq 1$, $u^4|p^*(u)|^2$ is not integrable and we have $\Delta^2 \int_{|u|\leq \pi\mu_n} u^4 |p^*(u)|^2 du = \Delta^2 O(\mu_n) = O(\Delta^{3/2})$. The best rate for estimating p is $O((n\Delta)^{-(2\delta+4)/(2\delta+5)})$ for a choice $\breve{m} = O((n\Delta)^{1/(2\delta+5)})$. Note that $\Delta^{3/2} \leq (n\Delta)^{-(2\delta+4)/(2\delta+5)}$ for $n\Delta^2 \leq 1$ and $-1/2 < \delta \leq 0$.
We have $\int x^r n(x)dx < \infty$ for $r > 1/2 - \delta$. Hence, to estimate σ^r using $\hat{\sigma}_n^{(r)}$, we must choose $1/2 - \delta < r < 1$.

Example 3. Drift + Brownian motion + Pure jump martingale.
Consider $L_t = bt + \sigma W_t + B_{\Gamma_t}$ where W, B, Γ are independent processes, W, B are standard Brownian motion, and Γ is a pure-jump subordinator with Lévy density $n_\Gamma(\gamma) = \beta\gamma^{\delta-1/2}\gamma^{-1}e^{-\alpha\gamma}\mathbf{1}_{\gamma>0}$ as above (assuming $\delta > -1$). The Lévy density $n(.)$ of (L_t) (and of $(X_t = B_{\Gamma_t})$) is linked with n_Γ (see (33)) and can be computed as the norming constant of a Generalized Inverse Gaussian

distribution:

$$n(x) = \frac{2\beta}{\sqrt{2\pi}} K_{\delta-1}(\sqrt{2\alpha}|x|)(\frac{|x|}{\sqrt{2\alpha}})^{\delta-1},$$

where K_ν is a Bessel function of third kind (MacDonald function) (see *e.g.* Barndorff-Nielsen and Shephard (2001)). For $\delta = 1/2$, B_{Γ_t} is a symmetric bilateral Lévy Gamma process (see Madan and Seneta (1990), Küchler and Tappe (2008)). For $\delta = 0$, B_{Γ_t} is a normal inverse Gaussian Lévy process (see Barndorff-Nielsen and Shephard (2001)). The relation (33) allows to check that the function $p(x) = x^3 n(x)$ belongs to $\mathbb{L}^1 \cap \mathbb{L}^2$ and satisfies (H6) for $\delta > -3/4$. Moreover, we can obtain

$$p^*(u) = -i\beta \left(\frac{u^3 \Gamma(\delta + 5/2)}{(\alpha + u^2/2)^{5/2}} - 3\frac{u\Gamma(\delta + 3/2)}{(\alpha + u^2/2)^{3/2}} \right).$$

Thus, $\int_{|u| \geq \pi m} |p^*(u)|^2 du = O(m^{-3})$ and $\Delta^2 \int_{|u| \leq \pi\mu_n} u^4 |p^*(u)|^2 du = \Delta^2 O(\mu_n) = O(\Delta^{3/2})$. The best rate for estimating p is $O((n\Delta)^{-3/4})$ obtained for $\breve{m} = O((n\Delta)^{1/4}))$. We have $\Delta^{3/2} \leq (n\Delta)^{-3/4}$ as $n\Delta^2 \leq 1$. As $\int \gamma^{r/2} n_\Gamma(\gamma) d\gamma < \infty$ for $r > 1 - \delta/2$, the estimation of σ^r by $\hat{\sigma}_n^{(r)}$ requires $1 - \delta/2 < r < 1$. Therefore, we must have $\delta < 0$.

7. Simulations

In this section, we present numerical results for simulated Lévy processes corresponding to Examples 1 and 2 (see Section 6). For these models, the functions $g(x) = xn(x)$, h and p belong to $\mathbb{L}^1 \cap \mathbb{L}^2(\mathbb{R})$. Thus, we can apply the method of Comte and Genon-Catalot (2009a), to estimate g when $b_0 = 0$, $\sigma = 0$, and the method developed here to estimate h when $\sigma = 0$ and p when $\sigma \neq 0$. For simplicity, we have implemented the estimators $\bar{h}_{\bar{m}}$, $\bar{p}_{\bar{m}}$ defined by (10)-(20) and (27)-(32). The numerical constant κ' appearing in the penalties has been set to 7.5 for g, 4 for h and 3 for p; its calibration is done by preliminary experiments. The cutoff $\bar{m}$ is chosen among 100 equispaced values between 0 and 10.

Figure 1 shows estimated curves for models with jump part coming from compound Poisson processes (see (39)) where the Y_i's are standard Gaussian, Exponential $\mathcal{E}(1)$, $\beta(1,3)$, and $\beta(3,3)$ rescaled on $[-4,4]$. The intensity c is equal to 0.5, except for the $\beta(1,3)$ Y_i's where $c = 4$.

Figure 2 shows estimated curves for jump part of Lévy Gamma and bilateral Lévy Gamma type. The bilateral Lévy Gamma process is the difference $\Gamma_t - \Gamma'_t$ of two independent Lévy Gamma processes.

On top of each graph, we give the mean value of the selected cutoff with its standard deviation in parentheses. This value is surprisingly small. As expected, the presence of a Gaussian component deteriorates the estimation, which remains satisfactory on the whole.

Tables 2 and 3 show the means of the estimation results for $b = \mathbb{E}(L_1) = b_0 + \int xn(x)dx$ (see (34)) and σ, with standard deviations in parentheses. We have set $\hat{\sigma}(r) = [\hat{\sigma}_n^{(r)}]^{1/r}$ to denote the estimator of σ, see (35).

The estimation of b is good in all cases, and especially when $n\Delta$ is large. The estimation of σ is clearly more difficult, with noticeable differences according to the values of n and Δ. When Δ is not small enough, the estimation can be heavily biased. In accordance with the theory, when r is smaller, the estimator of σ is slightly better (smaller bias). Table 4 shows the values of $n\Delta^2$ and $n\Delta^{2-r}$, which should be small for the performance of the estimator to be satisfactory. It is worth noting that σ is constantly over-estimated. We also implemented $\hat{\sigma}_M$ (see (38)) on some of the same simulated data, see Table 5; the performances of the estimator are strongly related to the asymptotic conditions: $n\Delta$ large, and $n\Delta^2$ small. This is why the results when $n\Delta^2 = 125, 5$ (see Table 4) are not reported. When negative values occured, the estimate is

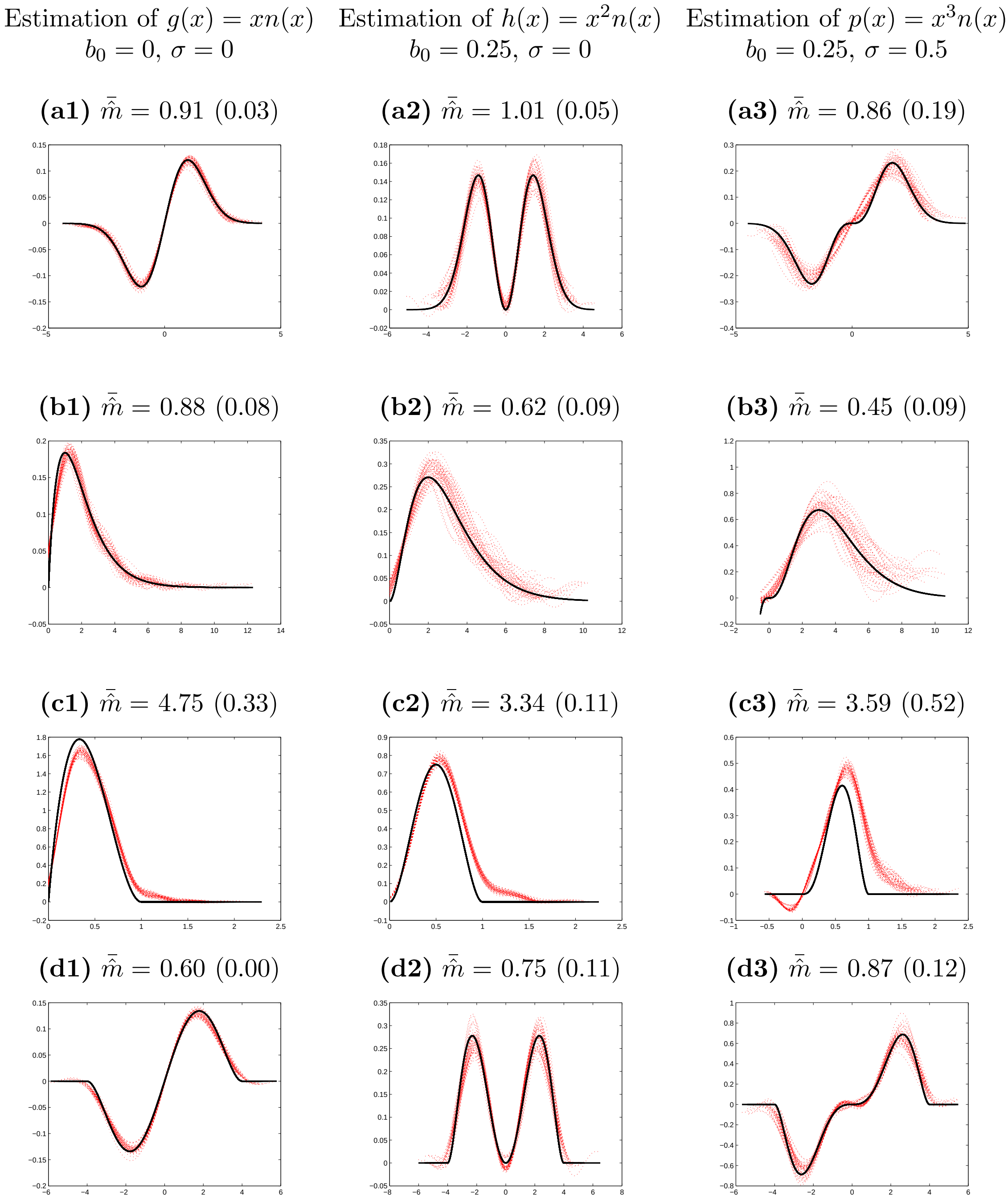

FIGURE 1. Variability bands for the estimation of g, h, p for a compound Poisson process with Gaussian (first line), Exponential $\mathcal{E}(1)$ (second line), $\beta(1,3)$ (third line) and rescaled on $[-4,4]$ $\beta(3,3)$ (fourth line) Y_i's,with $c = 0.5$, except for the third line $c = 4$. True (bold black line) and 50 estimated curves (dotted red), $\Delta = 0.05$, $n = 5.10^4$.

set to 0, which explains the large standard deviations. In such case, the median is preferable. Contrary to $\hat{\sigma}(r)$, $\hat{\sigma}_M$ always under-estimates σ.

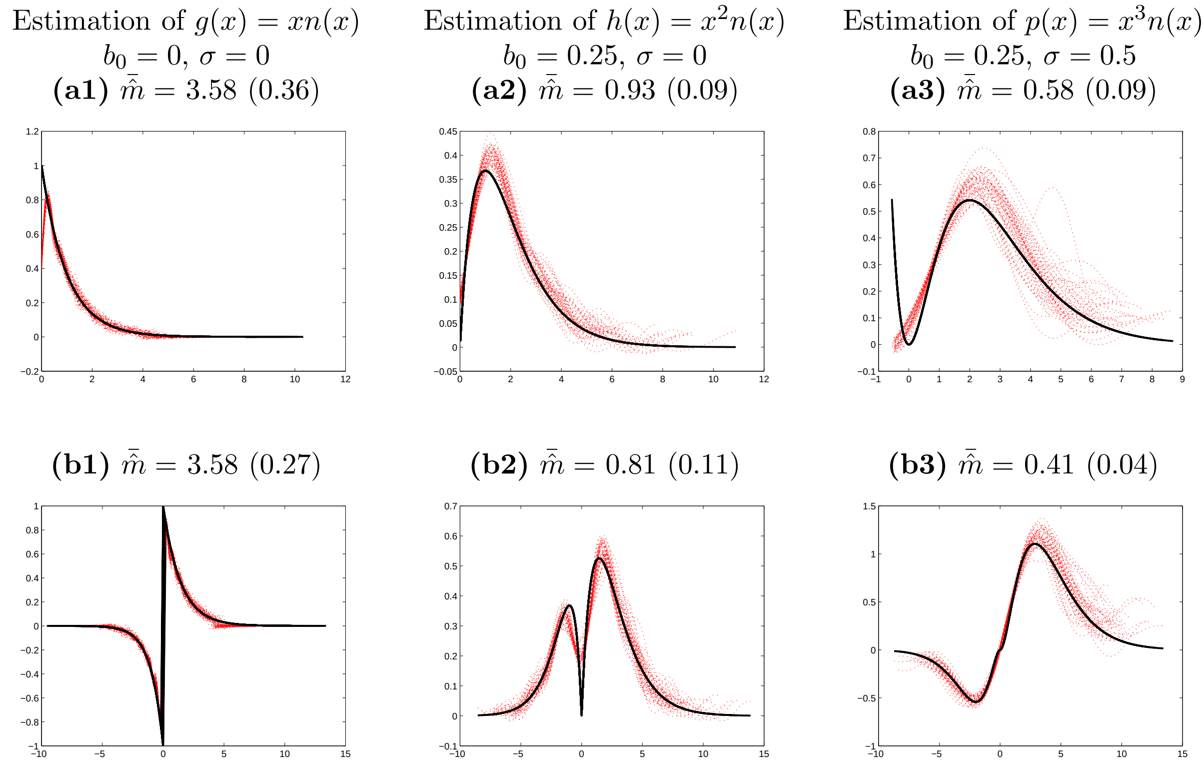

FIGURE 2. Variability bands for the estimation of g, h, p for jumps from a Lévy-Gamma process with $\beta = 1, \alpha = 1$ (first line), a bilateral Lévy-Gamma process with $(\beta, \alpha) = (0.7, 1), (\beta', \alpha') = (1, 1)$ (second line). True (bold black line) and 50 estimated curves (dotted red), $\Delta = 0.05$, $n = 5.10^4$.

Model	(n, Δ)	$(5.10^4, 0.05)$	$(5.10^4, 0.01)$	$(5.10^4, 10^{-3})$	$(10^4, 10^{-3})$
Poisson	$\hat{b}$ $(b = 1)$	1.000 (0.02)	0.997 (0.04)	0.995 (0.123)	1.001 (0.280)
Gaussian	$\hat{\sigma}(1/2)$	0.602 (0.03)	0.527 (0.002)	0.504 (0.002)	0.504 (0.005)
	$\hat{\sigma}(1/4)$	0.589 (0.03)	0.521 (0.002)	0.503 (0.002)	0.503 (0.002)
Poisson	$\hat{b}$ $(b = 1.5)$	1.502 (0.05)	1.502 (0.051)	1.494 (0.142)	1.461 (0.359)
Exp(1)	$\hat{\sigma}(1/2)$	0.611 (0.003)	0.530 (0.003)	0.505 (0.002)	0.505 (0.005)
	$\hat{\sigma}_{1/4}$	0.594 (0.003)	0.522 (0.003)	0.503 (0.002)	0.503 (0.005)
Gamma	$\hat{b}$ $(b = 2)$	2.001 (0.02)	2.000 (0.05)	1.998 (0.177)	2.018 (0.335)
(1,1)	$\hat{\sigma}(1/2)$	0.705 (0.004)	0.562 (0.003)	0.512 (0.002)	0.513 (0.005)
	$\hat{\sigma}(1/4)$	0.677 (0.004)	0.548 (0.003)	0.508 (0.002)	0.508 (0.005)
Bilateral	$\hat{b}$ $(b = 1.4286)$	1.426 (0.035)	1.4286 (0.076)	1.4493 (0.264)	1.405 (0.619)
Gamma	$\hat{\sigma}(1/2)$	0.862 (0.005)	0.628 (0.004)	0.526 (0.003)	0.526 (0.006)
(0.7,1), (1.1)	$\hat{\sigma}(1/4)$	0.798 (0.004)	0.593 (0.003)	0.516 (0.002)	0.515 (0.006)

TABLE 2. Estimation of (b, σ), $b_0 = 1$, the true value of b in parenthesis, $\sigma = 0.5$, $K = 200$ replications.

Model	(n,Δ)	$(5.10^4, 0.05)$	$(5.10^4, 0.01)$	$(5.10^4, 10^{-3})$	$(10^4, 10^{-3})$
Poisson	$\hat{b}$ (1)	0.999 (0.025)	1.005 (0.059)	0.998 (0.178)	1.025 (0.85)
Gaussian	$\hat{\sigma}(1/2)$	1.082 (0.005)	1.026 (0.004)	1.006 (0.004)	1.005 (0.009)
	$\hat{\sigma}(1/4)$	1.072 (0.005)	1.020 (0.005)	1.004 (0.004)	1.003 (0.01)
Poisson	$\hat{b}$ (1.5)	1.510 (0.026)	1.498 (0.06)	1.481 (0.190)	1.485 (0.442)
Exp(1)	$\hat{\sigma}(1/2)$	1.096 (0.005)	1.030 (0.004)	1.006 (0.004)	1.006 (0.009)
	$\hat{\sigma}(1/4)$	1.080 (0.005)	1.022 (0.004)	1.003 (0.004)	1.003 (0.010)
Gamma	$\hat{b}$ (2)	2.00 (0.026)	1.995 (0.068)	1.991 (0.196)	2.023 (0.195)
(1,1)	$\hat{\sigma}(1/2)$	1.172 (0.005)	1.062 (0.005)	1.014 (0.004)	1.014 (0.004)
	$\hat{\sigma}(1/4)$	1.152 (0.005)	1.050 (0.005)	1.010 (0.005)	1.010 (0.004)
Bilateral	$\hat{b}$ (1.4286)	1.425 (0.04)	1.431 (0.10)	1.429 (0.28)	1.492 (0.63)
Gamma	$\hat{\sigma}(1/2)$	1.330 (0.006)	1.136 (0.005)	1.033 (0.005)	1.033 (0.01)
(0.7,1), (1.1)	$\hat{\sigma}(1/4)$	1.284 (0.006)	1.105 (0.005)	1.022 (0.005)	1.022 (0.01)

TABLE 3. Estimation of (b, σ), $b_0 = 1$, the true value of b in parenthesis, $\sigma = 1$, power variation method for estimation of σ, $K = 200$ replications.

(n,Δ)	$(5.10^4, 0.05)$	$(5.10^4, 0.01)$	$(5.10^4, 10^{-3})$	$(10^4, 10^{-3})$
$n\Delta$	2500	500	50	10
$n\Delta^2$	125	5	0.05	0.01
$n\Delta^{2-1/2}$	559	50	1.6	0.3
$n\Delta^{2-1/4}$	264	16	0.3	0.06

TABLE 4. Values of n, Δ, $n\Delta$, $n\Delta^2$, $n\Delta^{2-r}$ for $r = 1/2$ and $r = 1/4$.

(n,Δ)	Model	Poisson-Gaussian	Poisson-Expon.	Gamma $(1,1)$
$(10^4, 10^{-3})$	mean (std)	0.95 (0.18)	0.91 (0.34)	0.93 (0.30)
	median	0.98	0.96	0.96
$(5.10^4, 10^{-3})$	mean(std)	0.98 (0.06)	0.94 (0.20)	0.96 (0.13)
	median	0.98	0.97	0.96

TABLE 5. Estimation of σ with second method, $b_0 = 1$, $\sigma = 1$, $K = 200$ replications.

8. Concluding remarks

In this paper, we consider a general Lévy process and discrete observations of the sample paths, with sampling interval Δ. The asymptotic framework $\Delta \to 0$, $n\Delta \to +\infty$ is especially well fitted to produce simple adaptive nonparametric estimators of the Lévy density, showing nice performances on simulated data. For the other parameters of the characteristic triple, central limit theorems for empirical means based on the increments (including power variations) are given. The practical implementation confirms the theoretical results: the drift is very well estimated, the variance of the Gaussian component requires a large number of observations.

9. Proofs

9.1. **Proof of Theorem 3.1.** We only study $\hat{h}_{\hat{m}}$ as the result for $\bar{h}_{\bar{m}}$ can be proved analogously (and is even simpler).

The proof is given in two steps. We define, for some ϱ, $0 < \varrho < 1$,

$$\Omega_\varrho := \left\{ \left| \frac{[(1/n\Delta)\sum_{k=1}^n Z_k^2][(1/n\Delta)\sum_{k=n+1}^{2n} Z_k^2]}{(\mathbb{E}(Z_1^2/\Delta))^2} - 1 \right| \le \varrho/2 \right\} \bigcap \left\{ \left| \frac{[(1/n\Delta)\sum_{k=1}^n Z_k^4]}{(\mathbb{E}(Z_1^4/\Delta))} - 1 \right| \le \varrho/2 \right\},$$

so that $\mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2) = \mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2 \mathbf{I}_{\Omega_\varrho}) + \mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2 \mathbf{I}_{\Omega_\varrho^c})$.

Step 1. For the study of $\mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2 \mathbf{I}_{\Omega_\varrho^c})$, we refer to the analogous proof given in Comte and Genon-Catalot (2009) (see Section A4 therein). Using that $\mathbb{E}(Z_1^{16}) < +\infty$, we can prove $\mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2 \mathbf{I}_{\Omega_\varrho^c}) \le C/(n\Delta)$. For this, we make use of the Rosenthal inequality recalled in the Appendix.

Step 2. Study of $\mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2 \mathbf{I}_{\Omega_\varrho})$.
The proof relies on the following decomposition of γ_n

$$\begin{aligned} \gamma_n(t) - \gamma_n(s) &= \|t-h\|^2 - \|s-h\|^2 + 2\langle t-s, h\rangle - \frac{1}{\pi}\langle \hat{h}^*, t^* - s^*\rangle \\ &= \|t-h\|^2 - \|s-h\|^2 - 2\nu_n(t-s) - 2R_n(t-s), \end{aligned}$$

where

$$\nu_n(t) = \frac{1}{2\pi}\langle \hat{h}^* - \mathbb{E}(\hat{h}^*), t^*\rangle, \quad R_n(t) = \frac{1}{2\pi}\langle \mathbb{E}(\hat{h}^*) - h^*, t^*\rangle.$$

As $\gamma_n(\hat{h}_m) = -\|\hat{h}_m\|^2$, we deduce from (19) that, for all $m \in \mathcal{M}_n$,

$$\gamma_n(\hat{h}_{\hat{m}}) + \text{pen}(\hat{m}) \le \gamma_n(h_m) + \text{pen}(m).$$

This yields:

$$\|\hat{h}_{\hat{m}} - h\|^2 \le \|h - h_m\|^2 + \text{pen}(m) - \text{pen}(\hat{m}) + 2\nu_n(\hat{h}_{\hat{m}} - h_m) + 2R_n(\hat{h}_{\hat{m}} - h_m)$$

Then, for $\phi_n = \nu_n, R_n$, we use the inequality:

$$\begin{aligned} 2\phi_n(\hat{h}_{\hat{m}} - h_m) &\le 2\|\hat{h}_{\hat{m}} - h_m\| \sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |\phi_n(t)| \\ &\le \frac{1}{8}\|\hat{h}_{\hat{m}} - h_m\|^2 + 8 \sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |\phi_n(t)|^2 \end{aligned}$$

Using that $\|\hat{h}_{\hat{m}} - h_m\|^2 \le 2\|\hat{h}_{\hat{m}} - h\|^2 + 2\|\hat{h}_m - h\|^2$ and some algebra, we find

$$\begin{aligned} \frac{1}{4}\|\hat{h}_{\hat{m}} - h\|^2 &\le \frac{7}{4}\|h - h_m\|^2 + \text{pen}(m) - \text{pen}(\hat{m}) + \\ &\quad + 8 \sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |R_n(t)|^2 + 8 \sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |\nu_n(t)|^2 \end{aligned} \tag{40}$$

We have to study the terms containing a supremum, which are of different nature. First, for $R_n(t)$, we have:

Lemma 9.1. *We have:* $\sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |R_n(t)|^2 \le C\Delta^2 \int_{-\pi m_n}^{\pi m_n} u^2 |h^*(u)|^2 du$.

Proof. We have $R_n(t) = \frac{1}{2\pi}\langle t^*, (1-\psi_\Delta^2)h^*\rangle$. By using Lemma 2.2, we find

$$\begin{aligned}\sup_{t\in S_m+S_{\hat m}, \|t\|=1} |\langle t^*, (1-\psi_\Delta^2)h^*\rangle|^2 &\le \sup_{t\in S_{m_n}, \|t\|=1} |\langle t^*, (1-\psi_\Delta^2)h^*\rangle|^2 \\ &\le 2\pi\|(1-\psi_\Delta^2)h^*\mathbf{1}_{[-\pi m_n, \pi m_n]}\|^2 \\ &\le C\Delta^2\int_{-\pi m_n}^{\pi m_n} u^2|h^*(u)|^2du.\end{aligned}$$

□

On the other hand, ν_n must be decomposed into $\nu_n(t) = \sum_{j=1}^4 \nu_{n,j}(t) + r_n(t)$ with

$$\begin{aligned} r_n(t) &= \frac{1}{2\pi\Delta}\langle t^*, (\hat\psi^{(1)}_{\Delta,1}(u) - \psi'_\Delta(u))(\hat\psi^{(1)}_{\Delta,2}(u) - \psi'_\Delta(u))\rangle \\ &\quad -\frac{1}{2\pi\Delta}\langle t^*, (\hat\psi^{(2)}_{\Delta,1}(u) - \psi''_\Delta(u))(\hat\psi^{(0)}_{\Delta,2}(u) - \psi_\Delta(u))\rangle, \end{aligned} \tag{41}$$

and

$$\nu_{n,1}(t) = \frac{1}{2\pi\Delta}\langle t^*, (\psi''_\Delta - \hat\psi^{(2)}_{\Delta,1})\psi_\Delta\rangle, \quad \nu_{n,2}(t) = \frac{1}{2\pi\Delta}\langle t^*, (\psi_\Delta - \hat\psi^{(0)}_{\Delta,2})\psi''_\Delta\rangle.$$
$$\nu_{n,3}(t) = \frac{1}{2\pi\Delta}\langle t^*, (\hat\psi^{(1)}_{\Delta,1} - \psi'_\Delta)\psi'_\Delta\rangle, \quad \nu_{n,4}(t) = \frac{1}{2\pi\Delta}\langle t^*, (\hat\psi^{(1)}_{\Delta,2} - \psi'_\Delta)\psi'_\Delta\rangle,$$

Lemma 9.2. *We have:* $\mathbb{E}\left(\sup_{t\in S_m+S_{\hat m}, \|t\|=1}|r_n(t)|^2\right) \le \frac{C}{n}$.

Proof. Using the independence of the subsamples, we can write:

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in S_m+S_{\hat m}, \|t\|=1}|r_n(t)|^2\right) &\le \mathbb{E}\left(\sup_{t\in S_{m_n}, \|t\|=1}|r_n(t)|^2\right) \\ &\le \frac{1}{2\pi^2\Delta^2}\mathbb{E}[\|(\hat\psi^{(1)}_{\Delta,1} - \psi'_\Delta)(\hat\psi^{(1)}_{\Delta,2} - \psi'_\Delta)\mathbf{1}_{[-\pi m_n,\pi m_n]}\|^2 \\ &\qquad + \|(\hat\psi^{(2)}_{\Delta,1} - \psi''_\Delta)(\hat\psi^{(0)}_{\Delta,2} - \psi_\Delta)\mathbf{1}_{[-\pi m_n,\pi m_n]}\|^2] \\ &\le \frac{1}{2\pi^2\Delta^2}\int_{-\pi m_n}^{\pi m_n}\mathbb{E}[|\hat\psi^{(1)}_{\Delta,1}(u) - \psi'_\Delta(u)|^2]\mathbb{E}[|\hat\psi^{(1)}_{\Delta,2}(u) - \psi'_\Delta(u)|^2]du \\ &\quad + \frac{1}{2\pi^2\Delta^2}\int_{-\pi m_n}^{\pi m_n}\mathbb{E}[|\hat\psi^{(2)}_{\Delta,1}(u) - \psi''_\Delta(u)|^2]\mathbb{E}[|\hat\psi^{(0)}_{\Delta,2}(u) - \psi_\Delta(u)|^2]du \\ &\le \frac{m_n}{\pi\Delta^2}\left(\frac{[\mathbb{E}(Z_1^2)]^2}{n^2} + \frac{\mathbb{E}(Z_1^4)}{n^2}\right) \le \frac{C}{n} \end{aligned} \tag{42}$$

because $m_n \le n\Delta$ and $\mathbb{E}(Z_1^2)$ and $\mathbb{E}(Z_1^4)$ have order Δ. □

Now, the study of the $\nu_{n,j}$'s relies on Lemma 10.1. Let us first study the process $\nu_{n,1}$. We must split $Z_k^2 = Z_k^2\mathbf{1}_{Z_k^2\le k_n\sqrt{\Delta}} + Z_k^2\mathbf{1}_{Z_k^2>k_n\sqrt{\Delta}}$ with k_n to be defined later. This implies that $\nu_{n,1}(t) = \nu^P_{n,1}(t) + \nu^R_{n,1}(t)$ (P for Principal, R for residual) with

$$\nu^P_{n,1}(t) = \frac{1}{n}\sum_{k=1}^n[f_t(Z_k) - \mathbb{E}(f_t(Z_k))] \text{ with } f_t(z) = \frac{1}{2\pi\Delta}z^2\mathbf{1}_{z^2\le k_n\sqrt{\Delta}}\langle t^*, e^{iz\cdot}\psi_\Delta\rangle, \tag{43}$$

and $\nu^R_{n,1}(t) = \nu_{n,1}(t) - \nu^P_{n,1}(t)$.

We prove the following result:

Proposition 9.1. *Under the assumptions of Theorem 3.1, choose* $k_n = C\frac{\sqrt{n}}{\ln(n\Delta)}$ *and define*

$$p(m,m') = 4\mathbb{E}(Z_1^4/\Delta)\frac{m \vee m'}{\Delta}, \tag{44}$$

then

$$\mathbb{E}\left(\sup_{t\in S_m+S_{\hat{m}}, \|t\|=1}[\nu_{n,1}^P(t)]^2 - p(m,\hat{m})\right)_+ + \mathbb{E}\left[\sup_{t\in S_{m_n}, \|t\|=1}|\nu_{n,1}^{(R)}(t)|^2\right] \le C\frac{\ln^2(n\Delta)}{n\Delta},$$

where C *is a constant.*

For $\nu_{n,2}$, we have

Proposition 9.2. *Under the assumptions of Theorem 3.1,*

$$\mathbb{E}\left(\sup_{t\in S_m+S_{\hat{m}}, \|t\|=1}[\nu_{n,2}(t)]^2 - p(m,\hat{m})\right)_+ \le \frac{C}{n\Delta},$$

where C *is a constant.*

For both $\nu_{n,3}$ and $\nu_{n,4}$, which are similar, we have to split again $Z_k = Z_k \mathbf{I}_{|Z_k|\le k_n\sqrt{\Delta}} + Z_k \mathbf{I}_{|Z_k|>k_n\sqrt{\Delta}}$ with the same k_n as above. We define $\nu_{n,j}(t) = \nu_{n,j}^P(t) + \nu_{n,j}^R(t)$ as previously, for $j = 3, 4$.

Proposition 9.3. *Under the assumptions of Theorem 3.1, define for* $j = 3, 4$

$$q(m,m') = 4\mathbb{E}^2(Z_1^2)/\Delta)\frac{m \vee m'}{\Delta}, \tag{45}$$

then

$$\mathbb{E}\left(\sup_{t\in S_m+S_{\hat{m}}, \|t\|=1}[\nu_{n,j}^P(t)]^2 - q(m,\hat{m})\right)_+ + \mathbb{E}\left[\sup_{t\in S_{m_n}, \|t\|=1}|\nu_{n,j}^{(R)}(t)|^2\right] \le C\frac{\ln^2(n\Delta)}{n\Delta},$$

where C *is a constant.*

Now, on Ω_ϱ, the following inequality holds (by bounding the indicator by 1), for any choice of κ,

$$(1-\varrho)\text{pen}_{th}(m) \le \text{pen}(m) \le (1+\varrho)\text{pen}_{th}(m),$$

where $\text{pen}_{th}(m) = \mathbb{E}(\text{pen}(m))$. It follows from (40) that

$$\begin{aligned} \frac{1}{4}\mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2 \mathbf{I}_{\Omega_\varrho}) &\le \frac{7}{4}\|h - h_m\|^2 + \text{pen}_{th}(m) - \mathbb{E}(\text{pen}(\hat{m})\mathbf{I}_{\Omega_\varrho}) + \\ &+ C\Delta^2 \int_{-\pi m_n}^{\pi m_n} u^2 |h^*(u)|^2 du + 8\mathbb{E}(\sup_{t\in S_m+S_{\hat{m}}, \|t\|=1} |\nu_n(t)|^2 \mathbf{I}_{\Omega_\varrho}). \end{aligned} \tag{46}$$

Recalling that

$$\nu_n(t) = r_n(t) + \nu_{n,1}^P(t) + \nu_{n,1}^R(t) + \nu_{n,2}(t) + \nu_{n,3}^P(t) + \nu_{n,3}^R(t) + \nu_{n,4}^P(t) + \nu_{n,4}^R(t),$$

we have

$$\begin{aligned}
\mathbb{E}(\sup_{t\in S_m+S_{\hat m},\|t\|=1}|\nu_n(t)|^2\mathbf{I}_{\Omega_\varrho}) &\le 8\left(\frac{C}{n\Delta}+\sum_{j\in\{1,3,4\}}\mathbb{E}(\sup_{t\in S_m+S_{\hat m},\|t\|=1}|\nu_{n,j}^P(t)|^2\mathbf{I}_{\Omega_\varrho})\right.\\
&\qquad\left.+\mathbb{E}(\sup_{t\in S_m+S_{\hat m},\|t\|=1}|\nu_{n,2}(t)|^2\mathbf{I}_{\Omega_\varrho})\right)\\
&\le 8(\frac{C'}{n\Delta}+2\mathbb{E}[(p(m,\hat m)+q(m,\hat m))\mathbf{I}_{\Omega_\varrho}]) \qquad (47)
\end{aligned}$$

We note that

$$p(m,m')+q(m,m')=\frac{1}{4\kappa}(\mathrm{pen}_{th}(m)+\mathrm{pen}_{th}(m')).$$

Thus

$$\begin{aligned}
&\mathrm{pen}_{th}(m)-\mathbb{E}(\mathrm{pen}(\hat m)\mathbf{I}_{\Omega_\varrho})+128\mathbb{E}[(p(m,\hat m)+q(m,\hat m))\mathbf{I}_{\Omega_\varrho}]\\
\le\ &\mathrm{pen}_{th}(m)-(1-\varrho)\mathbb{E}(\mathrm{pen}_{th}(\hat m)\mathbf{I}_{\Omega_\varrho})+\frac{32}{\kappa}\mathbb{E}[(\mathrm{pen}_{th}(m)+\mathrm{pen}_{th}(\hat m))\mathbf{I}_{\Omega_\varrho}]\\
\le\ &(1+\frac{32}{\kappa})\mathrm{pen}_{th}(m)+(\frac{32}{\kappa}-(1-\varrho))\mathbb{E}[\mathrm{pen}_{th}(\hat m))\mathbf{I}_{\Omega_\varrho}].
\end{aligned}$$

Therefore we choose κ such that $(32/\kappa-(1-\varrho))\le 0$, that is $\kappa\ge 32/(1-\varrho)$. This together with (46) and (47) yields

$$\frac14\mathbb{E}(\|\hat h_{\hat m}-h\|^2\mathbf{I}_{\Omega_\varrho}) \le \frac74\|h-h_m\|^2+(2-\varrho)\mathrm{pen}_{th}(m)+C\Delta^2\int_{-\pi m_n}^{\pi m_n}u^2|h^*(u)|^2du+\frac{C"}{n\Delta}.$$

9.2. **Proof of Propositions 9.1, 9.2, 9.3 .**

Proof of Proposition 9.1. Let $m''=m\vee m'$, and note that $S_m+S_{m'}=S_{m''}$. We evaluate the constants M,H,v to apply 10.1 to $\nu_{n,1}^P(t)$ (see (43)).

$$\begin{aligned}
\sup_{z\in\mathbb{R}}|f_t(z)| &\le \frac{k_n}{2\pi\sqrt\Delta}\sup_{z\in\mathbb{R}}|\int_{-\pi m''}^{\pi m''}t^*(-u)e^{iuz}\psi_\Delta(u)du|\\
&\le \frac{k_n}{2\pi\sqrt\Delta}\int_{-\pi m''}^{\pi m''}|t^*(u)|du\le\frac{k_n}{2\pi\sqrt\Delta}(2\pi m''\int_{-\pi m''}^{\pi m''}|t^*(u)|^2du)^{1/2}\\
&= \frac{k_n}{\sqrt\Delta}(m'')^{1/2}\|t\|=\frac{k_n\sqrt{m''}}{\sqrt\Delta}:=M
\end{aligned}$$

Moreover

$$\mathbb{E}(\sup_{t\in S_m+S_{m'},\|t\|=1}[\nu_{n,1}^P(t)]^2)\le\frac{1}{2\pi n\Delta^2}\int_{-\pi m''}^{\pi m''}\mathbb{E}(Z_1^4)\psi_\Delta^2(u)du\le\frac{m''\mathbb{E}(Z_1^4/\Delta)}{n\Delta}:=H^2.$$

The most delicate term is v.

$$\begin{aligned}
\mathrm{Var}(f_t(Z_1)) &= \frac{1}{4\pi^2\Delta^2}\mathbb{E}\left(Z_1^4\mathbf{I}_{Z_1^2\le k_n\sqrt\Delta}|\iint e^{ixZ_1}t^*(-x)\psi_\Delta(x)dx|^2\right)\\
&\le \frac{1}{4\pi^2\Delta^2}\mathbb{E}\left(Z_1^4\iint e^{i(x-y)Z_1}t^*(-x)t^*(y)\psi_\Delta(x)\psi_\Delta(-y)dxdy\right)\\
&= \frac{1}{4\pi^2\Delta^2}\iint\psi_\Delta^{(4)}(x-y)t^*(-x)t^*(y)\psi_\Delta(x)\psi_\Delta(-y)dxdy,
\end{aligned}$$

where we recall that $\psi_\Delta^{(4)}(x) = \mathbb{E}(Z_1^4 e^{ixZ_1})$. Making use of the basis $(\varphi_{m",j}, j \in \mathbb{Z})$ of $S_{m"}$, we have $t = \sum_{j\in\mathbb{Z}} t_j \varphi_{m'',j}$ with $\|t\|^2 = \sum_{j\in\mathbb{Z}} t_j^2 = 1$,

$$\begin{aligned}
\mathrm{Var}(f_t(Z_1)) &\leq \frac{1}{4\pi^2\Delta^2} \sum_{j,k\in\mathbb{Z}} t_j t_k \iint \psi_\Delta^{(4)}(x-y)\varphi_{m'',j}^*(-x)\varphi_{m'',k}^*(y)\psi_\Delta(x)\psi_\Delta(-y)dxdy \\
&\leq \frac{1}{4\pi^2\Delta^2}\left(\sum_{j,k\in\mathbb{Z}} \left|\iint \psi_\Delta^{(4)}(x-y)\varphi_{m'',j}^*(-x)\varphi_{m'',k}^*(y)\psi_\Delta(x)\psi_\Delta(-y)dxdy\right|^2\right)^{1/2} \\
&= \frac{1}{4\pi^2\Delta^2}\left(\iint_{[-\pi m'',\pi m'']^2} |\psi_\Delta^{(4)}(x-y)|^2|\psi_\Delta(x)|^2|\psi_\Delta(-y)|^2dxdy\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(f_t(Z_1)) &\leq \frac{1}{4\pi^2\Delta^2}\left(\iint_{[-\pi m'',\pi m'']^2} \psi_\Delta^{(4)}(x-y)|^2dxdy\right)^{1/2} \\
&\leq \frac{\sqrt{2\pi m''}}{4\pi^2\Delta^2}\left(\int_{[-2\pi m'',2\pi m'']} |\psi_\Delta^{(4)}(z)|^2dz\right)^{1/2}. \qquad (48)
\end{aligned}$$

Therefore, we need to study $\int_{[-2\pi m'',2\pi m'']} |\psi_\Delta^{(4)}(z)|^2dz$. Recall that $\phi(u) = ib - \int_0^u h^*(v)dv$. We have

$$\psi_\Delta^{(4)} = \Delta[\phi^{(3)} + \Delta(4\phi\phi'' + 3(\phi')^2) + 6\Delta^2\phi'\phi^2 + \Delta^3\phi^4]\psi_\Delta,$$

where

$$\phi'(u) = -h^*(u), \phi''(u) = -i\int e^{iux}x^3n(x)dx, \phi^{(3)}(u) = \int e^{iux}x^4n(x)dx$$

satisfy: $\int |\phi'(u)|^2du = \|h\|^2, |\phi'(u)| \leq |h|_1$,

$$\int |\phi''(u)|^2du = \int x^6n^2(x)dx = \int x^2h^2(x)dx, \int |\phi^{(3)}(u)|^2du = \int x^8n^2(x)dx = \int x^4h^2(x)dx.$$

By assumption, h^* is in $\mathbb{L}_1(\mathbb{R})$, thus, $|\phi(u)| \leq |b| + |h^*|_1 := M_\phi$. Therefore

$$|\psi_\Delta^{(4)}|^2 \leq C\Delta^2(|\phi^{(3)}|^2 + \Delta^2((\phi'')^2 + (\phi')^4) + \Delta^4(\phi')^2 + \Delta^6),$$

where C is a constant depending on M_ϕ and $|h|_1$. Therefore,

$$\begin{aligned}
\int_{-2\pi m''}^{2\pi m''} |\psi_\Delta^{(4)}(u)|^2du &\leq C\Delta^2[\int x^4h^2(x)dx + \Delta^2(\int x^2h^2(x)dx + 4\pi m''|h|_1^4) + \Delta^4\|h\|^2 + 4\pi m''\Delta^6] \\
&\leq C_1\Delta^2[\int x^4h^2(x)dx + \Delta^2\int x^2h^2(x)dx + \Delta^4\|h\|^2] + C_2m''\Delta^4
\end{aligned}$$

Thus, using Assumptions (H1), (H3), (H4),

$$\int_{[-2\pi m'',2\pi m'']} |\psi_\Delta^{(4)}(u)|^2du \leq K(\Delta^2 + m''\Delta^4).$$

As $m''\Delta^4 \leq n\Delta^5$ and $n\Delta^3 \leq 1$ we get

$$\int_{[-2\pi m'',2\pi m'']} |\psi_\Delta^{(4)}(u)|^2du \leq 2K\Delta^2.$$

This together with (48) yields

$$v = c\frac{\sqrt{m''}}{\Delta}$$

where c is a constant.

Applying Lemma 10.1 yields, for $\epsilon^2 = 1/2$ and $p(m,m')$ given by (44) yields

$$\mathbb{E}\left(\sup_{t\in S_m+S_{m'},\|t\|=1}[\nu_{n,1}^P(t)]^2 - p(m,m')\right)_+ \leq C_1\left(\frac{\sqrt{m"}}{n\Delta}e^{-C_2\sqrt{m"}} + \frac{k_n^2 m"}{n^2\Delta}e^{-C_3\sqrt{n}/k_n}\right)$$

as $p(m,m') = 4H^2$. We choose

$$k_n = \frac{C_3}{4}\frac{\sqrt{n}}{\ln(n\Delta)},$$

and as $m \leq n\Delta$, we get

$$\mathbb{E}\left(\sup_{t\in S_m+S_{m'},\|t\|=1}[\nu_{n,1}^P(t)]^2 - p(m,m')\right)_+ \leq C_1'\left(\frac{\sqrt{m"}}{n\Delta}e^{-C_2\sqrt{m"}} + \frac{1}{(\Delta n)^4\ln^2(n\Delta)}\right).$$

Therefore

$$\sum_{m'=1}^{m_n}\mathbb{E}\left(\sup_{t\in S_m+S_{m'},\|t\|=1}[\nu_{n,1}^P(t)]^2 - p(m,m')\right)_+ \leq C_1'\left(\frac{\sum_{m'=1}^{n\Delta}\sqrt{m"}e^{-C_2\sqrt{m"}}}{n\Delta} + \frac{1}{(n\Delta)^3\ln^2(n\Delta)}\right).$$

As $C_2xe^{-C_2x}$ is decreasing for $x \geq 1/C_2$, and its maximum is $1/(eC_2)$, we get

$$\begin{aligned}\sum_{m'=1}^{m_n}\sqrt{m"}e^{-C_2\sqrt{m"}} &\leq \sum_{\sqrt{m'}\leq 1/C_2}(eC_2)^{-1} + \sum_{\sqrt{m'}\geq 1/C_2}\sqrt{m'}e^{-C_2\sqrt{m'}}\\ &\leq \frac{1}{eC_2^3} + \sum_{m'=1}^{\infty}\sqrt{m'}e^{-C_2\sqrt{m'}} < +\infty.\end{aligned}$$

It follows that

$$\sum_{m'=1}^{m_n}\mathbb{E}\left(\sup_{t\in S_m+S_{m'},\|t\|=1}[\nu_{n,1}^P(t)]^2 - p(m,m')\right)_+ \leq \frac{C}{n\Delta}.$$

Let us now study the second term $\nu_{n,j}^{(R)}(t)$ in the decomposition of $\nu_{n,j}(t)$. The cases $j = 3, 4$ being similar, we consider only $\nu_{n,j}^{(R)}(t)$ for $j = 1$.

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in S_{m_n},\|t\|=1}|\nu_{n,1}^{(R)}(t)|^2\right]\\ \leq\ &\frac{1}{4\pi^2\Delta^2}\mathbb{E}\left(\int_{-\pi m_n}^{\pi m_n}\left|\frac{1}{n}\sum_{k=1}^{n}(Z_k^2\mathbf{1}_{Z_k^2>k_n\sqrt{\Delta}}e^{iuZ_k} - \mathbb{E}(Z_k^2\mathbf{1}_{Z_k^2>k_n\sqrt{\Delta}}e^{iuZ_k}))\right|^2|\psi_\Delta^2(u)|^2du\right)\\ \leq\ &\frac{\mathbb{E}(Z_1^4\mathbf{1}_{Z_1^2>k_n\sqrt{\Delta}})}{4n\pi^2\Delta^2}\int_{-\pi m_n}^{\pi m_n}du \leq \frac{m_n\mathbb{E}(Z_1^{4+2p})}{2\pi n\Delta^2(k_n\sqrt{\Delta})^p}\\ \leq\ &K\frac{\mathbb{E}(Z_1^{4+2p}/\Delta)\ln^p(n\Delta)}{2\pi(n\Delta)^{p/2}},\end{aligned}$$

using $m_n \leq n\Delta$ and recalling that $k_n = (C_3/4)(\sqrt{n}/\ln(n\Delta))$. Taking $p = 2$, which is possible because $\mathbb{E}(Z_1^8) < +\infty$, gives a bound of order $\ln^2(n\Delta)/(n\Delta)$.

Proposition 9.1 is proved. □

Proof of Proposition 9.2. For $\nu_{n,2}$, the variables are bounded without splitting, and the function f_t is replaced by $\tilde{f}_t = (2\pi\Delta)^{-1}\langle t^*, e^{iz\cdot}\psi''_\Delta\rangle$. We just check the orders of M, H^2 and v for the application of Lemma 10.1. For $t \in S_{m''} = S_m + S_{m'}$ and $\|t\| \leq 1$, we have

$$\begin{aligned}\sup_{z\in\mathbb{R}} |\tilde{f}_t(z)| &\leq \frac{1}{2\pi\Delta}\sqrt{\int_{-\pi m''}^{\pi m''} |t^*(-u)|^2 du \int_{-\pi m''}^{\pi m''} |\psi''_\Delta(u)|^2 du} \\ &\leq \sqrt{m''}\frac{\mathbb{E}(Z_1^2)}{\Delta} \leq C\sqrt{m''} := M.\end{aligned}$$

Next

$$\mathbb{E}(\sup_{t\in S_m+S_{m'},\|t\|=1} [\nu_{n,2}(t)]^2) \leq \frac{1}{2\pi n\Delta^2}\int_{-\pi m''}^{\pi m''} |\psi''_\Delta(u)|^2 du \leq \frac{m''\mathbb{E}^2(Z_1^2)/\Delta)}{n\Delta} := H^2.$$

Following the same line as previously for v, we get

$$\mathrm{Var}(\tilde{f}_t(Z_1)) \leq \frac{1}{4\pi^2\Delta^2}\left(\iint_{[-\pi m'',\pi m'']^2} |\psi_\Delta(u-v)|^2|\psi''_\Delta(u)|^2|\psi''_\Delta(-v)|^2 dudv\right)^{1/2}.$$

As $\psi''_\Delta = \Delta(\phi' + \Delta\phi^2]\psi_\Delta$, we get (recall that $M_\phi = |b| + |h^*|_1$ is the upper bound of $|\phi(u)|$):

$$\begin{aligned}\mathrm{Var}(\tilde{f}_t(Z_1)) &\leq \frac{1}{4\pi^2\Delta^2}\int_{-\pi m''}^{\pi m''} |\psi''_\Delta(x)|^2 dx \\ &\leq \frac{2\Delta^2(\|h^*\|^2 + 2\pi m''\Delta^2 M_\phi^2)}{4\pi^2\Delta^2} \leq \frac{1}{\pi}(\|h\|^2 + M_\phi^2 m_n\Delta^2) \\ &\leq \frac{\|h\|^2 + M_\phi^2}{\pi} := v\end{aligned}$$

as $m_n\Delta^2 \leq n\Delta^3 \leq 1$. □

Proof of Proposition 9.3. Here f_t is replaced by $\breve{f}_t(z) = z\mathbf{I}_{|z|\leq k'_n\sqrt{\Delta}}\langle t^*, e^{iz\cdot}\psi'_\Delta\rangle$. Using now that $|\psi'_\Delta(u)| \leq \mathbb{E}(|Z_1|) \leq \sqrt{\mathbb{E}(Z_1^2)}$, we obtain here that $M = k'_n\sqrt{m''}\sqrt{\mathbb{E}(Z_1^2/\Delta)}$. On the other hand, we find $H^2 = m''\mathbb{E}^2(Z_1^2)/(n\Delta^2)$. Lastly, we find

$$\mathrm{Var}(\breve{f}_t(Z_1)) \leq \frac{1}{4\pi^2\Delta^2}\left(\iint_{[-\pi m'',\pi m'']^2} |\psi_\Delta^{(2)}(u-v)|^2|\psi'_\Delta(u)|^2|\psi'_\Delta(-v)|^2 dudv\right)^{1/2}.$$

With the bounds for $|\psi'_\Delta|$ and $\int_{-2\pi m''}^{2\pi m''} |\psi''_\Delta(z)|^2 dz$, we obtain $v = c\mathbb{E}(Z_1^2/\Delta)\sqrt{m''}$. □

9.3. **Proof of Theorem 4.1.** The proof follows the same lines as for the adaptive estimator of h. We introduce, for $0 < \varrho < 1$,

$$\Omega_b := \left\{\left|\frac{[(1/(3n\Delta))\sum_{k=1}^{3n} Z_k^6]}{(\mathbb{E}(Z_1^6/\Delta))} - 1\right| \leq \varrho\right\}.$$

Provided that $\mathbb{E}(Z_1^{24}) < \infty$, we can make use of the Rosenthal inequality to obtain:

$$\mathbb{E}(\|\bar{p}_{\bar{m}} - p\|^2\mathbf{I}_{\Omega_\varrho^c}) \leq C/n\Delta.$$

For the study of $\mathbb{E}(\|\bar{p}_{\bar{m}} - p\|^2\mathbf{I}_{\Omega_\varrho})$, the decomposition is similar to the previous case (see (40)) where $\hat{h}_{\hat{m}}, h$ are now replaced by $\bar{p}_{\bar{m}}, p$). The processes $R_n(t)$ and $\nu_n(t)$ are given by:

$$\nu_n(t) = \frac{1}{2\pi}\langle \bar{p}^* - \mathbb{E}(\bar{p}^*), t^*\rangle, \quad R_n(t) = \frac{1}{2\pi}\langle \mathbb{E}(\bar{p}^*) - p^*, t^*\rangle.$$

The term $R_n(t)$ is dealt using (30). For the term containing $\nu_n(t)$, we need apply 10.1. So, ν_n is split into the sum of a principal and a residual term, respectively denoted by ν_n^P and ν_n^R with

$$\nu_n^P(t) = \frac{1}{3n}\sum_{k=1}^{3n}[f_t(Z_k) - \mathbb{E}(f_t(Z_k))] \text{ with } f_t(z) = \frac{1}{2\pi\Delta} z^3 \mathbf{1}_{|z|^3 \le k_n\sqrt{\Delta}} \langle t^*, e^{iz\cdot}\rangle, \tag{49}$$

and $\nu_n^R(t) = \nu_n(t) - \nu_n^P(t)$. Everything is analogous. The difference is that, for applying 10.1, we have to bound $\int_{-2\pi m"}^{2\pi m"} |\psi_\Delta^{(6)}(u)|^2 du$ (instead of $\int_{-2\pi m"}^{2\pi m"} |\psi_\Delta^{(4)}(u)|^2 du$ previously). Using $\psi'_\Delta = \Delta\tilde{\phi}\psi_\Delta$ (see (5)-(25)), we find

$$\begin{aligned}
\psi_\Delta^{(6)} \quad = \quad & \Delta\psi_\Delta\phi^{(5)} + \Delta^2\psi_\Delta[6\tilde{\phi}\phi^{(4)} + 15\phi^{(3)}(\phi'(u) - \sigma^2)] \\
& + \Delta^3\psi_\Delta[15\phi^{(3)}\tilde{\phi}^2 + 60\phi''(\phi'(u) - \sigma^2)\tilde{\phi} + 15(\phi'(u) - \sigma^2)^3] \\
& + \Delta^4\psi_\Delta[17\phi''\tilde{\phi}^{(3)} + 36\tilde{\phi}^{(2)}(\phi'(u) - \sigma^2)^2] \\
& + 12\Delta^5\psi_\Delta\tilde{\phi}^4(\phi'(u) - \sigma^2) + \Delta^6\psi_\Delta\tilde{\phi}^6.
\end{aligned}$$

Now, $\tilde{\phi}(u) \le C(1 + |u|)$ and all the derivatives of $\tilde{\phi}, \phi$ are bounded. Moreover, under (H6), $\int |\phi^{(5)}(u)|^2 du = \int x^6 |p(x)|^2 dx < +\infty$. Thus, we find the following bound

$$\int_{-2\pi m"}^{2\pi m"} |\psi_\Delta^{(6)}|^2 \le C\Delta^2(1 + \Delta^2 m^3 + \Delta^4 m^5 + \Delta^6 m^7 + \Delta^8 m^9 + \Delta^{10} m^{13}) = O(\Delta^2),$$

as $m \le \sqrt{n\Delta}$. The proof may then be completed as for $\hat{h}_{\hat{m}}$.□

9.4. **Proof of Proposition 5.1.**

Proof of (i). The assumptions and the fact that $r \le 1$ imply

$$|\Gamma_\Delta|^r = |\sum_{s\le\Delta} \Gamma_s - \Gamma_{s_-}|^r \le \sum_{s\le\Delta} |\Gamma_s - \Gamma_{s_-}|^r.$$

Taking expectations yields

$$\mathbb{E}|\Gamma_\Delta|^r \le \Delta \int |\gamma|^r n(\gamma) d\gamma.$$

Proof of (ii). Consider f a non-negative function such that $f(0) = 0$. We have:

$$\mathbb{E}\sum_{s\le t} f(X_s - X_{s_-}) = \mathbb{E}\sum_{s\le t} f(B_{\Gamma_s} - B_{\Gamma_{s_-}}).$$

Then,

$$\sum_{s\le t} \mathbb{E} f(B_{\Gamma_s} - B_{\Gamma_{s_-}}) = \sum_{s\le t}\int_{\mathbb{R}} f(x)\left(\mathbb{E} e^{(-x^2/2(\Gamma_s - \Gamma_{s_-}))}\frac{1}{\sqrt{2\pi(\Gamma_s - \Gamma_{s_-})}}\right) dx.$$

Since, for all x,

$$\mathbb{E}\sum_{s\le t} e^{(-x^2/2(\Gamma_s - \Gamma_{s_-}))}\frac{1}{\sqrt{2\pi(\Gamma_s - \Gamma_{s_-})}} = t\int_0^{+\infty} e^{-x^2/2\gamma}\frac{1}{\sqrt{2\pi\gamma}} n_\Gamma(\gamma) d\gamma,$$

we get the formula for n_X. Setting $m_\alpha = \mathbb{E}|X|^\alpha$, for X a standard Gaussian variable, yields

$$\int_{\mathbb{R}} |x|^\alpha n_X(x) dx = m_\alpha \int_0^{+\infty} \gamma^{\alpha/2} n_\Gamma(\gamma) d\gamma.$$

Thus
$$\mathbb{E}|X_\Delta|^r = m_r \mathbb{E}(\Gamma_\Delta^{r/2}).$$
As $r/2 \le 1$,
$$\Gamma_\Delta^{r/2} = (\sum_{s\le\Delta} \Gamma_s - \Gamma_{s_-})^{r/2} \le \sum_{s\le\Delta}(\Gamma_s - \Gamma_{s_-})^{r/2}.$$
Taking expectation gives the result.
Proof of *(iii)* The result is proved *e.g.* in Barndorff-Nielsen *et al.* (2006) (Theorem 1, p. 804) (see also Aït-Sahalia and Jacod (2007)). □

9.5. **Proof of proposition 5.3.** The study of (35) relies on the following result which is standard for $r = 2$.

Lemma 9.3. *Let $Y_t = \theta t + \sigma W_t$ for θ a constant and consider*
$$\tilde{\sigma}_n^{(r)} = \frac{1}{m_r n \Delta^{r/2}} \sum_{k=1}^n |Y_{k\Delta} - Y_{(k-1)\Delta}|^r.$$
Then, for all r, $\sqrt{n}(\tilde{\sigma}_n^{(r)} - \sigma^r)$ converges in distribution to a centered Gaussian distribution with variance $\sigma^{2r}(m_{2r}/m_r^2 - 1)$ as n tends to infinity, Δ tends to 0, $n\Delta$ tends to infinity, and $n\Delta^2$ tends to 0.

Proof. We have
$$\mathbb{E}\tilde{\sigma}_n^{(r)} = \frac{1}{m_r}\mathbb{E}|\theta\sqrt{\Delta} + \sigma X|^r,$$
for X a standard Gaussian variable. We get, after a change of variables,
$$\mathbb{E}\tilde{\sigma}_n^{(r)} = \frac{1}{m_r}\int |u|^r \exp[-\frac{(u-\theta\sqrt{\Delta})^2}{2\sigma^2}]\frac{du}{\sigma\sqrt{2\pi}}$$
Thus,
$$\mathbb{E}\tilde{\sigma}_n^{(r)} - \sigma^r = \sigma^r\left(e^{-\theta^2\Delta/2\sigma^2} - 1\right) + \frac{1}{m_r}e^{-\theta^2\Delta/2\sigma^2}\int |u|^r(e^{\theta u\sqrt{\Delta}/\sigma^2} - 1)e^{-\frac{u^2}{2\sigma^2}}\frac{du}{\sigma\sqrt{2\pi}}$$
Noting that
$$e^{\theta u\sqrt{\Delta}/\sigma^2} - 1 = \theta u\sqrt{\Delta}/\sigma^2 + \Delta\sum_{n\ge 2}\frac{1}{n!}(u\theta/\sigma^2]^n\Delta^{n/2-1}$$
and that $\int |u|^r u e^{-\frac{u^2}{2\sigma^2}}\frac{du}{\sigma\sqrt{2\pi}} = 0$, we easily obtain
$$|\mathbb{E}\tilde{\sigma}_n^{(r)} - \sigma^r| \le c\Delta$$
Thus, $\sqrt{n}|\mathbb{E}\tilde{\sigma}_n^{(r)} - \sigma^r| = o(1)$ if $\sqrt{n}\Delta = (n\Delta^2)^{1/2} = o(1)$. Noting that $\mathbb{E}|\theta\sqrt{\Delta} + \sigma X|^k$ converges to $\sigma^k m_k$ as Δ tends to 0, we get
$$n\mathrm{Var}\tilde{\sigma}_n^{(r)} \to \sigma^{2r}(m_{2r}/m_r^2 - 1).$$
Finally, we look at
$$\chi_{k,n} = \frac{1}{n}\left(|\theta\sqrt{\Delta} + \sigma(W_{k\Delta} - W_{(k-1)\Delta})/\sqrt{\Delta}|^r - \mathbb{E}|\theta\sqrt{\Delta} + \sigma X|^r\right),$$
which satisfies
$$n\mathbb{E}\chi_{k,n}^4 \le \frac{c}{n^3}.$$
Hence, $\sqrt{n}(\tilde{\sigma}_n^{(r)} - \mathbb{E}\tilde{\sigma}_n^{(r)})$ converges in distribution to the centered Gaussian with the announced variance which completes the proof. □

Proof of (i). As noted above, $L_t = b_0 t + \sigma W_t + \Gamma_t$ with $b_0 = b - \int x n(x)dx$. Using that, for $r \leq 1$, $||\sum a_i + b_i|^r - |\sum a_i|^r| \leq \sum |b_i|^r$, we get

$$|\hat{\sigma}_n^{(r)} - \tilde{\sigma}_n^{(r)}| \leq \frac{1}{m_r n \Delta^{r/2}} \sum_{k=1}^{n} |\Gamma_{k\Delta} - \Gamma_{(k-1)\Delta}|^r,$$

where $\tilde{\sigma}_n^{(r)}$ is built with $Y_t = b_0 t + \sigma W_t$ as in the previous Lemma. Thus, applying Proposition 5.1 (i),

$$\mathbb{E}\sqrt{n}|\hat{\sigma}_n^{(r)} - \tilde{\sigma}_n^{(r)}| \leq \frac{1}{m_r}\sqrt{n}\Delta^{1-r/2} \int |x|^r n(x)dx.$$

Since $r < 1$, the constraint $n\Delta^{2-r} = o(1)$ can be fulfilled and implies $n\Delta^2 = o(1)$. Hence, the result follows from the previous proposition.

Proof of (ii). The proof is analogous to the previous one (using Proposition 5.1 (ii)) and is omitted. □

9.6. **Proof of Proposition 5.4.** Let us set

$$\tilde{\sigma}_M^2 = \frac{1}{3n\Delta}\sum_{k=1}^{3n} Z_k^2 \cos MZ_k, \quad \hat{b}_1 = \frac{1}{n\Delta}\sum_{k=1}^{n} Z_k, \quad \hat{b}_2 = \frac{1}{n\Delta}\sum_{\ell=n+1}^{2n} Z_\ell.$$

We have

$$3n\Delta \mathrm{Var}\tilde{\sigma}_M^2 = \frac{1}{\Delta}\left[\mathbb{E}(Z_1^4 \cos^2 MZ_1) - \left(\mathbb{E}(Z_1^2 \cos MZ_1)\right)^2\right].$$

We need choose M tending to infinity in such a way that the right handside above tends to a limit. For this, we use:

$$\mathbb{E}(Z_1^2 \cos MZ_1) = -\mathrm{Re}\psi_\Delta''(M),$$

$$\mathbb{E}(Z_1^4 \cos^2 MZ_1) = \frac{1}{2}\mathbb{E}(Z_1^4(1 + \cos 2MZ_1)) = \frac{1}{2}(\mathbb{E}(Z_1^4) + \mathrm{Re}\psi_\Delta^{(4)}(2M)).$$

We know that $(1/\Delta)\mathbb{E}(Z_1^4$ tends to $\int x^4 n(x)dx$ as Δ tends to 0. Some elementary computations yield (see (5)-(25)):

$$\psi_\Delta''(u) = -\Delta\psi_\Delta(u)(h^*(u) + \sigma^2) + \Delta^2\psi_\Delta(u)\tilde{\phi}^2(u),$$

$$\psi_\Delta^{(4)}(u) = \psi_\Delta(u)\left[\Delta\phi^{(3)}(u) + \Delta^2\left(4\phi''(u)\tilde{\phi}(u)) + 3(h^*(u) + \sigma^2)^2\right) - 6\Delta^3\tilde{\phi}^2(u)(h^*(u) + \sigma^2) + \Delta^4\tilde{\phi}^4(u)\right].$$

We have $|\psi_\Delta| \leq 1$, $|\tilde{\phi}(u)| \leq |b| + |u|(\sigma^2 + |h|_1)$, $\phi', \phi'', \phi^{(3)}$ are Fourier transforms of integrable functions and thus tend to 0 as $|u|$ tends to infinity. Hence, as M tends to infinity,

$$\frac{1}{\Delta}|\psi_\Delta^{(4)}(2M)| \leq C(\Delta M + \Delta^2 M^2 + \Delta^3 M^4).$$

Choosing M such that $\Delta^3 M^4$ tends to 0, we obtain that the above term tends to 0. This choice implies also that $\Delta^{-1/2}\psi_\Delta''(M)$ tends to 0. Hence, $3n\Delta\mathrm{Var}\tilde{\sigma}_M^2$ tends to $\int x^4 n(x)dx/2$.

For the Lindeberg condition, we write

$$\sqrt{n\Delta}(\tilde{\sigma}_M^2 - \mathbb{E}(\tilde{\sigma}_M^2)) = \sum_{k=1}^{3n} \chi_{k,n},$$

where $n\mathbb{E}|\chi_{k,n}|^4 \leq \frac{C}{n\Delta}(\mathbb{E}Z_1^4/\Delta)$ tends to 0. So, we have the convergence in distribution result for $\sqrt{3n\Delta}(\tilde{\sigma}_M^2 - \mathbb{E}\tilde{\sigma}_M^2)$. Finally,

$$\sqrt{3n\Delta}(\hat{\sigma}_M^2 - \mathbb{E}\hat{\sigma}_M^2) - \sqrt{3n\Delta}(\tilde{\sigma}_M^2 - \mathbb{E}\tilde{\sigma}_M^2) = -\Delta\sqrt{3n\Delta}(\hat{b}_1\hat{b}_2 - b^2).$$

As the righthand side above tends to 0 in probability, we obtain the convergence in distribution of $\sqrt{3n\Delta}(\hat{\sigma}_M^2 - \mathbb{E}\hat{\sigma}_M^2)$.

It remains to study the bias of $\hat{\sigma}_M^2$. From definition (36), we deduce

$$\mathbb{E}(\hat{\sigma}_M^2) = \sigma^2 + h^*(0) - \int_0^M \mathrm{Im} p^*(v)dv + \int_0^M \mathrm{Im}(p^*(v) - \mathbb{E}(\bar{p}^*(v)))dv.$$

Therefore,

$$|\mathbb{E}(\hat{\sigma}_M^2) - \sigma^2| \leq \int_0^M |\mathbb{E}(\bar{p}^*(v)) - p^*(v)|dv + \int_M^{+\infty} |p^*(v)|dv.$$

It follows from (30) that

$$\begin{aligned} |\mathbb{E}(\hat{\sigma}_M^2) - \sigma^2| &\leq C\int_0^M \Delta|p^*(v)||v|(1+|v|)dv + \Delta(1+|v|)dv + \Delta^2(1+|v|)^3 dv + \int_M^{+\infty} |p^*(v)|dv \\ &\leq C'\left(\Delta\int_0^M |p^*(v)|(1+v^2)dv + \Delta M^2 + \Delta^2 M^4\right) + \int_M^{+\infty} |p^*(v)|dv \end{aligned}$$

Assume now that $p \in \mathcal{C}(a, L)$, then

$$\begin{aligned} |\mathbb{E}(\hat{\sigma}_M^2) - \sigma^2| &\leq C'\left(\Delta\int_0^M |p^*(v)|(1+v^2)^{a/2}(1+v^2)^{1-a/2}dv + \Delta M^2 + \Delta^2 M^4\right) \\ &\quad + \int_M^{+\infty} |p^*(v)|(1+v^2)^{a/2}(1+v^2)^{-a/2}dv \\ &\leq C'\Delta\sqrt{L}\left(\int_0^M (1+v^2)^{2-a}dv\right)^{1/2} + C'\Delta M^2 + C'\Delta^2 M^4 \\ &\quad + \sqrt{L}\left(\int_M^{+\infty} (1+v^2)^{-a}dv\right)^{1/2} \end{aligned}$$

If $a > 5/2$, we get

$$\begin{aligned} |\mathbb{E}(\hat{\sigma}_M^2) - \sigma^2| &\leq K(\Delta + \Delta M^2 + \Delta^2 M^4 + M^{-a+1/2}) \\ &\leq K(\Delta + \Delta M^2 + \Delta^2 M^4 + M^{-2}). \end{aligned}$$

Thus, choosing $M = \Delta^{-1/4}$ will give a bias of order $\Delta^{1/2}$. As $n\Delta^2$ tends to 0, and $M^4\Delta^3 = \Delta^2 \to 0$, we get the result. □

10. Appendix

10.1. **Two classical results. The Talagrand inequality.** The following result follows from the Talagrand concentration inequality given in Klein and Rio (2005) and arguments in Birgé and Massart (1998) (see the proof of their Corollary 2 page 354).

Lemma 10.1. *(Talagrand Inequality) Let $Y_1, \ldots, Y_n$ be independent random variables, let $\nu_{n,Y}(f) = (1/n)\sum_{i=1}^n [f(Y_i) - \mathbb{E}(f(Y_i))]$ and let $\mathcal{F}$ be a countable class of uniformly bounded measurable functions. Then for $\epsilon^2 > 0$*

$$\mathbb{E}\Big[\sup_{f\in\mathcal{F}} |\nu_{n,Y}(f)|^2 - 2(1+2\epsilon^2)H^2\Big]_+ \leq \frac{4}{K_1}\left(\frac{v}{n}e^{-K_1\epsilon^2\frac{nH^2}{v}} + \frac{98M^2}{K_1 n^2 C^2(\epsilon^2)}e^{-\frac{2K_1 C(\epsilon^2)\epsilon}{7\sqrt{2}}\frac{nH}{M}}\right),$$

with $C(\epsilon^2) = \sqrt{1+\epsilon^2} - 1$, $K_1 = 1/6$, and

$$\sup_{f\in\mathcal{F}} \|f\|_\infty \leq M, \quad \mathbb{E}\Big[\sup_{f\in\mathcal{F}} |\nu_{n,Y}(f)|\Big] \leq H, \ \sup_{f\in\mathcal{F}} \frac{1}{n}\sum_{k=1}^n \mathrm{Var}(f(Y_k)) \leq v.$$

By standard density arguments, this result can be extended to the case where $\mathcal{F}$ is a unit ball of a linear normed space, after checking that $f \mapsto \nu_n(f)$ is continuous and $\mathcal{F}$ contains a countable dense family.

The Rosenthal inequality. (see *e.g.* Hall and Heyde (1980, p.23)) Let $(X_i)_{1\leq i\leq n}$ be n independent centered random variables, such that $\mathbb{E}(|X_i|^p) < +\infty$ for an integer $p \geq 1$. Then there exists a constant $C(p)$ such that

$$\mathbb{E}\left(\left|\sum_{i=1}^n X_i\right|^p\right) \leq C(p)\left(\sum_{i=1}^n \mathbb{E}(|X_i|^p) + \left(\sum_{i=1}^n \mathbb{E}(X_i^2)\right)^{p/2}\right). \tag{50}$$

References

[1] Aït-Sahalia Y. and Jacod J. (2007). Volatility estimators for discretely sampled Lévy processes. *The Annals of Statistics* **35**, 355-392.

[2] Barndorff-Nielsen O.E. and Shephard N. (2001). Modelling by Lévy processes for financial econometrics. In: *Lévy processes. Theory and Applications* (Barndorff-Nielsen O.E., Mikosch T., Resnick S.L.), 283-318.

[3] Barndorff-Nielsen O.E., Shephard N. and Winkel M. (2006). Limit theorems for multipower variation in the presence of jumps. *Stochastic Process. Appl* **116**, 796-806.

[4] Barron, A.R., Birgé, L. and Massart, P. (1999). Risk bounds for model selection via penalization. *Probab. Theory Relat. Fields.* **97**, 113-150.

[5] Bertoin J. (1996). *Lévy processes.* Cambridge University Press.

[6] Birgé, L. and Massart, P. (1998). Minimum contrast estimators on sieves: Exponential bounds and rates of convergence. *Bernoulli* **4**, 329-375.

[7] Comte, F. and Genon-Catalot, V. (2008). Nonparametric adaptive estimation for pure jump Lévy processes. To appear in *Les Annales de l'I.H.P., Probab. & Statist.*.

[8] Comte, F. and Genon-Catalot, V. (2009a). Nonparametric estimation for pure jump Lévy processes based on high frequency data. *Stochastic Process. Appl* **119**, 4088-4123.

[9] Comte, F. and Genon-Catalot, V. (2009b). Nonparametric estimation for pure jump irregularly sampled or noisy Lévy processes. Preprint MAP5 2009-25. To appear in *Statistica Neerlandica*.

[10] van Es, B., Gugushvili, S. and Spreij, P. (2007). A kernel type nonparametric density estimator for decompounding. *Bernoulli* **13**, 672-694.

[11] Figueroa-López J.E. (2008). Small-time moment asymptotics for Lévy processes. *Statist. Probab. Lett.* **78**, 3355-3365.

[12] Figueroa-López J.E. (2009). Nonparametric estimation for Lévy models based on discrete-sampling. *IMS Lecture Notes-Monograph Series. Optimality: The Third Erich L. Lehmann Symposium*, **57**, 117-146.

[13] Figueroa-López J.E. and Houdré C. (2006). Risk bounds for the nonparametric estimation of Lévy processes. IMS Lecture Notes-Monograph Series High dimensional probability **51**, 96-116.

[14] Gugushvili, S. (2009). Nonparametric estimation of the characteristic triplet of a discretely observed Lévy process. *J. Nonparametr. Stat.*, **21**, 321-343.

[15] Hall, P. and Heyde, C.C. (1980). *Martingale limit theory and its applications.* Academic press, Inc. (London) LTD.

[16] Härdle, W., Kerkyacharian, G., Picard, D., and Tsybakov, A. (1998). *Wavelets, approximation, and statistical applications.* Lecture Notes in Statistics, 129. Springer-Verlag, New York.

[17] Jacod J. (2007). Asymptotic properties of power variations of Lévy processes. *ESAIM: Probability and Statistics*, **11**, 173-196.

[18] Jongbloed, G. and van der Meulen, F. H. (2006) Parametric estimation for subordinators and induced OU processes. *Scand. J. Statist.* **33**, 825-847.

[19] Jongbloed, G., van der Meulen, F. H. and van der Vaart, A. W. (2005) Nonparametric inference for Lévy-driven Ornstein-Uhlenbeck processes. *Bernoulli* **11**, 759-791.

[20] Klein, T. and Rio, E. (2005). Concentration around the mean for maxima of empirical processes. *Ann. Probab.* **33** 1060-1077.

[21] Küchler, U. and Tappe, S. (2008) Bilateral Gamma distributions and processes in financial mathematics. *Stochastic Process. Appl.*, **118**, 261-283.

[22] Lebedev, N.N. (1972). *Special functions and their applications.* Revised edition, translated from the Russian and edited by Richard A. Silverman. Dover Publications, Inc., New York.

[23] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach spaces. Isoperimetry and processes.* Springer-Verlag, Berlin, Ergebnisse der Mathematik und ihrer Grenzgebiete. **3**. Folge, 23.

[24] Madan, D.B. and Seneta, E. (1990) The Variance Gamma (V.G.) Model for Share Market Returns. *The Journal of Business* **63**, No. 4, 511-524.

[25] Massart, P. (2007) *Concentration inequalities and model selection.* Lectures from the 33rd Summer School on Probability Theory held in Saint-Flour, July 6–23, 2003. With a foreword by Jean Picard. Lecture Notes in Mathematics, 1896. Springer, Berlin

[26] Meyer, Y. (1990) *Ondelettes et opérateurs.* I. Hermann, Paris.

[27] Neumann, M. and Reiss, M. (2009). Nonparametric estimation for Lévy processes from low-frequency observations. *Bernoulli* **15**, 223-248.

[28] Sato K.I. (1999). *Lévy processes and infinitely divisible distributions.* Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge.

[29] Watteel R.N. and Kulperger R.J. (2003). Nonparametric estimation of the canonical measure for infinitely divisible distributions. *Journal of Statistical Computation and Simulation*, **73**, 7, 525-542.

[30] Woerner J.H.C. (2006). Power and multipower variation: Inference for high frequency data, in: A.N. Shiryaev, M. do Rosario Grossinho, P. 0liviera, M. Esquivel (Eds.), Proceedings of the International Conference on Stochastic Finance 2004, Springer Verlag, Berlin, 2006, 343-364.