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To cite this version:
Dehbia Achab. Construction process for simple Lie algebras. 2010. hal-00445795

HAL Id: hal-00445795
https://hal.archives-ouvertes.fr/hal-00445795
Preprint submitted on 11 Jan 2010

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CONSTRUCTION PROCESS FOR SIMPLE LIE ALGEBRAS

Dehbia ACHAB

Abstract

We give here a construction process for the complex simple Lie algebras and the non Hermitian type real forms which intersect the minimal nilpotent complex adjoint orbit, using a finite dimensional irreducible representation of the conformal group, or of some 2-fold covering of it, with highest weight a semi-invariant of degree 4. This process leads to a 5-graded simple complex Lie algebra and the underlying semi-invariant is intimately related to the structure of the minimal nilpotent orbit. We also describe a similar construction process for the simple real Lie algebras of Hermitian type.

Conformal and Meta-Conformal Groups

Let $V$ be a Euclidean complex vector space and $Q$ a degree 4 homogeneous polynomial on $V$. Let $L$ be defined by

$$L := \{ g \in GL(V) \mid \exists \gamma(g) \in \mathbb{C}, Q(gz) = \gamma(g)Q(z) \}$$

and suppose that it has an open orbit and that $L$ is self-adjoint:

$$\forall g \in L, g^* \in L.$$

More precisely, $V$ is a semi-simple complex Jordan algebra with rank $\leq 4$, $L$ is the structure group of $V$ and $Q$ semi-invariant for $L$.

If $V = \sum_{i=1}^{s} V_i$ is the decomposition of $V$ into simple ideals, then $Q(z) = \prod_{i=1}^{s} \Delta^{k_i}(z_i)$, where $\Delta$ is the determinant polynomial of $V_i$ and the $k_i$ are positive integers such that $\sum_{i=1}^{s} k_i r_i = 4$, $r_i$ being the rank of the Jordan algebra $V_i$. In the sequel, $e$ is the unit element of $V$.

Let $K$ be the conformal group of $V$, that is the set of rational transformations $g$ of $V$ such that, for each $z \in V$ where $g$ is defined, the differential $Dg(z) \in L$.

$K$ is a semi-simple Lie group. It is generated by $L$, the group $N$ of translations $\tau_a (a \in V)$ and the conformal inversion $\sigma(z) = \nabla logQ(z)$.

$P := L \ltimes N$ is a maximal parabolic subgroup of $K$ and $L$ is its Lévi factor.

The Lie algebra $\mathfrak{t}$ of $K$ writes $\mathfrak{t} = \mathfrak{t}_{-1} + \mathfrak{t}_0 + \mathfrak{t}_1$, with

$$\mathfrak{t}_{-1} = Lie(N), \mathfrak{t}_1 = Lie(\sigma N \sigma), \mathfrak{t}_0 = Lie(L).$$

Let $\mathfrak{p}$ be the complex vector space generated by the polynomials $Q(z - a)$ with $a \in V$.

We first suppose that there exists a character $\chi$ of $L$ such that

$$Q(lz) = \chi(l)^2 Q(z).$$

Then, the conformal group $K$ acts on $\mathfrak{p}$ by:

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\[
(\kappa(g)p)(z) = \mu(g, z)p(g^{-1}z)
\]
with
\[
\mu(g^{-1}, z) = \chi(Dg(z))^{-1}.
\]
In particular
\[
(\kappa(\tau_u)p)(z) = p(z - a) \quad (a \in V)
\]
\[
(\kappa(l)p)(z) = \chi(l)p(l^{-1}z) \quad (l \in L)
\]
\[
(\kappa(\sigma)p)(z) = Q(z)p(-z^{-1}).
\]
\(\kappa\) is the finite dimensional irreducible representation of \(K\) with highest weight \(Q\). It is also the representation \(Ind^K_L \chi\) obtained by parabolic induction from the character \(\chi\) of \(L\). The derived representation \(d\kappa\) can be obtained on the generators of \(\mathfrak{k}\) as follows:

For \(X \in \mathfrak{k}_{-1}\), let \(v \in V\) be such that \(exp(tX) = \tau_{tv}\), then
\[
(d\kappa(X)p)(z) = \frac{d}{dt} |_{t=0} p(z - tv) = -Dp(z)(v).
\]
For \(X \in \mathfrak{k}_0\),
\[
(d\kappa(X)p)(z) = \frac{d}{dt} |_{t=0} \gamma(exp(\frac{t}{2}X))p(exp(-tX)z)
\]
\[
= D\gamma(id) \circ D(exp)(0)(\frac{1}{2}X)p(z) - Dp(z)(D(exp)(0)(X)z)
\]
\[
= \frac{1}{2}D\gamma(id)(X)p(z) - Dp(z)(X.z).
\]
For \(X \in \mathfrak{k}_1\), \(exp(tX) = \sigma\tau_{tv}\sigma\) with \(u \in V\), then
\[
(d\kappa(X)p)(z) = \frac{d}{dt} |_{t=0} (\kappa(\sigma\tau_{tu}\sigma)p)(z)
\]
\[
= \frac{d}{dt} |_{t=0} (\kappa(\sigma)\kappa(\tau_u)[Q(z)p(\sigma(z))]) = \frac{d}{dt} |_{t=0} (\kappa(\sigma)[Q(z - tu)p(\sigma(z - tu))])
\]
\[
= \frac{d}{dt} |_{t=0} [Q(\sigma(z) - tu)p(\sigma(\sigma(z) - tu))]) = \frac{d}{dt} |_{t=0} Q(e - tu)p(\sigma(\sigma(e - tu)))
\]
\[
= -DQ(e)(zu)p(z) - Dp(z)(zu)
\]
If the character \(\chi\) of \(L\) does’nt exist, we consider the 2-fold-covering group of \(L\) defined by
\[
\tilde{L}^{(2)} = \{(l, \alpha) \in L \times \mathbb{C}^* \mid \alpha^2 = \gamma(l)^{-1}\}.
\]
\(\tilde{L}^{(2)}\) acts on \(V\) by \((l, \alpha).z = \alpha l.z\), then
\[
Q((l, \alpha).z) = \alpha^4 \gamma(l)Q(z) = \alpha^2 Q(z) = \tilde{\chi}(l, \alpha)^2 Q(z)
\]
where \(\tilde{\chi}(l, \alpha) = \alpha\) is a character of \(\tilde{L}^{(2)}\). Moreover, \(\tilde{L}^{(2)}\) is a subgroup of a 2-fold-covering group of the conformal group \(K\), which will be called the meta-conformal group, defined by:
\[
\tilde{K}^{(2)} = \{(g, \phi) \in K \times \mathcal{O}(V) \mid \phi(z)^2 = \gamma(Dg(z))^{-1}\}
\]
equipped with the group law given by:
\[
(g_1, \phi_1)(g_2, \phi_2) = (g_1g_2, \phi_3), \quad \text{with} \quad \phi_3(z) = \phi_1(g_2z)\phi_2(z).
\]
Proposition. For each \( g \in K \), the function \( \psi_g : z \mapsto \gamma(Dg(z))^{-1} \), holomorphic on \( V - \{ z \in V \mid Dg(z) = 0 \} \), has an analytic continuation to \( V \). Moreover, there exists \( \phi_g \in O(V) \), such that \( \psi_g = \phi_g^2 \).

Proof. Using the cocycle property \( D(g_1g_2)(z) = Dg_1(g_2(z))Dg_2(z) \) which implies \( \psi_{g_1g_2}(z) = \psi_{g_1}(g_2.z)\psi_{g_2}(z) \), it suffices to prove the proposition for the generators of \( K \).

For \( g = l \in L \), \( Dg(z) = l \) and \( \psi_g(z) = \gamma(l)^{-1} \) for \( z \in V \). For \( g = \tau_v \), \( Dg(z) = id_V \) and \( \psi_g(z) = 1 \) for \( z \in V \). If \( g = \sigma \), as for each invertible \( z, D\sigma(z) = P(z)^{-1} \) and \( \det(P(z)w) = \det(z)^2\det(w) \) (cf. [F-K] Proposition II.3.3), then we get \( \psi_g(z) = Q(z)^2 \) which is a polynomial, it follows that \( \psi_g \) has analytic continuation to \( V \) and \( \phi_g(z)^2 = \psi_g \).

Corollary. \( \tilde{K}^{(2)} \) is a 2-fold covering group of \( K \), which contains the covering \( \tilde{L}^{(2)} \) of \( L \). Moreover, \( \tilde{K}^{(2)} \) is generated by the elements
\[
(l, \alpha) \quad \text{with} \quad (l \in L, \alpha = \gamma(l)^{-2}),
\]
\[
(\tau_v, 1) \quad \text{with} \quad v \in V,
\]
\[
(\sigma, Q).
\]

The group \( \tilde{K}^{(2)} \) will be called the meta-conformal group of \( V \), associated to the semi-invariant \( Q \). The subgroup \( \tilde{P}^{(2)} \) generated by \( \tilde{L}^{(2)} \) and the \( (\tau_v, 1) \) is maximal parabolic of \( \tilde{K}^{(2)} \) with Levi factor \( \tilde{L}^{(2)} \).

We consider the representation \( \tilde{\kappa} \) of \( \tilde{K}^{(2)} \) in \( p \) defined by
\[
(\tilde{\kappa}(\tau_v, 1)p)(z) = p(z - v)
\]
\[
(\tilde{\kappa}(l, \alpha)p)(z) = \alpha^{-1}p(l^{-1}.z)
\]
\[
(\tilde{\kappa}(\sigma, Q)p)(z) = Q(z)p(-\sigma(z)).
\]

\( \tilde{\kappa} \) is the finite dimensional irreducible representation of \( \tilde{K}^{(2)} \) with highest weight \( Q \). It is also the representation \( Ind_{\tilde{P}^{(2)}}^{\tilde{K}^{(2)}} \tilde{\chi} \) obtained by parabolic induction from the chatacter \( \tilde{\chi} \) of \( \tilde{L}^{(2)} \). The derived representation \( d\tilde{\kappa} \) can be obtained on the generators of \( \mathfrak{g} \) as follows:

For \( X \in \mathfrak{f}_{-1} \), let \( v \in V \) be such that \( exp(tX) = (\tau_v, 1) \), then
\[
(d\tilde{\kappa}(X)p)(z) = \frac{d}{dt} \mid_{t=0} p(z - tv) = -Dp(z)(v).
\]

For \( X \in \mathfrak{f}_0 \),
\[
(d\tilde{\kappa}(X)p)(z) = \frac{d}{dt} \mid_{t=0} \gamma(exp(\frac{t}{2}X))p(exp(-tX)z)
\]
\[
= D\gamma(id) \circ D(exp)(0)(\frac{1}{2}X)p(z) - Dp(z)(D(exp)(0)(X)z)
\]
\[
= \frac{1}{2}D\gamma(id)(X)p(z) - Dp(z)(X.z).
\]
For $X \in \mathfrak{k}_1$,  
\[
(d\tilde{\kappa}(X)p)(z) = \frac{d}{dt} \big|_{t=0} (\tilde{\kappa}((\sigma,Q),(\tau_{tu},1),(\sigma,Q))p)(z) 
\]
\[
= \frac{d}{dt} \big|_{t=0} (\tilde{\kappa}((\sigma,Q))\tilde{\kappa}((\tau_{tu},1))\tilde{\kappa}((\sigma,Q))p)(z) 
\]
\[
= \frac{d}{dt} \big|_{t=0} (\tilde{\kappa}((\sigma,Q))\tilde{\kappa}((\tau_{tu},1))[Q(z)p(\sigma(z))]). 
\]
\[
= \frac{d}{dt} \big|_{t=0} [Q(z)Q(\sigma(z) - tu)p(\sigma(z) - tu)]. 
\]
\[
= \frac{d}{dt} \big|_{t=0} Q(e - tu)p(z\sigma(e - tu)) 
\]
\[
= -DQ(e)(zu)p(z) = Dp(z)(zu). 
\]

Notice that the infinitesimal representations $d\kappa$ and $d\tilde{\kappa}$ are equal. In the sequel, We denote this representation of $\mathfrak{k}$ in $\mathfrak{p}$ by $\rho$.

**Graduation of $\mathfrak{k}$ and $\mathfrak{p}$.**

The Lie algebra $\mathfrak{k} = \text{Lie}(K)$ is the Kantor-Koecher-Tits Lie algebra of $V$. We denote by $h_t$ the dilation of $V : h_t.z = e^{-t}z \quad (t \in \mathbb{R})$. Then $h_t \in L, h_t = e^{tH}$ with $H \in \text{Lie}(L)$ and $\chi(h_t) = e^{-2t}$. (In the case of the character $\tilde{\chi}$ of $\tilde{L}^{(2)}$, we consider $\tilde{h}_t = (h_t,e^{2t}) \in \tilde{L}^{(2)}$ in such a way that $\tilde{\chi}(\tilde{h}_t) = e^{-2t}$.) We can prove that $\rho(H) = E - 2$, where $E$ is the Euler operator $(Ep)(z) = \langle z, \nabla p(z) \rangle$. $H$ defines a graduation of $\mathfrak{k}$:

\[
\mathfrak{k} = \mathfrak{k}^{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1
\]

with

\[
\mathfrak{k}_j = \{ X \in \mathfrak{k} \mid \text{ad}(H)X = jX \} \quad j = -1,0,1.
\]

Notice that

\[
\text{Ad}(\sigma) : \mathfrak{k}_j \rightarrow \mathfrak{k}_{-j}, X \rightarrow \sigma X \sigma
\]

$\mathfrak{k}^{-1} = \text{Lie}(N) \simeq V, \mathfrak{k}_0 = \text{Lie}(L), \mathfrak{k}_1 = \text{Lie}(\sigma N \sigma) \simeq V$

and that

$H \in \mathfrak{z}(\mathfrak{k}_0) \quad \text{(centre of } \mathfrak{z}(\mathfrak{k}_0))$.

$H$ defines also a graduation of $\mathfrak{p}$:

$\mathfrak{p} = \mathfrak{p}^{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2$

with

$\mathfrak{p}_j = \{ p \in \mathfrak{p} \mid \rho(H)p = jp \}.$

$\mathfrak{p}_j$ is the set of homogeneous polynomials of degree $j + 2$ in $\mathfrak{p}$.

Notice that

\[
\kappa(\sigma) : \mathfrak{p}_j \rightarrow \mathfrak{p}_{-j}, p \mapsto \kappa(\sigma)p
\]

$\mathfrak{p}_{-2} = \mathbb{C}, \mathfrak{p}_2 = \mathbb{C}, \mathfrak{p}_{-1} \simeq V, \mathfrak{p}_1 \simeq V$. 


Construction process of simple Lie algebras

Let $\mathfrak{g}$ be the vector space defined by $\mathfrak{g} := \mathfrak{k} + \mathfrak{p}$. Let’s denote by $E = Q, F = 1$ ($E, F \in \mathfrak{p}$).

**Theorem 1.** There exists on $\mathfrak{g}$ a unique Lie algebra structure such that:

(S1) $[X, X'] = [X, X']_\mathfrak{k}$ $(X, X' \in \mathfrak{k})$

(S2) $[X, p] = \rho(X)p$ $(X \in \mathfrak{k}, p \in \mathfrak{p})$

(S3) $[E, F] = H$.

**Lemma 1.**

(a) $\forall X \in \mathfrak{k}_{-1}, \rho(X)F = 0$ et $\forall Y \in \mathfrak{k}_1, \rho(Y)E = 0$.

(b) $\forall X \in \mathfrak{k}_{-1}, \rho(X)E \in \mathfrak{p}_1$ et $\forall Y \in \mathfrak{k}_1, \rho(Y)F \in \mathfrak{p}_{-1}$.

(c) $\forall X \in \mathfrak{k}_0, \rho(X)E = \alpha(X)E$ and $\rho(X)F = -\alpha(X)F$ with $\alpha(X) = -\frac{1}{2}D\gamma(id)(X)$.

**Proof.**

(a) Let be $X \in \mathfrak{k}_{-1}$, then $(\rho(X)F)(z) = -DF(z)(v) = 0$.

Let be $Y \in \mathfrak{k}_1$, then

$$\frac{d}{dt}_{t=0} (\kappa(\sigma)\kappa(\tau v)\kappa(\sigma)E)(z)$$

$$= \frac{d}{dt}_{t=0} (\kappa(\sigma)\kappa(\tau v)F)(z) = \frac{d}{dt}_{t=0} (\kappa(\sigma)F)(z) = 0.$$

(b) Let be $X \in \mathfrak{k}_{-1}$. Then for $\lambda \in \mathbb{C}^*$,

$$(\rho(X)E)(\lambda z) = \frac{d}{dt}_{t=0} \lambda^2 E(z + \frac{t}{\lambda}v)$$

$$= \frac{d}{dt}_{t=0} \lambda^2 E(z + \frac{t}{\lambda}v) = \lambda^2 \frac{d}{ds}_{s=0} E(z + s\lambda^2 v)$$

$$= \lambda^2 (\rho(X)E)(z).$$

Let be $Y \in \mathfrak{k}_1$. Then $Y = \sigma X \sigma$ with $X \in \mathfrak{k}_{-1}$. It follows that $(\rho(Y)F) = \kappa(\sigma)\rho(X)E \in \kappa(\sigma)(\mathfrak{p}_1) = \mathfrak{p}_{-1}$.

(c) Let be $X \in \mathfrak{k}_0$ then

$$(\rho(X)E)(z) = \frac{d}{dt}_{t=0} \gamma(exp(\frac{t}{2}X)E(exp(-tX))z)$$

$$= \frac{d}{dt}_{t=0} \gamma(exp(-\frac{t}{2}X))E(z) = -\frac{1}{2}D\gamma(id)(X)E(z)$$

$$(\rho(X)F)(z) = \frac{d}{dt}_{t=0} \gamma(exp(\frac{t}{2}X)F(exp(-tX)z)$$

$$= \frac{d}{dt}_{t=0} \gamma(exp(\frac{t}{2}X))F(z) = \frac{1}{2}D\gamma(id)(X)F(z). \square$$
Lemma 2.
(a) \( \forall p \in p_1, \exists! X \in \mathfrak{t}_{-1}, p = \rho(X)E. \)
(b) \( \forall p \in p_{-1}, \exists! Y \in \mathfrak{t}_1, p = \rho(Y)F. \)

Proof. As \( p \) is a simple \( \mathfrak{t} \)-module with highest weight \( E \), we can write \( p = \rho(\mathcal{U}(\mathfrak{t}))E \), where \( \mathcal{U}(\mathfrak{t}) \) is the enveloping algebra of \( \mathfrak{t} \). This allows, using lemma 1, to prove that \( X,Y,X_1,X_2 \) exist. On another side, the linear maps 
\( \mathfrak{t}_{-1} \rightarrow p_1, X \mapsto \rho(X)E \) and \( \mathfrak{t}_1 \rightarrow p_{-1}, Y \mapsto \rho(Y)F \)
are injective. In fact, let \( X \in \mathfrak{t}_{-1} \) be such that \( d\kappa(X)E = 0 \). Then, for each \( z \in V \),
\( \frac{d}{dt} t = 0 E(z + tv) = 0. \)
As \( E(z) = \prod_{i=1}^{s} \Delta_{i}^{k_i}(z_i) \), then, denoting by \( z = (z_i) \) and \( v = (v_i) \), we get
\( \frac{d}{dt} t = 0 E(z + tv) = \frac{d}{dt} |_{t=0} \prod_{i=1}^{s} \Delta_{i}^{k_i}(z_i + tv_i) = 0. \)
Then if all the \( z_i \) are invertible,
\( \frac{d}{dt} t = 0 \prod_{i=1}^{s} \Delta_{i}^{k_i}(z_i + tv_i) = \sum_{i=1}^{s} k_i \Delta_{i}^{k_i}(z_i) tr(z_i^{-1}u_i) \prod_{j \neq i} \Delta_{j}^{k_j}(z_j) = E(z) \sum_{i=1}^{s} k_i tr(z_i^{-1}v_i), \)
then for each \( z_i \in V_i \) invertible, \( \sum_{i=1}^{s} k_i tr(z_i^{-1}v_i) = 0 \), which implies that for each \( z_i \in V_i, \)
\( \sum_{i=1}^{s} k_i tr(z_i v_i) = 0 \), and finally for each \( 1 \leq i \leq s, \forall z_i \in V_i, tr(z_i v_i) = 0 \), and as the bilinear form \( tr(xy) \) is non degenerated on each \( V_i \), then for each \( i, v_i = 0, i.e. v = 0 \) and \( X = 0. \)
Let \( Y \in \mathfrak{t}_1 \) be such that \( \rho(Y)F = 0 \), i.e. \( \kappa(\sigma): d\kappa(X)E = 0 \) where \( Y = \sigma X \sigma \) with \( X \in \mathfrak{t}_{-1} \). Then \( \rho(X)E = 0 \) and then \( X = 0 \) and \( Y = 0. \)

Lemma 3. The representation \( \rho : \mathfrak{t} \rightarrow \text{End}(p) \) is injective.

Proof. Let \( X \in \mathfrak{t} \) be such that \( \rho(X) = 0. \) We write \( X = X_{-1} + X_0 + X_1 \) with \( X_j \in \mathfrak{t}_j. \)
As \( \rho(\mathfrak{t}_j)(p_i) \subset p_{i+j} \), we obtain that \( \rho(X_{-1}) = \rho(X_0) = \rho(X_1) = 0. \)
\( \rho(X_{-1}) = 0 \) implies that for all \( p \in p \) and \( z \in V \), \( Dp(z)(v) = 0 \) (where \( \exp(tX_{-1}) = \tau_{tv} \)). In particular, if \( p(z) = tr(zu) \) with \( u \in V \), then \( tr(vu) = 0 \) for arbitrary \( u, \) and
, as \( V \) semisimple means that the bilinear form \( tr(uv) \) is non degenerate, then we get \( v = 0 \) and \( X_{-1} = 0 \).
\( \rho(X_0) = 0 \) implies \( \rho(X_0)E = 0 \), then \( D\gamma(id)(X_0) = 0 \). Now, as for each linear form \( p \) on \( V \), \( \rho(X_0)p(z) = \frac{1}{2} D\gamma(id)(X_0)p(z) + \frac{d}{dt} t = 0 p(exp(-tX_0)z) = 0, \)
and as \( D\gamma(id)(X_0) = 0, \) we have \( \frac{d}{dt} t = 0 p(exp(-tX_0)z) = p(-X_0) = 0. \) If \( p(z) = tr(zu) \) with \( u \in V \) then \( tr(-X_0zu) = 0 \) for each \( z, u \in V, \) then \( X_0 = 0. \)
As \( X_1 = \sigma Y_{-1} \sigma \) with \( Y_{-1} \in \mathfrak{t}_{-1} \), then it is clear that \( \rho(X_1) = 0 \) implies \( \rho(Y_{-1}) = 0, \)
implies \( Y_{-1} = 0 \) and \( X_1 = 0. \)
Proof of theorem 1. We define the Lie bracket \([p, p'] \in \mathfrak{t}\) for two elements \(p, p'\) of \(\mathfrak{p}\) in such a way that
\[
[E, F] = H, \ [p_1, p_j] \subset \mathfrak{k}_{i+j}
\]
and
\[
[X, [p, p']] = [(X, p), [p', X]] + [p, [X, p']]. \quad (\forall X \in \mathfrak{t}).
\]
It follows that
\[
\forall X \in \mathfrak{k}_{-1}, [\rho(X)E, F] = [X, [E, F]] = [X, H] = X
\]
and the bracket \([p_1, p_{-2}] \subset \mathfrak{k}_{-1}\) is well defined.

\[
\forall Y \in \mathfrak{k}_1, [\rho(Y)F, E] = [Y, [F, E]] = -[Y, H] = Y
\]
and the bracket \([p_{-1}, p_0] \subset \mathfrak{k}_1\) is well defined.

\[
\forall X \in \mathfrak{k}_{-1}, p \in \mathfrak{p}_0, [\rho(X)E, p] = [(X, E, p)] = [X, [E, p]] = [\rho(X)p, E] \in [p_{-1}, p_2]
\]
and the bracket \([p_{1}, p_0] \subset \mathfrak{k}_1\) is well defined.

\[
\forall Y \in \mathfrak{k}_1, p \in \mathfrak{p}_0, [\rho(Y)F, p] = [(Y, F, p)] = [Y, [F, p]] = [\rho(Y)p, F] \in [p_{1}, p_{-2}]
\]
and the bracket \([p_{-1}, p_0] \subset \mathfrak{k}_{-1}\) is well defined.

For \(X \in \mathfrak{k}_{-1}, Y \in \mathfrak{k}_1,
\[
[\rho(X)E, \rho(Y)F] = [X, [E, \rho(Y)F]] - [E, \rho(X)\rho(Y)F]
\]
\[
= [X, [E, \rho(Y)F]] - [E, \rho([X, Y])F]
\]
\[
= -[X, Y] + \alpha([X, Y])H \in \mathfrak{k}_0
\]
and the bracket \([p_1, p_{-1}] \subset \mathfrak{k}_0\) is well defined.

\[
\forall X, X' \in \mathfrak{k}_{-1}, [\rho(X)E, \rho(X')E] = [X, [E, \rho(X')E]] = [E, \rho(X)\rho(X')E] = 0,
\]
then \([p_1, p_1] = 0\).

\[
\forall Y, Y' \in \mathfrak{k}_1, [\rho(Y)F, \rho(Y')F] = [Y, [F, \rho(Y')F]] = [F, \rho(Y)\rho(Y')F] = 0,
\]
then \([p_{-1}, p_{-1}] = 0\).

The bracket \([p_0, p_0]\) is then well determined. In fact, for \(p, p' \in \mathfrak{p}_0\), as the restriction of \(\rho\) to \(\mathfrak{k}_0\) is injective, we define \(X_0 = [p_0, p'_0]\) as the unique element of \(\mathfrak{k}_0\) such that \(\rho(X_0)\) satisfies :
\[
\rho(X_0)E = 0, \rho(X_0)F = 0 \quad \text{and} \quad \rho(X_0)\phi = -[[\phi, p_0], p'_0] - [p_0, [\phi, p'_0]] \forall \phi \in p_{-1} \cup p_1
\]
and its restriction to \(\mathfrak{p}_0\) is then determined because \(\mathfrak{p}_0\) is generated by the brackets \([X, p]\) with \(X \in \mathfrak{k}_{-1}, p \in \mathfrak{p}_1\), and in this case we have
\[
\rho(X_0)([X, p]) = -[[X, p], X_0] = -[[X, p], [p_0, p'_0]] = -[[X, p], p_0, p'_0] - [p_0, [X, p], p'_0]
\]
\[
\in [[[p_0, X], p], p'_0] + [[X, [p_0, X], p'_0] + [p_0, [p'_0, X], p]] + [p_0, [X, [p_0, X], p'_0]]
\]
\[
\subset [[[p_0, \mathfrak{k}_{-1}], p_1], p_0] + [[[\mathfrak{k}_{-1}, [p_0, p_1]], p_0] + [p_0, [[p_0, \mathfrak{k}_{-1}], p_1]] + [p_0, [\mathfrak{k}_{-1}, [p_0, p_1]]])
\]
\[
\subset [p_{-1}, p_1, p_0] + [[\mathfrak{k}_{-1}, \mathfrak{k}_1], p_0] + [p_0, [p_{-1}, p_1]] + [p_0, [\mathfrak{k}_{-1}, \mathfrak{k}_1]]
\]
\[

\subset [\mathfrak{k}_0, p_0] + [\mathfrak{k}_{-1}, p_0] + [p_0, \mathfrak{k}_0] + [p_0, \mathfrak{k}_0] \subset [\mathfrak{k}_0, p_0] \subset \mathfrak{p}_0.
\]
Let the vector space $g = \mathfrak{k} + \mathfrak{p}$ be equipped with the bracket defined by

$$[X + p, X' + p'] = [X, X']_\mathfrak{k} + [p, p'] + \rho(X)p' - \rho(X')p \quad (X, X' \in \mathfrak{k}, p, p' \in \mathfrak{p}).$$

It remains to prove that the Jacobi identity holds. In fact, as $\mathfrak{k}$ is a Lie algebra and $\rho$ is a representation of $\mathfrak{k}$ in $\mathfrak{p}$, then it is clear that for $X, Y, Z \in \mathfrak{k},$

$$[X, [Y, Z]] = [X, [Y, Z]_\mathfrak{k}] + [X, [Y, Z]_\mathfrak{k}] + [X, [Y, Z]_\mathfrak{k}] = [[X, Y], Z] + [X, [Y, Z]]$$

and for $X, Y \in \mathfrak{k}, p \in \mathfrak{p},$

$$[X, [Y, p]] = d\kappa(X)d\kappa(Y)p + d\kappa(Y)d\kappa(X)p = [[X, Y], p] + [X, [Y, p]].$$

By another side, the Lie bracket has been defined such that for $X \in \mathfrak{k}, p, p' \in \mathfrak{p},$

$$[X, [p, p']] = [[X, p], p'] + [p, [X, p']].$$

It remains just to establish the identity

$$[p'', [p, p']] = [[p'', p], p'] + [p, [p'', p']] \quad \forall p, p', p'' \in \mathfrak{p} \quad (*)$$

We prove this identity step by step. Notice first that we can suppose $p'' = E.$

In the cases $(p, p' \in \mathfrak{p}_2), (p \in \mathfrak{p}_2, p' \in \mathfrak{p}_1), (p \in \mathfrak{p}_2, p' \in \mathfrak{p}_0), (p \in \mathfrak{p}_2, p' \in \mathfrak{p}_1), (p \in \mathfrak{p}_2, p' \in \mathfrak{p}_0), (p, p' \in \mathfrak{p}_0)$, the identity (*) is trivial.

If $p \in \mathfrak{p}_1, p' \in \mathfrak{p}_2$ then $p = \rho(X)E$ et $p' = \rho(X')F$ with $X \in \mathfrak{p}_-1$ and $X' \in \mathfrak{p}_1$, then

$$[E, [p, p']] = [E, [X', X]] - \alpha([X', X])[E, H]$$

$$= \rho([X', X])E - 2\alpha([X', X])E$$

$$= -\alpha([X', X])E$$

and as $[[E, p], p'] = 0$ and $[p, [E, p']] = [p, -X'] = \rho(X')d\kappa(X)E = \rho([X', X])E = -\alpha([X', X])E$, the identity (*) is satisfied.

If $p \in \mathfrak{p}_1, p' \in \mathfrak{p}_-2$ then $p = \rho(X)E$ with $X \in \mathfrak{p}_-1$ and $p' = F,$ then $[E, [p, p']] = [E, X] = -\rho(X)E, [[E, p], p'] = 0$, $[p, [E, p']] = [p, H] = -[H, p] = -\rho(X)E,$ then (*) is satisfied.

If $p \in \mathfrak{p}_0, p' \in \mathfrak{p}_-1$ then $p' = \rho(X')F$ with $X' \in \mathfrak{p}_1$, then

$$[E, [p, p']] = [E, [p, \rho(X')F]] = [E, [X', [p, F]] - [[X', p], F]]$$

$$= [E, [-\rho(X')p, F]] = -[\rho(X')p, [E, F]] = \rho(X)p$$

and $[[E, p], p'] = 0$ and $[p, [E, p']] = [p, [E, \rho(X')F]] = [X', p] = \rho(X)p,$ then (*) is satisfied.

If $p \in \mathfrak{p}_0, p' \in \mathfrak{p}_-2$ then $p' = F$ and $[E, [p, p']] = 0, [[E, p], p'] = 0$ and $[p, [E, p']] = -[p, H] = 0,$ then (*) is satisfied.

If $p, p' \in \mathfrak{p}_-1$ then $p = \rho(X)F, p' = \rho(X')F$ with $X, X' \in \mathfrak{p}_1$ and $[E, [p, p']] = 0,$ then

$$[[E, p], p'] = [[E, \rho(X)F], \rho(X')F] = \rho(X)\rho(X')F$$

and
\[ [p, [E, p']] = [\rho(X)F, [E, \rho(X')F]] = -\rho(X')\rho(X)F \]
and, as \([X, X'] = 0\), then (*) is satisfied.

If \(p \in p_{-1}, p' \in p_{-2}\) then \(p = \rho(Y)F\) with \(Y \in \mathfrak{t}_1\) and \(p' = F\), then \([E, [p, p']] = 0\),
\[
[[E, p], p'] = [[E, \rho(Y)F], F] = -[Y, F] = -\rho(Y)F
\]
and
\[
[p, [E, p']] = [\rho(Y)F, [E, F]] = -[\rho(Y)F, H] = -\rho(Y)F
\]
and (*) is satisfied.

If \(p, p' \in p_{-2}\) then \(p = p' = F\), then
\[
\]
and (*) is satisfied. \(\square\)

**Proposition.** \((E, F, H)\) is an \(sl_2\)-triple in \(\mathfrak{g}\):
\[
\]

\(ad(H)\) has eigenvalues \(-2, -1, 0, 1, 2\) with respective eigenspaces \(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2\) where
\[
\mathfrak{g}_{-2} = p_{-2}, \mathfrak{g}_{-1} = \mathfrak{t}_{-1} + p_{-1}, \mathfrak{g}_0 = \mathfrak{t}_0 + p_0, \mathfrak{g}_1 = \mathfrak{t}_1 + p_1, \mathfrak{g}_2 = p_2.
\]
The Lie algebra \(\mathfrak{g}\) is 5-graded:
\[
\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}
\]
such that
\[
\mathfrak{g}_0 = [\mathfrak{g}_{-2}, \mathfrak{g}_2] + [\mathfrak{g}_{-1}, \mathfrak{g}_1].
\]
Moreover, the map \(\tau: \mathfrak{g} \rightarrow \mathfrak{g}\) defined by
\[
\tau(X + p) = \sigma X \sigma + \kappa(\sigma)p \quad (X \in \mathfrak{t}, p \in \mathfrak{p})
\]
is an involution of the Lie algebra \(\mathfrak{g}\) such that \(\tau(\mathfrak{g}_i) = \mathfrak{g}_{-i}\).

\(\mathfrak{g}\) will be called the Lie algebra associated to the pair \((V, Q)\).

**Theorem 2.** \(\mathfrak{g}\) is a simple complex Lie algebra.

**Proof.** Let \(\mathcal{I} \neq \{0\}\) be an ideal of \(\mathfrak{g}\). For \(T \in \mathfrak{g}\), we write \(T = T_{\mathfrak{t}} + T_{\mathfrak{p}}\) with \(T_{\mathfrak{t}} \in \mathfrak{t}, T_{\mathfrak{p}} \in \mathfrak{p}\). We consider
\[
\mathcal{I}_p = \{p \in \mathfrak{p} \mid \exists T \in \mathcal{I}, T_p = p\}.
\]
\(\mathcal{I}_p\) is a a non trivial \(\rho(\mathcal{U}(\mathfrak{t}))\)-submodule of \(\mathfrak{p}\), then equal to \(\mathfrak{p}\). In fact, if \(\mathcal{I}_p = \{0\}\) then \(\mathcal{I} \subseteq \mathfrak{t}\) and then, for each \(X \in \mathcal{I}\) and \(p \in \mathfrak{p}\), \([X, p] = 0\), which means \(\rho(X) = 0\) and then, as \(\rho\) is injective (lemma 3), \(X = 0\). We deduce that \(\mathcal{I}_p \neq \{0\}\). Now, let be \(X \in \mathfrak{t}\) and \(p \in \mathcal{I}_p\), then there exists \(T \in \mathcal{I}\) tel que \(T_p = p\). Let be \(T = T_{\mathfrak{t}} + p \in \mathfrak{t} + \mathfrak{p}\), then
\[
[X, T] = [X, T_{\mathfrak{t}}] + [X, p] = [X, T_{\mathfrak{t}}] + \rho(X)p
\]
and as \([X, T] \in \mathcal{I}\), then \([X, p] = \rho(X)p \in \mathcal{I}_p\). Finally, as \(\mathfrak{p}\) is a simple \(\rho(\mathcal{U}(\mathfrak{t}))\)-module then \(\mathcal{I}_p = \mathfrak{p}\).
It follows that there exist \( T \in I \) and \( T' \in I \) such that \( T_p = E \) and \( T'_p = F \). We denote by \( T = T_1 + E \) and \( T' = T'_1 + F \) and with respect to the decomposition \( \mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1 \) we write

\[
T = T_{-1} + T_0 + T_1 \quad \text{and} \quad T'_1 = T'_{-1} + T'_0 + T'_1.
\]

Then

\[
[H, T] = -T_{-1} + T_1 + 2E \in I \quad \text{and} \quad [H, T'] = -T'_{-1} + T'_1 - 2F \in I.
\]

It follows that

\[
S = T + [H, T] = 2T_1 + T_0 + 3E \in I \quad \text{and} \quad S' = T' + [H, T'] = 2T'_1 + T'_0 - F \in I
\]

\[
[H, S] = 2T_1 + 6E \in I \quad \text{and} \quad [H, S'] = 2T'_1 + 2F \in I
\]

\[
[H, S] - S = -T_0 + 3E \in I \quad \text{and} \quad [H, S'] - S' = -T_0 + 3F \in I
\]

and finally

\[
[H, [H, S] - S] = 6E \in I \quad \text{and} \quad [H, [H, S'] - S'] = -6F \in I
\]

then

\[
E \in I, F \in I, H \in I.
\]

We deduce that for \( i \neq 0, g_i \subset I \). Moreover, as \( \mathfrak{p} = \rho(U(\mathfrak{k}))E \), then \( \mathfrak{p} \subset I \) and in particular \( \mathfrak{p}_0 \subset I \). At last, \( \mathfrak{k}_0 = [\mathfrak{k}_{-1}, \mathfrak{k}_1] \subset I \), and \( I = g \). \( \square \)

**Theorem 3.**

\[
dim g = 4\dim(V) + 2 + \dim k_0 + \dim p_0.
\]

Moreover, if \( h_\mathfrak{k} \) is a Cartan subalgebra of \( \mathfrak{k}_0 \) which contain \( H \), and if

\[
a_\mathfrak{p} = \{ p \in \mathfrak{p} \mid [X, p] = 0, \forall X \in h_\mathfrak{k} \}
\]

then \( h := h_\mathfrak{k} + a_\mathfrak{p} \) is a Cartan subalgebra of \( g \)

\[
rank(g) = rank(\mathfrak{k}_0) + \dim(a_\mathfrak{p}).
\]

**Proof.** \( \mathfrak{t}_1, \mathfrak{k}_{-1} \) are isomorphic to \( V \). Moreover, \( p_1 \) is generated by the first order partial derivatives of \( Q \), \( p_0 \) is generated by the second order partial derivatives of \( Q \) and \( p_{-1} \), which is genrated by the order 3 partial derivatives of \( Q \), is equal to the space of linear forms on \( V \). Then \( p_1 \) and \( p_{-1} \) are isomorphic to \( V \). As \( p_2 = \mathbb{C}Q \) et \( p_{-2} = \mathbb{C}1 \), the dimension of \( g \) is then given by

\[
dim g = 4\dim(V) + 2 + \dim \mathfrak{k}_0 + \dim p_0.
\]

Moreover, if \( h_\mathfrak{k} \) is a Cartan subalgebra of \( \mathfrak{k} \) containing \( H \), it is a Cartan subalgebra of \( \mathfrak{k}_0 \) and the subspace of \( \mathfrak{p} \), defined by

\[
a_\mathfrak{p} = \{ p \in \mathfrak{p} \mid [X, p] = 0, \forall X \in h_\mathfrak{k} \}
\]

is contained in \( p_0 \), and is Abelian. It follows that the subalgebra of \( g \) given by \( h := h_\mathfrak{k} + a_\mathfrak{p} \) is a Cartan subalgebra of \( g \) and \( rank(g) = rank(\mathfrak{k}_0) + \dim(a_\mathfrak{p}) \). \( \square \)
Cartan-Adapted Real Forms

Let $\hat{\mathfrak{e}}_R$ be the conformal Lie algebra of Euclidean real form $J$ of the Jordan algebra $V$. Then $\hat{\mathfrak{e}}_R$ is a real form of $\mathfrak{e}$. Notice that $H \in \hat{\mathfrak{e}}_R$.

The Cartan involution of $\hat{\mathfrak{e}}_R$ is given by $\theta : \hat{\mathfrak{e}}_R \to \hat{\mathfrak{e}}_R, X \mapsto -\sigma X \sigma$, and the Cartan decomposition of $\hat{\mathfrak{e}}_R$ writes $\hat{\mathfrak{e}}_R = \hat{\mathfrak{e}}_R^+ + \hat{\mathfrak{e}}_R^-$, with $\hat{\mathfrak{e}}_R^+ = \{ X \in \hat{\mathfrak{e}}_R \mid \sigma X \sigma = -X \}$ and $\hat{\mathfrak{e}}_R^- = \{ X \in \hat{\mathfrak{e}}_R \mid \sigma X \sigma = X \}$.

The compact real form $\mathfrak{k}_R$ of $\mathfrak{k}$ is then given by $\mathfrak{k}_R = \hat{\mathfrak{e}}_R^+ + \mathfrak{i}\hat{\mathfrak{e}}_R^-$. Notice that $\sigma H \sigma(z) = d_{t=0} \sigma(\exp(-t)\sigma(z)) = d_{t=0} \exp(t)z = -Hz$, then $\sigma H \sigma = -H$ and $H \in \mathfrak{e}_R$.

**Proposition.** Let $c_\mathfrak{k}$ be the conjugation of $\mathfrak{k}$ with respect to $\mathfrak{k}_R$. It is given by $c_\mathfrak{k}(X) = -\sigma \bar{X} \sigma$, where $X \mapsto \bar{X}$ is the conjugation of $\mathfrak{k}$ with respect to $\hat{\mathfrak{e}}_R$. Moreover $c_\mathfrak{k}(\mathfrak{k}) = \mathfrak{k}$.

**Proof.** Let $X \in \mathfrak{e}_R$, then $X = X_1 + iX_2$ with $X_1 \in \hat{\mathfrak{e}}_R^+$ and $X_2 \in \hat{\mathfrak{e}}_R^-$. Then $-\sigma \bar{X} \sigma = -\sigma \bar{X}_1 \sigma + i\sigma \bar{X}_2 \sigma = X_1 + iX_2 = X$. For $X \in \mathfrak{k}_j$, $[H, c_\mathfrak{k}(X)] = c_\mathfrak{k}([H, X]) = j c_\mathfrak{k}(X)$. □

We denote by $\mathfrak{u}$ the compact real form of $\mathfrak{g}$. It follows that

$\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$ and $\mathfrak{e}_R = \mathfrak{e} \cap \mathfrak{u}$.

We put $\mathfrak{p}_R = \mathfrak{p} \cap i\mathfrak{u}$ then the real Lie subalgebra of $\mathfrak{g}$ defined by $\mathfrak{g}_R = \mathfrak{e}_R + \mathfrak{p}_R$ is a real form of $\mathfrak{g}$. Its Cartan decomposition is just $\mathfrak{e}_R + \mathfrak{p}_R$. The Cartan signature of $\mathfrak{g}_R$ is then given by

$s_c = \text{dim}(\mathfrak{p}) - \text{dim}(\mathfrak{e}) = \text{dim}(\mathfrak{p}_0) - \text{dim}(\mathfrak{e}_0) + 2$.

The real form $\mathfrak{g}_R = \mathfrak{e}_R + \mathfrak{p}_R$ will be called the Cartan-Adapted real form of the Lie algebra $\mathfrak{g} = \mathfrak{e} + \mathfrak{p}$ associated to the pair $(V, Q)$. (cf. table 1 for the classification)

**Corollary.** The symmetric pair $(\mathfrak{g}, \mathfrak{e})$ is non Hermitian.

**Proof.** It is a consequence of the fact that the decomposition $\mathfrak{g} = \mathfrak{e} + \mathfrak{p}$ is the complexification of the Cartan decomposition of $\mathfrak{g}$ and the fact that $\mathfrak{p}$ is a simple $\mathfrak{k}$-module. □

**Remark.** It is possible to see the statement of theorem 1 as a special case of constructions of Lie algebras by Allison and Faulkner, using the Cayley-Dickson process to associate a 5-graded simple Lie algebra to some structurable algebra $W$, that is $W = \mathfrak{e}_0 + \mathfrak{p}_0$ in our terminology (cf. [A-F]).
The classification of the simple complex Lie algebras associated to the pair \((V, Q)\) where \(V\) is a semisimple Jordan algebra of rank \(\leq 4\) and \(Q\) is a degree 4 semi-invariant can be obtained by considering all the possible cases for \(V = \sum_{i=1}^{s} V_i\) and \(Q = \prod_{i=1}^{s} \Delta_i^{k_i}\) where \(k_i \in \mathbb{N}^*\) and \(\Delta_i\) is the polynomial determinant of the simple Jordan algebra \(V_i\). For each case, we determine the dimension and the rank of \(g\), and the Cartan signature of the real form \(g_{\mathbb{R}}\).

We obtain by this construction process all the simple real Lie algebras which intersect the minimal nilpotent complex adjoint orbit (cf.\[B.\]).

\[
\begin{array}{cccccccc}
V & Q & \mathfrak{t} & g & \mathfrak{k}_{\mathbb{R}} & g_{\mathbb{R}} \\
\mathbb{C} & z^4 & sl(2, \mathbb{C}) & sl(3, \mathbb{C}) & su(2) & sl(3, \mathbb{R}) \\
\mathbb{C}^{p-2} & \Delta(z)^2 & so(p, \mathbb{C}) & sl(p, \mathbb{C}) & so(p, \mathbb{R}) & sl(p, \mathbb{R}) \\
\mathbb{C}^{\oplus 2} & z^2 w^2 & sl(2, \mathbb{C})^{\oplus 2} & so(6, \mathbb{C}) & su(2)^{\oplus 2} & so(3, 3) \\
\mathbb{C}^{\oplus 3} & z^2 w w & sl(2, \mathbb{C})^{\oplus 3} & so(7, \mathbb{C}) & so(3)^{\oplus 3} & so(3, 4) \\
\mathbb{C}^{\oplus 4} & z w w w & sl(2, \mathbb{C})^{\oplus 4} & so(8, \mathbb{C}) & so(3)^{\oplus 4} & so(4, 4) \\
\mathbb{C}^{p-2} + \mathbb{C} & \Delta(z) w & so(p, \mathbb{C}) + sl(2, \mathbb{C}) & so(p + 3, \mathbb{C}) & so(p) + so(3) & so(p, 3) \\
\mathbb{C}^{p-2} + \mathbb{C}^{\oplus 2} & \Delta(z) w & so(p, \mathbb{C}) + sl(2, \mathbb{C})^{\oplus 2} & so(p + 4, \mathbb{C}) & so(p) + so(3)^{\oplus 2} & so(p, 4) \\
\mathbb{C}^{q-2} + \mathbb{C}^{\oplus 2} & \Delta(z) \Delta(w) & so(p, \mathbb{C}) + so(q, \mathbb{C}) & so(p + q, \mathbb{C}) & so(p) + so(q) & so(p, q) \\
Sym(4, \mathbb{C}) & det(z) & sp(8, \mathbb{C}) & E_6 & sp(8) & E_6(8) \\
M(4, \mathbb{C}) & det(z) & sl(8, \mathbb{C}) & E_7 & su(8) & E_7(7) \\
Asym(8, \mathbb{C}) & det(z) & so(16, \mathbb{C}) & E_8 & so(16) & E_8(8) \\
Sym(3, \mathbb{C}) + \mathbb{C} & det(z) w & sp(6, \mathbb{C}) + sl(2, \mathbb{C}) & f_4 & sp(6) + su(2) & F_4(4) \\
M(3, \mathbb{C}) + \mathbb{C} & det(z) w & sl(6, \mathbb{C}) + sl(2, \mathbb{C}) & E_6 & su(6) + su(2) & E_6(2) \\
Asym(6, \mathbb{C}) + \mathbb{C} & det(z) w & so(12, \mathbb{C}) + sl(2, \mathbb{C}) & E_7 & so(12) + su(2) & E_7(-5) \\
Herm(3, \mathbb{C})^{\oplus 2} + \mathbb{C} & det(z) w & E_7 + sl(2, \mathbb{C}) & E_8 & E_7^+ + su(2) & E_8(-24) \\
\mathbb{C}^{\oplus 2} & z^3 w & sl(2, \mathbb{C})^{\oplus 2} & G_2 & so(3)^{\oplus 2} & G_2(2) \\
\end{array}
\]

where \(p, q \geq 5\).
About the minimal nilpotent orbit

Let $G$ be a connected complex Lie group with Lie algebra $\mathfrak{g}$. Then, the adjoint group $G_{ad}$ of $\mathfrak{g}$ is given by $G_{ad} = G/Z(G)$. Denote by $G_{adR}$ the connected subgroup of $G_{ad}$ with Lie algebra $\mathfrak{g}_R$, $K_{ad}$ the connected subgroup of $G_{ad}$ with Lie algebra $\mathfrak{k}$, and by $K_{adR}$ the maximal compact subgroup of both $\mathfrak{g}_R$ and $K_{ad}$ with Lie algebra $\mathfrak{k}_R$. It is well known, by the Kostant-Sekiguchi correspondence (cf. [S.]), that there is a bijection between the set of nilpotent $K_{adR}$-orbits in $\mathfrak{p}$ and the set of nilpotent $G_{adR}$-orbits in $\mathfrak{g}_R$. Moreover, if $O_{min}$ is the minimal nilpotent adjoint orbit in $\mathfrak{g}$, then, as $\mathfrak{g}$ is non Hermitian, $O_{min} \cap \mathfrak{p}$ is equal to the $K_{ad}$-orbit of the highest weight vector in $\mathfrak{p}$ and $O_{min} \cap \mathfrak{g}_R$ is the corresponding orbit in $\mathfrak{g}_R$ by the Sekiguchi bijection.

As $E$ is the highest weight vector of the adjoint action of $\mathfrak{k}$ in $\mathfrak{p}$, then

$$O_{p} := O_{min} \cap \mathfrak{p} = K_{ad}E.$$

Now, following the terminology of Sekiguchi (cf. [S.]), the $sl_2$-triple $(H, E, F)$ is normal for the symmetric pair $(\mathfrak{g}, \mathfrak{k})$, that means $H \in \mathfrak{k}, E, F \in \mathfrak{p}$. But the conditions of strict normality, i.e. $H \in i\mathfrak{k}_R, E+F \in \mathfrak{p}_R, i(E-F) \in \mathfrak{p}_R$ (which are equivalent to $\tilde{\theta}(H) = -H$ and $\tilde{\theta}(E) = -F$ where $\tilde{\theta}$ is the Cartan involution of $\mathfrak{g}$) are not satisfied (because $H \in \mathfrak{k}_R \subset \mathfrak{u}$). However, by a lemma of Sekiguchi (cf. [S.]), there exists $k_0 \in K_{ad}$ such that the the normal $sl_2$-triple $(k_0H, k_0E, k_0F)$ is also strictly normal. Then, the real nilpotent orbit in $\mathfrak{g}_R$ associated to the orbit $K_{adR}.E$ in $\mathfrak{p}$ by the Sekiguchi bijection is given by

$$O_{R} := O_{min} \cap \mathfrak{g}_R = G_{adR}.(k_0[\frac{1}{2}(E + F + iH)]).$$

A theorem of M. Vergne (cf. [V.]) gives a canonical $K_{adR}$-equivariant diffeomorphism (the Vergne-Kronheimer diffeomorphism) from $O_R$ onto $O_p$. (a kind of generalization of the realisation of $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ as $\mathbb{C}^n$.)

In our context, where $K$ is the conformal group and $\tilde{K}^{(2)}$ the meta-conformal group, it is natural to consider the $K$-orbit $\Xi = \kappa(K).E$ or the $\tilde{K}^{(2)}$-orbit $\tilde{\Xi}^{(2)} = \tilde{\kappa}(\tilde{K}^{(2)}).E$. As the groups $K$ (resp. $\tilde{K}^{(2)}$) and $K_{ad}$ (with same Lie algebra $\mathfrak{k}$) are very closer (they may be equal or one may be a 2-fold covering of the other), and as the stabilizer of $E$ in $K$ (resp. $\tilde{K}^{(2)}$) is equal to $L' \rtimes \sigma N \sigma$, where $L'$ is the kernel of the character $\chi$ (resp. $\tilde{\chi}$), then these orbits are finite order coverings of the minimal orbit $O_{min} \cap \mathfrak{p}$. 
Application to Representation Theory

**Theorem 4.** Let $\tilde{G}_R$ be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}_R$. Assume that $(\pi, \mathcal{H})$ is a unitary representation of $K_R$ such that its differential $d\pi$ extends to an irreducible representation $\tilde{\rho}$ of $\mathfrak{g}$ in the space $\mathcal{H}^{K_R}$ of $K_R$-finite vectors. Assume that the operators $\tilde{\rho}(p)$ are antisymmetric for $p \in \mathfrak{p}$. Then there exists a unique unitary irreducible representation $\tilde{\pi}$ of $\tilde{G}_R$ such that $d\tilde{\pi} = \tilde{\rho}$.

**Proof.** In fact, by the Nelson criterion, it is enough to prove that $\tilde{\rho}(L)$ is essentially self-adjoint for the Laplacian $L$ of $\mathfrak{g}_R$. Let’s consider a basis $\{X_1, \ldots, X_k\}$ of $\mathfrak{k}_R$ and a basis $\{p_1, \ldots, p_l\}$ of $\mathfrak{p}_R$. As $\mathfrak{g}_R = \mathfrak{k}_R + \mathfrak{p}_R$ is the Cartan decomposition of $\mathfrak{g}_R$, then the Laplacian and the Casimir operators of $\mathfrak{g}_R$ are respectively given by

$$L = X_1^2 + \ldots + X_k^2 + p_1^2 + \ldots + p_l^2$$

and

$$C = X_1^2 + \ldots + X_k^2 - p_1^2 - \ldots - p_l^2.$$

It follows that $L = 2(X_1^2 + \ldots + X_k^2) - C$ and $\tilde{\rho}(L) = 2\tilde{\rho}(X_1^2 + \ldots + X_k^2) - \tilde{\rho}(C)$. As $\tilde{\rho}(X_1^2 + \ldots + X_k^2) = d\pi(X_1^2 + \ldots + X_k^2)$ and as $\pi$ is a unitary representation of $K_R$, then the image $\tilde{\rho}(X_1^2 + \ldots + X_k^2)$ of the Laplacian of $\mathfrak{k}_R$ is essentially self-adjoint. Moreover, $\tilde{\rho}(C)$ is scalar, because the dimension of $\mathcal{H}^{K_R}$ being countable, then the commutant of $\tilde{\rho}$, which is a division algebra over $\mathbb{C}$ has a countable dimension too, and then is equal to $\mathbb{C}$.

It follows that $\tilde{\rho}(L)$ is essentially self-adjoint and that $\tilde{\rho}$ integrates to a unitary representation of $\tilde{G}_R$. \qed
The case of degree 2 semi-invariant

Suppose that $E = Q$ has degree 2. Then $\chi(h_t) = e^{-t}$ and $H$ defines a graduation on $p$ given by

$$p = p_{-1} + p_0 + p_1$$

with $p_{-1} = \mathbb{C}F$, $p_1 = \mathbb{C}E$ and $p_0 \simeq V$ is generated by the first order partial derivatives of $Q$. We consider the vector space $g = \mathfrak{k} + p$ and, as in the preceding sections, we can prove the following:

**Theorem 5.** There exists on $g$ a unique Lie algebra structure such that

$(S_1)$ $[X, X'] = [X, X']_{\mathfrak{k}} \quad \forall X, X' \in \mathfrak{k}$

$(S_2)$ $[X, p] = \rho(X)p \quad \forall X \in \mathfrak{k}, p \in p$

$(S_3)$ $[E, F] = H.$

$g$, endowed with this structure, is a simple 3-graded Lie algebra. Moreover

$$\dim g = 3\dim(V) + 2 + \dim(\mathfrak{k}_0)$$

and, if $\mathfrak{h}_\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}_0$ which contains $H$, and if

$$\mathfrak{a}_p = \{p \in \mathfrak{p} \mid [X, p] = 0 \quad \forall X \in \mathfrak{h}_\mathfrak{k}\}$$

then $\mathfrak{h} := \mathfrak{h}_\mathfrak{k} + \mathfrak{a}_p$ is a Cartan subalgebra of $g$ and

$$\text{rank}(g) = \text{rank}(\mathfrak{k}_0) + \dim(\mathfrak{a}_p).$$

Moreover, if $\mathfrak{k}_R$ is the compact real form of $\mathfrak{k}$ and $\mathfrak{u}$ is the compact real form of $g$, then $g = \mathfrak{u} + i\mathfrak{u}$ is the Cartan decomposition of $\tilde{g}$ and $\mathfrak{l}_R = \mathfrak{u} \cap \mathfrak{t}$. Moreover, if we denote by $\mathfrak{p}_R = \mathfrak{p} \cap i\mathfrak{u}$, then the real Lie subalgebra of $g$ defined by $g_R = \mathfrak{t}_R + \mathfrak{p}_R$ is a real form of $g$. Its Cartan decomposition is just $\mathfrak{t}_R + \mathfrak{p}_R$. The Cartan signature of $g_R$ is then given by

$$s_c = \dim(\mathfrak{p}) - \dim(\mathfrak{t}) = 2 - \dim(\mathfrak{k}_0) - \dim(V).$$

The real form $g_R = \mathfrak{t}_R + \mathfrak{p}_R$ will be called the Cartan-Adapted real form of the Lie algebra $g = \mathfrak{k} + \mathfrak{p}$ associated to the pair $(V, Q)$. The symmetric pair $(g, \mathfrak{k})$ is non-Hermitian (cf. table 2 for the classification).

**Table 2**

<table>
<thead>
<tr>
<th>$V$</th>
<th>$Q$</th>
<th>$\mathfrak{t}$</th>
<th>$\mathfrak{g}$</th>
<th>$\mathfrak{t}_R$</th>
<th>$\mathfrak{g}_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{C}$</td>
<td>$z^2$</td>
<td>$sl(2, \mathbb{C})$</td>
<td>$so(4, \mathbb{C})$</td>
<td>$su(2)$</td>
<td>$so(3, 1)$</td>
</tr>
<tr>
<td>$\mathbb{C}^{p-2}$</td>
<td>$\Delta(z)$</td>
<td>$so(p, \mathbb{C})$</td>
<td>$so(p + 1, \mathbb{C})$</td>
<td>$so(p)$</td>
<td>$so(p, 1)$</td>
</tr>
<tr>
<td>$\mathbb{C} + \mathbb{C}$</td>
<td>$zw$</td>
<td>$sl(2, \mathbb{C}) + sl(2, \mathbb{C})$</td>
<td>$so(5, \mathbb{C})$</td>
<td>$su(2) + su(2)$</td>
<td>$so(4, 1)$</td>
</tr>
</tbody>
</table>

where $p \geq 5$. 
Analogy with the Hermitian case

Now, let \( V \) be a semi-simple Jordan algebra with rank \( r \), \( Q \) semi-invariant of degree \( 2r \) and we denote \( L \) the structure group of \( V \) given by
\[
L := \{ g \in GL(V) \mid \exists \gamma(g) \in \mathbb{C}, Q(g z) = \gamma(g) Q(z) \}.
\]
Let \( p \) be the complex vector space generated by the polynomials \( Q(z - a) \) with \( a \in V \).

As in the part I, if there exists a character \( \chi \) of \( L \) such that
\[
Q(l, z) = \chi(l)^2 Q(z)
\]
then the conformal group \( K \) of \( V \) acts on \( p \) by the representation \( \kappa \) and if the character \( \chi \) does not exist, then we consider the action in \( p \) given by the representation \( \tilde{\kappa} \) of the meta-conformal group associated to the semi-invariant \( Q \). We denote by \( \rho = d \kappa = d \tilde{\kappa} \).

In particular, \( L \) acts in \( p \) by the restriction of \( \kappa \) and for \( X \in \mathfrak{l} = \text{Lie}(L) \),
\[
(\rho(X)p)(z) = \frac{d}{dt} \big|_{t=0} \gamma(\exp(\frac{1}{2} t X)) p(\exp(-t X) z).
\]
In this case, if \( h_t \in K_o \) is the dilation of \( V : h_t, z = e^{-t} z \) and \( H \in \mathfrak{t}_o \) the corresponding infinitesimal element, then \( \chi(h_t) = e^{-rt} \) and \( H \) defines a graduation of \( p \) given by
\[
p = p_{-r} + p_{-r+1} + \ldots + p_{0} + \ldots + p_{r-1} + p_r
\]
with
\[
p_j = \{ p \in p \mid \rho(H)p = j p \}
\]
\( p_j \) is the set of homogeneous polynomials of degree \( j + r \) in \( p \) and in particular,
\[
p_{-r} = \mathbb{C}, p_r = \mathbb{C}.Q, p_{-r+1} \simeq p_{r-1} \simeq V.
\]

We denote by \( \mathcal{V}^+ = p_{-r+1}, \mathcal{V}^- = p_{r-1} \). They are two simple \( \mathfrak{l} \)-modules. Denote by \( \mathcal{V} = \mathcal{V}^+ + \mathcal{V}^- \) and then we consider the complex vector space defined by \( \tilde{\mathfrak{g}} = \mathfrak{l} + \mathcal{V} \).

**Theorem 6.** There exists on \( \tilde{\mathfrak{g}} \) a Lie algebra structure such that
\[
\begin{align*}
(\mathcal{S}_1) & \quad [X, X'] = [X, X']_{\tilde{\mathfrak{l}}} \quad \forall X, X' \in \mathfrak{l} \\
(\mathcal{S}_2) & \quad [X, p] = \rho(X)p \quad \forall X \in \mathfrak{l}, p \in \mathcal{V}.
\end{align*}
\]
\( \tilde{\mathfrak{g}} \), endowed with this structure, is a simple 3-graded Lie algebra.

Moreover
\[
dim \mathfrak{g} = 2 \dim(V) + \dim(\mathfrak{l})
\]
and, if \( \mathfrak{h}_\mathfrak{l} \) is a Cartan subalgebra of \( \mathfrak{l} \), then it is a Cartan subalgebra of \( \tilde{\mathfrak{g}} \) and \( \text{rank}(\tilde{\mathfrak{g}}) = \text{rank}(\tilde{\mathfrak{l}}) \).

**Proof.** In fact, as for each \( p \in p_{r-1} \) and for each \( p' \in p_{-r+1} \), there exists a unique \( X \in \mathfrak{t}_{r-1} \) and a unique \( X' \in \mathfrak{t}_1 \) such that \( p = \rho(X)F \) and \( p' = \rho(X')E \) (where \( F = 1 \) and \( E = Q \)), then we define \( [p, p'] = [X, X']_{\tilde{\mathfrak{l}}} \). Also, for \( p_1 = \rho(X_1)E \) and \( p_2 = \rho(X_2)E \) (with \( X_1, X_2 \in \mathfrak{t}_{-1} \)), we define the bracket as \( [p_1, p_2] = [X_1, X_2]_{\tilde{\mathfrak{l}}} = 0 \) and for \( p'_1 = \rho(X'_1)E \) and \( p'_2 = \rho(X'_2)E \) (with \( X'_1, X'_2 \in \mathfrak{t}_1 \)), we define the bracket as \( [p'_1, p'_2] = [X'_1, X'_2]_{\tilde{\mathfrak{l}}} = 0 \). It becomes then easy to show that the Jacobi identity holds and that \( \tilde{\mathfrak{g}} \) is isomorphic to the conformal Lie algebra of \( V \).

Moreover, denote by \( \mathfrak{l}_R \) the compact real form of \( \mathfrak{l} \) and by \( \tilde{\mathfrak{u}} \) the compact real form of \( \tilde{\mathfrak{g}} \). Then \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{u}} + i\tilde{\mathfrak{u}} \) is the Cartan decomposition of \( \tilde{\mathfrak{g}} \) and \( \mathfrak{l}_R = \tilde{\mathfrak{u}} \cap \mathfrak{l} \). Moreover, if we
denote by \( V_R = V \cap \mathfrak{u} \), then the real Lie subalgebra of \( \tilde{\mathfrak{g}} \) defined by \( \tilde{\mathfrak{g}}_R = \mathfrak{l}_R + V_R \) is a real form of \( \tilde{\mathfrak{g}} \). Its Cartan decomposition is just \( \mathfrak{l}_R + V_R \). The Cartan signature of \( \tilde{\mathfrak{g}}_R \) is then given by

\[
s_c = \text{dim}(V) - \text{dim}(l) = 2\text{dim}(V) - \text{dim}(l).
\]

The real form \( \tilde{\mathfrak{g}}_R = \mathfrak{l}_R + V_R \) will be called the Cartan-Adapted real form of the Lie algebra \( \tilde{\mathfrak{g}} = \mathfrak{l} + V \) associated to the pair \((V,Q)\). (cf. table 3 for the classification)

**Corollary.** The real forms \( \tilde{\mathfrak{g}}_R \) are of Hermitian type.

**Proof.** It is a consequence of the fact that the decomposition \( \tilde{\mathfrak{g}} = \mathfrak{l} + V \) is the complexification of the Cartan decomposition of \( \tilde{\mathfrak{g}}_R \) and the fact that \( V \) is a sum of two irreducible representations, \( V^+ \) and \( V^- \) of \( l \). □

\[\text{(Table 3)}\]

<table>
<thead>
<tr>
<th>( V )</th>
<th>( Q )</th>
<th>( l )</th>
<th>( \tilde{\mathfrak{g}} )</th>
<th>( \mathfrak{l}_R )</th>
<th>( \tilde{\mathfrak{g}}_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{C} )</td>
<td>( z^2 )</td>
<td>( \mathbb{C} )</td>
<td>( \mathfrak{sl}(2, \mathbb{C}) )</td>
<td>( \mathbb{R} )</td>
<td>( \mathfrak{sl}(2, \mathbb{R}) )</td>
</tr>
<tr>
<td>( \mathbb{C}^p )</td>
<td>( \Delta(z)^2 )</td>
<td>( \mathfrak{so}(p - 2, \mathbb{C}) + \mathbb{C} )</td>
<td>( \mathfrak{so}(p, \mathbb{C}) )</td>
<td>( \mathfrak{so}(p - 2) + \mathbb{R} )</td>
<td>( \mathfrak{so}(p - 2, 2) )</td>
</tr>
<tr>
<td>( \text{Sym}(r, \mathbb{C}) )</td>
<td>( \text{det}(z)^2 )</td>
<td>( \mathfrak{sl}(r, \mathbb{C}) + \mathbb{C} )</td>
<td>( \mathfrak{sp}(r, \mathbb{C}) )</td>
<td>( \mathfrak{su}(r) + \mathbb{R} )</td>
<td>( \mathfrak{sp}(r, \mathbb{R}) )</td>
</tr>
<tr>
<td>( \text{M}(r, \mathbb{C}) )</td>
<td>( \text{det}(z)^2 )</td>
<td>( \mathfrak{sl}(r, \mathbb{C}) \oplus \mathbb{C} )</td>
<td>( \mathfrak{su}(r) )</td>
<td>( \mathfrak{su}(2r, \mathbb{R}) )</td>
<td>( \mathfrak{sl}(2r, \mathbb{R}) )</td>
</tr>
<tr>
<td>( \text{Asym}(2r, \mathbb{C}) )</td>
<td>( \text{det}(z) )</td>
<td>( \mathfrak{sl}(2r, \mathbb{C}) + \mathbb{C} )</td>
<td>( \mathfrak{so}(4r, \mathbb{C}) )</td>
<td>( \mathfrak{so}(2r) )</td>
<td>( \mathfrak{so}^*(4r) )</td>
</tr>
<tr>
<td>( \text{Herm}(3, \mathbb{O})^C )</td>
<td>( \text{det}(z)^2 )</td>
<td>( \mathfrak{E}_6(\mathbb{C})^C )</td>
<td>( \mathfrak{E}_7(\mathbb{C}) )</td>
<td>( \mathfrak{E}_6(\mathbb{R}) + \mathbb{R} )</td>
<td>( \mathfrak{E}_7(-25) )</td>
</tr>
</tbody>
</table>

where \( p \geq 5 \).
References


[A.2] B. Allison (1990), Simple structurable algebras of skew dimension one, Comm. in Alg. 18, 1245-1279.


