Very weak solutions for the stationary Oseen and Navier-Stokes equations.

Chérif Amrouche, M.A. Rodriguez-Bellido

Very weak solutions for the stationary Oseen and Navier-Stokes equations

Chérif Amrouche[a], María Ángeles Rodríguez-Bellido[b,1]

[a] *Laboratoire de Mathématiques Appliquées, CNRS UMR 5142, Université de Pau et des Pays de l'Adour, IPRA, Avenue de l'Université- 64000 Pau (France)*

[b] *Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Aptdo. de Correos 1160 - 41080 Sevilla (Spain)*

Received *****; accepted after revision +++++

Presented by

Abstract

We consider the stationary Oseen and Navier-Stokes equations in a bounded domain of class $\mathcal{C}^{1,1}$ of $\mathbb{R}^3$. Here we give a new and simpler proof of the existence of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ corresponding to boundary data in $\mathbf{W}^{-1/p,p}(\Gamma)$. These solutions are obtained without imposing smallness assumptions on the exterior forces. We also obtain regularity results in fractional Sobolev spaces.

To cite this article: C. Amrouche, M. A. Rodríguez-Bellido, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Résumé

Solutions très faibles pour les équations stationnaires d'Oseen et de Navier-Stokes. Nous considérons les équations stationnaires d'Oseen et de Navier-Stokes dans un ouvert borné connexe et de classe $\mathcal{C}^{1,1}$de $\mathbb{R}^3$. Nous donnons ici une nouvelle preuve plus simple de l'existence de solutions très faibles $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ correspondant à des données au bord dans $\mathbf{W}^{-1/p,p}(\Gamma)$. Ces solutions sont obtenues sans hypothèse de petitesse des forces extérieures. On obtient aussi des résultats de régularité dans des espaces de Sobolev fractionnaires.

Pour citer cet article : C. Amrouche, M. A. Rodríguez-Bellido, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Version française abrégée

L'objet de cette note consiste essentiellement à étudier l'existence de solutions très faibles $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ pour les équations d'Oseen (O) et de Navier-Stokes (NS). L'une des difficultés consiste

Email addresses: `cherif.amrouche@univ-pau.fr` (Chérif Amrouche), `angeles@us.es` (María Ángeles Rodríguez-Bellido).

[1] Partially supported by M.E.C. (Spain), Project MTM2006-07932, and by Junta de Andalucía, Project P06-FQM-02373.

à donner un sens aux conditions aux limites de Dirichlet. Le terme de convection rend les choses plus difficiles pour le problème (O) et complique sérieusement la situation pour l'étude du problème non linéaire (NS). Les résultats concernant l'existence de solutions très faibles sont donnés dans le théorème 2.4 pour (O) et les théorèms 3.1 et 3.2 pour (NS). Les autres résultats concernent la régularité de telles solutions sous des hypothèses adéquates. Nons considérons en particulier le cas où les données et donc les solutions appartiennent à des espaces de Sobolev fractionnaires.

1. Introduction

Let Ω be a bounded connected open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ with boundary Γ. We are interested in some questions concerning the stationary Oseen and Navier-Stokes equations, that generally can be written as:

$$(O) \quad -\Delta \boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \nabla\cdot\boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma,$$

$$(NS) -\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \nabla\cdot\boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma,$$

where $\boldsymbol{u}$ denotes the velocity field and q the pressure, both being unknown, and $\boldsymbol{f}$, h, $\boldsymbol{g}$ and $\boldsymbol{v}$ are given.

In the case of incompressible fluids, $h = 0$, it has been well-known since Leray [6] that if $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ with $p \geq 2$, Γ_i are the connected components of the boundary Γ, $i = 0, \ldots, I$, and

$$\int_{\Gamma_i} \boldsymbol{g}\cdot\boldsymbol{n}\, d\sigma = 0, \qquad \forall i = 0, ..., I, \tag{1}$$

then there exists a solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying (NS). Serre proved [8] the existence of weak solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ for any $\frac{3}{2} < p < 2$ when $h = 0$ and $\boldsymbol{g}$ satisfies the above conditions. Recently, Kim [5] improves Serre's existence and regularity results on weak solutions of (NS) for any $\frac{3}{2} \leq p < 2$, when Γ is connected ($I = 0$) provided h is small in an appropriate norm (due to (2), see below, $\boldsymbol{g}$ is also small in the corresponding appropriate norm).

Existence of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$, for $h = 0$, arbitrary large $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$ and large $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$, without assuming condition (1), was proved first by Marusic-Paloka in [7] (see Theorem 5) with Ω a bounded simply-connected open set of class $\mathcal{C}^{1,1}$. But the proof of Theorem 5 becomes correct only if either condition (1) or condition (12) hold. The same result was proved by Kim [5] for arbitrary large external forces $\boldsymbol{f} \in [\mathbf{W}_0^{1,3/2}(\Omega) \cap W^{2,3}(\Omega)]'$, for small $h \in [W^{1,3/2}(\Omega)]'$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$, with Γ supposed connected ($I = 0$). Observe that the space chosen for h and for $\boldsymbol{f}$ are not correct either and led us to some erros (in particular, the equivalence given in Theorem 5 there does not work).

The purpose of our work is to generalize the theory of very weak solutions of the Dirichlet problem from the Stokes equations to the Oseen and Navier-Stokes equations, defining rigorously the traces of the vector functions which are living in subspaces of $\mathbf{L}^p(\Omega)$ (see [1], [2]), and the spaces for the data. We prove existence and regularity of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ of Oseen equations for any $p \in (1, +\infty)$ with arbitrary large data in Sobolev spaces of negative order. In the Navier-Stokes case, the existence of very weak solution is proved for arbitrary large external forces, but with a smallness condition for both h and $\boldsymbol{g}$. Uniqueness of very weak solutions is also proved for small enough data. The detailed proofs of the results announced in this Note are given in [2].

2. Oseen Equations

For any $1 < r, p < \infty$, we define the spaces: $\mathbf{H}_p(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla\cdot\boldsymbol{v} = 0\}$, $\mathbf{X}_{r,p}(\Omega) = \{\boldsymbol{\varphi} \in \mathbf{W}_0^{1,r}(\Omega);\ \nabla\cdot\boldsymbol{\varphi} \in W_0^{1,p}(\Omega)\}$, and $\mathbf{T}_{p,r}(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \Delta\boldsymbol{v} \in (\mathbf{X}_{r',p'}(\Omega))'\}$, endowed with the topology given by the norm $\|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} = \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}$.

As for the Navier-Stokes system, we can prove that if $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$, $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$, $h \in L^2(\Omega)$ and $\boldsymbol{g} \in \mathbf{H}^{1/2}(\Gamma)$ with h and $\boldsymbol{g}$ verifying the compatibility condition

$$\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n}\, d\sigma, \tag{2}$$

then the problem (O) has a unique solution $(\boldsymbol{u}, q) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ verifying the following estimate:

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\ (\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)})\Big).$$

Theorem 2.1 (Strong solutions) *Consider $p \geq \frac{6}{5}$, $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$, $h \in W^{1,p}(\Omega)$, $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$, with $s = 3$ if $p < 3$, $s = p$ if $p > 3$, or $s = 3 + \varepsilon$ if $p = 3$, for some arbitrary $\varepsilon > 0$, and satisfying the compatibility condition (2). Then, the unique solution of (O) verifies $(\boldsymbol{u}, q) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Moreover, there exists a constant $C > 0$ such that*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big)\left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\Big).$$

Proof: First, let $(\boldsymbol{u}, q) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ be the unique solution of Problem (O). For a given $\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ $(\lambda > 0)$ such that $\nabla \cdot \boldsymbol{v}_\lambda = 0$ and $\|\boldsymbol{v}_\lambda - \boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} \leq \lambda$, let $(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ be the unique solution of the problem (O_λ): $-\Delta \boldsymbol{u}_\lambda - \boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda + \nabla q_\lambda = \boldsymbol{f}$ and $\nabla \cdot \boldsymbol{u}_\lambda = h$ in Ω, $\boldsymbol{u}_\lambda = \boldsymbol{g}$ on Γ (use the Stokes regularity and a bootstrap argument). Secondly, we focus on the obtention of a strong estimate for $(\boldsymbol{u}_\lambda, q_\lambda)$. If $\widetilde{\boldsymbol{v}}$ is the extension by zero of $\boldsymbol{v}$ to $\mathbb{R}^3$ and ρ_ε the classical mollifier, we consider

$$\boldsymbol{v}_\lambda = \boldsymbol{v}_1^\varepsilon + \boldsymbol{v}_{\lambda,2}^\varepsilon \quad \text{where} \quad \boldsymbol{v}_1^\varepsilon = \widetilde{\boldsymbol{v}} \star \rho_{\varepsilon/2}, \quad \boldsymbol{v}_{\lambda,2}^\varepsilon = \boldsymbol{v}_\lambda - \widetilde{\boldsymbol{v}} \star \rho_{\varepsilon/2} \quad \text{for } \varepsilon > 0, \text{ and } 0 < \lambda < \varepsilon/2. \tag{3}$$

By regularity estimates for the Stokes problem, we have

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} + \|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}). \tag{4}$$

In order to estimate the term $\|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}$, we use (3) and Sobolev embeddings. First:

$$\|\boldsymbol{v}_{\lambda,2}^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq \|\boldsymbol{v}_{\lambda,2}^\varepsilon\|_{\mathbf{L}^s(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^k(\Omega)} \leq C\,\varepsilon \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)}, \quad \text{with } \frac{1}{k} = \frac{1}{p} - \frac{1}{s}. \tag{5}$$

For the estimate on $\boldsymbol{v}_1^\varepsilon$, we consider two cases: If $p \leq 2$, let $r \in]3, \infty]$ be such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{2}$, and $t \geq 1$ such that $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ satisfying: $\|\boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^r(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)} \leq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)}$. Using the estimate (5), we deduce from (4) that

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)})).$$

If $p > 2$, using the compact embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,q}(\Omega)$, with $q < p^*$, for any $\varepsilon' > 0$, we known that there exists $C_{\varepsilon'} > 0$ such that $\|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \leq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1(\Omega)}$. Considering the case $p < 3$ and then the case $p \geq 3$, we can choose the exponent q and fix $\varepsilon > 0$ and $\varepsilon' > 0$ small enough to obtain

$$\begin{aligned}\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\Big(&\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} \\ &+ C_{\varepsilon'} \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\Omega)} \Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\big)\ (\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)})\Big)\Big).\end{aligned}$$

Thus, we deduce that there exists a sequence of real numbers k_λ such that $(\boldsymbol{u}_\lambda, q_\lambda + k_\lambda)$ converges weakly in $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ to $(\boldsymbol{u}, q)$, solution of Problem (O) with the corresponding estimate. ■

Theorem 2.2 *Let $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$, $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ verify the compatibility condition:*

$$\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}. \tag{6}$$

Then, the problem (O) has a unique solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. *Moreover, there exists some constant* $C > 0$ *such that, for* $\alpha = 1$ *if* $p \geq 2$ *and* $\alpha = 1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}$ *if* $p < 2$, *we have*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \alpha\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right). \quad (7)$$

Sketch of the proof: We split it in two cases. If $p \geq 2$, we decompose the solution $(\boldsymbol{u}, q)$ as $(\boldsymbol{z}, \theta) + (\boldsymbol{u}_0, q_0)$, being $(\boldsymbol{u}_0, q_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying $-\Delta \boldsymbol{u}_0 + \nabla q_0 = \boldsymbol{f}$ and $\nabla \cdot \boldsymbol{u}_0 = h$ in Ω, $\boldsymbol{u}_0 = \boldsymbol{g}$ on Γ, and $(\boldsymbol{z}, \theta) \in \mathbf{W}^{2,t}(\Omega) \times W^{1,t}(\Omega)$ satisfying $-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla\theta = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0$ and $\nabla \cdot \boldsymbol{z} = 0$ in Ω, $\boldsymbol{z} = \mathbf{0}$ on Γ, where $\frac{1}{t} = \frac{1}{3} + \frac{1}{p}$. The corresponding estimates (see Theorem 2.1) and the embedding $\mathbf{W}^{2,t}(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$ conclude the proof in this case. Secondly, if $p < 2$, we are able to conclude by a duality argument. ■

Remark 1 Estimate (7) can be improved for $p \in [\frac{6}{5}, 6]$, and for any $p > 1$ if $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ as:

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \leq C\big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\big(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\big)\Big).$$

Corollary 2.3 *Consider* $1 < p < 6/5$ *and* $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$, $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$, $h \in W^{1,p}(\Omega)$ *and* $\boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$ *verifying the compatibility condition (6). Then, the solution given by Theorem 2.2 satisfies* $(\boldsymbol{u}, q) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ *and the following estimate holds:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\Big).$$

Using the previous results and following arguments in [2], we obtain:

Theorem 2.4 (Very weak solution of Oseen equations) *Let* $\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$, $h \in L^r(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$, *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{s}$, *be given, satisfying the compatibility condition (6), and* $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ *with* $s = 3$ *if* $p > 3/2$, $s = p'$ *if* $p < 3/2$, *or* $s = 3 + \varepsilon$ *if* $p = 3/2$. *Then, the Oseen problem (O) has a unique solution* $(\boldsymbol{u}, q) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ *verifying the estimates*

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\Big), \quad (8)$$
$$\|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)^2\Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\Big).$$

Concerning the regularity of solutions for the Oseen equations in fractional Sobolev spaces, we obtain:

Theorem 2.5 (Regularity for Oseen equations) *Consider* $\sigma \in (1/p, 2]$. *Let* $\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega)$, $h \in W^{\sigma-1,p}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma)$ *be given satisfying the compatibility condition* (6), *and* $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ *with* s *as in Theorem 2.4. Then, the Oseen problem (O) has a unique solution* $(\boldsymbol{u},\ q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)/\mathbb{R}$ *satisfying*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma,p}(\Omega)} + \|q\|_{W^{\sigma-1,p}(\Omega)/\mathbb{R}} \leq C\left(\|\boldsymbol{f}\|_{\mathbf{W}^{\sigma-2,p}(\Omega)}) + \|h\|_{W^{\sigma-1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma-1/p,p}(\Omega)}\right).$$

3. Navier-Stokes Equations

Now, we present two theorems giving existence of very weak solutions for the Navier-Stokes equations in $\mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$, first one for the small data case, and second one for arbitrary large $\boldsymbol{f}$ but h and $\boldsymbol{g}$ small enough in a domain possibly multiply-connected.

Theorem 3.1 (Very weak solution for Navier-Stokes, small data case) *Let* $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h \in L^{3/2}(\Omega)$ *and* $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ *verify* (6). *Then,*

i) there exists a constant $\alpha_1 > 0$ *such that, if* $\|\ \boldsymbol{f}\ \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|\ h\ \|_{L^{3/2}(\Omega)} + \|\ \boldsymbol{g}\ \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \alpha_1$, *then, there exists a very weak solution* $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ *to problem* (NS) *verifying the estimates*

$$\|\ \boldsymbol{u}\ \|_{\mathbf{L}^3(\Omega)} \leq C\left(\|\ \boldsymbol{f}\ \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|\ h\ \|_{L^{3/2}(\Omega)} + \|\ \boldsymbol{g}\ \|_{\mathbf{W}^{-1/3,3}(\Gamma)}\right), \quad (9)$$
$$\|\ q\ \|_{W^{-1,3}/\mathbb{R}} \leq C_1 \|\ \boldsymbol{f}\ \|_{[\mathbf{X}_{3,3/2})]'} + 2(1 + C_2)C\left(\|\ \boldsymbol{f}\ \|_{[\mathbf{X}_{3,3/2}]'} + \|\ h\ \|_{L^{3/2}} + \|\ \boldsymbol{g}\ \|_{\mathbf{W}^{-1/3,3}}\right), \quad (10)$$

where $C > 0$ is the constant given in (8), $\alpha_1 = \min\left\{(2C)^{-1}, (2C^2)^{-1}\right\}$, and C_1 and C_2 constants of Sobolev embeddings.

ii) Moreover, there exists a constant $\alpha_2 \in]0, \alpha_1]$ such that if $\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \alpha_2$, then this solution is unique, up to an additive constant for q.

Proof: We prove existence of a very weak solution by applying Banach's fixed point theorem over the Oseen equations. Indeed, let $T : \mathbf{H}_3(\Omega) \to \mathbf{H}_3(\Omega)$ be the application defined as $\boldsymbol{v} \mapsto T\boldsymbol{v} = \boldsymbol{u}$, where $\boldsymbol{u}$ is the unique solution of (O) provided by Theorem 2.4. We set $\mathbf{B}_r = \{\boldsymbol{v} \in \mathbf{H}_3(\Omega);\ \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \leq r\}$. We will prove that there exists $\theta \in]0, 1[$ such that

$$\|T\boldsymbol{v}_1 - T\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)} = \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)} \leq \theta \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}. \tag{11}$$

In order to estimate $\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}$, we observe that for each $i = 1, 2$, $(\boldsymbol{u}_i, q_i)$ is the solution of $-\Delta \boldsymbol{u}_i + \boldsymbol{v}_i \cdot \nabla \boldsymbol{u}_i + \nabla q_i = \boldsymbol{f}$ and $\nabla \cdot \boldsymbol{u}_i = h$ in Ω, $\boldsymbol{u}_i = \boldsymbol{g}$ on Γ, with the estimates

$$\|\boldsymbol{u}_i\|_{\mathbf{L}^3(\Omega)} \leq C \left(1 + \|\boldsymbol{v}_i\|_{\mathbf{L}^3(\Omega)}\right) \Big(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)}\Big),$$

being $C > 0$ the constant given in (8). However, to estimate the difference $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we have to argue differently. Consider the problem fulfilled by $(\boldsymbol{u}, q) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, q_1 - q_2)$, which is $-\Delta \boldsymbol{u} + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{u} + \nabla q = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_2$ and $\nabla \cdot \boldsymbol{u} = 0$ in Ω, $\boldsymbol{u} = \mathbf{0}$ on Γ, where $\boldsymbol{u}_1 = T\boldsymbol{v}_1$, $\boldsymbol{u}_2 = T\boldsymbol{v}_2$ and $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$. Using the very weak estimates (8) for the Oseen problem successively for $\boldsymbol{u}$ and for $\boldsymbol{u}_2$, we obtain that

$$\|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \leq C \left(1 + \|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}\right) \|(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}_2\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} \leq C^2 \beta \left(1 + \|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}\right) \left(1 + \|\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}\right) \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)},$$

where $\beta = \| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)}$. Thus, we obtain estimate (11) if we consider $C^2 \beta \,(1 + r)^2 < 1$, and (9)-(10) hold for C_1 the continuity constant of the Sobolev embedding $[\mathbf{X}_{3,3/2}(\Omega)]' \hookrightarrow \mathbf{W}^{-2,3}(\Omega)$ and C_2 the continuity constant of the Sobolev embedding $\mathbf{W}_0^{1,3/2}(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$. The uniqueness result is a simple consequence of Sobolev embeddings and the Stokes estimates. ■

Theorem 3.2 (Very weak solution for Navier-Stokes, arbitrary forces) *Let $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $h \in L^{3/2}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ be given, and satisfying the compatibility condition (6). There exists a constant $\delta > 0$ (depending only on Ω) such that the problem (NS) has a very weak solution $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ if*

$$\|h\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \leq \delta. \tag{12}$$

Sketch of the proof: We decompose (NS) into two problems. One system, denoted (NS_1), for small data:

$$-\Delta \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \nabla q_\varepsilon^1 = \boldsymbol{f} - \boldsymbol{f}_\varepsilon, \quad \nabla \cdot \boldsymbol{v}_\varepsilon = h - h_\varepsilon \quad \text{in } \Omega, \quad \text{and} \quad \boldsymbol{v}_\varepsilon = \boldsymbol{g} - \boldsymbol{g}_\varepsilon \quad \text{on } \Gamma.$$

with $\varepsilon > 0$ and the (NS_2) system:

$$-\Delta \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \nabla q_\varepsilon^2 = \boldsymbol{f}_\varepsilon, \quad \nabla \cdot \boldsymbol{z}_\varepsilon = h_\varepsilon \quad \text{in } \Omega, \quad \boldsymbol{z}_\varepsilon = \boldsymbol{g}_\varepsilon \quad \text{on } \Gamma.$$

where $\boldsymbol{f}_\varepsilon \in \mathbf{H}^{-1}(\Omega)$, $h_\varepsilon \in L^2(\Omega)$ and $\boldsymbol{g}_\varepsilon \in \mathbf{H}^{1/2}(\Gamma)$ satisfy

$$\|\boldsymbol{f} - \boldsymbol{f}_\varepsilon\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h - h_\varepsilon\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \varepsilon \text{ and } \|h_\varepsilon\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \leq 2\delta,$$

(here, we have used density arguments). Finally, we use an extension of Hopf's lemma: (see [3], Remark VIII.4.4 for instance) for any $\alpha > 0$, there exists $\boldsymbol{y}_\varepsilon \in \mathbf{H}^1(\Omega)$, depending on α, such that for $C_1 > 0$ depending only on Ω, $\nabla \cdot \boldsymbol{y}_\varepsilon = h_\varepsilon$ in Ω, $\boldsymbol{y}_\varepsilon = \boldsymbol{g}_\varepsilon$ on Γ and for any $\boldsymbol{w} \in \mathbf{H}_0^1(\Omega)$,

$$\left|\int_\Omega (\boldsymbol{w} \cdot \nabla)\boldsymbol{y}_\varepsilon \cdot \boldsymbol{w}\, d\boldsymbol{x}\right| \leq \left(\alpha + \|h_\varepsilon\|_{L^{3/2}(\Omega)} + C \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}|\right) \|\boldsymbol{w}\|^2_{\mathbf{H}^1(\Omega)} \leq (\alpha + 2C_1\delta)\|\boldsymbol{w}\|^2_{\mathbf{H}^1(\Omega)}. \blacksquare$$

To finish, we prove some regularity results on very weak solutions for the Navier-Stokes equations by using the regularity results for the Stokes and Oseen problems.

Theorem 3.3 (Regularity for Navier-Stokes equations) *Let $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ be the solution given by Theorem 3.2. Then, the following regularity results hold:*

i) If $\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$, $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$, with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $\max\{r, 3\} \leq p$, then $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$.

ii) Consider $r \geq 3/2$, $\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega)$, $h \in L^r(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma)$. Then $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$.

iii) For $r \in (1, +\infty)$, if $\boldsymbol{f} \in \mathbf{L}^r(\Omega)$, $h \in W^{1,r}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{2-1/r,r}(\Gamma)$, then $(\boldsymbol{u}, q) \in \mathbf{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$.

iv) Suppose that $3/2 \leq p \leq 3$, $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$ for $\mathbb{F}_0 \in \mathbf{W}^{\sigma,r}(\Omega)$ and $f_1 \in W^{\sigma-1,p}(\Omega)$, $h \in W^{\sigma,r}(\Omega)$, and $\boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma)$, with $\sigma = \frac{3}{p} - 1$, $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $r \leq p$. Then $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$.

v) Let σ be such that $1/p < \sigma \leq 1$ and $\sigma \geq 3/p - 1$. Suppose that $\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega)$, $h \in W^{\sigma-1,p}(\Omega)$, and $\boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma)$. Then $(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$.

Remark 2 i) Point i) shows in particular that for any $p \geq 3$, if $\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma)$, with $\frac{3p}{3+p} \leq r \leq p$, and $\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} = 0$ for any $i = 1, \ldots, I$ and $h = 0$, then Problem (NS) has a solution $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$. Serre [8] proves that for any $3/2 < r < 2$ (and then for $r > 3/2$), if $\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma)$, $h = 0$ and (1) is verified for any $i = 0, \ldots, I$, then (NS) has a solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega)$. Point ii) proves that this result holds if $r = 3/2$ without assuming h or the flux $\boldsymbol{g}$ through Γ_i to be equal to 0. Actually, it suffices to assume the smallness condition (12).

ii) From relation (6), condition (12) is automatically fulfilled when the norm $\|h\|_{L^{3/2}(\Omega)}$ is small enough and $I = 0$, that means that the boundary Γ is connected, which is the case considered by Kim [5].

iii) Marusic-Paloka [7] proves Theorem 3.2 with $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega) \subseteq (\mathbf{X}_{3,3/2}(\Omega))'$, $h = 0$ and $\boldsymbol{g} \in \mathbf{L}^2(\Gamma) \subseteq \mathbf{W}^{-1/3,3}(\Gamma)$ with $\|\boldsymbol{g}\|_{\mathbf{L}^2(\Gamma)}$ small, in a domain Ω simply-connected. In fact, the solution $\boldsymbol{u} \in \mathbf{L}^3(\Omega)$ obtained in [7] is more regular and belongs to $\mathbf{H}^{1/2}(\Omega)$ by point iv) with $p = 2$.

iv) Galdi et al. [4] prove Theorem 3.2 and Theorem 3.3 point i) with $\boldsymbol{f} = \operatorname{div} \mathbb{F}_0$, $\mathbb{F}_0 \in \mathbb{L}^r(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $\max\{2r, 3\} \leq p$, in a domain Ω of class $\mathcal{C}^{2,1}$, assuming that $\boldsymbol{f}$, h and $\boldsymbol{g}$ are small enough in their respective norms. The smallness condition on $\boldsymbol{f}$ is in fact unnecessary.

References

[1] C. Amrouche and V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension,* Czechoslovak Mathematical Journal, 44, 119 (1994), pp. 109–140.

[2] C. Amrouche and M. A. Rodríguez-Bellido, *Stokes, Oseen and Navier-Stokes equations with singular data,* Submitted.

[3] G. P. Galdi, *An Introduction to the Matematical Theory of the Navier-Stokes Equations, Vol 2: Nonlinear Steady Problems.* Springer Tracts in Natural Philosophy, vol. 39. Springer, New York (1994).

[4] G. P. Galdi, C. G. Simader and H. Sohr, *A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/q,q}$,* Math. Ann. , 331 (2005), pp. 41–74.

[5] H. Kim, *Existence and regularity of very weak solutions of the stationary Navier-Stokes equations,* Arch. Rational Mech. Anal., **193** (2009), 117-152.

[6] J. Leray, *Etude de divesres équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique,* J. Math. Pures Appl. 12 (1933), pp. 1–82.

[7] E. Marusič-Paloka, *Solvability of the Navier-Stokes system with L^2 boundary data*, Appl. Math. Optim. 41 (2000), pp. 365-375.

[8] D. Serre, *Équations de Navier-Stokes stationnaires avec données peu régulières*, Ann. Sc. Norm. Sup. Pisa 10-4, (1983), pp. 543-559.