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Weak vector and scalar potentials.
Applications to Poincaré’s theorem and Korn’s inequality in Sobolev spaces with negative exponents.

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Abstract

In this paper, we present several results concerning vector potentials and scalar potentials with data in Sobolev spaces with negative exponents, in a not necessarily simply-connected, three-dimensional domain. We then apply these results to Poincaré’s theorem and to Korn’s inequality.

1 Weak versions of a classical theorem of Poincaré

In this work, (the results of which were announced in \cite{2}), \( \Omega \) is a bounded open connected subset of \( \mathbb{R}^3 \) with a Lipschitz-continuous boundary \( \Gamma \). The notation \( X', \langle, \rangle_X \) denotes the duality pairing between a topological space \( X \) and its dual \( X' \). The letter \( C \) denotes a constant that is not necessarily the same at its various occurrences.

We begin with a weak version of a well-known theorem of Poincaré. Here as elsewhere in this paper, “weak” means that the result to which it is attached holds as well in Sobolev spaces with negative exponents.

\begin{theorem}
Let \( f \in H^{-m}(\Omega)^3 \) for some integer \( m \geq 0 \). Then the following properties are equivalent:
\begin{itemize}
\item[(i)] \( H^{-m}(\Omega)^3 \langle f, \varphi \rangle_{H^m_0(\Omega)^3} = 0 \) for all \( \varphi \in V_m = \{ \varphi \in H^m_0(\Omega)^3; \text{ div } \varphi = 0 \} \),
\item[(ii)] \( H^{-m}(\Omega)^3 \langle f, \varphi \rangle_{H^m(\Omega)^3} = 0 \) for all \( \varphi \in V = \{ \varphi \in H^m(\Omega)^3; \text{ div } \varphi = 0 \} \).
\end{itemize}
\end{theorem}

Proof. For the equivalence between (i), (ii) and (iii), we refer to \cite{4}. Since the implication (iii) \( \Rightarrow \) (iv) clearly holds, it remains to prove that (iv) \( \Rightarrow \) (iii).

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To begin with, let \( f \in H^{-m}(\Omega)^3 \) be such that \( \text{curl } f = 0 \) in \( \Omega \). We then use the same argument as in [8]: We know that there exist a unique \( u \in H^0_0(\Omega)^3 \) and a unique \( p \in H^{-m+1}(\Omega)/\mathbb{R} \) (see [5]) such that

\[
\Delta^m u + \text{grad } p = f \quad \text{and} \quad \text{div } u = 0 \quad \text{in } \Omega.
\]  
(1)

Hence \( \Delta^m \text{curl } u = 0 \) in \( \Omega \) so that the hypoellipticity (see [10]) of the polyharmonic operator \( \Delta^m \) implies that \( \text{curl } u \in C^\infty(\Omega)^3 \). Since \( \text{div } u = 0 \), we deduce that \( \Delta u = \text{curl } \text{curl } u \in C^\infty(\Omega)^3 \). This also implies that \( \Delta^m u \) belongs to \( C^\infty(\Omega)^3 \) and is an irrotational vector field. By the classical Poincaré theorem, there exists \( q \in C^\infty(\Omega)^3 \) such that \( \Delta^m u = \text{grad } q \). Thus, \( f = \text{grad } (p + q) \) and, thanks to [4] proposition 2.10, the function \( p + q \) belongs to the space \( H^{m+1}(\Omega) \).

We can give another proof of the implication (iv) \( \implies \) (iii) by using the following theorem:

**Theorem 1.2.** Assume that both \( \Omega \) and \( \mathbb{R}^3 \setminus \Omega \) are simply-connected. Let \( u \in H^m_0(\Omega)^3 \), \( m \geq 0 \), be a vector field that satisfies \( \text{div } u = 0 \) in \( \Omega \). Then there exists a vector potential \( \psi \) in \( H^{m+1}_0(\Omega)^3 \) such that

\[
u = \text{curl } \psi, \quad \text{div } \Delta^m \psi = 0 \quad \text{in } \Omega, \quad \text{and} \quad \|\psi\|_{H^{m+1}(\Omega)^3} \leq C\|u\|_{H^m(\Omega)^3}.
\]  
(2)

**Proof.** Let \( u \in H^m_0(\Omega)^3 \) be such that \( \text{div } u = 0 \) in \( \Omega \) and let \( \tilde{u} \) denote the extension of \( u \) by 0 in \( \mathbb{R}^3 \setminus \Omega \). Thus \( \tilde{u} \in H^m_0(\mathbb{R}^3)^3 \), \( \text{div } \tilde{u} = 0 \) in \( \mathbb{R}^3 \), and there exist an open ball \( B \) containing \( \overline{\Omega} \) and a vector field \( w \in H^{m+1}_0(B)^3 \) such that \( \tilde{u} = \text{curl } w \), \( \text{div } \Delta^m w = 0 \) in \( B \), and

\[
\|w\|_{H^{m+1}(B)^3} \leq C\|u\|_{H^m(B)^3}.
\]

The open set \( \Omega' := B \setminus \overline{\Omega} \) is bounded, has a Lipschitz-continuous boundary and is simply-connected. Furthermore, the vector field \( w' := w|_{\Omega'} \) belongs to \( H^{m+1}(\Omega')^3 \) and satisfies \( \text{curl } w' = 0 \) in \( \Omega' \). Therefore there exists a function \( \chi' \in H^1(\Omega') \) such that \( w' = \text{grad } \chi' \) in \( \Omega' \). Hence in fact \( \chi' \in H^{m+2}(\Omega') \) and the estimate

\[
\|\chi'\|_{H^{m+2}(\Omega')} \leq C\|w'\|_{H^{m+1}(\Omega')}^3
\]

holds. Since the function \( \chi' \in H^{m+2}(\Omega') \) can be extended to a function \( \tilde{\chi} \) in \( H^{m+2}(\mathbb{R}^3) \), with

\[
\|\tilde{\chi}\|_{H^{m+2}(\mathbb{R}^3)} \leq C\|\chi'\|_{H^{m+2}(\Omega')} \leq C\|w'\|_{H^{m+1}(\Omega')}^3,
\]

the vector field \( \tilde{\varphi} := w - \text{grad } \tilde{\chi} \) belongs to the space \( H^{m+1}(B)^3 \) and satisfies \( \tilde{\varphi}|_{\Omega'} = 0 \). Then the restriction \( \varphi := \tilde{\varphi}|_{\Omega} \) belongs to the space \( H^{m+1}_0(\Omega)^3 \), satisfies the estimate (2), and \( \text{curl } \varphi = \text{curl } w = \tilde{u} \) in \( B \). Thus \( u = \text{curl } \varphi \).
Let now \( p \) denote the unique solution in the space \( H_0^{m+2}(\Omega) \) of \( \Delta^{m+2}p = \text{div} \Delta^{m+1}\varphi \), so that the estimate
\[
\|p\|_{H^{m+2}(\Omega)} \leq C\|\varphi\|_{H^{m+1}(\Omega)}^3
\]
holds. Then the function \( \psi = \varphi - \text{grad} \ p \) satisfies (2).

We can give yet another proof of the above implication (iv) \( \implies \) (iii): Consider again the solution \( u \in H_0^m(\Omega)^3 \) to (1) and let \( v \in H_0^{m+1}(\Omega)^3 \) denote the vector potential of \( u \) as given by theorem 1.2. We then have \( \Delta^m \text{curl} \ u = 0 \) if \( m = 2k \), for some integer \( k \geq 1 \), then
\[
H^{-m-1}(\Omega)^3 \langle \Delta^m \text{curl} \ u, v \rangle_{H^m(\Omega)^3} = H^{-1}(\Omega)^3 \langle \Delta^k \text{curl} \ u, \Delta^k v \rangle_{H^0(\Omega)^3} = \int_{\Re^3} \Delta^k u \cdot \Delta^k \text{curl} v \, dx = \|\Delta^k u\|_{L^2(\Omega)^3}^2.
\]
This implies that \( \Delta^k u = 0 \) in \( \Omega \) and thus \( u = 0 \) since \( u \in H_0^m(\Omega)^3 \). The case \( m = 2k + 1 \) follows by a similar argument.

\[
\square
\]

2 Scalar Potentials

Let \( \Gamma_i, 0 \leq i \leq I \), denote the connected components of the boundary \( \Gamma \) of the domain \( \Omega \), \( \Gamma_0 \) being the boundary of the only unbounded connected component of \( \mathbb{R}^3 \setminus \overline{\Omega} \). We do not assume that \( \Omega \) is simply-connected, however we assume that there exist \( J \) connected and oriented surfaces \( \Sigma_j, 1 \leq j \leq J \) contained in \( \Omega \), with the following properties: each surface \( \Sigma_j \) is an open subset of a smooth manifold, the boundary of \( \Sigma_j \) is contained in \( \Gamma \) for \( 1 \leq j \leq J \), the intersection \( \Sigma_i \cap \Sigma_j \) is empty for \( i \neq j \), and finally the open set \( \Omega^0 = \Omega \setminus \bigcup_{j=1}^J \Sigma_j \) is simply-connected and pseudo-Lipschitz in the sense of [1]. Each such surface \( \Sigma_j \) is called a cut. Finally, let \( [:]_j \) denote the jump of a function over each cut \( \Sigma_j \), \( 1 \leq j \leq J \).

We then define the spaces
\[
H(\text{curl}, \Omega) = \{v \in L^2(\Omega)^3; \, \text{curl} \ v \in L^2(\Omega)^3\},
H(\text{div}, \Omega) = \{v \in L^2(\Omega)^3; \, \text{div} \ v \in L^2(\Omega)\},
\]
each one being equipped with the graph norm, and their subspaces
\[
H_0(\text{curl}, \Omega) = \{v \in H(\text{curl}, \Omega); \, v \times n = 0 \quad \text{on} \, \Gamma\},
H_0(\text{div}, \Omega) = \{v \in H(\text{div}, \Omega); \, v \cdot n = 0 \quad \text{on} \, \Gamma\}.
\]
For any function $q$ in $H^1(\Omega^c)$, $\text{grad } q$ denotes the gradient of $q$ in the sense of distributions in $\mathcal{D}'(\Omega^c)$. It belongs to $L^2(\Omega^c)^3$ and therefore can be extended to $L^2(\Omega)^3$. In order to distinguish this extension from the gradient of $q$ in $\mathcal{D}'(\Omega)$, we denote it by $\overline{\text{grad }} q$. Finally, we remark that the space

$$K_T(\Omega) := \{ w \in H(\text{curl}, \Omega) \cap H_0(\text{div}, \Omega); \text{ curl } w = 0 \text{ and div } w = 0 \text{ in } \Omega \}$$

is of dimension equal to $J$: As shown in [1] Prop. 3.14, it is spanned by the vector fields $\overline{\text{grad }} q_j^T$, $1 \leq j \leq J$, where each function $q_j^T \in H^1(\Omega^c)$, which is unique up to an additive constant, satisfies

$$\begin{align*}
\Delta q_j^T &= 0 \quad \text{in } \Omega^c, \\
\partial_n q_j^T &= 0, \quad \text{on } \Gamma, \\
[q_j^T]|_k &= \text{constant, } [\partial_n q_j^T]|_k = 0, \quad \langle \partial_n q_j^T, 1 \rangle_{\Sigma_k} = \delta_{jk} \quad \text{for } 1 \leq k \leq J.
\end{align*}$$

(3)

where $\langle \cdot, \cdot \rangle_{\Sigma_k}$ denotes the duality pairing between the spaces $H^{-1/2}(\Sigma_k)$ and $H^{1/2}(\Sigma_k)$.

**Theorem 2.1.** Given any function $f \in L^2(\Omega)^3$ that satisfies

$$\text{curl } f = 0 \quad \text{in } \Omega \quad \text{and} \quad \int_{\Omega} f \cdot v \; dx = 0 \quad \text{for all } v \in K_T(\Omega),$$

(4)

there exists a scalar potential $\chi$ in $H^1(\Omega)$ such that

$$f = \text{grad } \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)^3}. \quad (5)$$

**Proof.** It suffices to show that, given any vector field $v \in H_0(\text{div}, \Omega)$ such that $\text{div } v = 0$ in $\Omega$, there holds $\int_{\Omega} f \cdot v \; dx = 0$. Let

$$z = \sum_{j=1}^J \langle v \cdot n, 1 \rangle_{\Sigma_j} \overline{\text{grad }} q_j^T$$

and $w = v - z$. According to [1], theorem 3.17, there exists a vector potential $\psi \in L^2(\Omega)^3$ that satisfies $w = \text{curl } \psi$, $\text{div } \psi = 0$ in $\Omega$ and $\psi \times n = 0$ on $\Gamma$. Hence

$$\int_{\Omega} f \cdot v \; dx = \int_{\Omega} f \cdot \text{curl } \psi \; dx = 0.$$

The result is then a consequence of theorem 1.1: there exists a function $\chi \in H^1(\Omega)$ satisfying (5). \qed

**Remark 2.2.** (1) Any function $f \in L^2(\Omega)^3$ that satisfies $\text{curl } f = 0$ in $\Omega$ can be decomposed as:

$$f = \text{grad } \chi + \overline{\text{grad }} p, \quad \text{with } \chi \in H^1(\Omega) \quad \text{and} \quad \overline{\text{grad }} p \in K_T(\Omega).$$
Such a result was alluded to in [11].

(2) The second condition in (4) is trivially satisfied when \( \Omega \) is simply-connected, since \( \mathcal{K}_T(\Omega) = \{0\} \) in this case.

**Theorem 2.3.** Given any distribution \( f \in H_0(\text{div}, \Omega)' \) that satisfies

\[
\text{curl } f = 0 \quad \text{in } \Omega \quad \text{and} \quad H_0(\text{div}, \Omega) \langle f, v \rangle_{H_0(\text{div}, \Omega)} = 0 \quad \text{for all } v \in \mathcal{K}_T(\Omega),
\]

there exists a scalar potential \( \chi \) in \( L^2(\Omega) \) such that

\[
f = \text{grad } \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{L^2(\Omega)} \leq C\|f\|_{H_0(\text{div}, \Omega)'}.
\]

**Proof.** Let \( f \in H_0(\text{div}, \Omega)' \) be such that \( \text{curl } f = 0 \) in \( \Omega \). Hence (see proposition 1 of [6]) there exist \( \psi \in L^2(\Omega)^3 \) and \( \chi_0 \in L^2(\Omega) \) such that

\[
f = \psi + \text{grad } \chi_0 \quad \text{in } \Omega \quad \text{and} \quad \|\psi\|_{L^2(\Omega)^3} + \|\chi_0\|_{L^2(\Omega)} \leq C\|f\|_{H_0(\text{div}, \Omega)'}.
\]

Observe that, thanks to the density of \( \mathcal{D}(\Omega)^3 \) in \( H_0(\text{div}, \Omega) \),

\[
H_0(\text{div}, \Omega) \langle \text{grad } \chi_0, v \rangle_{H_0(\text{div}, \Omega)} = 0 \quad \text{for all } v \in \mathcal{K}_T(\Omega).
\]

Therefore, the function \( \psi \in L^2(\Omega)^3 \) satisfies relations (4). By theorem 2.1, there exists a function \( p \in H^1(\Omega) \) such that

\[
\psi = \text{grad } p \quad \text{in } \Omega \quad \text{and} \quad \|p\|_{H^1(\Omega)} \leq C\|\psi\|_{L^2(\Omega)^3} \leq C\|f\|_{H_0(\text{div}, \Omega)'}.
\]

Hence the function \( \chi = p + \chi_0 \) satisfies the announced properties. \( \square \)

**Remark 2.4.** Note that this theorem is an extension of the equivalence (iii) \( \iff \) (iv) in theorem 1.1 with \( m = 1 \) to the case where \( \Omega \) is not simply-connected.

More generally, let us introduce, for any integer \( m \geq 0 \), the space

\[
H_0^m(\text{div}, \Omega):=\{v \in H_0(\text{div}, \Omega); \text{ div } v \in H_0^m(\Omega)\},
\]

which coincides with \( H_0(\text{div}, \Omega) \) for \( m = 0 \). Its dual space, denoted by \( H^{-m}(\text{div}, \Omega) \), can then be characterized by

\[
H^{-m}(\text{div}, \Omega) = \{\psi + \text{grad } \chi; \psi \in H_0(\text{div}, \Omega)', \chi \in H^{-m}(\Omega)\}.
\]

One can also show that \( \mathcal{D}(\Omega)^3 \) is dense in \( H_0^m(\text{div}, \Omega) \) and that the following Green formula holds for any \( \chi \in H^{-m}(\text{div}, \Omega) \) and \( v \in H_0^m(\text{div}, \Omega) \):

\[
H^{-m}(\text{div}, \Omega) \langle \text{grad } \chi, v \rangle_{H_0^m(\text{div}, \Omega)} + H^{-m}(\Omega) \langle \chi, \text{ div } v \rangle_{H_0^m(\Omega)} = 0.
\]
As a consequence of theorem 2.3, it is easy to prove the following theorem, which shows that property (iv) in theorem 1.1 also holds when $\Omega$ is not simply-connected.

**Theorem 2.5.** For any distribution $f \in H^{-m}(\text{div}, \Omega)$ that satisfies (6), there exists a scalar potential $\chi$ in $H^{-m}(\Omega)$ such that

$$
\begin{equation}
\begin{aligned}
f &= \text{grad} \; \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{H^{-m}(\Omega)} \leq C \|f\|_{H^{-m}(\text{div}, \Omega)}.
\end{aligned}
\end{equation}
$$

**Proof.** We give the proof when $m = 1$; the general case is similar. Let $f \in H^{-1}(\text{div}, \Omega)$ satisfy (6). Then, there exist $\psi \in H_{0}(\text{div}, \Omega)^{\prime}$ and $\chi_{0} \in H^{-1}(\Omega)$ such that

$$
\begin{equation}
\begin{aligned}
f &= \psi + \text{grad} \; \chi_{0} \quad \text{in } \Omega \quad \text{and} \quad \|\psi\|_{H_{0}(\text{div}, \Omega)^{\prime}} + \|\chi_{0}\|_{H^{-1}(\Omega)} \leq C \|f\|_{H^{-1}(\text{div}, \Omega)}.
\end{aligned}
\end{equation}
$$

Observe that, thanks to (11), we have

$$
\begin{equation}
\begin{aligned}
H^{-1}(\text{div}, \Omega) \langle \text{grad} \; \chi_{0}, \; v \rangle_{H_{0}^{1}(\text{div}, \Omega)} = - H^{-1}(\Omega) \langle \chi_{0}, \; \text{div} \; v \rangle_{H_{0}^{1}(\Omega)} = 0
\end{aligned}
$$

for all $v \in K_{T}(\Omega)$. By theorem 2.3, there exists a function $p \in L^{2}(\Omega)$ such that $\psi = \text{grad} \; p$ and the estimate (7) holds. Then the function $\chi = \chi_{0} + p$ satisfies the announced properties. $\square$

### 3 Vector potentials in $H_{0}^{m}(\Omega)^{3}$

First, we recall some results concerning the existence of tangential vector potential (see [1] for proofs).

Below, $\langle \cdot, \cdot \rangle_{\Gamma_{i}}$ denotes the duality pairing between the spaces $H^{-1/2}(\Gamma_{i})$ and $H^{1/2}(\Gamma_{i})$. Given any function $u \in H(\text{div}, \Omega)$ that satisfies

$$
\begin{equation}
\begin{aligned}
\text{div} \; u &= 0 \quad \text{in } \Omega \quad \text{and} \quad \langle u \cdot n, \; 1 \rangle_{\Gamma_{i}} = 0, \quad 0 \leq i \leq I,
\end{aligned}
\end{equation}
$$

there exists a vector potential $\psi$ in $L^{2}(\Omega)^{3}$ such that

$$
\begin{equation}
\begin{aligned}
u &= \text{curl} \; \psi, \quad \text{div} \; \psi = 0 \quad \text{in } \Omega, \quad \text{and} \quad \psi \cdot n = 0 \quad \text{on } \Gamma,
\end{aligned}
\end{equation}
$$

satisfying the estimate

$$
\begin{equation}
\begin{aligned}
\|\psi\|_{L^{2}(\Omega)^{3}} \leq C \|u\|_{L^{2}(\Omega)^{3}}.
\end{aligned}
\end{equation}
$$

Moreover, there exists a unique vector field $\psi \in L^{2}(\Omega)^{3}$ satisfying (13) and such that

$$
\begin{equation}
\begin{aligned}
\langle \psi \cdot n, \; 1 \rangle_{\Sigma_{j}} = 0, \quad 1 \leq j \leq J,
\end{aligned}
\end{equation}
$$

6
and the estimate (14) holds. When \( \Omega \) is of class \( C^{1,1} \), then \( \psi \) belongs to \( H^1(\Omega)^3 \) and the estimate
\[
\| \psi \|_{H^1(\Omega)^3} \leq C \| u \|_{L^2(\Omega)^3}
\]
holds. If moreover \( u \in H^m(\Omega)^3 \) and \( \Omega \) is of class \( C^{m+1,1} \), for some integer \( m \geq 0 \), then \( \psi \) belongs to \( H^{m+1}(\Omega)^3 \) and the estimate
\[
\| \psi \|_{H^{m+1}(\Omega)^3} \leq C \| u \|_{H^m(\Omega)^3}
\]
holds. We also recall the result concerning the existence of normal vector potentials (see again [1] for proofs). For any vector field \( u \in H(\text{div}, \Omega) \) that satisfies
\[
\text{div } u = 0 \quad \text{in } \Omega, \quad u \cdot n = 0 \quad \text{on } \Gamma \quad \text{and } \langle u \cdot n, 1 \rangle_{\Sigma_j} = 0, \ 1 \leq j \leq J, \quad (18)
\]
there exists a vector potential \( \psi \) in \( L^2(\Omega)^3 \) such that
\[
u = \text{curl } \psi, \quad \text{div } \psi = 0 \quad \text{in } \Omega \quad \text{and } \psi \times n = 0 \quad \text{on } \Gamma,
\]
and the estimate
\[
\| \psi \|_{L^2(\Omega)^3} \leq C \| u \|_{L^2(\Omega)^3}
\]
holds. Moreover, there exists a unique vector field \( \psi \in L^2(\Omega)^3 \) satisfying (19) and such that
\[
\langle \psi \cdot n, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I, \quad (21)
\]
and the estimate (20) holds. When \( u \) is more regular, then (16) and (17) are also satisfied.

**Remark 3.1.** Let \( u \) be a vector field in \( H(\text{div}, \Omega) \) that satisfies:
\[
\text{div } u = 0 \quad \text{in } \Omega \quad \text{and } u \cdot n = 0 \quad \text{on } \Gamma.
\]
Using the same arguments as those of theorem 2.1, it is easy to verify that
\[
\langle u \cdot n, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J,
\]
if and only if
\[
\int_{\Omega} u \cdot \text{grad } q_j^T \, dx = 0 \quad \text{for all } 1 \leq j \leq J.
\]

Another kind of less standard but useful vector potential is given by the following theorem.

**Theorem 3.2.** Assume that the boundary of the domain \( \Omega \) is of class \( C^{1,1} \). For any function \( u \) in \( H(\text{div}, \Omega) \) satisfying (18), there exists a vector potential \( \psi \) in \( H^1(\Omega)^3 \), such that
\[
u = \text{curl } \psi \quad \text{and } \text{div } \Delta \psi = 0 \quad \text{in } \Omega, \quad \| \psi \|_{H^1(\Omega)^3} \leq C \| u \|_{L^2(\Omega)^3}. \quad (22)
\]
Proof. Given any vector field \( u \in H(\text{div}, \Omega) \) satisfying (18), we associate the vector potential \( \psi_0 \in H^1(\Omega)^3 \) satisfying (19) and the estimate
\[
\|\psi_0\|_{H^1(\Omega)^3} \leq C\|u\|_{L^2(\Omega)^3}.
\]

That \( \Gamma \) is of class \( C^{1,1} \) implies that the normal trace \( \psi_0 \cdot n \) belongs to \( H^{1/2}(\Gamma) \). Hence, the fourth-order problem
\[
\Delta^2 \chi = 0 \quad \text{in} \quad \Omega, \quad \chi = 0 \quad \text{and} \quad \partial_n \chi = \psi_0 \cdot n \quad \text{on} \quad \Gamma
\]
has a unique solution \( \chi \) in \( H^2(\Omega) \) satisfying the estimate
\[
\|\chi\|_{H^2(\Omega)} \leq C\|\psi_0 \cdot n\|_{H^{1/2}(\Gamma)} \leq C\|u\|_{L^2(\Omega)^3}.
\]
Then the vector field
\[
\psi = \psi_0 - \text{grad} \ \chi
\]
satisfies (22).

The vector field \( \psi \) given by the previous theorem is unique up to vector fields belonging to the space
\[
K^1_0(\Omega):=\{w \in H^1_0(\Omega)^3; \, \text{curl} \ w = 0 \ \text{and} \ \text{div} \ (\Delta w) = 0 \ \text{in} \ \Omega\}
\]
(see proposition 3.4 below).

**Corollary 3.3.** Assume that the boundary of the domain \( \Omega \) is of class \( C^{m+1,1} \), for some integer \( m \geq 0 \). For any vector field \( u \in H^m(\Omega)^3 \) that satisfies (18), there exists a vector potential \( \psi \) in \((H^{m+1}(\Omega) \cap H^1_0(\Omega))^3\) satisfying
\[
u = \text{curl} \ \psi \quad \text{and} \quad \text{div} \ \Delta \psi = 0 \quad \text{in} \ \Omega \quad \text{and} \quad \|\psi\|_{H^{m+1}(\Omega)^3} \leq C\|u\|_{H^m(\Omega)^3}.
\]

**Proof.** Under the given assumptions, the vector potential \( \psi \) given by the previous theorem belongs to \( H^{m+1}(\Omega)^3 \) and its normal trace \( \psi \cdot n \) belongs to \( H^{m+1/2}(\Gamma) \), on the one hand. On the other hand, the solution \( \chi \) to the fourth-order problem found in the previous belongs to \( H^{m+2}(\Omega)^3 \). 

We now characterize the space \( K^1_0(\Omega) \).

**Proposition 3.4.** Assume that the boundary of the domain \( \Omega \) is of class \( C^{1,1} \). Then the space \( K^1_0(\Omega) \) is spanned by the vector fields \( \text{grad} \ q^1_i, 1 \leq i \leq I \), where
each \( q_i^1 \) is the unique solution in \( H^2(\Omega) \) to the problem

\[
\begin{align*}
\Delta^2 q_i^1 &= 0 \quad \text{in } \Omega, \\
q_i^1 |_{\Gamma_0} &= 0 \quad \text{and } \quad q_i^1 |_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I, \\
\partial_n q_i^1 &= 0 \quad \text{on } \Gamma, \\
\langle \partial_n \Delta q_i^1, 1 \rangle_{\Gamma_k} &= \delta_{ik} \quad \text{and} \quad \langle \partial_n \Delta q_i^1, 1 \rangle_{\Gamma_0} = -1, \quad 1 \leq k \leq I.
\end{align*}
\]  

(23)

**Proof.** First, we prove that the space \( K_0^1(\Omega) \) and the space

\[
G^1 := \{ \text{grad } q \in H^1_0(\Omega)^3; \quad \Delta^2 q = 0 \quad \text{in } \Omega \}
\]

coincide. First, it is clear that \( G^1 \) is included in \( K_0^1(\Omega) \). Second, given \( w \in K_0^1(\Omega) \), let \( \tilde{w} \) denote the extension by zero of \( w \) to an open ball \( B \) containing \( \Omega \). Since \( \text{curl } \tilde{w} = 0 \) in \( B \), \( \tilde{w} \) is the gradient of a function \( q \in H^2(B) \). Moreover, \( q = 0 \) in \( B \setminus \overline{\Omega} \), so that \( q' := q|\Omega \) belongs to \( H^2_0(\Omega) \). Since \( w = \text{grad } q' \), one finds that \( w \) belongs to \( G^1 \).

Moreover, it is clear that the set of vector fields \( \text{grad } q_i, 1 \leq i \leq I \), where \( q_i \in H^2(\Omega) \) is the unique solution to

\[
\begin{align*}
\Delta^2 q_i &= 0 \quad \text{in } \Omega, \\
q_i |_{\Gamma_0} &= 0 \quad \text{and } \quad q_i |_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I, \\
\partial_n q_i &= 0 \quad \text{on } \Gamma,
\end{align*}
\]  

(24)

spans \( G^1 (= K_0^1(\Omega)) \).

One still has to check the last line of (23). Introduce now

\[
M_2 := \{ r \in H^2(\Omega); \quad r |_{\Gamma_0} = 0 \quad \text{and } r |_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I, \quad \partial_n r = 0 \quad \text{on } \Gamma \}.
\]

For \( 1 \leq i \leq I \), the problem: find \( q_i^1 \) in \( M_2 \) such that

\[
\forall r \in M_2, \quad \int_{\Omega} \Delta q_i^1 \Delta r \, dx = -r |_{\Gamma_i}^1,
\]  

(25)

has a unique solution. Furthermore, the following Green’s formula can be proven by a density argument, for any functions \( q \) and \( r \) in \( M_2 \) with \( \Delta^2 q \) in \( L^2(\Omega) \):

\[
\int_{\Omega} (\Delta^2 q) r \, dx = \int_{\Omega} \Delta q \Delta r \, dx + \sum_{i=1}^{I} r |_{\Gamma_i} \langle \partial_n (\Delta q), 1 \rangle_{\Gamma_i}.
\]

This formula implies that the solution \( q_i^1 \) to (25) satisfies (23). The vector fields \( \text{grad } q_i^1, 1 \leq i \leq I \), are clearly linearly independent and they belong to \( K_0^1(\Omega) \). Consequently, they form a basis of \( K_0^1(\Omega) \). \( \square \)
Proposition 3.5. Assume that the boundary of the domain $\Omega$ is of class $C^{1,1}$. Given any function $u$ in $H(\text{div}, \Omega)$ satisfying (18), there exists a unique vector potential $\psi$ in $H^1_0(\Omega)^3$ satisfying

$$u = \text{curl } \psi, \quad \text{div } \Delta \psi = 0 \text{ in } \Omega \text{ and } \langle \partial_n(\text{div } \Delta \psi), 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I.$$  

Moreover, the estimate (16) holds.

Proof. Let $(\psi_0 - \text{grad } \chi)$ be the potential vector of $u$ given in the proof of theorem 3.2. Then the vector field

$$\psi = \psi_0 - \text{grad } \chi + \sum_{i=1}^I \langle \partial_n(\Delta \chi), 1 \rangle_{\Gamma_i} \text{grad } q_i^1$$

satisfies (26) (note that the quantities $\langle \partial_n(\Delta \chi), 1 \rangle_{\Gamma_i}$ are well defined since $\Delta^2 \chi = 0$). \hfill $\Box$

Corollary 3.6. Assume that the boundary of the domain $\Omega$ is of class $C^{m+1,1}$ for some integer $m \geq 0$. Given any function $u$ in $H^m(\Omega)^3$ that satisfies (18), there exists a unique vector potential $\psi$ in $(H^{m+1} \cap H^1_0(\Omega))^3$ satisfying

$$u = \text{curl } \psi, \quad \text{div } \Delta \psi = 0 \text{ in } \Omega \quad \text{and} \quad \langle \partial_n(\text{div } \Delta \psi), 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I$$

and the estimate (17).

Theorem 3.7. Assume that the boundary of the domain $\Omega$ is of class $C^{2,1}$. Given any function $u$ in $H^1_0(\Omega)^3$ that satisfies

$$\text{div } u = 0 \text{ in } \Omega \quad \text{and} \quad \langle u \cdot n, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J,$$  

there exists a vector potential $\psi$ in $H^2_0(\Omega)^3$ such that

$$u = \text{curl } \psi \quad \text{and} \quad \text{div } \Delta^2 \psi = 0 \text{ in } \Omega \quad \text{and} \quad \|\psi\|_{H^2(\Omega)^3} \leq C\|u\|_{H^1(\Omega)^3}. $$  

(28)

Proof. Given $u$ in $H^1_0(\Omega)^3$ that satisfies (27), let $\varphi \in (H^2(\Omega) \cap H^1_0(\Omega))^3$ denote the vector potential given by corollary 3.6. The sixth-order problem

$$\Delta^3 \chi = 0 \text{ in } \Omega, \quad \chi = \frac{\partial \chi}{\partial n} = 0 \quad \text{and} \quad \frac{\partial^2 \chi}{\partial n^2} = \frac{\partial \varphi}{\partial n} \cdot n \quad \text{on } \Gamma,$$  

(29)

has a unique solution $\chi \in H^3(\Omega)$ that satisfies the estimate

$$\|\chi\|_{H^3(\Omega)} \leq C\|\frac{\partial \varphi}{\partial n}\|_{H^{1/2}(\Gamma)^3} \leq C\|\varphi\|_{H^2(\Omega)^3} \leq C\|u\|_{H^1(\Omega)^3}.$$

10
Note that the last boundary condition in (29) can be written as
\[
\left( \frac{\partial}{\partial n} \text{grad } \chi \right) \cdot n = \frac{\partial \varphi}{\partial n} \cdot n.
\]

For any unit tangent vector \( \tau \) on \( \Gamma \), we have:
\[
\frac{\partial \varphi}{\partial n} \cdot \tau = \frac{\partial \varphi}{\partial x_j} n_j \tau_i = \frac{\partial \varphi_j}{\partial x_i} \tau_i n_j = \frac{\partial \varphi_j}{\partial \tau} n_j = 0.
\]

Also, one can show that \( (\partial_n \text{grad } \chi) \cdot \tau = 0 \), which implies that the relation \( \partial_n \text{grad } \chi = \partial_n \varphi \) holds. So, the vector field \( \psi = \varphi - \text{grad } \chi \) belongs to \( H^2(\Omega)^3 \) and satisfies (28).

The vector field \( \psi \) given by Theorem 3.7 is unique up to vector fields in the space
\[
K_0^2(\Omega) := \{ w \in H_0^2(\Omega)^3; \text{curl } w = 0 \text{ and } \text{div } \Delta^2 w = 0 \text{ in } \Omega \},
\]
which we now characterize.

**Proposition 3.8.** Assume that the boundary of the domain \( \Omega \) is of class \( C^{2,1} \).
Then the space \( K_0^2(\Omega) \) is spanned by the vector fields \( \text{grad } q_i^2 \), \( 1 \leq i \leq I \), where each function \( q_i^2 \) is the unique solution in \( H^3(\Omega) \) to the problem
\[
\begin{align*}
\Delta^3 q_i^2 &= 0 & \text{in } \Omega, \\
q_i^2 \big|_{\Gamma_0} &= 0 & \text{and } q_i^2 \big|_{\Gamma_k} = \delta_{ik}, & 1 \leq k \leq I, \\
\partial_n q_i^2 &= \partial^2_{n} q_i^2 = 0 & \text{on } \Gamma, \\
\langle \partial_n (\Delta^2 q_i^2), 1 \rangle_{\Gamma_k} &= \delta_{ik} \text{ and } \langle \partial_n (\Delta^2 q_i^2), 1 \rangle_{\Gamma_0} = -1, 1 \leq k \leq I.
\end{align*}
\]

**Proof.** First, we prove that the space \( K_0^2(\Omega) \) coincides with the space
\[
G^2 := \{ \text{grad } q \in H_0^2(\Omega)^3; \Delta^3 q = 0 \text{ in } \Omega \},
\]
using the same argument as in proposition 3.4. We next note that the set of vector fields \( \text{grad } q_i \), \( 1 \leq i \leq I \), where \( q_i \in H^3(\Omega) \) is the unique solution to the problem
\[
\begin{align*}
\Delta^3 q_i &= 0 & \text{in } \Omega, \\
q_i \big|_{\Gamma_0} &= 0 & \text{and } q_i \big|_{\Gamma_k} = \delta_{ik}, & 1 \leq k \leq I, \\
\partial_n q_i &= \partial^2_{n} q_i = 0 & \text{on } \Gamma,
\end{align*}
\]
spans \( K_0^2(\Omega) \).
Let now
\[ M_3 := \{ r \in H^3(\Omega); r \big|_{\Gamma_0} = 0, \quad r \big|_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I, \quad \partial_n r = \partial^2_{nn} r = 0 \text{ on } \Gamma \}. \]

For \( 1 \leq i \leq I \), the problem: find \( q_i^2 \) in \( M_3 \) such that
\[ \forall r \in M_3, \quad \int_\Omega \text{grad} \Delta q_i^2 \cdot \text{grad} \Delta r \, dx = r \big|_{\Gamma_i}, \quad (32) \]
has a unique solution. Furthermore, the following Green’s formula can be proved by a density argument, for any functions \( q \) and \( r \) in \( M_3 \) with \( \Delta^3 q \) in \( L^2(\Omega) \):
\[ \int_\Omega (\Delta^3 q) r \, dx = -\int_\Omega \text{grad} q \cdot \text{grad} r \, dx + \sum_{i=1}^{I} r \big|_{\Gamma_i} \langle \partial_n(\Delta^2 q), \rangle_{\Gamma_i}. \]
This formula shows that the solution \( q_i^2 \) of (32) satisfies (30). The vector fields \( \text{grad} q_i^2, \quad 1 \leq i \leq I \), are clearly linearly independent and they belong to \( K_0^2(\Omega) \). Consequently, they form a basis of \( K_0^2(\Omega) \).

**Corollary 3.9.** Assume that the boundary of the domain \( \Omega \) is of class \( C^{2,1} \). Given any function \( u \) in \( H^1_0(\Omega)^3 \) such that (27) holds, there exists a unique vector potential \( \psi \) in \( H^2_0(\Omega)^3 \) satisfying
\[ u = \text{curl} \ \psi, \quad \text{div} \Delta^2 \psi = 0 \text{ in } \Omega \quad \text{and} \langle \partial_n(\text{div} \Delta \psi), 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I, \]
with the corresponding estimate.

More generally, we can prove using the same arguments:

**Theorem 3.10.** Assume that boundary of the domain \( \Omega \) is of class \( C^{m+1,1} \) for some integer \( m \geq 1 \). Given any vector field \( u \) in \( H^m_0(\Omega)^3 \) that satisfies (27), there exists a vector potential \( \psi \) in \( H^{m+1}_0(\Omega)^3 \) such that
\[ u = \text{curl} \ \psi \quad \text{and} \quad \text{div} \Delta^{m+1} \psi = 0 \text{ in } \Omega \quad \text{and} \quad \| \psi \|_{H^{m+1}(\Omega)^3} \leq C \| u \|_{H^m(\Omega)^3}. \quad (33) \]
Moreover, there exists a unique vector potential \( \psi \) in \( H^{m+1}_0(\Omega)^3 \), satisfying (33) and
\[ \langle \partial_n(\text{div} \Delta \psi^{m+1}), 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I. \quad (34) \]

**Remark 3.11.** Similar results are found in Borchers & Sohr [7], but with different proof.

Let \( \Omega \) be a domain with a boundary of class \( C^{m+1,1} \) for some integer \( m \geq 1 \) and let \( u \) in \( H^m_0(\Omega)^3 \) be such that \( \text{div} \ u = 0 \). If \( \Omega \) is simply-connected \( (J = 0) \), and \( \Gamma \) is connected \( (I = 0) \), then there exists a unique vector potential \( \psi \) in \( H^{m+1}_0(\Omega)^3 \) satisfying (33).
4 Weak vector potentials

First, we note that the continuous embeddings $H_0(\text{curl}, \Omega)' \hookrightarrow H^{-1}(\Omega)^3$ and $H_0(\text{div}, \Omega)' \hookrightarrow H^{-1}(\Omega)^3$ hold. Besides, given any $f \in H^{-1}(\Omega)^3$, we know that there exist a unique $u \in H_0^1(\Omega)^3$ and $\chi \in L^2(\Omega)$ such that

\[ f = -\Delta u + \nabla \chi \quad \text{and} \quad \text{div} \ u = 0 \quad \text{in} \ \Omega, \quad (35) \]

and satisfying the estimate

\[ \|u\|_{H^1(\Omega)^3} + \|\chi\|_{L^2(\Omega)^3/R} \leq C\|f\|_{H^{-1}(\Omega)^3}. \]

Letting $\xi = \text{curl} \ u$, we obtain the decomposition $f = \text{curl} \ \xi + \text{grad} \ \chi$ with $\text{div} \ \xi = 0$ in $\Omega$ and $\xi \cdot n = 0$ on $\Gamma$. Since $\xi \in L^2(\Omega)^3$ and $\chi \in L^2(\Omega)$, it follows that $\text{curl} \ \xi \in H_0(\text{curl}, \Omega)'$ and $\text{grad} \ \chi \in H_0(\text{div}, \Omega)'$, so that $H^{-1}(\Omega)^3 = H_0(\text{curl}, \Omega)' + H_0(\text{div}, \Omega)'$. \hspace{1cm} (36)

**Proposition 4.1.** Assume that the boundary of the domain $\Omega$ is of class $C^{1.1}$. Then, for any $f$ in the dual space $H_0(\text{div}, \Omega)'$, there exist a unique $u \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ and $\chi \in L^2(\Omega)$ solution to (35) and satisfying the estimate

\[ \|u\|_{H^2(\Omega)^3} + \|\chi\|_{L^2(\Omega)^3} \leq C\|f\|_{H_0(\text{div}, \Omega)'} \]

Proof. Let $f$ be in the dual space of $H_0(\text{div}, \Omega)$. We know (see proposition 1 of [6]) that there exist $\psi \in L^2(\Omega)^3$ and $\chi_0 \in L^2(\Omega)$ such that

\[ f = \psi + \text{grad} \ \chi_0 \quad \text{and} \quad \|\psi\|_{L^2(\Omega)^3} + \|\chi_0\|_{L^2(\Omega)} \leq C\|f\|_{H_0(\text{div}, \Omega)'} \hspace{1cm} (37) \]

Thanks to the regularity of $\Gamma$, there exist $u \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ and $p \in H^1(\Omega)$ satisfying

\[ \psi = -\Delta u + \text{grad} \ p \quad \text{and} \quad \text{div} \ u = 0 \quad \text{in} \ \Omega, \quad (38) \]

with

\[ \|u\|_{H^2(\Omega)^3} + \|p\|_{H^1(\Omega)^3/R} \leq C\|\psi\|_{L^2(\Omega)^3}. \]

Then $u$ and $\chi = p + \chi_0$ satisfy the announced properties. \hspace{1cm} \square

We next consider the space

\[ K_N(\Omega) := \{ w \in H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega); \ \text{curl} \ w = 0 \text{ and div} \ w = 0 \text{ in} \ \Omega \} \]

which is of dimension $I$. As shown in proposition 3.18 of [1], this space is spanned by the vector fields $\text{grad} \ q_i^N$, $1 \leq i \leq N$, where each function $q_i^N \in$
$H^1(\Omega)$ is the unique solution to the problem

$$\begin{align*}
\Delta q_i^N &= 0 \quad \text{in } \Omega, \\
q_i^N &= 0 \quad \text{on } \Gamma_0, \quad \langle \partial_n q_i^N, 1 \rangle_{\Gamma_0} = -1, \\
q_i^N &= \text{constant} \quad \text{on } \Gamma_k, \quad \langle \partial_n q_i^N, 1 \rangle_{\Gamma_k} = \delta_{ik}, \text{ for } 1 \leq k \leq I.
\end{align*}$$

(39)

**Theorem 4.2.** Given any distribution $f$ in the dual space $H_0(\text{curl}, \Omega)'$ that satisfies

$$\begin{align*}
\text{div } f &= 0 \quad \text{in } \Omega \text{ and } H_0(\text{curl}, \Omega)'(f, \ v)_{H_0(\text{curl}, \Omega)} = 0 \quad \text{for all } \ v \in K_N(\Omega),
\end{align*}$$

(40)

there exists a vector potential $\xi$ in $L^2(\Omega)^3$ such that

$$f = \text{curl } \xi, \quad \text{with } \text{div } \xi = 0 \quad \text{in } \Omega \quad \text{and } \xi \cdot n = 0 \quad \text{on } \Gamma,$$

(41)

and such that the following estimate holds:

$$\|\xi\|_{L^2(\Omega)^3} \leq C \|f\|_{H_0(\text{curl}, \Omega)'}.$$  

(42)

**Proof.** Let $f$ be in the dual space $H_0(\text{curl}, \Omega)'$. According to corollary 5 of [6], there exist $\psi \in L^2(\Omega)^3$ and $\xi_0 \in L^2(\Omega)^3$ with $\text{div } \xi_0 = 0$ in $\Omega$ and $\xi_0 \cdot n = 0$ on $\Gamma$, such that $f = \psi + \text{curl } \xi_0$ and such that the estimate

$$\|\psi\|_{L^2(\Omega)^3} + \|\xi_0\|_{L^2(\Omega)^3} \leq C \|f\|_{H_0(\text{curl}, \Omega)'}$$

holds. Thanks to the density of $\mathcal{D}(\Omega)^3$ in $H_0(\text{curl}, \Omega)$, we deduce that for all $\ v \in K_N(\Omega)$, we have

$$H_0(\text{curl}, \Omega)'(\text{curl } \xi_0, \ v)_{H_0(\text{curl}, \Omega)} = 0.$$  

Since $\text{div } f = 0$, it follows that $\text{div } \psi = 0$. Then, thanks to the orthogonality relations

$$H_0(\text{curl}, \Omega)'(f, \ \text{grad } q_i^N)_{H_0(\text{curl}, \Omega)} = 0 \quad \text{for all } i = 1, \ldots, I,$$

the relations $\langle \psi \cdot n, 1 \rangle_{\Gamma_i} = 0$ are satisfied for all $i = 1, \ldots, I$. There thus exists a vector potential $\varphi \in L^2(\Omega)^3$ (see theorem 3.12 of [1]) such that $\psi = \text{curl } \varphi$, with $\text{div } \varphi = 0$ in $\Omega$ and $\varphi \cdot n = 0$ on $\Gamma$, and such that

$$\|\varphi\|_{L^2(\Omega)^3} \leq C \|\psi\|_{L^2(\Omega)^3}.$$  

Hence, the vector field $\xi = \xi_0 + \varphi$ possesses the announced properties. \hfill \Box

**Remark 4.3.** The previous theorem has been established in [6] when $\Gamma$ is connected, in which case $K_N = \{0\}$.  

14
For any integer \( m \geq 0 \), let us introduce the space
\[
H^m_0(\text{curl}, \Omega) := \{ \mathbf{v} \in H_0(\text{curl}, \Omega); \text{ curl } \mathbf{v} \in H^m_0(\Omega)^3 \}.
\]

We can easily characterize its dual space, as:
\[
H^{-m}(\text{curl}, \Omega) = \{ \psi + \text{curl } \xi; \ \psi \in H_0(\text{curl}, \Omega)', \ \xi \in H^{-m}(\Omega)^3 \}.
\]

We can prove that \( \mathcal{D}(\Omega)^3 \) is dense in \( H^m_0(\text{curl}, \Omega) \) and that the following Green formula holds: for any \( \xi \in H^{-m}(\text{curl}, \Omega) \) and \( \mathbf{v} \in H^m_0(\text{curl}, \Omega) \)
\[
\langle \text{curl} \xi, \mathbf{v} \rangle_{H^m_0(\text{curl}, \Omega)} + H^{-m}(\Omega)^3 \langle \xi, \text{curl } \mathbf{v} \rangle_{H^m_0(\Omega)^3} = 0. \tag{43}
\]

By using the decomposition (1) with \( (m + 1) \) instead of \( m \), it is easy to prove (as in Section 2) that
\[
H^{-m-1}(\Omega)^3 = H^{-m}(\text{curl}, \Omega) + H^{-m}(\text{div }, \Omega), \quad \text{for } m \geq 1.
\]

**Theorem 4.4.** For any distribution \( \mathbf{f} \) in the dual space \( H^{-1}(\text{curl}, \Omega) \) that satisfies
\[
\text{div } \mathbf{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \mathbf{f}, \mathbf{v} \rangle = 0, \quad \text{for all } \mathbf{v} \in K_N(\Omega) \tag{44}
\]
there exists a vector potential \( \xi \) in \( H^{-1}(\Omega)^3 \) such that
\[
\mathbf{f} = \text{curl } \xi, \quad \text{div } \xi = 0 \quad \text{in } \Omega, \quad \text{and} \quad \| \xi \|_{H^{-1}(\Omega)^3} \leq C \| \mathbf{f} \|_{H^{-1}(\text{curl}, \Omega)}. \tag{45}
\]

**Proof.** Given \( \mathbf{f} \) in the dual space \( H^{-1}(\text{curl}, \Omega) \), there exist \( \mathbf{f}_0 \in H_0(\text{curl}, \Omega)' \) and \( \xi_0 \in H^{-1}(\Omega)^3 \) such that \( \mathbf{f} = \mathbf{f}_0 + \text{curl } \xi_0 \), and satisfying the estimate
\[
\| \mathbf{f}_0 \|_{H_0(\text{curl}, \Omega)'} + \| \xi_0 \|_{H^{-1}(\Omega)^3} \leq C \| \mathbf{f} \|_{H^{-1}(\text{curl}, \Omega)}.
\]

Since \( \xi_0 \in H^{-1}(\Omega)^3 \), there exists \( \mathbf{\theta}_0 \in L^2(\Omega)^3 \) satisfying \( \text{div } \mathbf{\theta}_0 = 0 \) in \( \Omega \), \( \mathbf{\theta}_0 \cdot \mathbf{n} = 0 \) on \( \Gamma \), and there exists \( \chi \in L^2(\Omega) \) such that \( \xi_0 = \text{curl } \mathbf{\theta}_0 + \text{grad } \chi \) and
\[
\| \mathbf{\theta}_0 \|_{L^2(\Omega)^3} + \| \chi \|_{L^2(\Omega)/R} \leq C \| \xi_0 \|_{H^{-1}(\Omega)^3}.
\]

Since \( \mathbf{f}_0 \in H_0(\text{curl}, \Omega)' \), then \( \mathbf{f}_0 = \mathbf{\psi}_0 + \text{curl } \varphi_0 \), with \( \mathbf{\psi}_0 \in L^2(\Omega)^3 \), \( \varphi_0 \in L^2(\Omega)^3 \), \( \text{div } \varphi_0 = 0 \) in \( \Omega \), \( \varphi_0 \cdot \mathbf{n} = 0 \) on \( \Gamma \) and
\[
\| \mathbf{\psi}_0 \|_{L^2(\Omega)^3} + \| \varphi_0 \|_{L^2(\Omega)^3} \leq C \| \mathbf{f}_0 \|_{H_0(\text{curl}, \Omega)'}.
\]

Then \( \mathbf{f} = \mathbf{\psi}_0 + \text{curl } \varphi_0 + \text{curl } \text{curl } \mathbf{\theta}_0 = \mathbf{\psi}_0 + \text{curl } \mathbf{\mu} \), with \( \mathbf{\mu} = \varphi_0 + \text{curl } \mathbf{\theta}_0 \), \( \text{div } \mathbf{\mu} = 0 \) in \( \Omega \), and the estimate
\[
\| \mathbf{\psi}_0 \|_{L^2(\Omega)^3} + \| \mathbf{\mu} \|_{H^{-1}(\Omega)^3} \leq C \| \mathbf{f} \|_{H^{-1}(\text{curl}, \Omega)}
\]
holds.

Thanks to the density of $\mathcal{D}(\Omega)^3$ in $H_0^1(\text{curl}, \Omega)$, we infer that
\[ H^{-1}(\text{curl}, \Omega) \langle \text{curl} \ \mu, \ v \rangle_{H_0^1(\text{curl}, \Omega)} = 0, \quad \text{for all } v \in K_N(\Omega). \]

Since $\text{div} \ f = 0$, $\text{div} \ \psi_0 = 0$ and therefore the condition $\langle \psi_0 \cdot n, 1 \rangle_{\Gamma_i} = 0$ is automatically satisfied for any $i = 0, \ldots, I$. Then by (12), there exists a vector potential $\varphi \in L^2(\Omega)^3$ such that
\[ \psi_0 = \text{curl} \ \varphi, \quad \text{div} \ \varphi = 0 \quad \text{in } \Omega \quad \text{and} \quad \varphi \cdot n = 0 \quad \text{on } \Gamma, \]
and
\[ ||\varphi||_{L^2(\Omega)^3} \leq C ||\psi_0||_{L^2(\Omega)^3}. \]

Hence, the vector field $\xi = \mu + \varphi$ satisfies the announced properties. \hfill \Box

More generally, we can prove:

**Theorem 4.5.** Given any integer $m \geq 0$ and any distribution $f$ in the dual space $H^{-m}(\text{curl}, \Omega)$ that satisfies (44), there exists a vector potential $\xi$ in $H^{-m}(\Omega)^3$ such that
\[ f = \text{curl} \ \xi, \quad \text{with} \quad \text{div} \ \xi = 0 \quad \text{in } \Omega, \quad \text{and} \quad ||\xi||_{H^{-m}(\Omega)^3} \leq C ||f||_{H^{-m}(\text{curl}, \Omega)}. \]

5 Weak versions of Korn’s inequality

Finally, we consider tensor fields. The next theorem generalizes theorem 3.2 of [8] and theorem 7 of [3] to Sobolev spaces with negative exponents.

In what follows, the subscript $^s$ denotes a space of symmetrix matrix fields.

**Theorem 5.1.** Assume that $\Omega$ is simply-connected. Given an integer $m \geq 0$, let $e = (e_{ij}) \in H^{-m}(\Omega)^{3 \times 3}$ be a symmetric matrix field that satisfies the following compatibility conditions for all $i, j, k, l \in \{1, 2, 3\}$:
\[ R_{ijkl} := \frac{\partial^2 e_{ik}}{\partial x_j \partial x_l} + \frac{\partial^2 e_{jl}}{\partial x_k \partial x_i} - \frac{\partial^2 e_{jk}}{\partial x_l \partial x_i} - \frac{\partial^2 e_{il}}{\partial x_k \partial x_j} = 0 \quad \text{in } H^{-m-2}(\Omega). \quad (46) \]

Then there exists a vector field $v \in H^{-m+1}(\Omega)^3$ such that $e_{ij} = \frac{1}{2} (\partial_i v_j + \partial_j v_i)$ and $v$ is unique up to vector fields in the space $R(\Omega) = \{a + b \wedge \text{id}_\Omega; \ a, b \in \mathbb{R}^3\}$. 


Proof. Given $\mathbf{e} = (e_{ij}) \in H^{-m}_s(\Omega)^{3 \times 3}$, let $f_{ijk} := \partial_j e_{ik} - \partial_i e_{jk}$. Then $f_{ijk} \in H^{-m-1}(\Omega)$ and, thanks to the compatibility conditions (46), we have

$$\frac{\partial}{\partial x_l} f_{ijk} = \frac{\partial}{\partial x_k} f_{ijl}.$$ 

Hence the implication (iii) $\implies$ (iv) in theorem 1.1 shows that there exist distributions $p_{ij} \in H^{-m}(\Omega)$, unique up to additive constants, such that $\partial_k p_{ij} = f_{ijk}$.

Besides, since $\partial_k p_{ij} = -\partial_k p_{ji}$, we can choose the distributions $p_{ij}$ in such a way that $p_{ij} + p_{ji} = 0$. Noting that the distributions $q_{ij} := e_{ij} + p_{ij}$ belong to $H^{-m}(\Omega)$ and satisfy $\partial_k q_{ij} = \partial_j q_{ik}$, we again resort to theorem 1.1 to assert the existence of distributions $v_i \in H^{-m+1}(\Omega)$, unique up to additive constants, such that $\partial_j v_i = q_{ij}$. □

For any integer $m \geq 0$, let

$$E(\Omega) := \{ \mathbf{e} \in H^{-m}_s(\Omega)^{3 \times 3}, \mathcal{R}_{ijkl}(\mathbf{e}) = 0 \}$$

and

$$\dot{H}^{-m+1}(\Omega)^3 := H^{-m+1}(\Omega)^3 / R(\Omega).$$

By the previous theorem, given any $\mathbf{e} = (e_{ij}) \in E(\Omega)$, there exists a unique $\mathbf{v} = (v_i) \in \dot{H}^{-m+1}(\Omega)^3$ such that $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$. We may thus define a linear mapping $\mathcal{F} : E(\Omega) \to \dot{H}^{-m+1}(\Omega)^3$ by $\mathcal{F}(\mathbf{e}) = \mathbf{v}$. Using the same method as in [8], we can then prove the following Korn’s inequality in Sobolev spaces with negative exponents:

**Theorem 5.2.** The linear mapping $\mathcal{F} : E(\Omega) \to \dot{H}^{-m+1}(\Omega)^3$ is an isomorphism. Besides, there exists a constant $C \geq 0$ such that

$$\inf_{r \in R(\Omega)} \| \mathbf{v} + r \|_{H^{-m+1}(\Omega)^3} \leq C \sum_{i,j} \| \varepsilon_{ij}(\mathbf{v}) \|_{H^{-m}(\Omega)} \quad \text{for all } \mathbf{v} \in H^{-m+1}(\Omega)^3,$$

and

$$\| \mathbf{v} \|_{H^{-m+1}(\Omega)^3} \leq C (\| \mathbf{v} \|_{H^{-m}(\Omega)} + \sum_{i,j} \| \varepsilon_{ij}(\mathbf{v}) \|_{H^{-m}(\Omega)}) \quad \text{for all } \mathbf{v} \in H^{-m+1}(\Omega)^3$$

where $\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$.

**Remark 5.3.** Analogous techniques would likewise extend to Sobolev spaces with negative exponents the results obtained for non-simply connected domains in [9], [12] and [13].

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References


18