Weak vector and scalar potentials. Applications to Poincaré's theorem and Korn's inequality in Sobolev spaces with negative exponents.

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Abstract

In this paper, we present several results concerning vector potentials and scalar potentials with data in Sobolev spaces with negative exponents, in a not necessarily simply-connected, three-dimensional domain. We then apply these results to Poincaré's theorem and to Korn's inequality.

1 Weak versions of a classical theorem of Poincaré

In this work, (the results of which were announced in [2]), Ω is a bounded open connected subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary Γ. The notation ${}_{X'}\langle , \rangle_X$ denotes the duality pairing between a topological space X and its dual X'. The letter C denotes a constant that is not necessarily the same at its various occurrences.

We begin with a weak version of a well-known theorem of Poincaré. Here as elsewhere in this paper, "weak" means that the result to which it is attached holds as well in Sobolev spaces with negative exponents.

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Theorem 1.1. *Let $\boldsymbol{f} \in H^{-m}(\Omega)^3$ for some integer $m \geq 0$. Then the following properties are equivalent:*

(i) ${}_{H^{-m}(\Omega)^3}\langle \boldsymbol{f},\ \boldsymbol{\varphi} \rangle_{H_0^m(\Omega)^3} = 0$ for all $\boldsymbol{\varphi} \in V_m = \{\boldsymbol{\varphi} \in H_0^m(\Omega)^3;\ \mathrm{div}\ \boldsymbol{\varphi} = 0\}$,

(ii) ${}_{H^{-m}(\Omega)^3}\langle \boldsymbol{f},\ \boldsymbol{\varphi} \rangle_{H_0^m(\Omega)^3} = 0$ for all $\boldsymbol{\varphi} \in \mathcal{V} = \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^3;\ \mathrm{div}\ \boldsymbol{\varphi} = 0\}$

To begin with, let $\boldsymbol{f} \in H^{-m}(\Omega)^3$ be such that $\mathbf{curl}\ \boldsymbol{f} = \mathbf{0}$ in Ω. We then use the same argument as in [8]: We know that there exist a unique $\boldsymbol{u} \in H_0^m(\Omega)^3$ and a unique $p \in H^{-m+1}(\Omega)/\mathbb{R}$ (see [5]) such that

$$\Delta^m \boldsymbol{u} + \mathbf{grad}\ p = \boldsymbol{f} \quad \text{and} \quad \text{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega. \tag{1}$$

Hence $\Delta^m \mathbf{curl}\ \boldsymbol{u} = \mathbf{0}$ in Ω so that the hypoellipticity (see [10]) of the polyharmonic operator Δ^m implies that $\mathbf{curl}\ \boldsymbol{u} \in \mathcal{C}^\infty(\Omega)^3$. Since div $\boldsymbol{u} = 0$, we deduce that $\Delta \boldsymbol{u} = \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{u} \in \mathcal{C}^\infty(\Omega)^3$. This also implies that $\Delta^m \boldsymbol{u}$ belongs to $\mathcal{C}^\infty(\Omega)^3$ and is an irrotational vector field. By the classical Poincaré theorem, there exists $q \in \mathcal{C}^\infty(\Omega)^3$ such that $\Delta^m \boldsymbol{u} = \mathbf{grad}\ q$. Thus, $\boldsymbol{f} = \mathbf{grad}\ (p+q)$ and, thanks to [4] proposition 2.10, the function $p+q$ belongs to the space $H^{-m+1}(\Omega)$. □

We can give another proof of the implication (iv) $\Longrightarrow$ (iii) by using the following theorem:

Theorem 1.2. *Assume that both Ω and $\mathbb{R}^3 \setminus \Omega$ are simply-connected. Let $\boldsymbol{u} \in H_0^m(\Omega)^3$, $m \geq 0$, be a vector field that satisfies* div $\boldsymbol{u} = 0$ *in Ω. Then there exists a vector potential $\boldsymbol{\psi}$ in $H_0^{m+1}(\Omega)^3$ such that*

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}, \quad \text{div}\ \Delta^{m+1}\boldsymbol{\psi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C\|\boldsymbol{u}\|_{H^m(\Omega)^3}. \tag{2}$$

Proof. Let $\boldsymbol{u} \in H_0^m(\Omega)^3$ be such that div $\boldsymbol{u} = 0$ in Ω and let $\widetilde{\boldsymbol{u}}$ denote the extension of $\boldsymbol{u}$ by $\mathbf{0}$ in $\mathbb{R}^3 \setminus \Omega$. Thus $\widetilde{\boldsymbol{u}} \in H_0^m(\mathbb{R}^3)^3$, div $\widetilde{\boldsymbol{u}} = 0$ in $\mathbb{R}^3$, and there exist an open ball B containing $\overline{\Omega}$ and a vector field $\boldsymbol{w} \in H_0^{m+1}(B)^3$ such that $\widetilde{\boldsymbol{u}} = \mathbf{curl}\ \boldsymbol{w}$, div $\Delta^{m+1}\boldsymbol{w} = 0$ in B, and

$$\|\boldsymbol{w}\|_{H^{m+1}(B)^3} \leq C\|\boldsymbol{u}\|_{H^m(B)^3}.$$

The open set $\Omega' := B \setminus \overline{\Omega}$ is bounded, has a Lipschitz-continuous boundary and is simply-connected. Furthermore, the vector field $\boldsymbol{w}' := \boldsymbol{w}|_{\Omega'}$ belongs to $H^{m+1}(\Omega')^3$ and satisfies $\mathbf{curl}\ \boldsymbol{w}' = \mathbf{0}$ in Ω'. Therefore there exists a function $\chi' \in H^1(\Omega')$ such that $\boldsymbol{w}' = \mathbf{grad}\ \chi'$ in Ω'. Hence in fact $\chi' \in H^{m+2}(\Omega')$ and the estimate

$$\|\chi'\|_{H^{m+2}(\Omega')} \leq C\|\boldsymbol{w}'\|_{H^{m+1}(\Omega')^3}$$

holds. Since the function $\chi' \in H^{m+2}(\Omega')$ can be extended to a function $\widetilde{\chi}$ in $H^{m+2}(\mathbb{R}^3)$, with

$$\|\widetilde{\chi}\|_{H^{m+2}(\mathbb{R}^3)} \leq C\|\chi'\|_{H^{m+2}(\Omega')} \leq C\|\boldsymbol{w}'\|_{H^{m+1}(\Omega')^3},$$

the vector field $\widetilde{\boldsymbol{\varphi}} := \boldsymbol{w} - \mathbf{grad}\ \widetilde{\chi}$ belongs to the space $H^{m+1}(B)^3$ and satisfies $\widetilde{\boldsymbol{\varphi}}|_{\Omega'} = \mathbf{0}$. Then the restriction $\boldsymbol{\varphi} := \widetilde{\boldsymbol{\varphi}}|_{\Omega}$ belongs to the space $H_0^{m+1}(\Omega)^3$, satisfies the estimate (2), and $\mathbf{curl}\ \widetilde{\boldsymbol{\varphi}} = \mathbf{curl}\ \boldsymbol{w} = \widetilde{\boldsymbol{u}}$ in B. Thus $\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\varphi}$

in Ω. Let now p denote the unique solution in the space $H_0^{m+2}(\Omega)$ of $\Delta^{m+2}p = \text{div}\ \Delta^{m+1}\boldsymbol{\varphi}$, so that the estimate

$$\|p\|_{H^{m+2}(\Omega)} \leq C\|\boldsymbol{\varphi}\|_{H^{m+1}(\Omega)^3}$$

holds. Then the function $\boldsymbol{\psi} = \boldsymbol{\varphi} - \mathbf{grad}\ p$ satisfies (2). □

We can give yet another proof of the above implication (iv) $\Longrightarrow$ (iii): Consider again the solution $\boldsymbol{u} \in H_0^m(\Omega)^3$ to (1) and let $\boldsymbol{v} \in H_0^{m+1}(\Omega)^3$ denote the vector potential of $\boldsymbol{u}$ as given by theorem 1.2. We then have $\Delta^m\mathbf{curl}\ \boldsymbol{u} = \mathbf{0}$. If $m = 2k$, for some integer $k \geq 1$, then

$$\begin{aligned} {}_{H^{-m-1}(\Omega)^3}\langle \Delta^m\mathbf{curl}\ \boldsymbol{u},\ \boldsymbol{v}\rangle_{H_0^{m+1}(\Omega)^3} &= {}_{H^{-1}(\Omega)^3}\langle \Delta^k\mathbf{curl}\ \boldsymbol{u},\ \Delta^k\boldsymbol{v}\rangle_{H_0^1(\Omega)^3} \\ &= \int_\Omega \Delta^k\boldsymbol{u} \cdot \Delta^k\mathbf{curl}\ \boldsymbol{v}\ d\boldsymbol{x} \\ &= \|\Delta^k\boldsymbol{u}\|^2_{L^2(\Omega)^3}. \end{aligned}$$

This implies that $\Delta^k\boldsymbol{u} = \mathbf{0}$ in Ω and thus $\boldsymbol{u} = \mathbf{0}$ since $\boldsymbol{u} \in H_0^m(\Omega)^3$. The case $m = 2k + 1$ follows by a similar argument. □

2 Scalar Potentials

Let Γ_i, $0 \leq i \leq I$, denote the connected components of the boundary Γ of the domain Ω, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$. We do not assume that Ω is simply-connected, however we assume that there exist J connected and oriented surfaces Σ_j, $1 \leq j \leq J$ contained in Ω, with the following properties: each surface Σ_j is an open subset of a smooth manifold, the boundary of Σ_j is contained in Γ for $1 \leq j \leq J$, the intersection $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \neq j$, and finally the open set $\Omega^\circ = \Omega \setminus \bigcup_{j=1}^J \Sigma_j$ is simply-connected and pseudo-Lipschitz in the sense of [1]. Each such surface Σ_j is called a cut. Finally, let $[\cdot]_j$ denote the jump of a function over each cut Σ_j, $1 \leq j \leq J$.

We then define the spaces

$$\begin{aligned} &H(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathbf{curl}\ \boldsymbol{v} \in L^2(\Omega)^3\}, \\ &H(\text{div}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \text{div}\ \boldsymbol{v} \in L^2(\Omega)\}, \end{aligned}$$

each one being equipped with the graph norm, and their subspaces

$$\begin{aligned} &H_0(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in H(\mathbf{curl}, \Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma\}, \\ &H_0(\text{div}, \Omega) = \{\boldsymbol{v} \in H(\text{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma\}. \end{aligned}$$

For any function q in $H^1(\Omega^\circ)$, $\mathbf{grad}\ q$ denotes the gradient of q in the sense of distributions in $\mathscr{D}'(\Omega^\circ)$. It belongs to $L^2(\Omega^\circ)^3$ and therefore can be extended to $L^2(\Omega)^3$. In order to distinguish this extension from the gradient of q in $\mathscr{D}'(\Omega)$, we denote it by $\widetilde{\mathbf{grad}}\ q$. Finally, we remark that the space

$$K_T(\Omega) := \{\boldsymbol{w} \in H(\mathbf{curl},\Omega) \cap H_0(\mathrm{div},\Omega);\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ and div } \boldsymbol{w} = 0 \text{ in } \Omega\}$$

is of dimension equal to J: As shown in [1] Prop. 3.14, it is spanned by the vector fields $\widetilde{\mathbf{grad}}\ q_j^T$, $1 \le j \le J$, where each function $q_j^T \in H^1(\Omega^\circ)$, which is unique up to an additive constant, satisfies

$$\begin{aligned} &\Delta q_j^T = 0 && \text{in } \Omega^\circ,\\ &\partial_n q_j^T = 0, && \text{on } \Gamma,\\ &[q_j^T]_k = \text{constant},\ [\partial_n q_j^T]_k = 0,\ \langle \partial_n q_j^T\, ,\, 1\rangle_{\Sigma_k} = \delta_{jk} && \text{for } 1 \le k \le J. \end{aligned} \tag{3}$$

where $\langle\cdot,\cdot\rangle_{\Sigma_k}$ denotes the duality pairing between the spaces $H^{-1/2}(\Sigma_k)$ and $H^{1/2}(\Sigma_k)$.

Theorem 2.1. *Given any function $\boldsymbol{f} \in L^2(\Omega)^3$ that satisfies*

$$\mathbf{curl}\ \boldsymbol{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\ d\boldsymbol{x} = 0 \qquad \text{for all } \boldsymbol{v} \in K_T(\Omega), \tag{4}$$

there exists a scalar potential χ in $H^1(\Omega)$ such that

$$\boldsymbol{f} = \mathbf{grad}\ \chi \quad \text{in } \Omega \quad \text{and} \qquad \|\chi\|_{H^1(\Omega)} \le C\|\boldsymbol{f}\|_{L^2(\Omega)^3}. \tag{5}$$

Proof. It suffices to show that, given any vector field $\boldsymbol{v} \in H_0(\mathrm{div},\Omega)$ such that div $\boldsymbol{v} = 0$ in Ω, there holds $\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\ d\boldsymbol{x} = 0$. Let

$$\boldsymbol{z} = \sum_{j=1}^{J} \langle \boldsymbol{v}\cdot\boldsymbol{n},\ 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\ q_j^T$$

and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{z}$. According to [1], theorem 3.17, there exists a vector potential $\boldsymbol{\psi} \in L^2(\Omega)^3$ that satisfies $\boldsymbol{w} = \mathbf{curl}\ \boldsymbol{\psi}$, div $\boldsymbol{\psi} = 0$ in Ω and $\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0}$ on Γ. Hence

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\ d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\mathbf{curl}\ \boldsymbol{\psi}\ d\boldsymbol{x} = 0.$$

The result is then a consequence of theorem 1.1: there exists a function $\chi \in H^1(\Omega)$ satisfying (5). □

Remark 2.2. **(1)** Any function $\boldsymbol{f} \in L^2(\Omega)^3$ that satisfies $\mathbf{curl}\ \boldsymbol{f} = 0$ in Ω can be decomposed as:

$$\boldsymbol{f} = \mathbf{grad}\ \chi + \widetilde{\mathbf{grad}}\ p, \quad \text{with} \quad \chi \in H^1(\Omega) \quad \text{and} \quad \widetilde{\mathbf{grad}}\ p \in K_T(\Omega).$$

Such a result was alluded to in [11].

(2) The second condition in (4) is trivially satisfied when Ω is simply-connected, since $K_T(\Omega) = \{\mathbf{0}\}$ in this case.

Theorem 2.3. *Given any distribution* $\boldsymbol{f} \in H_0(\mathrm{div}, \Omega)'$ *that satisfies*

$$\mathbf{curl}\, \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega \quad \text{and} \quad {}_{H_0(\mathrm{div},\Omega)'}\langle \boldsymbol{f}\ ,\ \boldsymbol{v}\rangle_{H_0(\mathrm{div},\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in K_T(\Omega), \tag{6}$$

there exists a scalar potential χ *in* $L^2(\Omega)$ *such that*

$$\boldsymbol{f} = \mathbf{grad}\ \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{7}$$

Proof. Let $\boldsymbol{f} \in H_0(\mathrm{div}, \Omega)'$ be such that $\mathbf{curl}\ \boldsymbol{f} = \mathbf{0}$ in Ω. Hence (see proposition 1 of [6]) there exist $\boldsymbol{\psi} \in L^2(\Omega)^3$ and $\chi_0 \in L^2(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\ \chi_0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\chi_0\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{8}$$

Observe that, thanks to the density of $\mathscr{D}(\Omega)^3$ in $H_0(\mathrm{div}, \Omega)$,

$${}_{H_0(\mathrm{div},\Omega)'}\langle \mathbf{grad}\ \chi_0,\ \boldsymbol{v}\rangle_{H_0(\mathrm{div},\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in K_T(\Omega).$$

Therefore, the function $\boldsymbol{\psi} \in L^2(\Omega)^3$ satisfies relations (4). By theorem 2.1, there exists a function $p \in H^1(\Omega)$ such that

$$\boldsymbol{\psi} = \mathbf{grad}\ p \quad \text{in } \Omega \quad \text{and} \quad \|p\|_{H^1(\Omega)} \leq C\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}.$$

Hence the function $\chi = p + \chi_0$ satisfies the announced properties. □

Remark 2.4. Note that this theorem is an extension of the equivalence (iii) $\Longleftrightarrow$ (iv) in theorem 1.1 with $m = 1$ to the case where Ω is not simply-connected.

More generally, let us introduce, for any integer $m \geq 0$, the space

$$H_0^m(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in H_0(\mathrm{div}, \Omega);\ \mathrm{div}\, \boldsymbol{v} \in H_0^m(\Omega)\},$$

which coincides with $H_0(\mathrm{div}, \Omega)$ for $m = 0$. Its dual space, denoted by $H^{-m}(\mathrm{div}, \Omega)$, can then be characterized by

$$H^{-m}(\mathrm{div}, \Omega) = \{\boldsymbol{\psi} + \mathbf{grad}\ \chi;\ \boldsymbol{\psi} \in H_0(\mathrm{div}, \Omega)',\ \chi \in H^{-m}(\Omega)\}.$$

One can also show that $\mathscr{D}(\Omega)^3$ is dense in $H_0^m(\mathrm{div}, \Omega)$ and that the following Green formula holds for any $\chi \in H^{-m}(\mathrm{div}, \Omega)$ and $\boldsymbol{v} \in H_0^m(\mathrm{div}, \Omega)$:

$${}_{H^{-m}(\mathrm{div},\Omega)}\langle \mathbf{grad}\ \chi,\ \boldsymbol{v}\rangle_{H_0^m(\mathrm{div},\Omega)} + {}_{H^{-m}(\Omega)}\langle \chi,\ \mathrm{div}\boldsymbol{v}\rangle_{H_0^m(\Omega)} = 0. \tag{9}$$

As a consequence of theorem 2.3, it is easy to prove the following theorem, which shows that property (iv) in theorem 1.1 also holds when Ω is not simply-connected.

Theorem 2.5. *For any distribution* $\boldsymbol{f} \in H^{-m}(\mathrm{div}, \Omega)$ *that satisfies* (6), *there exists a scalar potential* χ *in* $H^{-m}(\Omega)$ *such that*

$$\boldsymbol{f} = \mathbf{grad}\ \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{H^{-m}(\Omega)} \leq C\|\boldsymbol{f}\|_{H^{-m}(\mathrm{div},\Omega)}. \tag{10}$$

Proof. We give the proof when $m = 1$; the general case is similar. Let $\boldsymbol{f} \in H^{-1}(\mathrm{div}, \Omega)$ satisfy (6). Then, there exist $\boldsymbol{\psi} \in H_0(\mathrm{div}, \Omega)'$ and $\chi_0 \in H^{-1}(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\ \chi_0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H_0(\mathrm{div},\Omega)'} + \|\chi_0\|_{H^{-1}(\Omega)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\mathrm{div},\Omega)}. \tag{11}$$

Observe that, thanks to (11), we have

$$_{H^{-1}(\mathrm{div},\Omega)}\langle \mathbf{grad}\ \chi_0,\ \boldsymbol{v}\rangle_{H_0^1(\mathrm{div},\Omega)} = -\ _{H^{-1}(\Omega)}\langle \chi_0,\ \mathrm{div}\, \boldsymbol{v}\rangle_{H_0^1(\Omega)} = 0$$

for all $\boldsymbol{v} \in K_T(\Omega)$. By theorem 2.3, there exists a function $p \in L^2(\Omega)$ such that $\boldsymbol{\psi} = \mathbf{grad}\ p$ and the estimate (7) holds. Then the function $\chi = \chi_0 + p$ satisfies the announced properties. □

3 Vector potentials in $H_0^m(\Omega)^3$

First, we recall some results concerning the existence of tangential vector potential (see [1] for proofs).

Below, $\langle \cdot, \cdot \rangle_{\Gamma_i}$ denotes the duality pairing between the spaces $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$. Given any function $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$ that satisfies

$$\mathrm{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \boldsymbol{u} \cdot \boldsymbol{n}\ ,\ 1\rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I, \tag{12}$$

there exists a vector potential $\boldsymbol{\psi}$ in $L^2(\Omega)^3$ such that

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}, \qquad \mathrm{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{13}$$

satisfying the estimate

$$\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{u}\|_{L^2(\Omega)^3}. \tag{14}$$

Moreover, there exists a unique vector field $\boldsymbol{\psi} \in L^2(\Omega)^3$ satisfying (13) and such that

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{n}\ ,\ 1\rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \tag{15}$$

and the estimate (14) holds. When Ω is of class $\mathscr{C}^{1,1}$, then $\boldsymbol{\psi}$ belongs to $H^1(\Omega)^3$ and the estimate

$$\|\boldsymbol{\psi}\|_{H^1(\Omega)^3} \leq C\|\boldsymbol{u}\|_{L^2(\Omega)^3} \tag{16}$$

holds. If moreover $\boldsymbol{u} \in H^m(\Omega)^3$ and Ω is of class $\mathscr{C}^{m+1,1}$, for some integer $m \geq 0$, then $\boldsymbol{\psi}$ belongs to $H^{m+1}(\Omega)^3$ and the estimate

$$\|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C\|\boldsymbol{u}\|_{H^m(\Omega)^3} \tag{17}$$

holds. We also recall the result concerning the existence of normal vector potentials (see again [1] for proofs). For any vector field $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$ that satisfies

$$\mathrm{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma \quad \text{and } \langle \boldsymbol{u} \cdot \boldsymbol{n}\ ,\ 1\rangle_{\Sigma_j} = 0,\ 1 \leq j \leq J, \tag{18}$$

there exists a vector potential $\boldsymbol{\psi}$ in $L^2(\Omega)^3$ such that

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}, \quad \mathrm{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma, \tag{19}$$

and the estimate

$$\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{u}\|_{L^2(\Omega)^3} \tag{20}$$

holds. Moreover, there exists a unique vector field $\boldsymbol{\psi} \in L^2(\Omega)^3$ satisfying (19) and such that

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{n}\ ,\ 1\rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I\,, \tag{21}$$

and the estimate (20) holds. When $\boldsymbol{u}$ is more regular, then (16) and (17) are also satisfied.

Remark 3.1. Let $\boldsymbol{u}$ be a vector field in $H(\mathrm{div}, \Omega)$ that satisfies:

$$\mathrm{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma.$$

Using the same arguments as those of theorem 2.1, it is easy to verify that

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}\ ,\ 1\rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J,$$

if and only if

$$\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\ q_j^T\ d\boldsymbol{x} = 0 \quad \text{for all } 1 \leq j \leq J.$$

Another kind of less standard but useful vector potential is given by the following theorem.

Theorem 3.2. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{1,1}$. For any function $\boldsymbol{u}$ in $H(\mathrm{div}, \Omega)$ satisfying* (18)*, there exists a vector potential $\boldsymbol{\psi}$ in $H_0^1(\Omega)^3$, such that*

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\ \Delta\boldsymbol{\psi} = 0 \quad \text{in } \Omega, \quad \|\boldsymbol{\psi}\|_{H^1(\Omega)^3} \leq C\|\boldsymbol{u}\|_{L^2(\Omega)^3}. \tag{22}$$

Proof. Given any vector field $\boldsymbol{u} \in H(\mathrm{div}, \Omega)$ satisfying (18), we associate the vector potential $\boldsymbol{\psi}_0 \in H^1(\Omega)^3$ satisfying (19) and the estimate

$$\|\boldsymbol{\psi}_0\|_{H^1(\Omega)^3} \leq C\|\boldsymbol{u}\|_{L^2(\Omega)^3}.$$

That Γ is of class $\mathscr{C}^{1,1}$ implies that the normal trace $\boldsymbol{\psi}_0 \cdot \boldsymbol{n}$ belongs to $H^{1/2}(\Gamma)$. Hence, the fourth-order problem

$$\Delta^2\chi = 0 \quad \text{in } \Omega, \quad \chi = 0 \quad \text{and} \quad \partial_n\chi = \boldsymbol{\psi}_0 \cdot \boldsymbol{n} \quad \text{on } \Gamma$$

has a unique solution χ in $H^2(\Omega)$ satisfying the estimate

$$\|\chi\|_{H^2(\Omega)} \leq C\|\boldsymbol{\psi}_0 \cdot \boldsymbol{n}\|_{H^{1/2}(\Gamma)} \leq C\|\boldsymbol{u}\|_{L^2(\Omega)^3}.$$

Then the vector field

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{grad}\ \chi$$

satisfies (22). □

The vector field $\boldsymbol{\psi}$ given by the previous theorem is unique up to vector fields belonging to the space

$$K_0^1(\Omega) := \{\boldsymbol{w} \in H_0^1(\Omega)^3;\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ and div } (\Delta\boldsymbol{w}) = 0 \text{ in } \Omega\}$$

(see proposition 3.4 below).

Corollary 3.3. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{m+1,1}$, for some integer $m \geq 0$. For any vector field $\boldsymbol{u} \in H^m(\Omega)^3$ that satisfies* (18)*, there exists a vector potential $\boldsymbol{\psi}$ in $(H^{m+1}\Omega) \cap H_0^1(\Omega))^3$ satisfying*

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi} \quad \text{and} \quad \text{div}\ \Delta\boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \leq C\|\boldsymbol{u}\|_{H^m(\Omega)^3}.$$

Proof. Under the given assumptions, the vector potential $\boldsymbol{\psi}$ given by the previous theorem belongs to $H^{m+1}(\Omega)^3$ and its normal trace $\boldsymbol{\psi} \cdot \boldsymbol{n}$ belongs to $H^{m+1/2}(\Gamma)$, on the one hand. On the other hand, the solution χ to the fourth-order problem found in the previous belongs to $H^{m+2}(\Omega)^3$. □

We now characterize the space $K_0^1(\Omega)$.

Proposition 3.4. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{1,1}$. Then the space $K_0^1(\Omega)$ is spanned by the vector fields* $\mathbf{grad}\ q_i^1$*, $1 \leq i \leq I$, where* *each q_i^1 is the unique solution in $H^2(\Omega)$ to the problem*

$$
\begin{aligned}
&\Delta^2 q_i^1 = 0 && \text{in } \Omega, && \\
&q_i^1\Big|_{\Gamma_0} = 0 && \text{and} \quad q_i^1\Big|_{\Gamma_k} = \delta_{ik}, && 1 \le k \le I, \\
&\partial_n q_i^1 = 0 && \text{on } \Gamma, && \\
&\langle \partial_n \Delta q_i^1\,,\, 1\rangle_{\Gamma_k} = \delta_{ik} && \text{and} \quad \langle \partial_n \Delta q_i^1\,,\, 1\rangle_{\Gamma_0} = -1,\ 1 \le k \le I. &&
\end{aligned}
\tag{23}
$$

Proof. First, we prove that the space $K_0^1(\Omega)$ and the space

$$G^1 := \{\mathbf{grad}\ q \in H_0^1(\Omega)^3; \quad \Delta^2 q = 0 \quad \text{in } \Omega\}$$

coincide. First, it is clear that G^1 is included in $K_0^1(\Omega)$. Second, given $\boldsymbol{w} \in K_0^1(\Omega)$, let $\widetilde{\boldsymbol{w}}$ denote the extension by zero of $\boldsymbol{w}$ to an open ball B containing $\overline{\Omega}$. Since $\mathbf{curl}\ \widetilde{\boldsymbol{w}} = \mathbf{0}$ in B, $\widetilde{\boldsymbol{w}}$ is the gradient of a function $q \in H^2(B)$. Moreover, $q = 0$ in $B \setminus \overline{\Omega}$, so that $q' := q|_\Omega$ belongs to $H_0^2(\Omega)$. Since ,$\boldsymbol{w} = \mathbf{grad}\ q'$, one finds that $\boldsymbol{w}$ belongs to G^1.
Moreover, it is clear that the set of vector fields $\mathbf{grad}\ q_i$, $1 \le i \le I$, where $q_i \in H^2(\Omega)$ is the unique solution to

$$
\begin{aligned}
&\Delta^2 q_i = 0 \quad \text{in } \Omega, \\
&q_i\Big|_{\Gamma_0} = 0 \quad \text{and} \quad q_i\Big|_{\Gamma_k} = \delta_{ik},\ 1 \le k \le I, \\
&\partial_n q_i = 0 \quad \text{on } \Gamma,
\end{aligned}
\tag{24}
$$

spans G^1 $(= K_0^1(\Omega))$.

One still has to check the last line of (23). Introduce now

$$M_2 := \{r \in H^2(\Omega);\ r\Big|_{\Gamma_0} = 0 \text{ and } r\Big|_{\Gamma_k} = \delta_{ik},\ 1 \le k \le I,\ \ \partial_n r = 0 \text{ on } \Gamma\}.$$

For $1 \le i \le I$, the problem: find q_i^1 in M_2 such that

$$\forall r \in M_2, \quad \int_\Omega \Delta q_i^1\, \Delta r\, d\boldsymbol{x} = -r\Big|_{\Gamma_i}, \tag{25}$$

has a unique solution. Furthermore, the following Green's formula can be proven by a density argument, for any functions q and r in M_2 with $\Delta^2 q$ in $L^2(\Omega)$:

$$\int_\Omega (\Delta^2 q) r\, d\boldsymbol{x} = \int_\Omega \Delta q\, \Delta r\, d\boldsymbol{x} + \sum_{i=1}^{I} r\Big|_{\Gamma_i} \langle \partial_n(\Delta q), 1\rangle_{\Gamma_i}.$$

This formula implies that the solution q_i^1 to (25) satisfies (23). The vector fields $\mathbf{grad}\ q_i^1$, $1 \le i \le I$, are clearly linearly independent and they belong to $K_0^1(\Omega)$. Consequently, they form a basis of $K_0^1(\Omega)$. □

Proposition 3.5. *Assume that the boundary of the domain* Ω *is of class* $\mathscr{C}^{1,1}$. *Given any function* $\boldsymbol{u}$ *in* $H(\text{div}, \Omega)$ *satisfying* (18), *there exists a unique vector potential* $\boldsymbol{\psi}$ *in* $H_0^1(\Omega)^3$ *satisfying*

$$\boldsymbol{u} = \textbf{curl}\ \boldsymbol{\psi}, \quad \text{div}\ \Delta\boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and } \langle \partial_n(\text{div}\ \Delta\boldsymbol{\psi})\,,\,1\rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I. \tag{26}$$

Moreover, the estimate (16) *holds.*

Proof. Let $(\boldsymbol{\psi}_0 - \textbf{grad}\ \chi)$ be the potential vector of $\boldsymbol{u}$ given in the proof of theorem 3.2. Then the vector field

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \textbf{grad}\ \chi + \sum_{i=1}^{I} \langle \partial_n(\Delta\chi)\,,\,1\rangle_{\Gamma_i}\, \textbf{grad}\ q_i^1$$

satisfies (26) (note that the quantities $\langle \partial_n(\Delta\chi), 1\rangle_{\Gamma_i}$ are well defined since $\Delta^2\chi = 0$). □

Corollary 3.6. *Assume that the boundary of the domain* Ω *is of class* $\mathscr{C}^{m+1,1}$ *for some integer* $m \geq 0$. *Given any function* $\boldsymbol{u}$ *in* $H^m(\Omega)^3$ *that satisfies* (18), *there exists a unique vector potential* $\boldsymbol{\psi}$ *in* $(H^{m+1}\Omega) \cap H_0^1(\Omega))^3$ *satisfying*

$$\boldsymbol{u} = \textbf{curl}\ \boldsymbol{\psi}, \ \text{div}\ \Delta\boldsymbol{\psi} = 0 \text{ in } \Omega \quad \text{and} \quad \langle \partial_n(\text{div}\ \Delta\boldsymbol{\psi})\,,\,1\rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I$$

and the estimate (17).

Theorem 3.7. *Assume that the boundary of the domain* Ω *is of class* $\mathscr{C}^{2,1}$. *Given any function* $\boldsymbol{u}$ *in* $H_0^1(\Omega)^3$ *that satisfies*

$$\text{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega \qquad \text{and} \quad \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \tag{27}$$

there exists a vector potential $\boldsymbol{\psi}$ *in* $H_0^2(\Omega)^3$ *such that*

$$\boldsymbol{u} = \textbf{curl}\ \boldsymbol{\psi} \quad \text{and} \quad \text{div}\ \Delta^2\boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^2(\Omega)^3} \leq C\|\boldsymbol{u}\|_{H^1(\Omega)^3}. \tag{28}$$

Proof. Given $\boldsymbol{u}$ in $H_0^1(\Omega)^3$ that satisfies (27), let $\boldsymbol{\varphi} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ denote the vector potential given by corollary 3.6. The sixth-order problem

$$\Delta^3\chi = 0 \quad \text{in } \Omega, \quad \chi = \frac{\partial\chi}{\partial\boldsymbol{n}} = 0 \quad \text{and} \quad \frac{\partial^2\chi}{\partial\boldsymbol{n}^2} = \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} \quad \text{on } \Gamma, \tag{29}$$

has a unique solution $\chi \in H^3(\Omega)$ that satisfies the estimate

$$\|\chi\|_{H^3(\Omega)} \leq C\|\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\|_{H^{1/2}(\Gamma)^3} \leq C\|\boldsymbol{\varphi}\|_{H^2(\Omega)^3} \leq C\|\boldsymbol{u}\|_{H^1(\Omega)^3}.$$

Note that the last boundary condition in (29) can be written as

$$\left(\frac{\partial}{\partial \boldsymbol{n}}\mathbf{grad}\ \chi\right)\cdot \boldsymbol{n} = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}\cdot \boldsymbol{n}.$$

For any unit tangent vector $\boldsymbol{\tau}$ on Γ, we have:

$$\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}\cdot \boldsymbol{\tau} = \frac{\partial \boldsymbol{\varphi}_i}{\partial x_j}n_j\tau_i = \frac{\partial \boldsymbol{\varphi}_j}{\partial x_i}\tau_i n_j = \frac{\partial \boldsymbol{\varphi}_j}{\partial \boldsymbol{\tau}}n_j = 0.$$

Also, one can show that $(\partial_n \mathbf{grad}\ \chi)\cdot \boldsymbol{\tau} = 0$, which implies that the relation $\partial_n \mathbf{grad}\ \chi = \partial_n \boldsymbol{\varphi}$ holds. So, the vector field $\boldsymbol{\psi} = \boldsymbol{\varphi} - \mathbf{grad}\ \chi$ belongs to $H^2(\Omega)^3$ and satisfies (28). □

The vector field $\boldsymbol{\psi}$ given by Theoerm 3.7 is unique up to vector fields in the space

$$K_0^2(\Omega) := \{\boldsymbol{w} \in H_0^2(\Omega)^3;\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \quad \text{and} \quad \operatorname{div} \Delta^2 \boldsymbol{w} = 0 \text{ in } \Omega\},$$

which we now characterize.

Proposition 3.8. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{2,1}$. Then the space $K_0^2(\Omega)$ is spanned by the vector fields* $\mathbf{grad}\ q_i^2$*, $1 \le i \le I$, where each function q_i^2 is the unique solution in $H^3(\Omega)$ to the problem*

$$\begin{array}{lll}
\Delta^3 q_i^2 = 0 & \text{in } \Omega, & \\
q_i^2\big|_{\Gamma_0} = 0 & \text{and} \quad q_i^2\big|_{\Gamma_k} = \delta_{ik}, & 1 \le k \le I, \\
\partial_n q_i^2 = \partial_n^2 q_i^2 = 0 & \text{on } \Gamma, & \\
\langle \partial_n(\Delta^2 q_i^2)\,,\,1\rangle_{\Gamma_k} = \delta_{ik} & \text{and} \quad \langle \partial_n(\Delta^2 q_i^2)\,,\,1\rangle_{\Gamma_0} = -1,\ 1 \le k \le I. &
\end{array} \tag{30}$$

Proof. First, we prove that the space $K_0^2(\Omega)$ coincides with the space

$$G^2 := \{\mathbf{grad}\ q \in H_0^2(\Omega)^3; \quad \Delta^3 q = 0 \quad \text{in } \Omega\},$$

using the same argument as in proposition 3.4. We next note that the set of vector fields $\mathbf{grad}\ q_i$, $1 \le i \le I$, where $q_i \in H^3(\Omega)$ is the unique solution to the problem

$$\begin{array}{ll}
\Delta^3 q_i = 0 & \text{in } \Omega, \\
q_i\big|_{\Gamma_0} = 0 & \text{and} \quad q_i\big|_{\Gamma_k} = \delta_{ik},\ 1 \le k \le I, \\
\partial_n q_i = \partial_n^2 q_i = 0 & \text{on } \Gamma,
\end{array} \tag{31}$$

spans $K_0^2(\Omega)$.

Let now

$$M_3{:=}\{r \in H^3(\Omega);\ r\big|_{\Gamma_0} = 0,\ r\big|_{\Gamma_k} =\ \delta_{ik},\ 1 \le k \le I\ ,\ \partial_n r = \partial_n^2 r = 0 \text{ on } \Gamma\}.$$

For $1 \le i \le I$, the problem: find q_i^2 in M_3 such that

$$\forall r \in M_3, \quad \int_\Omega \mathbf{grad}\ \Delta q_i^2 \cdot \mathbf{grad}\ \Delta r\, d\boldsymbol{x} = r\big|_{\Gamma_i}, \tag{32}$$

has a unique solution. Furthermore, the following Green's formula can be proved by a density argument, for any functions q and r in M_3 with $\Delta^3 q$ in $L^2(\Omega)$:

$$\int_\Omega (\Delta^3 q) r\, d\boldsymbol{x} = -\int_\Omega \mathbf{grad}\ \Delta q \cdot \mathbf{grad}\ \Delta r\, d\boldsymbol{x} + \sum_{i=1}^{I} r\big|_{\Gamma_i} \langle \partial_n(\Delta^2 q), \rangle_{\Gamma_i}.$$

This formula shows that the solution q_i^2 of (32) satisfies (30). The vector fields $\mathbf{grad}\ q_i^2$, $1 \le i \le I$, are clearly linearly independent and they belong to $K_0^2(\Omega)$. Consequently, they form a basis of $K_0^2(\Omega)$. □

Corollary 3.9. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{2,1}$. Given any function $\boldsymbol{u}$ in $H_0^1(\Omega)^3$ such that (27) holds, there exists a unique vector potential $\boldsymbol{\psi}$ in $H_0^2(\Omega)^3$ satisfying*

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi},\ \text{div}\ \Delta^2 \boldsymbol{\psi} = 0 \text{ in } \Omega \quad \text{and} \langle \partial_n (\text{div}\ \Delta \boldsymbol{\psi})\ , 1\rangle_{\Gamma_i} = 0, \quad 0 \le i \le I,$$

with the corresponding estimate.

More generally, we can prove using the same arguments:

Theorem 3.10. *Assume that boundary of the domain Ω is of class $\mathscr{C}^{m+1,1}$ for some integer $m \ge 1$. Given any vector field $\boldsymbol{u}$ in $H_0^m(\Omega)^3$ that satisfies (27), there exists a vector potential $\boldsymbol{\psi}$ in $H_0^{m+1}(\Omega)^3$ such that*

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi} \quad \text{and} \quad \text{div}\ \Delta^{m+1} \boldsymbol{\psi} = 0 \quad \text{in } \Omega \text{ and } \|\boldsymbol{\psi}\|_{H^{m+1}(\Omega)^3} \le C \|\boldsymbol{u}\|_{H^m(\Omega)^3}. \tag{33}$$

Moreover, there exists a unique vector potential $\boldsymbol{\psi}$ in $H_0^{m+1}(\Omega)^3$, satisfying (33) and

$$\langle \partial_n \text{div}\ \Delta \boldsymbol{\psi}^{m+1}\ , 1\rangle_{\Gamma_i} = 0, \quad 0 \le i \le I. \tag{34}$$

Remark 3.11. Similar results are found in Borchers & Sohr [7], but with different proof.

Let Ω be a domain with a boundary of class $\mathscr{C}^{m+1,1}$ for some integer $m \ge 1$ and let $\boldsymbol{u}$ in $H_0^m(\Omega)^3$ be such that div $\boldsymbol{u} = 0$. If Ω is simply-connected ($J = 0$), and Γ is connected ($I = 0$), then there exists a unique vector potential $\boldsymbol{\psi}$ in $H_0^{m+1}(\Omega)^3$ satisfying (33).

4 Weak vector potentials

First, we note that the continuous embeddings $H_0(\mathbf{curl},\Omega)' \hookrightarrow H^{-1}(\Omega)^3$ and $H_0(\mathrm{div},\Omega)' \hookrightarrow H^{-1}(\Omega)^3$ hold. Besides, given any $\boldsymbol{f} \in H^{-1}(\Omega)^3$, we know that there exist a unique $\boldsymbol{u} \in H_0^1(\Omega)^3$ and $\chi \in L^2(\Omega)$ such that

$$\boldsymbol{f} = -\Delta \boldsymbol{u} + \mathbf{grad}\ \chi \quad \text{and} \quad \mathrm{div}\ \boldsymbol{u} = 0 \ \text{ in } \Omega, \tag{35}$$

and satisfying the estimate

$$\|\boldsymbol{u}\|_{H^1(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)^3}.$$

Letting $\boldsymbol{\xi} = \mathbf{curl}\ \boldsymbol{u}$, we obtain the decomposition $\boldsymbol{f} = \mathbf{curl}\ \boldsymbol{\xi} + \mathbf{grad}\ \chi$ with $\mathrm{div}\ \boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi} \cdot \boldsymbol{n} = 0$ on Γ. Since $\boldsymbol{\xi} \in L^2(\Omega)^3$ and $\chi \in L^2(\Omega)$, it follows that $\mathbf{curl}\ \boldsymbol{\xi} \in H_0(\mathbf{curl},\Omega)'$ and $\mathbf{grad}\ \chi \in H_0(\mathrm{div},\Omega)'$, so that

$$H^{-1}(\Omega)^3 = H_0(\mathbf{curl},\Omega)' + H_0(\mathrm{div},\Omega)'. \tag{36}$$

Proposition 4.1. *Assume that the boundary of the domain Ω is of class $\mathscr{C}^{1,1}$. Then, for any $\boldsymbol{f}$ in the dual space $H_0(\mathrm{div},\Omega)'$, there exist a unique $\boldsymbol{u} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ and $\chi \in L^2(\Omega)$ solution to* (35) *and satisfying the estimate*

$$\|\boldsymbol{u}\|_{H^2(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}.$$

Proof. Let $\boldsymbol{f}$ be in the dual space of $H_0(\mathrm{div},\Omega)$. We know (see proposition 1 of [6]) that there exist $\boldsymbol{\psi} \in L^2(\Omega)^3$ and $\chi_0 \in L^2(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\ \chi_0 \quad \text{and } \|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\chi_0\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{37}$$

Thanks to the regularity of Γ, there exist $\boldsymbol{u} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ and $p \in H^1(\Omega)$ satisfying

$$\boldsymbol{\psi} = -\Delta \boldsymbol{u} + \mathbf{grad}\ p \quad \text{and} \quad \mathrm{div}\ \boldsymbol{u} = 0 \ \text{ in } \Omega, \tag{38}$$

with

$$\|\boldsymbol{u}\|_{H^2(\Omega)^3} + \|p\|_{H^1(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{\psi}\|_{L^2(\Omega)^3}.$$

Then $\boldsymbol{u}$ and $\chi = p + \chi_0$ satisfy the announced properties. □

We next consider the space

$$K_N(\Omega) := \{\boldsymbol{w} \in H_0(\mathbf{curl},\Omega) \cap H(\mathrm{div},\Omega);\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ and } \mathrm{div}\ \boldsymbol{w} = 0 \text{ in } \Omega\}$$

which is of dimension I. As shown in proposition 3.18 of [1], this space is spanned by the vector fields $\mathbf{grad}\ q_i^N$, $1 \leq i \leq N$, where each function $q_i^N \in$

$H^1(\Omega)$ is the unique solution to the problem

$$\begin{aligned}
&\Delta q_i^N = 0 && \text{in } \Omega, \\
&q_i^N = 0 && \text{on } \Gamma_0, \quad \langle \partial_n q_i^N \,,\, 1\rangle_{\Gamma_0} = -1, \\
&q_i^N = \text{constant} && \text{on } \Gamma_k, \quad \langle \partial_n q_i^N \,,\, 1\rangle_{\Gamma_k} = \delta_{ik}, \text{ for } 1 \le k \le I.
\end{aligned} \tag{39}$$

Theorem 4.2. *Given any distribution $\boldsymbol{f}$ in the dual space $H_0(\mathbf{curl}, \Omega)'$ that satisfies*

$$\operatorname{div} \boldsymbol{f} = 0 \quad \text{in } \Omega \text{ and } {}_{H_0(\mathbf{curl},\Omega)'}\langle \boldsymbol{f} \,,\, \boldsymbol{v}\rangle_{H_0(\mathbf{curl},\Omega)} = 0 \quad \text{for all} \quad \boldsymbol{v} \in K_N(\Omega), \tag{40}$$

there exists a vector potential $\boldsymbol{\xi}$ in $L^2(\Omega)^3$ such that

$$\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\xi}, \quad \text{with} \quad \operatorname{div} \boldsymbol{\xi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\xi} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{41}$$

and such that the following estimate holds:

$$\|\boldsymbol{\xi}\|_{L^2(\Omega)^3} \le C \|\boldsymbol{f}\|_{H_0(\mathbf{curl},\Omega)'}. \tag{42}$$

Proof. Let $\boldsymbol{f}$ be in the dual space $H_0(\mathbf{curl}, \Omega)'$. According to corollary 5 of [6], there exist $\boldsymbol{\psi} \in L^2(\Omega)^3$ and $\boldsymbol{\xi}_0 \in L^2(\Omega)^3$ with $\operatorname{div} \boldsymbol{\xi}_0 = 0$ in Ω and $\boldsymbol{\xi}_0 \cdot \boldsymbol{n} = 0$ on Γ, such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\, \boldsymbol{\xi}_0$ and such that the estimate

$$\|\boldsymbol{\psi}\|_{L^2(\Omega)^3} + \|\boldsymbol{\xi}_0\|_{L^2(\Omega)^3} \le C \|\boldsymbol{f}\|_{H_0(\mathbf{curl},\Omega)'}$$

holds. Thanks to the density of $\mathscr{D}(\Omega)^3$ in $H_0(\mathbf{curl}, \Omega)$, we deduce that for all $\boldsymbol{v} \in K_N(\Omega)$, we have

$${}_{H_0(\mathbf{curl},\Omega)'}\langle \mathbf{curl}\, \boldsymbol{\xi}_0 \,,\, \boldsymbol{v}\rangle_{H_0(\mathbf{curl},\Omega)} = 0.$$

Since $\operatorname{div} \boldsymbol{f} = 0$, it follows that $\operatorname{div} \boldsymbol{\psi} = 0$. Then, thanks to the orthogonality relations

$${}_{H_0(\mathbf{curl},\Omega)'}\langle \boldsymbol{f} \,,\, \mathbf{grad}\ q_i^N\rangle_{H_0(\mathbf{curl},\Omega)} = 0 \quad \text{for all } i = 1, \dots, I,$$

the relations $\langle \boldsymbol{\psi} \cdot \boldsymbol{n} \,,\, 1\rangle_{\Gamma_i} = 0$ are satisfied for all $i = 1, \dots, I$. There thus exists a vector potential $\boldsymbol{\varphi} \in L^2(\Omega)^3$ (see theorem 3.12 of [1]) such that $\boldsymbol{\psi} = \mathbf{curl}\, \boldsymbol{\varphi}$, with $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on Γ, and such that

$$\|\boldsymbol{\varphi}\|_{L^2(\Omega)^3} \le C \|\boldsymbol{\psi}\|_{L^2(\Omega)^3}.$$

Hence, the vector field $\boldsymbol{\xi} = \boldsymbol{\xi}_0 + \boldsymbol{\varphi}$ possesses the announced properties. □

Remark 4.3. The previous theorem has been established in [6] when Γ is connected, in which case $K_N = \{\mathbf{0}\}$.

For any integer $m \geq 0$, let us introduce the space

$$H_0^m(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in H_0(\mathbf{curl}, \Omega);\ \mathbf{curl}\ \boldsymbol{v} \in H_0^m(\Omega)^3\}.$$

We can easily characterize its dual space, as:

$$H^{-m}(\mathbf{curl}, \Omega) = \{\boldsymbol{\psi} + \mathbf{curl}\ \boldsymbol{\xi};\ \boldsymbol{\psi} \in H_0(\mathbf{curl}, \Omega)',\ \boldsymbol{\xi} \in H^{-m}(\Omega)^3\}.$$

We can prove that $\mathscr{D}(\Omega)^3$ is dense in $H_0^m(\mathbf{curl}, \Omega)$ and that the following Green formula holds: for any $\boldsymbol{\xi} \in H^{-m}(\mathbf{curl}, \Omega)$ and $\boldsymbol{v} \in H_0^m(\mathbf{curl}, \Omega)$

$${}_{H^{-m}(\mathbf{curl},\Omega)}\langle \mathbf{curl}\boldsymbol{\xi},\ \boldsymbol{v}\rangle_{H_0^m(\mathbf{curl},\Omega)} + {}_{H^{-m}(\Omega)^3}\langle \boldsymbol{\xi},\ \mathbf{curl}\boldsymbol{v}\rangle_{H_0^m(\Omega)^3} = 0. \tag{43}$$

By using the decomposition (1) with $(m+1)$ instead of m, it is easy to prove (as in Section 2) that

$$H^{-m-1}(\Omega)^3 = H^{-m}(\mathbf{curl}, \Omega) + H^{-m}(\mathrm{div}\ , \Omega), \quad \text{for } m \geq 1.$$

Theorem 4.4. *For any distribution $\boldsymbol{f}$ in the dual space $H^{-1}(\mathbf{curl}, \Omega)$ that satisfies*

$$\mathrm{div}\, \boldsymbol{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \boldsymbol{f}\ ,\ \boldsymbol{v}\rangle = 0, \quad \text{for all } \boldsymbol{v} \in K_N(\Omega) \tag{44}$$

there exists a vector potential $\boldsymbol{\xi}$ in $H^{-1}(\Omega)^3$ such that

$$\boldsymbol{f} = \mathbf{curl}\ \boldsymbol{\xi}, \quad \mathrm{div}\ \boldsymbol{\xi} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \|\boldsymbol{\xi}\|_{H^{-1}(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H^{-1}(\mathbf{curl},\Omega)}. \tag{45}$$

Proof. Given $\boldsymbol{f}$ in the dual space $H^{-1}(\mathbf{curl}, \Omega)$, there exist $\boldsymbol{f}_0 \in H_0(\mathbf{curl}, \Omega)'$ and $\boldsymbol{\xi}_0 \in H^{-1}(\Omega)^3$ such that $\boldsymbol{f} = \boldsymbol{f}_0 + \mathbf{curl}\ \boldsymbol{\xi}_0$, and satisfying the estimate

$$\|\boldsymbol{f}_0\|_{H_0(\mathbf{curl},\Omega)'} + \|\boldsymbol{\xi}_0\|_{H^{-1}(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H^{-1}(\mathbf{curl},\Omega)}.$$

Since $\boldsymbol{\xi}_0 \in H^{-1}(\Omega)^3$, there exists $\boldsymbol{\theta}_0 \in L^2(\Omega)^3$ satisfying $\mathrm{div}\ \boldsymbol{\theta}_0 = 0$ in Ω, $\boldsymbol{\theta}_0 \cdot \boldsymbol{n} = 0$ on Γ, and there exists $\chi \in L^2(\Omega)$ such that $\boldsymbol{\xi}_0 = \mathbf{curl}\ \boldsymbol{\theta}_0 + \mathbf{grad}\ \chi$ and

$$\|\boldsymbol{\theta}_0\|_{L^2(\Omega)^3} + \|\chi\|_{L^2(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{\xi}_0\|_{H^{-1}(\Omega)^3}.$$

Since $\boldsymbol{f}_0 \in H_0(\mathbf{curl}, \Omega)'$, then $\boldsymbol{f}_0 = \boldsymbol{\psi}_0 + \mathbf{curl}\ \boldsymbol{\varphi}_0$, with $\boldsymbol{\psi}_0 \in L^2(\Omega)^3$, $\boldsymbol{\varphi}_0 \in L^2(\Omega)^3$, $\mathrm{div}\ \boldsymbol{\varphi}_0 = 0$ in Ω, $\boldsymbol{\varphi}_0 \cdot \boldsymbol{n} = 0$ on Γ and

$$\|\boldsymbol{\psi}_0\|_{L^2(\Omega)^3} + \|\boldsymbol{\varphi}_0\|_{L^2(\Omega)^3} \leq C\|\boldsymbol{f}_0\|_{H_0(\mathbf{curl},\Omega)'}.$$

Then $\boldsymbol{f} = \boldsymbol{\psi}_0 + \mathbf{curl}\ \boldsymbol{\varphi}_0 + \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{\theta}_0 = \boldsymbol{\psi}_0 + \mathbf{curl}\ \boldsymbol{\mu}$, with $\boldsymbol{\mu} = \boldsymbol{\varphi}_0 + \mathbf{curl}\ \boldsymbol{\theta}_0$, $\mathrm{div}\ \boldsymbol{\mu} = 0$ in Ω, and the estimate

$$\|\boldsymbol{\psi}_0\|_{L^2(\Omega)^3} + \|\boldsymbol{\mu}\|_{H^{-1}(\Omega)^3} \leq C\|\boldsymbol{f}\|_{H^{-1}(\mathbf{curl},\Omega)}$$

holds.

Thanks to the density of $\mathscr{D}(\Omega)^3$ in $H^1_0(\mathbf{curl},\Omega)$, we infer that

$$_{H^{-1}(\mathbf{curl},\Omega)}\langle \mathbf{curl}\ \boldsymbol{\mu}\ ,\ \boldsymbol{v}\rangle_{H^1_0(\mathbf{curl},\Omega)}\ = 0, \quad \text{for all } \boldsymbol{v}\in K_N(\Omega).$$

Since div $\boldsymbol{f} = 0$, div $\boldsymbol{\psi}_0 = 0$ and therefore the condition $\langle \boldsymbol{\psi}_0\cdot \boldsymbol{n}\ ,\ 1\rangle_{\Gamma_i} = 0$ is automatically satisfied for any $i = 0,\ldots, I$. Then by (12), there exists a vector potential $\boldsymbol{\varphi}\in L^2(\Omega)^3$ such that

$$\boldsymbol{\psi}_0 = \mathbf{curl}\ \boldsymbol{\varphi}, \quad \operatorname{div}\boldsymbol{\varphi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma,$$

and

$$\|\boldsymbol{\varphi}\|_{L^2(\Omega)^3} \le C\|\boldsymbol{\psi}_0\|_{L^2(\Omega)^3}.$$

Hence, the vector field $\boldsymbol{\xi} = \boldsymbol{\mu}+\boldsymbol{\varphi}$ satisfies the announced properties. □

More generally, we can prove:

Theorem 4.5. *Given any integer $m \ge 0$ and any distribution $\boldsymbol{f}$ in the dual space $H^{-m}(\mathbf{curl},\Omega)$ that satisfies (44), there exists a vector potential $\boldsymbol{\xi}$ in $H^{-m}(\Omega)^3$ such that*

$$\boldsymbol{f} = \mathbf{curl}\ \boldsymbol{\xi}, \quad \text{with} \qquad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in } \Omega, \quad \text{and } \|\boldsymbol{\xi}\|_{H^{-m}(\Omega)^3} \le C\|\boldsymbol{f}\|_{H^{-m}(\mathbf{curl},\Omega)}.$$

5 Weak versions of Korn's inequality

Finally, we consider tensor fields. The next theorem generalizes theorem 3.2 of [8] and theorem 7 of [3] to Sobolev spaces with negative exponents.

In what follows, the subscript $_s$ denotes a space of symmetrix matrix fields.

Theorem 5.1. *Assume that Ω is simply-connected. Given an integer $m \ge 0$, let $\boldsymbol{e} = (e_{ij}) \in H^{-m}_s(\Omega)^{3\times 3}$ be a symmetric matrix field that satisfies the following compatibility conditions for all $i, j, k, l \in \{1, 2, 3\}$:*

$$\mathscr{R}_{ijkl} := \frac{\partial^2 e_{ik}}{\partial x_l\partial x_j} + \frac{\partial^2 e_{jl}}{\partial x_k\partial x_i} - \frac{\partial^2 e_{jk}}{\partial x_l\partial x_i} - \frac{\partial^2 e_{il}}{\partial x_k\partial x_j} = 0 \quad \text{in} \quad H^{-m-2}(\Omega). \tag{46}$$

Then there exists a vector field $\boldsymbol{v}\in H^{-m+1}(\Omega)^3$ such that $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ and $\boldsymbol{v}$ is unique up to vector fields in the space $R(\Omega) = \{\boldsymbol{a} + \boldsymbol{b}\wedge \mathbf{id}_\Omega;\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}$.

Proof. Given $\boldsymbol{e} = (e_{ij}) \in H_s^{-m}(\Omega)^{3\times 3}$, let $f_{ijk} := \partial_j e_{ik} - \partial_i e_{jk}$. Then $f_{ijk} \in H^{-m-1}(\Omega)$ and, thanks to the compatibility conditions (46), we have

$$\frac{\partial}{\partial x_l} f_{ijk} = \frac{\partial}{\partial x_k} f_{ijl}.$$

Hence the implication (iii) $\Longrightarrow$ (iv) in theorem 1.1 shows that there exist distributions $p_{ij} \in H^{-m}(\Omega)$, unique up to additive constants, such that $\partial_k p_{ij} = f_{ijk}$.

Besides, since $\partial_k p_{ij} = -\partial_k p_{ji}$, we can choose the distributions p_{ij} in such a way that $p_{ij} + p_{ji} = 0$. Noting that the distributions $q_{ij} := e_{ij} + p_{ij}$ belong to $H^{-m}(\Omega)$ and satisfy $\partial_k q_{ij} = \partial_j q_{ik}$, we again resort to theorem 1.1 to assert the existence of distributions $v_i \in H^{-m+1}(\Omega)$, unique up to additive constants, such that $\partial_j v_i = q_{ij}$. □

For any integer $m \geq 0$, let

$$E(\Omega) := \{\boldsymbol{e} \in H_s^{-m}(\Omega)^{3\times 3},\ \mathscr{R}_{ijkl}(\boldsymbol{e}) = 0\}$$

and

$$\dot{H}^{-m+1}(\Omega)^3 := H^{-m+1}(\Omega)^3 / R(\Omega).$$

By the previous theorem, given any $\boldsymbol{e} = (e_{ij}) \in E(\Omega)$, there exists a unique $\dot{\boldsymbol{v}} = (\dot{v}_i) \in \dot{H}^{-m+1}(\Omega)^3$ such that $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$. We may thus define a linear mapping $\mathcal{F} : E(\Omega) \to \dot{H}^{-m+1}(\Omega)^3$ by $\boldsymbol{\mathcal{F}(e)} = \dot{\boldsymbol{v}}$. Using the same method as in [8], we can then prove the following Korn's inequality in Sobolev spaces with negative exponents:

Theorem 5.2. *The linear mapping $\mathcal{F} : E(\Omega) \to \dot{H}^{-m+1}(\Omega)^3$ is an isomorphism. Besides, there exists a constant $C \geq 0$ such that*

$$\inf_{\boldsymbol{r}\in R(\Omega)} \|\boldsymbol{v} + \boldsymbol{r}\|_{H^{-m+1}(\Omega)^3} \leq C \sum_{i,j} \|\varepsilon_{ij}(\boldsymbol{v})\|_{H^{-m}(\Omega)} \quad \text{for all } \boldsymbol{v} \in H^{-m+1}(\Omega)^3,$$

and

$$\|\boldsymbol{v}\|_{H^{-m+1}(\Omega)^3} \leq C(\|\boldsymbol{v}\|_{H^{-m}(\Omega)^3} + \sum_{i,j} \|\varepsilon_{ij}(\boldsymbol{v})\|_{H^{-m}(\Omega)}) \quad \text{for all } \boldsymbol{v} \in H^{-m+1}(\Omega)^3$$

where $\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$.

Remark 5.3. Analogous techniques would likewise extend to Sobolev spaces with negative exponents the results obtained for non-simply connected domains in [9], [12] and [13].

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