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CANONICAL BASES AND AFFINE
HECKE ALGEBRAS OF TYPE D

P. SHAN, M. VARAGNOLO, E. VASSEROT

Abstract. We prove a conjecture of Miemietz and Kashiwara on canonical bases
and branching rules of affine Hecke algebras of type D. The proof is similar to the
proof of the type B case in [VV].

INTRODUCTION

Let $f$ be the negative part of the quantized enveloping algebra of type $A^{(1)}$.
Lusztig’s description of the canonical basis of $f$ implies that this basis can be natu-
really identified with the set of isomorphism classes of simple objects of a category of
modules of the affine Hecke algebras of type A. This identification was mentioned
in [G], and was used in [A]. More precisely, there is a linear isomorphism between
$f$ and the Grothendieck group of finite dimensional modules of the affine Hecke
algebras of type A, and it is proved in [A] that the induction/restriction functors
for affine Hecke algebras are given by the action of the Chevalley generators and
their transposed operators with respect to some symmetric bilinear form on $f$.

The branching rules for affine Hecke algebras of type B have been investigated
quite recently, see [E], [EK1,2,3], [M] and [VV]. In particular, in [E], [EK1,2,3] an
analogue of Ariki’s construction is conjectured and studied for affine Hecke algebras
of type B. Here $f$ is replaced by a module $\theta V(\lambda)$ over an algebra $\theta B$. More precisely it
is conjectured there that $\theta V(\lambda)$ admits a canonical basis which is naturally identified
with the set of isomorphism classes of simple objects of a category of modules of
the affine Hecke algebras of type B. Further, in this identification the branching
rules of the affine Hecke algebras of type B should be given by the $\theta B$-action on
$\theta V(\lambda)$. This conjecture has been proved [VV]. It uses both the geometric picture
introduced in [E] (to prove part of the conjecture) and a new kind of graded algebras
similar to the KLR algebras from [KL], [R].

A similar description of the branching rules for affine Hecke algebras of type D
has also been conjectured in [KM]. In this case $f$ is replaced by another module $\circ V$
over the algebra $\theta B$ (the same algebra as in the type B case). The purpose of this
paper is to prove the type D case. The method of the proof is the same as in [VV].
First we introduce a family of graded algebras $\circ R_m$ for $m$ a non negative integer.
They can be viewed as the Ext-algebras of some complex of constructible sheaves
naturally attached to the Lie algebra of the group $SO(2m)$, see Remark 2.8. This
complex enters in the Kazhdan-Lusztig classification of the simple modules of the
affine Hecke algebra of the group $Spin(2m)$. Then we identify $\circ V$ with the sum of
the Grothendieck groups of the graded algebras $\circ R_m$.

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The plan of the paper is the following. In Section 1 we introduce a graded algebra \( {}^\circ R(\Gamma)_\nu \). It is associated with a quiver \( \Gamma \) with an involution \( \theta \) and with a dimension vector \( \nu \). In Section 2 we consider a particular choice of pair \( (\Gamma, \theta) \). The graded algebras \( {}^\circ R(\Gamma)_\nu \) associated with this pair \( (\Gamma, \theta) \) are denoted by the symbol \( {}^\circ R_m \).

Next we introduce the affine Hecke algebra of type D, more precisely the affine Hecke algebra associated with the group \( SO(2m) \), and we prove that it is Morita equivalent to \( {}^\circ R_m \). In Section 3 we categorify the module \( {}^\circ V \) from [KM] using the graded algebras \( {}^\circ R_m \), see Theorem 3.28. The main result of the paper is Theorem 3.33.

0. Notation

0.1. Graded modules over graded algebras. Let \( k \) be an algebraically closed field of characteristic 0. By a graded \( k \)-algebra \( R = \bigoplus_d R_d \) we’ll always mean a \( \mathbb{Z} \)-graded associative \( k \)-algebra. Let \( R\text{-mod} \) be the category of finitely generated graded \( R \)-modules, \( R\text{-fmod} \) be the full subcategory of finite-dimensional graded modules and \( R\text{-proj} \) be the full subcategory of projective objects. Unless specified otherwise all modules are left modules. We’ll abbreviate

\[
K(R) = [R\text{-proj}], \quad G(R) = [R\text{-fmod}].
\]

Here \( \mathcal{C} \) denotes the Grothendieck group of an exact category \( \mathcal{C} \). Assume that the \( k \)-vector spaces \( R_d \) are finite dimensional for each \( d \). Then \( K(R) \) is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in \( R\text{-proj} \), and \( G(R) \) is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in \( R\text{-fmod} \). Given an object \( M \) of \( R\text{-proj} \) or \( R\text{-fmod} \) let \([M]\) denote its class in \( K(R) \), \( G(R) \) respectively. When there is no risk of confusion we abbreviate \( M = [M] \). We’ll write \([M : N]\) for the composition multiplicity of the \( R \)-module \( N \) in the \( R \)-module \( M \). Consider the ring \( \mathcal{A} = \mathbb{Z}[v, v^{-1}] \). If the grading of \( R \) is bounded below then the \( \mathcal{A} \)-modules \( K(R) \), \( G(R) \) are free. Here \( \mathcal{A} \) acts on \( G(R), K(R) \) as follows

\[
vM = M[1], \quad v^{-1}M = M[-1].
\]

For any \( M, N \) in \( R\text{-mod} \) let

\[
\text{hom}_R(M, N) = \bigoplus_d \text{Hom}_R(M[N], N[d])
\]

be the \( \mathbb{Z} \)-graded \( k \)-vector space of all \( R \)-module homomorphisms. If \( R = k \) we’ll omit the subscript \( R \) in hom’s and in tensor products. For any graded \( k \)-vector space \( M = \bigoplus_d M_d \) we’ll write

\[
g\dim(M) = \sum_d v^d \dim(M_d),
\]

where \( \dim \) is the dimension over \( k \).
0.2. Quivers with involutions. Recall that a quiver $\Gamma$ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where $I$ is the set of vertices, $H$ is the set of arrows and for each $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of $h$ respectively. Note that the set $I$ may be infinite. We'll assume that no arrow may join a vertex to itself. For each $i, j \in I$ we write $H_{i,j} = \{h \in H; h' = i, h'' = j\}$. We'll abbreviate $i \to j$ if $H_{i,j} \neq \emptyset$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$ 

An involution $\theta$ on $\Gamma$ is a pair of involutions on $I$ and $H$, both denoted by $\theta$, such that the following properties hold for each $h \in H$

- $\theta(h)' = \theta(h''')$ and $\theta(h)'' = \theta(h')$,
- $\theta(h)'' = h''$ iff $\theta(h) = h$.

We'll always assume that $\theta$ has no fixed points in $I$, i.e., there is no $i \in I$ such that $\theta(i) = i$. To simplify we'll say that $\theta$ has no fixed point. Let

$$\thetaNI = \{\nu = \sum_i \nu_i i \in NI : \nu_{\theta(i)} = \nu_i, \forall i\}.$$ 

For any $\nu \in \thetaNI$ set $|\nu| = \sum_i \nu_i$. It is an even integer. Write $|\nu| = 2m$ with $m \in \mathbb{N}$. We'll denote by $\theta^\nu$ the set of sequences $\i = (i_{1-1}, \ldots, i_{m-1}, i_m)$ of elements in $I$ such that $\theta(i_{ij}) = i_{ij-1}$ and $\sum_k i_k = \nu$. For any such sequence $i$ we'll abbreviate $\theta(i) = (\theta(i_{1-m}), \ldots, \theta(i_{m-1}), \theta(i_m))$. Finally, we set

$$\theta^\nu = \bigcup_{\nu} \theta^\nu, \quad \nu \in \thetaNI, \quad |\nu| = 2m.$$ 

0.3. The wreath product. Given a positive integer $m$, let $\mathfrak{S}_m$ be the symmetric group, and $\mathbb{Z}_2 = \{-1, 1\}$. Consider the wreath product $W_m = \mathfrak{S}_m \wr \mathbb{Z}_2$. Write $s_1, \ldots, s_{m-1}$ for the simple reflections in $\mathfrak{S}_m$. For each $l = 1, 2, \ldots, m$ let $\varepsilon_l \in (\mathbb{Z}_2)^m$ be $-1$ placed at the $l$-th position. There is a unique action of $W_m$ on the set $\{1 - m, \ldots, m - 1, m\}$ such that $\mathfrak{S}_m$ permutes $1, 2, \ldots, m$ and such that $\varepsilon_l$ fixes $k$ if $k \neq l, 1 - l$ and switches $1$ and $1 - l$. The group $W_m$ acts also on $\theta^\nu$. Indeed, view a sequence $\i$ as the map

$$\{1 - m, \ldots, m - 1, m\} \to I, \quad l \mapsto i_l.$$ 

Then we set $w(\i) = \i \circ w^{-1}$ for $w \in W_m$. For each $\nu$ we fix once for all a sequence $I_\nu = (i_{1-1}, \ldots, i_m) \in \theta^\nu$. Let $W_\nu$ be the centralizer of $I_\nu$ in $W_m$. Then there is a bijection

$$W_\nu \backslash W_m \to \theta^\nu, \quad W_\nu w \mapsto w^{-1}(I_\nu).$$ 

Now, assume that $m > 1$. We set $s_0 = \varepsilon_1 s_1 \varepsilon_1$. Let $^0W_m$ be the subgroup of $W_m$ generated by $s_0, \ldots, s_{m-1}$. We'll regard it as a Weyl group of type $D_m$ such that $s_0, \ldots, s_{m-1}$ are the simple reflections. Note that $W_\nu$ is a subgroup of $^0W_m$. Indeed, if $W_\nu \not\subset ^0W_m$ there should exist $l$ such that $\varepsilon_l$ belongs to $W_\nu$. This would imply that $i_l = \theta(i_l)$, contradicting the fact that $\theta$ has no fixed point. Therefore $\theta^\nu$ decomposes into two $^0W_m$-orbits. We'll denote them by $\theta^\nu_+$ and $\theta^\nu_-$. For $m = 1$ we set $^0W_1 = \{e\}$ and we choose again $\theta^\nu_+$ and $\theta^\nu_-$ in an obvious way.
1. The graded \( k \)-algebra \( \nu \mathcal{R}(\Gamma)_\nu \)

Fix a quiver \( \Gamma \) with set of vertices \( I \) and set of arrows \( H \). Fix an involution \( \theta \) on \( \Gamma \). Assume that \( \Gamma \) has no 1-loops and that \( \theta \) has no fixed points. Fix a dimension vector \( \nu \neq 0 \) in \( \mathbb{N}I \). Set \( |\nu| = 2m \).

1.1. Definition of the graded \( k \)-algebra \( \nu \mathcal{R}(\Gamma)_\nu \). Assume that \( m > 1 \). We define a graded \( k \)-algebra \( \nu \mathcal{R}(\Gamma)_\nu \) with 1 generated by \( l_1, \sigma_2, \sigma_k \), with \( i \in \mathbb{N}I \), \( l = 1, 2, \ldots, m \), \( k = 0, 1, \ldots, m - 1 \) modulo the following defining relations

(a) \( l_1 v_1 = \delta_{1,l} v_1, \quad \sigma_k l_1 = l_{s_k} \sigma_k, \quad \sigma_1 l_1 = l_1 \sigma_1 \),

(b) \( \sigma_1 \nu \theta = \nu \theta \sigma_1 \),

(c) \( \sigma_k^2 l_1 = Q_{i_k, i_{s_1}(k)}(\sigma_{s_1}(k), \tau_{s_1}(k)) l_1 \),

(d) \( \sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k \) if \( 1 \leq k < k' - 1 < m - 1 \) or \( 0 = k < k' \neq 2 \),

(e) \( (\sigma_{s_1}(k) \sigma_k \sigma_{s_1}(k) - \sigma_k \sigma_{s_1}(k) \sigma_k) l_1 = \)

\[
\begin{cases} 
Q_{k, s_1(k)}(\tau_{s_1}(k), \tau_{s_1}(k)) l_1 & \text{if } i_k = i_{s_1(k)}, \\
0 & \text{else},
\end{cases}
\]

(f) \( (\sigma_k \nu - \tau_{s_1}(k) \sigma_{s_1}(k)) l_1 = \)

\[
\begin{cases} 
- l_1 & \text{if } l = k, i_k = i_{s_1(k)}, \\
l_1 & \text{if } l = s_1(k), i_k = i_{s_1(k)}, \\
0 & \text{else}.
\end{cases}
\]

Here we have set \( \nu_{1-l} = -\nu_l \) and

\[
(1.1) \quad Q_{i,j}(u, v) = \begin{cases} 
(-1)^{b_{i,j}}(u - v)^{i-j} & \text{if } i \neq j, \\
0 & \text{else}.
\end{cases}
\]

If \( m = 0 \) we set \( \nu \mathcal{R}(\Gamma)_{\nu} = k \). If \( m = 1 \) then we have \( \nu = i + \theta(i) \) for some \( i \in I \). Write \( \mathbf{i} = i \theta(i) \), and

\( \nu \mathcal{R}(\Gamma)_{\nu} = k[\mathcal{X}_1]_{\nu} \oplus k[\mathcal{X}_{\nu}]_{\theta(i)} \).

We’ll abbreviate \( \nu_{1:k} = \sigma_k l_1 \) and \( \nu_{1:l} = \sigma_1 l_1 \). The grading on \( \nu \mathcal{R}(\Gamma)_0 \) is the trivial one. For \( m \geq 1 \) the grading on \( \nu \mathcal{R}(\Gamma)_\nu \) is given by the following rules:

\[
\begin{align*}
\deg(l_1) &= 0, \\
\deg(\nu_{1:k}) &= 2, \\
\deg(\nu_{1:l}) &= -i_k s_1(k).
\end{align*}
\]

We define \( \omega \) to be the unique involution of the graded \( k \)-algebra \( \nu \mathcal{R}(\Gamma)_\nu \) which fixes \( l_1, \sigma_1, \sigma_k \). We set \( \omega \) to be identity on \( \nu \mathcal{R}(\Gamma)_0 \).

1.2. Relation with the graded \( k \)-algebra \( \nu \mathcal{R}(\Gamma)_\nu \). A family of graded \( k \)-algebra \( \nu \mathcal{R}(\Gamma)_{\lambda, \nu} \) was introduced in [VV, sec. 5.1], for \( \lambda \) an arbitrary dimension vector in \( \mathbb{N}I \). Here we’ll only consider the special case \( \lambda = 0 \), and we abbreviate \( \nu \mathcal{R}(\Gamma)_\nu = \nu \mathcal{R}(\Gamma)_{0, \nu} \). Recall that if \( \nu \neq 0 \) then \( \nu \mathcal{R}(\Gamma)_\nu \) is the graded \( k \)-algebra with 1 generated
by $l_1$, $x_l$, $s_k$, $\pi_1$, with $i \in \nu^l$, $l = 1, 2, \ldots, m$, $k = 1, \ldots, m - 1$ such that $l_1$, $x_l$ and $s_k$ satisfy the same relations as before and

$$
\pi_1^2 = 1, \quad \pi_1 l_1 \pi_1 = 1_{x_1}, \quad \pi_1 x_l \pi_1 = x_{x_1(l)}, \quad (\pi_1 \sigma_1)^2 = (\sigma_1 \pi_1)^2, \quad \pi_1 x_k \pi_1 = x_k \text{ if } k \neq 1. 
$$

If $\nu = 0$ then $^\nu R(\Gamma)_0 = k$. The grading is given by setting $\deg(l_1)$, $\deg(x_l)$, $\deg(s_k)$ to be as before and $\deg(\pi_1 l_1) = 0$. In the rest of Section 1 we'll assume $m > 0$. Then there is a canonical inclusion of graded $k$-algebras

$$
^\nu R(\Gamma) \subset ^\nu R(\Gamma)_\nu
$$

such that $l_1, x_l, s_k \mapsto l_1, x_l, s_k$ for $i \in \nu^l$, $l = 1, \ldots, m$, $k = 1, \ldots, m - 1$ and such that $\sigma_0 \mapsto \pi_1 \sigma_1 \pi_1$. From now on we'll write $\sigma_0 = \pi_1 \sigma_1 \pi_1$ whenever $m > 1$. The assignment $x \mapsto \pi_1 x \pi_1$ defines an involution of the graded $k$-algebra $^\nu R(\Gamma)_\nu$ which normalizes $^\nu R(\Gamma)_\nu$. Thus it yields an involution

$$
\gamma : ^\nu R(\Gamma)_\nu \to ^\nu R(\Gamma)_\nu.
$$

Let $(\gamma)$ be the group of two elements generated by $\gamma$. The smash product $^\nu R(\Gamma)_\nu \times (\gamma)$ is a graded $k$-algebra such that $\deg(\gamma) = 0$. There is an unique isomorphism of graded $k$-algebras

$$
^\nu R(\Gamma)_\nu \times (\gamma) \to ^\nu R(\Gamma)_\nu
$$

which is identity on $^\nu R(\Gamma)_\nu$ and which takes $\gamma$ to $\pi_1$.

1.3. The polynomial representation and the PBW theorem. For any $i$ in $\nu^l$ let $^\nu F_i$ be the subalgebra of $^\nu R(\Gamma)_\nu$ generated by $l_1$ and $x_l$ with $l = 1, 2, \ldots, m$. It is a polynomial algebra. Let

$$
^\nu F_\nu = \bigoplus_{i \in \nu^l} ^\nu F_i.
$$

The group $W_m$ acts on $^\nu F_\nu$ via $w(x_l) = x_{w(l)}$ for any $w \in W_m$. Consider the fixed points set

$$
^\nu S_\nu = (^\nu F_\nu)^{W_m}.
$$

Regard $^\nu R(\Gamma)_\nu$ and $End(^\nu F_\nu)$ as $^\nu F_\nu$-algebras via the left multiplication. In [VV, prop. 5.4] is given an injective graded $^\nu F_\nu$-algebra morphism $^\nu R(\Gamma)_\nu \to End(^\nu F_\nu)$. It restricts via (1.2) to an injective graded $^\nu F_\nu$-algebra morphism

$$
^\nu R(\Gamma)_\nu \to End(^\nu F_\nu).
$$

Next, recall that $^\nu W_m$ is the Weyl group of type $D_m$ with simple reflections $s_0, \ldots, s_{m - 1}$. For each $w$ in $^\nu W_m$ we choose a reduced decomposition $\tilde{w}$ of $w$. It has the following form

$$
w = s_{k_1} s_{k_2} \cdots s_{k_r}, \quad 0 \leq k_1, k_2, \ldots, k_r \leq m - 1.
$$

We define an element $\sigma_{\tilde{w}}$ in $^\nu R(\Gamma)_\nu$ by

$$
\sigma_{\tilde{w}} = \sum_{i} 1_i \sigma_{\tilde{w}}, \quad 1_i \sigma_{\tilde{w}} = \begin{cases} 1_i, & \text{if } r = 0 \\ 1_i s_{k_1}, s_{k_2} \cdots s_{k_r}, & \text{else}. \end{cases}
$$

Observe that the element $\sigma_{\tilde{w}}$ may depend on the choice of the reduced decomposition $\tilde{w}$. 

1.4. Proposition. The $k$-algebra $\mathcal{R}(\Gamma)_\nu$ is a free (left or right) $\mathcal{F}_\nu$-module with basis $\{\sigma_w; w \in \mathcal{W}_m\}$. Its rank is $2^{m-1}m!$. The operator $1_\nu \sigma_w$ is homogeneous and its degree is independent of the choice of the reduced decomposition $\nu$.

Proof: The proof is the same as in [VV, prop. 5.5]. First, we filter the algebra $\mathcal{R}(\Gamma)_\nu$ with $1_\nu$, $x_{i,l}$ in degree 0 and $\sigma_{1,k}$ in degree 1. The Nil Hecke algebra of type $D_m$ is the $k$-algebra $\mathcal{NH}_m$ generated by $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ with relations

$$\sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k$$

if $1 \leq k < k' - 1 < m - 1$ or $0 = k < k' \neq 2$,

$$\sigma_{s_k(k)} \sigma_{s_k(k)} = \sigma_k \sigma_{s_k(k)} \sigma_k, \quad \sigma_k^2 = 0.$$

We can form the semidirect product $\mathcal{F}_\nu \rtimes \mathcal{NH}_m$, which is generated by $1_\nu, \bar{x}_l, \sigma_k$ with the relations above and

$$\sigma_k \bar{x}_l = \bar{x}_{s_k(l)} \sigma_k, \quad \bar{x}_l \sigma_k = \bar{x}_l \sigma_k.$$

One proves as in [VV, prop. 5.5] that the map

$$\mathcal{F}_\nu \rtimes \mathcal{NH}_m \to \text{gr}(\mathcal{R}(\Gamma)_\nu), \quad 1_\nu \mapsto 1_\nu, \quad \bar{x}_l \mapsto \bar{x}_l, \quad \sigma_k \mapsto \sigma_k.$$

is an isomorphism of $k$-algebras.

Let $\mathcal{F}'_\nu = \bigoplus_i \mathcal{F}'_i$, where $\mathcal{F}'_i$ is the localization of the ring $\mathcal{F}_i$ with respect to the multiplicative system generated by

$$\{x_{i,l} \pm x_{i,l}; 1 \leq l \neq l' \leq m\} \cup \{x_{i,l}; l = 1, 2, \ldots, m\}.$$

1.5. Corollary. The inclusion $\mathcal{R}(\Gamma)_\nu \subset \text{End}(\mathcal{F}_\nu)$ yields an isomorphism of $\mathcal{F}'_\nu$-algebras $\mathcal{F}'_\nu \otimes_{\mathcal{F}_\nu} \mathcal{R}(\Gamma)_\nu \to \mathcal{F}'_\nu \rtimes \mathcal{W}_m$, such that for each $i$ and each $l = 1, 2, \ldots, m$, $k = 0, 1, 2, \ldots, m - 1$ we have

$$1_\nu \mapsto 1_\nu, \quad x_{i,l} \mapsto x_{i,l}, \quad \sigma_{1,k} \mapsto \begin{cases} (x_k - x_{s_k(k)})^{-1}(s_k - 1)1_\nu & \text{if } i_k = i_{s_k(k)}; \\ (x_k - x_{s_k(k)})^{b_{i_k,s_k(k)}} s_k 1_\nu & \text{if } i_k \neq i_{s_k(k)}. \end{cases}$$

Proof: Follows from [VV, cor. 5.6] and Proposition 1.4.

Restricting the $\mathcal{F}_\nu$-action on $\mathcal{R}(\Gamma)_\nu$ to the $k$-subalgebra $\mathcal{S}_\nu$ we get a structure of graded $\mathcal{S}_\nu$-module on $\mathcal{R}(\Gamma)_\nu$.

1.6. Proposition. (a) $\mathcal{S}_\nu$ is isomorphic to the center of $\mathcal{R}(\Gamma)_\nu$.

(b) $\mathcal{R}(\Gamma)_\nu$ is a free graded module over $\mathcal{S}_\nu$ of rank $(2^{m-1}m!)^2$.

Proof: Part (a) follows from Corollary 1.5. Part (b) follows from (a) and Proposition 1.4.
2. Affine Hecke algebras of type D

2.1. Affine Hecke algebras of type D. Fix \( p \) in \( k^\times \). For any integer \( m \geq 0 \) we define the extended affine Hecke algebra \( H_m \) of type \( D_m \) as follows. If \( m > 1 \) then \( H_m \) is the \( k \)-algebra with 1 generated by

\[
T_k, \quad X_i^\pm 1, \quad k = 0, 1, \ldots, m - 1, \quad l = 1, 2, \ldots, m
\]
satisfying the following defining relations:

(a) \( X_i X_j = X_j X_i \),

(b) \( T_k T_{s_k(a)} T_k = T_{s_k(b)} T_k T_{s_k(a)} \), \( T_k T_{k'} = T_{k} T_{k'} \) if \( 1 \leq k < k' - 1 \) or \( k = 0, k' \neq 2 \),

(c) \( (T_k - p)(T_k + p^{-1}) = 0 \),

(d) \( T_0 X_i^{-1} p = X_2, T_k X_i T_k = X_{\pm k} \) if \( k \neq 0 \), \( T_k X_l = X_l T_k \) if \( k \neq 0, l, l - 1 \) or \( k = 0, l \neq 1, 2 \).

Finally, we set \( H_0 = k \oplus k \) and \( H_1 = k[X_i^\pm 1] \).

2.2. Remarks. (a) The extended affine Hecke algebra \( H_0^B \) of type \( B_m \) with parameters \( p, q \in k^\times \) such that \( q = 1 \) is generated by \( P, T_k, X_i^\pm 1, k = 1, \ldots, m - 1, l = 1, \ldots, m \) such that \( T_k, X_i^\pm 1 \) satisfy the relations as above and

\[
P^2 = 1, \quad (PT_1)^2 = (T_1 P)^2, \quad PT_k = T_k P \text{ if } k \neq 1,
\]

\[
PX_i^{-1} P = X_i, \quad PX_l = X_l P \text{ if } l \neq 1.
\]

See e.g., [VV, sec. 6.1]. There is an obvious \( k \)-algebra embedding \( H_m \subset H_m^B \). Let \( \gamma \) denote also the involution \( H_m \to H_m, a \mapsto PaP \). We have a canonical isomorphism of \( k \)-algebras

\[
H_m \cong \langle \gamma \rangle \simeq H_m^B.
\]

Compare Section 1.2.

(b) Given a connected reductive group \( G \) we call affine Hecke algebra of \( G \) the Hecke algebra of the extended affine Weyl group \( W \ltimes P \), where \( W \) is the Weyl group of \( (G, T) \), \( P \) is the group of characters of \( T \), and \( T \) is a maximal torus of \( G \). Then \( H_m \) is the affine Hecke algebra of the group \( SO(2m) \). Let \( H_m^c \) be the affine Hecke algebra of the group \( Spin(2m) \). It is generated by \( H_m \) and an element \( X_0 \) such that

\[
X_0^2 = X_1 X_2 \cdots X_m, \quad T_k X_0 = X_0 T_k \text{ if } k \neq 0, \quad T_0 X_0 X_1^{-1} X_2^{-1} T_0 = X_0.
\]

Thus \( H_m \) is the fixed point subset of the \( k \)-algebra automorphism of \( H_m^c \) taking \( T_k, X_l \) to \( T_k, (-1)^{\delta_{kl}} X_l \) for all \( k, l \). Therefore, if \( p \) is not a root of 1 the simple \( H_m \)-modules can be recovered from the Kazhdan-Lusztig classification of the simple \( H_m^c \)-modules via Clifford theory, see e.g., [Re].

2.3. Intertwiners and blocks of \( H_m \). We define

\[
A = k[X_1^\pm 1, X_2^\pm 1, \ldots, X_m^\pm 1], \quad A' = A[\Sigma^{-1}], \quad H_m' = A' \otimes_A H_m,
\]
where $\Sigma$ is the multiplicative set generated by

$$
1 - X_i X_{i'}^{-1}, \quad 1 - p^2 X_i X_{i'}^{-1}, \quad l \neq l'.
$$

For $k = 0, \ldots, m-1$ the intertwiner $\varphi_k$ is the element of $H'_m$ given by the following formulas

$$(2.1) \quad \varphi_k - 1 = \frac{X_k - X_{s_k(k)}}{p X_k - p^{-1} X_{s_k(k)}} (T_k - p).$$

The group $^\circ W_m$ acts on $A'$ as follows

$$(s_{0,0})(X_1, \ldots, X_m) = a(X_1, \ldots, X_{k+1}, X_k, \ldots, X_m) \text{ if } k \neq 0,$$

$$(s_{0,0})(X_1, \ldots, X_m) = a(X_1^{-1}, X_2^{-1}, \ldots, X_m).$$

There is an isomorphism of $^\circ$-algebras

$$A' \times ^\circ W_m \to H'_m, \quad s_k \mapsto \varphi_k.$$  

The semi-direct product group $Z \times Z = Z \times \{ -1, 1 \}$ acts on $k^\times$ by $(n, \varepsilon) : z \mapsto z^n p^{\varepsilon n}$. Given a $Z \times Z$-invariant subset $I$ of $k^\times$ we denote by $H_m{-\bf Mod}_I$ the category of all $H_m$-modules such that the action of $X_1, X_2, \ldots, X_m$ is locally finite with eigenvalues in $I$. We associate to the set $I$ a quiver $\Gamma$ as follows. The set of vertices is $I$, and there is one arrow $p^2 i \to i$ whenever $i$ lies in $I$. We equip $\Gamma$ with an involution $\theta$ such that $\theta(i) = i^{-1}$ for each vertex $i$ and such that $\theta$ takes the arrow $p^2 i \to i$ to the arrow $i^{-1} \to p^{-2} i^{-1}$. We’ll assume that the set $I$ does not contain $1$ or $-1$ and that $p \neq 1, -1$. Thus the involution $\theta$ has no fixed points and no arrow may join a vertex of $\Gamma$ to itself.

2.4. Remark. We may assume that $I = \pm\{ p^n; \ n \in \mathbb{Z}_{\text{odd}} \}$. See the discussion in [KM]. Then $\Gamma$ is of type $A_{\infty}$ if $p$ has infinite order and $\Gamma$ is of type $A_r^{(1)}$ if $p^2$ is a primitive $r$-th root of unity.

2.5. $H_m$-modules versus $^\circ R_m$-modules. Assume that $m \geq 1$. We define the graded $k$-algebra

$$^\theta R_{I,m} = \bigoplus_\nu ^\theta R_{I,\nu}, \quad ^\theta R_{I,\nu} = ^\theta R(\Gamma)_\nu, \quad ^\circ R_{I,m} = \bigoplus_\nu ^\circ R_{I,\nu}, \quad ^\circ R_{I,\nu} = ^\circ R(\Gamma)_\nu,$$

$${}^\theta I^m = \bigcup_\nu {}^\theta I^\nu,$$

where $\nu$ runs over the set of all dimension vectors in $^\theta N I$ such that $|\nu| = 2m$. When there is no risk of confusion we abbreviate

$${}^\theta R_\nu = ^\theta R_{I,\nu}, \quad ^\theta R_m = ^\theta R_{I,m}, \quad ^\circ R_\nu = ^\circ R_{I,\nu}, \quad ^\circ R_m = ^\circ R_{I,m}.$$  

Note that $^\theta R_\nu$ and $^\theta R_m$ are the same as in [VV, sec. 6.4], with $\lambda = 0$. Note also that the $k$-algebra $^\circ R_m$ may not have 1, because the set $I$ may be infinite. We define $^\circ R_m{-\bf Mod}_0$ as the category of all (non-graded) $^\circ R_m$-modules such that the elements $x_1, x_2, \ldots, x_m$ act locally nilpotently. Let $^\circ R_m{-\bf fMod}_0$ and $H_m{-\bf fMod}_I$ be the full subcategories of finite dimensional modules in $^\circ R_m{-\bf Mod}_0$ and $H_m{-\bf Mod}_I$ respectively. Fix a formal series $f(x)$ in $k[[x]]$ such that $f(x) = 1 + x$ modulo $(x^2)$. 
2.6. Theorem. We have an equivalence of categories

\[ \mathcal{R}_m - \text{-Mod}_0 \to \mathcal{H}_m - \text{-Mod}_I, \quad M \mapsto M \]

which is given by
(a) \( X_i \) acts on \( 1_{i1} \) by \( i_{i1}^{-1} f(x_l) \) for each \( l = 1, 2, \ldots, m \),
(b) if \( m > 1 \) then \( T_k \) acts on \( 1_{i1} \) as follows for each \( k = 0, 1, \ldots, m - 1 \),

\[ \frac{(pf(x_k) - p^{-1}f(x_{k}(k)))/(x_k - x_{k}(k))}{f(x_k) - f(x_{k}(k))} \sigma_k + p \quad \text{if } i_{x_{k}(k)} = i_k, \]

\[ \frac{f(x_k) - f(x_{k}(k))}{(p^{-1}f(x_k) - p(f(x_{k}(k)))/(x_k - x_{k}(k))} \sigma_k + \frac{(p^{-2} - 1)f(x_{k}(k))}{pf(x_k) - p^{-1}f(x_{k}(k))} \quad \text{if } i_{x_{k}(k)} = p^2 i_k, \]

\[ \frac{pi_k f(x_k) - p^{-1}i_{x_{k}(k)} f(x_{k}(k))}{i_k f(x_k) - i_{x_{k}(k)} f(x_{k}(k))} \sigma_k + \frac{(p^{-1} - p)i_k f(x_{k}(k))}{i_k f(x_k) - i_{x_{k}(k)} f(x_{k}(k))} \quad \text{if } i_{x_{k}(k)} \neq i_k, p^2 i_k. \]

Proof: This follows from [VV, thm. 6.5] by Section 1.2 and Remark 2.2(a). One can also prove it by reproducing the arguments in loc. cit. by using (1.5) and (2.1).

2.7. Corollary. There is an equivalence of categories

\[ \Psi : \mathcal{R}_m - \text{-fMod}_0 \to \mathcal{H}_m - \text{-fMod}_I, \quad M \mapsto M. \]

2.8. Remarks. (a) Let \( g \) be the Lie algebra of \( G = SO(2m) \). Fix a maximal torus \( T \subset G \). The group of characters of \( T \) is the lattice \( \bigoplus_{i=1}^m \mathbb{Z} \varepsilon_i \), with Bourbaki’s notation. Fix a dimension vector \( \nu \in \mathfrak{g}^* \). Recall the sequence \( \iota_{
u} = (i_{1-\nu}, \ldots, i_{m-\nu}, i_{\nu}) \) from Section 0.3. Let \( g \in T \) be the element such that \( \varepsilon_i(g) = i_{i}^{-1} \) for each \( l = 1, 2, \ldots, m \). Recall also the notation \( \mathfrak{g}_G, \mathfrak{g}_G V \), and \( \mathfrak{g}_G V \) from [VV]. Then \( V \) is an object of \( \mathfrak{g}_G V \), \( \mathfrak{g}_G V = G_g \) is the centralizer of \( g \) in \( G \), and

\[ \mathfrak{g}_G V = g_{g, \nu}, \quad g_{g, \nu} = \{ x \in g; ad_g(x) = p^2 x \}. \]

Let \( F_g \) be the set of all Borel Lie subalgebras of \( g \) fixed by the adjoint action of \( g \). It is a non connected manifold whose connected components are labelled by \( \mathfrak{g}_g^* \). In Section 3.14 we’ll introduce two central idempotents \( 1_{\nu_+, \nu_-} \) of \( \mathfrak{g}_R \). This yields a graded \( k \)-algebra decomposition

\[ \mathfrak{g}_R = \mathfrak{g}_R 1_{\nu_+, \nu_-} \oplus \mathfrak{g}_R 1_{\nu_-, \nu_+}. \]

By [VV, thm. 5.8] the graded \( k \)-algebra \( \mathfrak{g}_R 1_{\nu_+, \nu_-} \) is isomorphic to \( \text{Ext} \mathcal{L}_{g, \nu} \mathcal{L}_{g, \nu} \),

where \( \mathcal{L}_{g, \nu} \) is the direct image of the constant perverse sheaf by the projection

\[ \{ (b, x) \in F_g \times g_{g, \nu}; x \in b \} \mapsto g_{g, \nu}, \quad (b, x) \mapsto x. \]

The complex \( \mathcal{L}_{g, \nu} \) has been extensively studied by Lusztig, see e.g., [L1], [L2]. We hope to come back to this elsewhere.

(b) The results in Section 2.5 hold true if \( k \) is any field. Set \( f(x) = 1 + x \) for instance.
2.9. Induction and restriction of $H_m$-modules. For $i \in I$ we define functors
\begin{align}
E_i : H_{m+1}f\text{-Mod}_I &\to H_mf\text{-Mod}_I, \\
F_i : H_mf\text{-Mod}_I &\to H_{m+1}f\text{-Mod}_I,
\end{align}
where $E_i M \subset M$ is the generalized $i^{-1}$-eigenspace of the $X_{m+1}$-action, and where $F_i M = \text{Ind}_{H_{m+1}}^{H_m}(M \otimes k_i)$. Here $k_i$ is the 1-dimensional representation of $k[X_{m+1}^\pm]$ defined by $X_{m+1} \mapsto i$. 

3. Global bases of $^\circ V$ and projective graded $^\circ R$-modules

3.1. The Grothendieck groups of $^\circ R_m$. The graded $k$-algebra $^\circ R_m$ is free of finite rank over its center by Proposition 1.6, a commutative graded $k$-subalgebra. Therefore any simple object of $^\circ R_m$-mod is finite-dimensional and there is a finite number of isomorphism classes of simple modules in $^\circ R_m$-mod. The Abelian group $G(^\circ R_m)$ is free with a basis formed by the classes of the simple objects of $^\circ R_m$-mod. The Abelian group $K(^\circ R_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both $G(^\circ R_m)$ and $K(^\circ R_m)$ are free $A$-modules, where $v$ shifts the grading by 1. We consider the following $A$-modules
\begin{align}
^\circ K_I = \bigoplus_{m \geq 0} ^\circ K_{I,m}, \quad ^\circ K_{I,m} = K(^\circ R_m), \\
^\circ G_I = \bigoplus_{m \geq 0} ^\circ G_{I,m}, \quad ^\circ G_{I,m} = G(^\circ R_m).
\end{align}
We’ll also abbreviate
\begin{align}
^\circ K_{I,*} = \bigoplus_{m \geq 0} ^\circ K_{I,m}, \quad ^\circ G_{I,*} = \bigoplus_{m \geq 0} ^\circ G_{I,m}.
\end{align}
From now on, to unburden the notation we may abbreviate $^\circ R = ^\circ R_m$, hoping it will not create any confusion. For any $M, N$ in $^\circ R$-mod we set
\begin{align}
(M : N) = \text{gdim}(M^\omega \otimes_R N), \quad \langle M : N \rangle = \text{gdim} \text{hom}_R(M, N),
\end{align}
where $\omega$ is the involution defined in Section 1.1. The Cartan pairing is the perfect $A$-bilinear form
\begin{align}
^\circ K_I \times ^\circ G_I \to A, \quad (P, M) \mapsto \langle P : M \rangle.
\end{align}

First, we concentrate on the $A$-module $^\circ G_I$. Consider the duality
\begin{align}
^\circ R\text{-fmod} \to ^\circ R\text{-fmod}, \quad M \mapsto M^\flat = \text{hom}(M, k),
\end{align}
with the action and the grading given by
\begin{align}
(xf)(m) = f(\omega(x)m), \quad (M^\flat)_d = \text{Hom}(M_{-d}, k).
\end{align}
This duality functor yields an $A$-antilinear map
\begin{align}
^\circ G_I \to ^\circ G_I, \quad M \mapsto M^\flat.
\end{align}
Let $^\circ B$ denote the set of isomorphism classes of simple objects of $^\circ R\text{-fMod}_0$. We can now define the upper global basis of $^\circ G_I$ as follows. The proof is given in Section 3.21.
3.2. Proposition/Definition. For each $b$ in $\mathcal{B}$ there is a unique selfdual irreducible graded $\mathcal{R}$-module $\mathcal{G}^{up}(b)$ which is isomorphic to $b$ as a (non graded) $\mathcal{R}$-module. We set $\mathcal{G}^{up}(0) = 0$ and $\mathcal{G}^{up} = \{ \mathcal{G}^{up}(b); b \in \mathcal{B} \}$. Hence $\mathcal{G}^{up}$ is a $\mathcal{A}$-basis of $\mathcal{G}_I$.

Now, we concentrate on the $\mathcal{A}$-module $\mathcal{K}_I$. We equip $\mathcal{K}_I$ with the symmetric $\mathcal{A}$-bilinear form

$$\left( \begin{array}{l} (1) \\
\end{array} \right)$$

Consider the duality $\mathcal{R}^{proj} \to \mathcal{R}^{proj}, \ P \mapsto P^\sharp = \text{hom}_{\mathcal{R}}(P, \mathcal{R})$, with the action and the grading given by

$$(xf)(p) = f(p)\omega(x), \quad (P^\sharp)_d = \text{Hom}_{\mathcal{R}}(P[-d], \mathcal{R})$$

This duality functor yields an $\mathcal{A}$-antilinear map

$$\mathcal{K}_I \to \mathcal{K}_I, \quad P \mapsto P^\sharp.$$ 

Set $\mathcal{K} = \mathbb{Q}(v)$. Let $\mathcal{K} \to \mathcal{K}, \ f \mapsto \tilde{f}$ be the unique involution such that $\tilde{v} = v^{-1}$.

3.3. Definition. For each $b$ in $\mathcal{B}$ let $\mathcal{G}^{low}(b)$ be the unique indecomposable graded module in $\mathcal{R}^{proj}$ whose top is isomorphic to $\mathcal{G}^{up}(b)$. We set $\mathcal{G}^{low}(0) = 0$ and $\mathcal{G}^{low} = \{ \mathcal{G}^{low}(b); b \in \mathcal{B} \}$, a $\mathcal{A}$-basis of $\mathcal{K}_I$.

3.4. Proposition. (a) We have $\langle \mathcal{G}^{low}(b); \mathcal{G}^{up}(b') \rangle = \delta_{b,b'}$ for each $b, b'$ in $\mathcal{B}$.

(b) We have $\langle P^\sharp; M \rangle = \langle P; M^\flat \rangle$ for each $P, M$.

(c) We have $\mathcal{G}^{low}(b)^\sharp = \mathcal{G}^{low}(b)$ for each $b$ in $\mathcal{B}$.

The proof is the same as in [VV, prop. 8.4].

3.5. Example. Set $\nu = i + \theta(i)$ and $i = \theta(i)$. Consider the graded $\mathcal{R}_\nu$-modules

$$\mathcal{R}_i = \mathcal{R}_1 = \mathcal{R}_1^\nu, \quad \mathcal{L}_i = \text{top}(\mathcal{R}_i).$$

The global bases are given by

$$\mathcal{G}^{low}_\nu = \{ \mathcal{R}_i, \mathcal{R}_{\theta(i)} \}, \quad \mathcal{G}^{up}_\nu = \{ \mathcal{L}_i, \mathcal{L}_{\theta(i)} \}.$$ 

For $m = 0$ we have $\mathcal{R}_0 = \mathcal{L}_0 = \mathcal{R}_1 \oplus \mathcal{L}_1$. Set $\phi_+ = \mathcal{R}_1 \oplus 0$ and $\phi_- = 0 \oplus \mathcal{L}_1$. We have

$$\mathcal{G}^{low}_0 = \{ \phi_+, \phi_- \}.$$
3.6. Definition of the operators \(e_i, f_i, e'_i, f'_i\). In this section we’ll always assume \(m > 0\) unless specified otherwise. First, let us introduce the following notation. Let \(D_{m,1}\) be the set of minimal representative in \(^oW_{m+1}\) of the cosets in \(^oW_m \setminus ^oW_{m+1}\). Write
\[
D_{m,1; m,1} = D_{m,1} \cap (D_{m,1})^{-1}.
\]
For each element \(w\) of \(D_{m,1; m,1}\) we set
\[
W(w) = ^oW_m \cap w(^oW_m)w^{-1}.
\]
Let \(R_1\) be the \(k\)-algebra generated by elements \(1_i, \kappa_i, i \in I\), satisfying the defining relations \(1_i 1_{i'} = \delta_{i,i'} 1_i\) and \(\kappa_i = 1, \kappa_i 1_i\). We equip \(R_1\) with the grading such that \(\deg(1_i) = 0\) and \(\deg(\kappa_i) = 2\). Let
\[
R_i = 1_i R_1 = R_1 1_i, \quad L_i = \text{top}(R_i) = R_i/(\kappa_i).
\]
Then \(R_i\) is a graded projective \(R_1\)-module and \(L_i\) is simple. We abbreviate
\[
^oR_{m,1} = ^oR_m \otimes R_1, \quad ^oR_{m,1} = ^oR_m \otimes R_1.
\]
There is an unique inclusion of graded \(k\)-algebras
\[
^oR_{m,1} \rightarrow ^oR_{m+1},
\]
\[
1_i \otimes 1_i \mapsto 1_i',
\]
\[
1_i \otimes \kappa_i \mapsto \kappa_i',
\]
\[
1_i \otimes \pi_i, l \mapsto \pi_i', l,
\]
\[
1_i \otimes \sigma_i, k \mapsto \sigma_i', k,
\]
(3.2)
where, given \(i \in \theta I_m\) and \(i \in I\), we have set \(i' = \theta(i)i_i\), a sequence in \(\theta I_{m+1}\). This inclusion restricts to an inclusion \(^oR_{m,1} \subset ^oR_{m+1}\).

3.7. Lemma. The graded \(^oR_{m,1}\)-module \(^oR_{m+1}\) is free of rank \(2(m+1)\).

Proof: For each \(w\) in \(D_{m,1}\) we have the element \(\sigma_w\) in \(^oR_{m+1}\) defined in (1.5). Using filtered/graded arguments it is easy to see that
\[
^oR_{m+1} = \bigoplus_{w \in D_{m,1}} ^oR_{m,1} \sigma_w.
\]
\[\square\]

We define a triple of adjoint functors \((\psi^!, \psi^*, \psi^\vee)\) where
\[
\psi^* : \text{mod}^oR_{m+1} \rightarrow \text{mod}^oR_m \times R_1\text{-mod}
\]
is the restriction and \( \psi_1, \psi_* \) are given by

\[
\psi_1 : \begin{cases} 
\circ R_m \text{-mod} \times R_1 \text{-mod} \to \circ R_{m+1} \text{-mod}, \\
(M, M') \mapsto \circ R_{m+1} \otimes_{\circ R_m} (M \otimes M'), 
\end{cases}
\]

\[
\psi_* : \begin{cases} 
\circ R_m \text{-mod} \times R_1 \text{-mod} \to \circ R_{m+1} \text{-mod}, \\
(M, M') \mapsto \text{hom}_{R_m,1}(\circ R_{m+1}, M \otimes M'). 
\end{cases}
\]

First, note that the functors \( \psi_1, \psi_* \) commute with the shift of the grading.

Next, the functor \( \psi_* \) is exact, and it takes finite dimensional graded modules to finite dimensional ones. The right graded \( \circ R_{m,1} \)-module \( \circ R_{m+1} \) is free of finite rank. Thus \( \psi_1 \) is exact, and it takes finite dimensional graded modules to finite dimensional ones. The left graded \( \circ R_{m,1} \)-module \( \circ R_{m+1} \) is also free of finite rank. Thus the functor \( \psi_* \) is exact, and it takes finite dimensional graded modules to finite dimensional ones. Further \( \psi_1 \) and \( \psi_* \) take projective graded modules to projective ones, because they are left adjoint to the exact functors \( \psi^*, \psi_* \) respectively. To summarize, the functors \( \psi_1, \psi_* \) are exact and take finite dimensional graded modules to finite dimensional ones, and the functors \( \psi_1, \psi_* \) take projective graded modules to projective ones.

For any graded \( \circ R_m \)-module \( M \) we write

\[
f_i(M) = \circ R_{m+1} 1_{m,i} \otimes_{\circ R_m} M, \\
e_i(M) = \circ R_{m-1} \otimes_{\circ R_{m-1}} 1_{m-1,i} M.
\]

Let us explain these formulas. The symbols \( 1_{m,i} \) and \( 1_{m-1,i} \) are given by

\[
1_{m-1,i} M = \bigoplus_i 1_{\theta(i)} 1_i M, \quad i \in \theta^{m-1}.
\]

Note that \( f_i(M) \) is a graded \( \circ R_{m+1} \)-module, while \( e_i(M) \) is a graded \( \circ R_{m-1} \)-module. The tensor product in the definition of \( e_i(M) \) is relative to the graded \( k \)-algebra homomorphism

\[
\circ R_{m-1,1} \to \circ R_{m-1} \otimes R_1 \to \circ R_{m-1} \otimes R_i \to \circ R_{m-1} \otimes L_i = \circ R_{m-1}.
\]

In other words, let \( e'_i(M) \) be the graded \( \circ R_{m-1} \)-module obtained by taking the direct summand \( 1_{m-1,i} M \) and restricting it to \( \circ R_{m-1} \). Observe that if \( M \) is finitely generated then \( e'_i(M) \) may not lie in \( \circ R_{m-1} \text{-mod} \). To remedy this, since \( e'_i(M) \) affords a \( \circ R_{m-1} \otimes R \)-action we consider the graded \( \circ R_{m-1} \)-module

\[
e_i(M) = e'_i(M)/\alpha e'_i(M).
\]

3.8. Definition. The functors \( e_i, f_i \) preserve the category \( \circ R \text{-proj} \), yielding \( A \)-linear operators on \( \circ K_I \) which act on \( \circ K_{1,*} \) by the formula (3.3) and satisfy also

\[
f_i(\phi_+) = \circ R_{\theta(i)}, \quad f_i(\phi_-) = \circ R_{\theta(i)}, \quad e_i(\circ R_{\theta(j)}) = \delta_{i,j} \phi_+ + \delta_{i,j} \phi_-.
\]

Let \( e_i, f_i \) denote also the \( A \)-linear operators on \( \circ G_I \) which are the transpose of \( f_i, e_i \) with respect to the Cartan pairing.

Note that the symbols \( e_i(M), f_i(M) \) have a different meaning if \( M \) is viewed as an element of \( \circ K_I \) or if \( M \) is viewed as an element of \( \circ G_I \). We hope this will not create any confusion. The proof of the following lemma is the same as in [VV, lem. 8.9].
3.9. Lemma. (a) The operators $e_i$, $f_i$ on $\mathcal{G}_I$ are given by

$$e_i(M) = 1_{m-1,i}M, \quad f_i(M) = \text{hom}_{\mathcal{R}_{m+1}}(\mathcal{R}_{m+1}, M \otimes L_i), \quad M \in \mathcal{G}_I.$$

(b) For each $M \in \mathcal{R}_m$-mod, $M' \in \mathcal{R}_{m+1}$-mod we have

$$(e_i(M') : M) = (M' : f_i(M)).$$

(c) We have $f_i(P) = f_i(P')$ for each $P \in \mathcal{R}$-proj.

(d) We have $e_i(M) = e_i(M')$ for each $M \in \mathcal{G}_I$.

3.10. Induction of $H_m$-modules versus induction of $\mathcal{R}_m$-modules. Recall the functors $E_i$, $F_i$ on $\mathcal{G}_I$ defined in (2.2). We have also the functors

for : $\mathcal{G}_I \to \mathcal{G}_I$, $\Psi : \mathcal{G}_I \to H_m$-mod,

where for is the forgetting of the grading. Finally we define functors

$$E_i : \mathcal{R}_m$-mod \to \mathcal{R}_{m-1}$-mod, $E_i M = 1_{m-1,i}M,$

$$F_i : \mathcal{R}_m$-mod \to \mathcal{R}_{m+1}$-mod, $F_i M = \psi_i(M, L_i).$

3.11. Proposition. There are canonical isomorphisms of functors

$$E_i \circ \Psi = \Psi \circ E_i, \quad F_i \circ \Psi = \Psi \circ F_i, \quad E_i \circ \text{for} = \text{for} \circ e_i, \quad F_i \circ \text{for} = \text{for} \circ f_{\theta(i)}.$$

Proof: The proof is the same as in [VV, prop. 8.17].

3.12. Proposition. (a) The functor $\Psi$ yields an isomorphism of Abelian groups

$$\bigoplus_{m \geq 0} [\mathcal{R}_m$-mod] = \bigoplus_{m \geq 0} [H_m$-mod].$$

The functors $E_i$, $F_i$ yield endomorphisms of both sides which are intertwined by $\Psi$.

(b) The functor for factors to a group isomorphism

$$\mathcal{G}_I/(v-1) = \bigoplus_{m \geq 0} [\mathcal{R}_m$-mod].$$

Proof: Claim (a) follows from Corollary 2.7 and Proposition 3.11. Claim (b) follows from Proposition 3.2.
3.13. Type D versus type B. We can compare the previous constructions with their analogues in type B. Let $\theta K$, $\theta B$, $\theta G^{\text{low}}$, etc, denote the type B analogues of $\theta K$, $\theta B$, $\theta G^{\text{low}}$, etc. See [VV] for details. We'll use the same notation for the functors $\psi^*$, $\psi_*$, $\psi_+$, $f_+$, etc, on the type B side and on the type D side. Fix $m > 0$ and $\nu \in \theta N$ such that $|\nu| = 2m$. The inclusion of graded $k$-algebras $\theta R_\nu \subset \theta R_\nu$ in (1.2) yields a restriction functor

$$\text{res} : \theta R_\nu \text{-mod} \rightarrow \theta R_\nu \text{-mod}$$

and an induction functor

$$\text{ind} : \theta R_\nu \text{-mod} \rightarrow \theta R_\nu \text{-mod}, \quad M \mapsto \theta R_\nu \otimes_{R_\nu} M.$$ 

Both functors are exact, they map finite dimensional graded modules to finite dimensional ones, and they map projective graded modules to projective ones. Thus, they yield morphisms of $A$-modules

$$\text{res} : \theta K_{I,m} \rightarrow \theta K_{I,m}, \quad \text{res} : \theta G_{I,m} \rightarrow \theta G_{I,m},$$

$$\text{ind} : \theta K_{I,m} \rightarrow \theta K_{I,m}, \quad \text{ind} : \theta G_{I,m} \rightarrow \theta G_{I,m}.$$ 

Moreover, for any $P \in \theta K_{I,m}$ and any $L \in \theta G_{I,m}$ we have

$$\text{res}(P^\gamma) = (\text{res} P)^\gamma, \quad \text{ind}(P^\gamma) = (\text{ind} P)^\gamma$$

$$\text{res}(L^\gamma) = (\text{res} L)^\gamma, \quad \text{ind}(L^\gamma) = (\text{ind} L)^\gamma.$$ 

Note also that $\text{ind}$ and $\text{res}$ are left and right adjoint functors, because

$$\theta R_\nu \otimes_{R_\nu} M = \text{hom}_{R_\nu}(\theta R_\nu, M)$$

as graded $\theta R_\nu$-modules.

3.14. Definition. For any graded $\theta R_\nu$-module $M$ we define the graded $\theta R_\nu$-module $M^\gamma$ with the same underlying graded $k$-vector space as $M$ such that the action of $\theta R_\nu$ is twisted by $\gamma$, i.e., the graded $k$-algebra $\theta R_\nu$ acts on $M^\gamma$ by $am = \gamma(a)m$ for $a \in \theta R_\nu$ and $m \in M$. Note that $(M^\gamma)^\gamma = M$, and that $M^\gamma$ is simple (resp. projective, indecomposable) if $M$ has the same property.

For any graded $\theta R_m$-module $M$ we have canonical isomorphisms of $\theta R$-modules

$$(f_!(M))^\gamma = f_!(M^\gamma), \quad (e_!(M))^\gamma = e_!(M^\gamma).$$

The first isomorphism is given by

$$\theta R_{m+1_{m,i}} \otimes_{R_m} M \rightarrow \theta R_{m+1_{m,i}} \otimes_{R_m} M, \quad a \otimes m \mapsto \gamma(a) \otimes m.$$ 

The second one is the identity map on the vector space $1_{m,i} M$.

Recall that $\theta I^\nu$ is the disjoint union of $\theta I^\nu_+$ and $\theta I^\nu_-$. We set

$$l_{\nu,+} = \sum_{i \in \theta I^\nu_+} l_i, \quad l_{\nu,-} = \sum_{i \in \theta I^\nu_-} l_i.$$
3.15. Lemma. Let $M$ be a graded $^0\mathcal{R}_\nu$-module.

(a) Both $1_{\nu, +}$ and $1_{\nu, -}$ are central idempotents in $^0\mathcal{R}_\nu$. We have $1_{\nu, +} = \gamma(1_{\nu, -})$.
(b) There is a decomposition of graded $^0\mathcal{R}_\nu$-modules $M = 1_{\nu, +}M \oplus 1_{\nu, -}M$.
(c) We have a canonical isomorphism of $^0\mathcal{R}_\nu$-modules $\text{res} \circ \text{ind}(M) = M \oplus M^\gamma$.
(d) If there exists $a \in \{+, -\}$ such that $1_{\nu, -a}M = 0$, then there are canonical isomorphisms of graded $^0\mathcal{R}_\nu$-modules

$$M = 1_{\nu, a}M, \quad 0 = 1_{\nu, a}M^\gamma, \quad M^\gamma = 1_{\nu, -a}M^\gamma.$$

Proof: Part (a) follows from Proposition 1.6 and the equality $\varepsilon_1(\theta_{I_{\nu}^\nu}) = \theta_{I_{\nu}^\nu}$. Part (b) follows from (a), (c) is given by definition, and (d) follows from (a), (b).

Now, let $m$ and $\nu$ be as before. Given $i \in I$, we set $\nu' = \nu + i + \theta(i)$. There is an obvious inclusion $W_m \subset W_{m+1}$. Thus the group $W_m$ acts on $^\theta I_{\nu'}$, and the map

$$^\theta I_{\nu'} \to ^\theta I_{\nu}, \quad i \mapsto \theta(i)i_i$$

is $W_m$-equivariant. Thus there is $a_i \in \{+, -\}$ such that the image of $^\theta I_{\nu'}$ is contained in $^\theta I_{a_i}$, and the image of $^\theta I_{-\nu'}$ is contained in $^\theta I_{-a_i}$.

3.16. Lemma. Let $M$ be a graded $^0\mathcal{R}_\nu$-module such that $1_{\nu, -a}M = 0$, with $a = +, -$. Then we have

$$1_{\nu', -a, a}f_i(M) = 0, \quad 1_{\nu', a, a}f_{\theta(i)}(M) = 0.$$

Proof: We have

$$1_{\nu', -a, a}f_i(M) = 1_{\nu', -a, a}^0\mathcal{R}_{\nu'}1_{\nu, i} \otimes ^0\mathcal{R}_{\nu'} M$$

$$= ^0\mathcal{R}_{\nu'}1_{\nu', -a, a}1_{\nu, i}1_{\nu, a} \otimes ^0\mathcal{R}_{\nu'}M.$$ 

Here we have identified $1_{\nu, a}$ with the image of $(1_{\nu, a}, 1_i)$ via the inclusion (3.2). The definition of this inclusion and that of $a_i$ yield that

$$1_{\nu', a, a}1_{\nu, a}, \quad 1_{\nu', -a, a}1_{\nu, a} = 0.$$

The first equality follows. Next, note that for any $i \in ^\theta I_{\nu'}$, the sequences $\theta(i)i_i$ and $i_i\theta(i) = \varepsilon_{m+1}(\theta(i)i_i)$ always belong to different $^0W_{m+1}$-orbits. Thus, we have $a_{\theta(i)} = -a_i$. So the second equality follows from the first.

Consider the following diagram

$$\xymatrix{ ^0\mathcal{R}_{\nu'}-\text{mod} \times ^0\mathcal{R}_{\nu'}-\text{mod} \ar[r]_-{\psi} & ^0\mathcal{R}_{\nu'}-\text{mod} \ar[l]^-{\psi^*} \\
\text{res} \times \text{id} \ar[u] & \text{ind} \times \text{id} \ar[u] \ar[l] \ar[r] & \text{res} \ar[u] \ar[r] & \text{ind} \ar[u] \\
^0\mathcal{R}_{\nu'}-\text{mod} \times \mathcal{R}_{\nu'}-\text{mod} \ar[r]_-{\psi} & ^0\mathcal{R}_{\nu'}-\text{mod} \ar[l]^-{\psi^*}.}$$
3.17. **Lemma.** There are canonical isomorphisms of functors
\[
\text{ind} \circ \psi_! = \psi_! \circ (\text{ind} \times \text{id}), \quad \psi^* \circ \text{id} = (\text{ind} \times \text{id}) \circ \psi^*, \quad \text{ind} \circ \psi_* = \psi_* \circ (\text{id} \times \text{id}), \\
\text{res} \circ \psi_! = \psi_! \circ (\text{res} \times \text{id}), \quad \psi^* \circ \text{res} = (\text{res} \times \text{id}) \circ \psi^*, \quad \text{res} \circ \psi_* = \psi_* \circ (\text{res} \times \text{id}).
\]

**Proof:** The functor ind is left and right adjoint to res. Therefore it is enough to prove the first two isomorphisms in the first line. The isomorphism
\[
\text{ind} \circ \psi_! = \psi_! \circ (\text{ind} \times \text{id})
\]
comes from the associativity of the induction. Let us prove that
\[
\psi^* \circ \text{id} = (\text{ind} \times \text{id}) \circ \psi^*.
\]
For any $M$ in $^\theta R_{\nu'} \text{-mod}$, the obvious inclusion $^\theta R_{\nu} \otimes R_i \subset ^\theta R_{\nu'}$ yields a map
\[
(\text{ind} \times \text{id}) \psi^* (M) = (^\theta R_{\nu} \otimes R_i) \otimes_{^\theta R_{\nu} \otimes R_i} R_{\nu'} \psi^* (M) \rightarrow \psi^* (^\theta R_{\nu'} \otimes_{^\theta R_{\nu} \otimes R_i} R_{\nu'} M).
\]
Combining it with the obvious map $\psi^* \circ (\text{id} \times \text{id}) \circ \psi^*$ we get a morphism of $^\theta R_{\nu'} \otimes_{^\theta R_{\nu} \otimes R_i} R_{\nu'} M$.

We need to show that it is bijective. This is obvious because at the level of vector spaces, the map above is given by
\[
M \oplus (\pi_{1,\nu} \otimes M) \rightarrow M \oplus (\pi_{1,\nu'} \otimes M), \quad m + \pi_{1,\nu} \otimes n \mapsto m + \pi_{1,\nu'} \otimes n.
\]
Here $\pi_{1,\nu}$ and $\pi_{1,\nu'}$ denote the element $\pi_1$ in $^\theta R_{\nu}$ and $^\theta R_{\nu'}$ respectively.

3.18. **Corollary.** (a) The operators $e_i, f_i$ on $^\theta K_{I,*}$ and on $^\theta K_{I,*}$ are intertwined by the maps $\text{ind}, \text{res}$, i.e., we have
\[
e_i \circ \text{ind} = \text{ind} \circ e_i, \quad f_i \circ \text{ind} = \text{ind} \circ f_i, \quad e_i \circ \text{res} = \text{res} \circ e_i, \quad f_i \circ \text{res} = \text{res} \circ f_i.
\]
(b) The same result holds for the operators $e_i, f_i$ on $^\theta G_{I,*}$ and on $^\theta G_{I,*}$.

3.19. Now, we concentrate on non graded irreducible modules. First, let
\[
\text{Res} : {}^\theta R_{\nu} \text{-Mod} \rightarrow {}^\theta R_{\nu'} \text{-Mod}, \quad \text{Ind} : {}^\theta R_{\nu'} \text{-Mod} \rightarrow {}^\theta R_{\nu} \text{-Mod}
\]
be the (non graded) restriction and induction functors. We have
\[
\text{for} \circ \text{res} = \text{Res} \circ \text{for}, \quad \text{for} \circ \text{ind} = \text{Ind} \circ \text{for}.
\]
3.20. Lemma. Let $L, L'$ be irreducible $^0\mathcal{R}_v$-modules.

(a) The $^0\mathcal{R}_v$-modules $L$ and $L'$ are not isomorphic.
(b) $\text{Ind}(L)$ is an irreducible $^0\mathcal{R}_v$-module, and every irreducible $^0\mathcal{R}_v$-module is obtained in this way.
(c) $\text{Ind}(L) \simeq \text{Ind}(L')$ iff $L' \simeq L$ or $L'$.

Proof: For any $^0\mathcal{R}_v$-module $M \neq 0$, both $1_{\nu,+}M$ and $1_{\nu,-}M$ are nonzero. Indeed, we have $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$, and we may suppose that $1_{\nu,+}M \neq 0$. The automorphism $M \to M$, $m \mapsto \pi_1 m$ takes $1_{\nu,+}M$ to $1_{\nu,-}M$ by Lemma 3.15(a). Hence $1_{\nu,-}M \neq 0$.

Now, to prove part (a), suppose that $\phi : L \to L'$ is an isomorphism of $^0\mathcal{R}_v$-modules. We can regard $\phi$ as a $\gamma$-antilinear map $L \to L$. Since $L$ is irreducible, by Schur’s lemma we may assume that $\phi^2 = \text{id}_L$. Then $L$ admits a $^0\mathcal{R}_v$-module structure such that the $^0\mathcal{R}_v$-action is as before and $\pi_1$ acts as $\phi$. Thus, by the discussion above, neither $1_{\nu,+}L$ nor $1_{\nu,-}L$ is zero. This contradicts the fact that $L$ is an irreducible $^0\mathcal{R}_v$-module.

Parts (b), (c) follow from (a) by Clifford theory, see e.g., [RR, appendix].

We can now prove Proposition 3.2.

3.21. Proof of Proposition 3.2. Let $b \in \mathcal{B}$. We may suppose that $b = 1_{\nu,+}b$. By Lemma 3.20(b) the module $\% = \text{Ind}(b)$ lies in $^0\mathcal{B}$. By [VV, prop. 8.2] there exists a unique self-dual irreducible graded $^0\mathcal{R}$-module $^0\mathcal{G}^\text{up}(\%)$ which is isomorphic to $\%$ as a non graded module. Set

\[ ^0\mathcal{G}^\text{up}(b) = 1_{\nu,+}\text{res}(^0\mathcal{G}^\text{up}(\%)). \]

By Lemma 3.15(d) we have $^0\mathcal{G}^\text{up}(b) = b$ as a non graded $^0\mathcal{R}$-module, and by (3.5) it is self-dual. This proves existence part of the proposition. To prove the uniqueness, suppose that $M$ is another module with the same properties. Then $\text{ind}(M)$ is a self-dual graded $^0\mathcal{R}$-module which is isomorphic to $\%$ as a non graded $^0\mathcal{R}$-module. Thus we have $\text{ind}(M) = ^0\mathcal{G}^\text{up}(\%)$ by loc. cit. By Lemma 3.15(d) we have also

\[ M = 1_{\nu,+}\text{res}(^0\mathcal{G}^\text{up}(\%)). \]

So $M$ is isomorphic to $^0\mathcal{G}^\text{up}(b)$.

3.22. The crystal operators on $^0\mathcal{G}_I$ and $^0\mathcal{B}$. Fix a vertex $i$ in $I$. For each irreducible graded $^0\mathcal{R}_m$-module $M$ we define

\[ \tilde{e}_i(M) = \text{soc}(e_i(M)), \quad \tilde{f}_i(M) = \text{top}\psi_i(M, L_i), \quad e_i(M) = \max\{n \geq 0; e_i^n(M) \neq 0\}. \]

3.23. Lemma. Let $M$ be an irreducible graded $^0\mathcal{R}$-module such that $e_i(M), f_i(M)$ belong to $^0\mathcal{G}_I,^*$. We have

\[ \text{ind}(\tilde{e}_i(M)) = \tilde{e}_i(\text{ind}(M)), \quad \text{ind}(\tilde{f}_i(M)) = \tilde{f}_i(\text{ind}(M)), \quad e_i(M) = e_i(\text{ind}(M)). \]

In particular, $\tilde{e}_i(M)$ is irreducible or zero and $\tilde{f}_i(M)$ is irreducible.
Proof: By Corollary 3.18 we have \( \text{ind}(e_i(M)) = e_i(\text{ind}(M)) \). Thus, since \( \text{ind} \) is an exact functor we have \( \text{ind}(\tilde{e}_i(M)) \subset e_i(\text{ind}(M)) \). Since \( \text{ind} \) is an additive functor, by Lemma 3.20(b) we have indeed

\[
\text{ind}(\tilde{e}_i(M)) \subset e_i(\text{ind}(M)).
\]

Note that \( \text{ind}(M) \) is irreducible by Lemma 3.20(b), thus \( e_i(\text{ind}(M)) \) is irreducible by [VV, prop. 8.21]. Since \( \text{ind}(\tilde{e}_i(M)) \) is nonzero, the inclusion is an isomorphism. The fact that \( \text{ind}(\tilde{e}_i(M)) \) is irreducible implies in particular that \( \tilde{e}_i(M) \) is simple. The proof of the second isomorphism is similar. The third equality is obvious.

\[\square\]

Similarly, for each irreducible \( \mathcal{R} \)-module \( b \) in \( \mathcal{B} \) we define

\[
\tilde{E}_i(b) = \text{soc}(E_i(b)), \quad \tilde{F}_i(b) = \text{top}(F_i(b)), \quad \varepsilon_i(b) = \max\{n \geq 0; E_i^n(b) \neq 0\}.
\]

Hence we have

\[
\text{for } o \tilde{e}_i = \tilde{E}_i \circ o, \quad \text{for } o \tilde{f}_i = \tilde{F}_i \circ o, \quad \varepsilon_i = \varepsilon_i \circ o.
\]

3.24. Proposition. For each \( b, b' \) in \( \mathcal{B} \) we have

(a) \( \tilde{F}_i(b) \in \mathcal{B} \),
(b) \( \tilde{E}_i(b) \in \mathcal{B} \cup \{0\} \),
(c) \( \tilde{F}_i(b) = b' \iff \tilde{E}_i(b) = b \),
(d) \( \varepsilon_i(b) = \max\{n \geq 0; \tilde{E}_i^n(b) \neq 0\} \),
(e) \( \varepsilon_i(\tilde{F}_i(b)) = \varepsilon_i(b) + 1 \),
(f) \( \text{if } \tilde{E}_i(b) = 0 \text{ for all } i \text{ then } b = \phi_\varepsilon \).

Proof: Part (c) follows from adjunction. The other parts follow from [VV, prop. 3.14] and Lemma 3.23.

\[\square\]

3.25. Remark. For any \( b \in \mathcal{B} \) and any \( i \neq j \) we have \( \tilde{F}_i(b) \neq \tilde{F}_j(b) \). This is obvious if \( j \neq \theta(i) \). Because in this case \( \tilde{F}_i(b) \) and \( \tilde{F}_j(b) \) are \( \mathcal{R}_\nu \)-modules for different \( \nu \). Now, consider the case \( j = \theta(i) \). We may suppose that \( \tilde{F}_i(b) = 1_{\nu, \theta_i} \tilde{F}_i(b) \) for certain \( \nu \). Then by Lemma 3.16 we have \( 1_{\nu, \theta_i} \tilde{F}_i(b) = 0 \). In particular \( \tilde{F}_i(b) \) is not isomorphic to \( \tilde{F}_{\theta_i}(b) \).

3.26. The algebra \( \mathcal{B} \) and the \( \mathcal{B} \)-module \( \mathcal{V} \). Following [EK1,2,3] we define a \( \mathcal{K} \)-algebra \( \mathcal{B} \) as follows.

3.27. Definition. Let \( \mathcal{B} \) be the \( \mathcal{K} \)-algebra generated by \( e_i, f_i \) and invertible elements \( t_i, i \in I \), satisfying the following defining relations

(a) \( t_it_j = t_jt_i \) and \( t_{\theta(i)} = t_i \) for all \( i, j \),
(b) \( t_iet_i^{-1} = v^{i+j+\theta(i)\cdot j} \epsilon_j \) and \( t_ift_i^{-1} = v^{-i-j-\theta(i)\cdot j} f_j \) for all \( i, j \),
(c) \( e_i f_j = v^{i+j} f_j e_i + \delta_{i,j} + \delta_{\theta(i),j} t_i \) for all \( i, j \).
Here and below we use the following notation

\[
\begin{aligned}
\theta(a) &= \theta^a/(a)!, \\
(a) &= \sum_{l=1}^{a} a^{a+1-2l}, \\
\langle a! \rangle &= \prod_{l=1}^{m}(l).
\end{aligned}
\]

We can now construct a representation of \( \theta \mathcal{B} \) as follows. By base change, the operators \( e_i, f_i \) in Definition 3.8 yield \( \mathcal{K} \)-linear operators on the \( \mathcal{K} \)-vector space

\[
\mathcal{V} = \mathcal{K} \otimes \mathcal{A} \mathcal{K}_I.
\]

We equip \( \mathcal{V} \) with the \( \mathcal{K} \)-bilinear form given by

\[
(M : N)_{\mathcal{K}X} = (1 - v^2)^m (M : N), \quad \forall M, N \in \mathcal{V}.
\]

3.28. Theorem. (a) The operators \( e_i, f_i \) define a representation of \( \mathcal{B} \) on \( \mathcal{V} \).

The \( \mathcal{B} \)-module \( \mathcal{V} \) is generated by linearly independent vectors \( \phi_+ \) and \( \phi_- \) such that for each \( i \) we have

\[
e_i \phi_\pm = 0, \quad t_i \phi_\pm = \phi_\mp, \quad \{x \in \mathcal{V}; e_jx = 0, \forall j\} = k \phi_+ \oplus k \phi_-.
\]

(b) The symmetric bilinear form on \( \mathcal{V} \) is non-degenerate. We have \( (\phi_a : \phi_{a'})_{\mathcal{K}X} = \delta_{a,a'} \) for \( a, a' = +, - \), and \( (e_i x : y) = (x : f_j y)_{\mathcal{K}X} \) for \( i \) in \( I \) and \( x, y \in \mathcal{V} \).

Proof: For each \( i \) in \( I \) we define the \( \mathcal{A} \)-linear operator \( t_i \) on \( \mathcal{V} \) by setting

\[
t_i \phi_\pm = \phi_\mp \quad \text{and} \quad t_i \mathcal{P} = v^{-\nu(i+\theta(i))} \mathcal{P}, \quad \forall \mathcal{P} \in \mathcal{V}.
\]

We must prove that the operators \( e_i, f_i, \) and \( t_i \) satisfy the relations of \( \mathcal{B} \). The relations \( (a), (b) \) are obvious. The relation \( (d) \) is standard. It remains to check \( (c) \). For this we need a version of the Mackey’s induction-restriction theorem. Note that for \( m > 1 \) we have

\[
D_{m,1;m,1} = \{e, s_m, \varepsilon_{m+1} \partial_1\},
\]

\[
W(e) = \mathcal{V}_m, \quad W(s_m) = \mathcal{V}_{m-1}, \quad W(\varepsilon_{m+1} \partial_1) = \mathcal{V}_m.
\]

Recall also that for \( m = 1 \) we have set \( \mathcal{V}_1 = \{e\} \).

3.29. Lemma. Fix \( i, j \) in \( I \). Let \( \nu, \theta \) in \( \mathbb{N}I \) be such that \( \nu + i + \theta(i) = \mu + j + \theta(j) \).

Put \(|\nu| = |\mu| = 2m\). The graded \( \mathcal{V} \)-bimodule \( \mathcal{V}_{\mu,j} \mathcal{V}_{\mu+1,j} \mathcal{V}_{\mu+2,j} \) has a filtration by graded bimodules whose associated graded is isomorphic to

\[
\delta_{i,j} \left( \mathcal{V}_{\nu} \otimes \mathcal{R}_i \right) \oplus \delta_{\theta(i),j} \left( \mathcal{V}_{\nu} \otimes \mathcal{R}_{\theta(i)} \right) \left[ \partial' \right] \oplus \mathcal{A}[d],
\]

where \( A \) is equal to

\[
\begin{aligned}
(\mathcal{V}_{m,1} \mathcal{V}_{\nu,i} \otimes \mathcal{R}_i) \otimes \left( \mathcal{V}_{\nu,i} \mathcal{V}_{m} \otimes \mathcal{R}_i \right) & \quad \text{if} \ m > 1, \\
(\mathcal{V}_{\theta(i)} \otimes \mathcal{R}_i \otimes \mathcal{V}_{\theta(i)} \otimes \mathcal{R}_j) \oplus (\mathcal{V}_j \otimes \mathcal{R}_i \otimes \mathcal{V}_{\theta(i)} \otimes \mathcal{R}_j) & \quad \text{if} \ m = 1.
\end{aligned}
\]
Here we have set $\nu' = \nu - j - \theta(j)$, $R' = \mathcal{O}_{m-1,1} \otimes R_1$, $i = i\theta(i)$, $j = j\theta(j)$, $d = -i \cdot j$, and $d' = -\nu \cdot (i + \theta(i))/2$.

The proof is standard and is left to the reader. Now, recall that for $m > 1$ we have

$$f_j(P) = \mathcal{O}_{m+1,m,j} \otimes \mathcal{O}_{m-1} (P \otimes R_1), \quad e'_j(P) = 1_{m-1,i} P,$$

where $1_{m-1,i} P$ is regarded as a $\mathcal{O}_{m-1}$-module. Therefore we have

$$e'_j f_j(P) = 1_{m,1} \mathcal{O}_{m+1,m,j} \otimes \mathcal{O}_{m-1} (P \otimes R_1),$$

$$f_j e'_j(P) = \mathcal{O}_{m,1} 1_{m-1,j} \otimes \mathcal{O}_{m-1,1} (1_{m-1,i} P \otimes R_1).$$

Therefore, up to some filtration we have the following identities

- $e'_j f_i(P) = P \otimes R_i + f_i e'_j(P)[-2],$
- $e'_j f_{\theta(i)}(P) = P^{\prime} \otimes R_{\theta(i)}[-\nu \cdot (i + \theta(i))/2] + f_{\theta(i)} e'_j(P)[-i \cdot \theta(i)],$
- $e'_j f_j(P) = f_j e'_j(P)[-i \cdot j]$ if $i \neq j, \theta(j).$

These identities also hold for $m = 1$ and $P = \mathcal{O}_{\theta(i)j}$ for any $i \in I$. The first claim of part (a) follows because $R_i = k \oplus R_i[2]$. The fact that $\mathcal{O}_V$ is generated by $\phi_{\pm}$ is a corollary of Proposition 3.31 below. Part (b) of the theorem follows from [KM, prop. 2.2(ii)] and Lemma 3.9(b).

\[ \square \]

3.30. Remarks. (a) The $\mathcal{O}_B$-module $\mathcal{O}_V$ is the same as the $\mathcal{O}_B$-module $\mathcal{O}_V$ from [KM, prop. 2.2]. The involution $\sigma : \mathcal{O}_V \to \mathcal{O}_V$ in [KM, rem. 2.5(ii)] is given by $\sigma(P) = P^{\prime}$. It yields an involution of $\mathcal{O}_B$ in the obvious way. Note that Lemma 3.20(a) yields $\sigma(b) \neq b$ for any $b \in \mathcal{O}_B$.

(b) Let $\mathcal{O}_V$ be the $\mathcal{O}_B$-module $\mathcal{O} \otimes \mathcal{O}_B$ and let $\phi$ be the class of the trivial $\mathcal{O}_B$-module $k$, see [VV, thm. 8.30]. We have an inclusion of $\mathcal{O}_B$-modules

$$\mathcal{O}_V \to \mathcal{O}_V, \quad \phi \mapsto \phi_+ \oplus \phi_- \quad P \mapsto \text{res}(P).$$

3.31. Proposition. For any $b \in \mathcal{O}_B$ the following holds

(a) $$f_i(\mathcal{O}_b^{\text{low}}(b)) = \langle \varepsilon_i(b) + 1 \rangle \mathcal{O}_b^{\text{low}}(F_i b) + \sum_{b'} f_{b,b'} \mathcal{O}_b^{\text{low}}(b'),$$

$$b' \in \mathcal{O}_B, \quad \varepsilon_i(b') > \varepsilon_i(b) + 1, \quad f_{b,b'} \in v^{2 - \varepsilon_i(b')} \mathbb{Z}[v].$$

(b) $$e_i(\mathcal{O}_b^{\text{low}}(b)) = v^{1 - \varepsilon_i(b)} \mathcal{O}_b^{\text{low}}(E_i b) + \sum_{b'} e_{b,b'} \mathcal{O}_b^{\text{low}}(b'),$$

$$b' \in \mathcal{O}_B, \quad \varepsilon_i(b') \geq \varepsilon_i(b), \quad e_{b,b'} \in v^{1 - \varepsilon_i(b')} \mathbb{Z}[v].$$

Proof: We prove part (a), the proof for (b) is similar. If $\mathcal{O}_b^{\text{low}}(b) = \phi_{\pm}$ this is obvious. So we assume that $\mathcal{O}_b^{\text{low}}(b)$ is a $\mathcal{O}_B$-module for $m \geq 1$. Fix $\nu \in \mathcal{O}_M$
such that $f_i(\upgamma G_{\text{low}}(b))$ is a $\upalpha R_{\nu}$-module. We'll abbreviate $1_{\nu,a} = 1_a$ for $a \in \{+,-\}$. Since $\upgamma G_{\text{low}}(b)$ is indecomposable, it fulfills the condition of Lemma 3.16. So there exists $a \in \{+,-\}$ such that $1_{-a}f_i(\upgamma G_{\text{low}}(b)) = 0$. Thus, by Lemma 3.15(c), (d) and Corollary 3.18 we have
\[
f_i(\upgamma G_{\text{low}}(b)) = 1_a \text{res ind } f_i(\upgamma G_{\text{low}}(b)) = 1_a \text{res } f_i(\upgamma G_{\text{low}}(b)).
\]
Note that $\theta b = \text{Ind}(b)$ belongs to $\theta B$ by Lemma 3.20. Hence (3.5) yields
\[
\text{ind}(\upgamma G_{\text{low}}(b)) = \theta G_{\text{low}}(\theta b).
\]
We deduce that $f_i(\upgamma G_{\text{low}}(b)) = 1_a \text{res } f_i(\theta G_{\text{low}}(b))$.

Now, write $f_i(\theta G_{\text{low}}(b)) = \sum f_{\theta, \theta'} G_{\text{low}}(\theta'')$ with $\theta'' \in \theta B$. Then we have
\[
f_i(\upgamma G_{\text{low}}(b)) = \sum f_{\theta, \theta'} 1_a \text{res } (\theta G_{\text{low}}(\theta'')).
\]
For any $\theta' \in \theta B$ the $\upalpha R$-module $1_a \text{Res}(\theta')$ belongs to $\upalpha B$. Thus we have
\[
1_a \text{res } (\theta G_{\text{low}}(\theta')) = \theta G_{\text{low}}(1_a \text{Res}(\theta')).
\]
If $\theta' \neq \theta''$ then $1_a \text{Res}(\theta'') \neq 1_a \text{Res}(\theta''')$, because $\theta' = \text{Ind}(1_a \text{Res}(\theta'))$. Thus
\[
f_i(\upgamma G_{\text{low}}(b)) = \sum f_{\theta, \theta'} G_{\text{low}}(1_a \text{Res}(\theta')),
\]
and this is the expansion of the lhs in the lower global basis. Finally, we have
\[
\varepsilon_i(1_a \text{Res}(\theta')) = \varepsilon_i(\theta')
\]
by Lemma 3.23. So part (a) follows from [VV, prop. 10.11(b), 10.16].

**3.32. The global bases of $\upgamma V$.** Since the operators $e_i, f_i$ on $\upgamma V$ satisfy the relations $e_i f_i = v^{-2} f_i e_i + 1$, we can define the modified root operators $\tilde{e}_i, \tilde{f}_i$ on the $\theta B$-module $\upgamma V$ as follows. For each $u \in \upgamma V$ we write
\[
u = \sum_{n \geq 0} f_i^{(n)} u_n \text{ with } e_i u_n = 0,
\]
\[
\tilde{e}_i(u) = \sum_{n \geq 1} f_i^{(n-1)} u_n, \quad \tilde{f}_i(u) = \sum_{n \geq 0} f_i^{(n+1)} u_n.
\]
Let $R \subset \mathcal{K}$ be the set of functions which are regular at $v = 0$. Let $\upgamma L$ be the $R$-submodule of $\upgamma V$ spanned by the elements $f_i^l \ldots f_{i_l}(\phi_{l})$ with $l \geq 0$, $i_1, \ldots, i_l \in I$. The following is the main result of the paper.
3.33. Theorem. (a) We have

$$
\mathcal{L} = \bigoplus_{b \in \mathbb{B}} R \mathcal{G}^\text{low}(b), \quad \bar{e}_1(\mathcal{L}) \subset \mathcal{L}, \quad \bar{f}_1(\mathcal{L}) \subset \mathcal{L},
$$

$$
\bar{e}_i(\mathcal{G}^\text{low}(b)) = \mathcal{G}^\text{low}(\tilde{E}_i(b)) \mod \nu \mathcal{L}, \quad \bar{f}_i(\mathcal{G}^\text{low}(b)) = \mathcal{G}^\text{low}(\tilde{F}_i(b)) \mod \nu \mathcal{L}.
$$

(b) The assignment \( b \mapsto \mathcal{G}^\text{low}(b) \mod \nu \mathcal{L} \) yields a bijection from \( \mathbb{B} \) to the subset of \( \mathcal{L}/\nu \mathcal{L} \) consisting of the \( \tilde{E}_1, \ldots, \tilde{E}_i \)'s. Further \( \mathcal{G}^\text{low}(b) \) is the unique element \( x \in \mathcal{V} \) such that \( x^2 = x \) and \( x = \mathcal{G}^\text{low}(b) \mod \nu \mathcal{L} \).

(c) For each \( b, b' \) in \( \mathbb{B} \) let \( \mathcal{E}_{i,b,b'}, \mathcal{F}_{i,b,b'} \in A \) be the coefficients of \( \mathcal{G}^\text{low}(b') \) in \( e_i(\mathcal{G}^\text{low}(b')), f_i(\mathcal{G}^\text{low}(b')) \) respectively. Then we have

$$
\mathcal{E}_{i,b,b'}|_{v=1} = [F_i \Psi \mathbf{for}(\mathcal{G}^\text{up}(b')) : \Psi \mathbf{for}(\mathcal{G}^\text{up}(b))],
$$

$$
\mathcal{F}_{i,b,b'}|_{v=1} = [E_i \Psi \mathbf{for}(\mathcal{G}^\text{up}(b')) : \Psi \mathbf{for}(\mathcal{G}^\text{up}(b))].
$$

Proof: Part (a) follows from [EK3, thm. 4.1, cor. 4.4], [E, Section 2.3], and Proposition 3.11. The first claim in (b) follows from (a). The second one is obvious. Part (c) follows from Proposition 3.11. More precisely, by duality we can regard \( \mathcal{E}_{i,b,b'}, \mathcal{F}_{i,b,b'} \) as the coefficients of \( \mathcal{G}^\text{up}(b) \) in \( e_i(\mathcal{G}^\text{low}(b')) \) and \( f_i(\mathcal{G}^\text{low}(b')) \) respectively. Therefore, by Proposition 3.11 we can regard \( \mathcal{E}_{i,b,b'}|_{v=1}, \mathcal{F}_{i,b,b'}|_{v=1} \) as the coefficients of \( \Psi \mathbf{for}(\mathcal{G}^\text{up}(b')) \) in \( F_i \Psi \mathbf{for}(\mathcal{G}^\text{up}(b')) \) and \( E_i \Psi \mathbf{for}(\mathcal{G}^\text{up}(b')) \) respectively.

\[ \square \]

References


[EK3] Enomoto, N., Kashiwara, M., Symmetric Crystals for \( \mathfrak{g}_\mathbb{C} \), Publications of the Research Institute for Mathematical Sciences, Kyoto University 44 (2008), 837-891.


