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On the Laplace transform of perpetuities with thin tails

Jean-Baptiste Bardet, Hélène Guérin, Florent Malrieu

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Abstract

We consider the random variables \( R \) which are solutions of the distributional equation \( R \overset{\mathcal{L}}{=} MR + Q \), where \((Q, M)\) is independent of \( R \) and \( |M| \leq 1 \). Goldie and Gröbel showed that the tails of \( R \) are no heavier than exponential. Alsmeyer and al provide a complete description of the domain of the Laplace transform of \( R \). We present here a simple proof in a particular case and an extension to the Markovian case.

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1 Introduction

We define on some probability space \((\Omega, \mathcal{A}, \mathbb{P})\) a couple of random variables \((M, Q)\), a sequence \((M_n, Q_n)_{n \geq 0}\) of independent and identically distributed random vectors with the same law as \((M, Q)\), and \(R_0\) a random variable independent of the sequence \((M_n, Q_n)_{n \geq 0}\). Define the sequence \((R_n)_{n \geq 0}\) by

\[
    R_{n+1} = M_n R_n + Q_n,
\]

for any \( n \geq 0 \). This sequence has been extensively studied in the last decades. Under weak assumptions (see [8]) which are obviously fulfilled in our setting, it can be shown that the sequence \((R_n)_{n \geq 0}\) converges almost surely to a random variable \( R \) such that

\[
    R \overset{\mathcal{L}}{=} MR + Q,
\]

where \( R \) is independent of \((M, Q)\).

In [7], Kesten established that \( R \) is in general heavy-tailed (i.e. not all the moments of \( R \) are finite) even if \( Q \) is light-tailed as soon as \( |M| \) can be greater than 1. Nevertheless, Goldie and Gröbel [4] have shown that \( R \) can have some exponential moments if \( |M| \leq 1 \). In particular, if \( Q \) and \( M \) are nonnegative the following result holds.

**Theorem 1.1** (Goldie, Gröbel [4]). Assume that

\[
    \mathbb{P}(Q \geq 0, 0 \leq M \leq 1) = 1, \quad \mathbb{P}(M < 1) > 0
\]

and that there is \( v_Q > 0 \) (possibly infinite) such that

\[
    \mathbb{E}(e^{vQ}) \begin{cases} < +\infty & \text{if } v < v_Q, \\ = +\infty & \text{if } v > v_Q. \end{cases}
\]

Then, the Laplace transform \( v \mapsto \mathbb{E}(e^{vR}) \) of the solution \( R \) of (2) is finite on the set \((-\infty, v_{GG})\) with \( v_{GG} = v_Q \vee \sup \{ v \geq 0, \mathbb{E}(e^{vQ}M) < 1 \} \).
In fact, the domain of the Laplace transform of $R$ is larger than $(-\infty, v_{GG})$. In [1], a full description of this domain is established. Let us provide a simple proof under the assumptions of Theorem 1.1.

2 The main result

**Theorem 2.1.** Under the assumptions of Theorem 1.1, assuming furthermore that $R_0$ is non-negative and has all its exponential moments finite, then

$$\sup_{n \geq 0} \mathbb{E}(e^{vR_n}) < +\infty \quad \text{and} \quad \mathbb{E}(e^{vR}) < +\infty$$

for any $v < v_c$, where

$$v_c = v_Q \wedge \sup\{v \geq 0, \mathbb{E}(e^{vQ1_{\{M=1\}}}) < 1\}.$$  

Moreover, for any $v > v_c$, $\sup_{n \geq 0} \mathbb{E}(e^{vR_n}) = +\infty$ and $\mathbb{E}(e^{vR}) = +\infty$.

For other recent generalizations of [4], the interested reader is referred to [6], where the authors give sharper results than ours on the tails for some specific examples.

**Proof of Theorem 2.1.** Let us start this section with the main lines of the proof of Theorem 1.1 of Goldie and Gr"ubel [4]. For $\rho > 0$, let $\mathcal{M}_\rho$ be the set of probability measures on $\mathbb{R}_+$ with finite exponential moment of order $\rho$, and $d_\rho$ a distance defined on $\mathcal{M}_\rho$ by:

$$d_\rho(\mu, \nu) = \int_0^\infty e^{\rho u} |\mu[u, \infty) - \nu[u, \infty)| \, du.$$  

Define the application $T$ on $\mathcal{M}_\rho$ as follows: for $X$ with law $\mu \in \mathcal{M}_\rho$, $T\mu$ is the law of $Q + M X$ with $(M, Q)$ independent of $X$. It is shown in [4] that,

$$d_\rho(T\mu, T\nu) \leq \mathbb{E}(e^{\rho Q} M) d_\rho(\mu, \nu).$$

Since

$$\mathbb{E}(e^{vX}) = v \int_0^\infty e^{v u} \mathbb{P}(X \geq u) \, du,$$

one can show that, for any $n \geq 0$ and $v < \min(v_0, v_Q)$ with $v_0 = \sup\{v \geq 0, \mathbb{E}(e^{vQ} M) < 1\},$

$$\mathbb{E}(e^{vR_n}) \leq v \frac{1 - \mathbb{E}(e^{vQ} M^n)}{1 - \mathbb{E}(e^{vQ} M)} d_\rho(T\mu_0, \mu_0) + \mathbb{E}(e^{vR_0}).$$

In others words, Goldie and Gr"ubel [4] established that for any $v < \min(v_0, v_Q)$, $(\mathbb{E}(e^{vR_n}))_n$ is uniformly bounded. This estimate can be extended to a larger domain.

Let us define $v_1 = \sup\{v \geq 0, \mathbb{E}(e^{vQ1_{\{M=1\}}}) < 1\}$ and $v_c = \min(v_1, v_Q)$. Let us fix $v < v_c$ and choose $\varepsilon > 0$ such that

$$\rho := \mathbb{E}(e^{vQ1_{\{1-\varepsilon < M \leq 1\}}}) < 1.$$
Let us notice that we have in fact more: for the same \( \varepsilon \), the statement
\[
\mathbb{E}(e^{vR_{n+1}}) = \mathbb{E}(e^{v(M_n R_n + Q_n)})
\]
implies that
\[
\mathbb{E}(e^{v((1-\varepsilon) R_n + Q_n)}) 1_{\{M_n \leq 1 - \varepsilon\}} + \mathbb{E}(e^{v(R_n + Q_n)}) 1_{\{1 - \varepsilon < M_n \leq 1\}}
\]
where \( L_Q(v) = \mathbb{E}(e^{vQ}) \). By iteration of this estimate, one gets for any \( n \geq 0 \)
\[
L_n(v) \leq \left( \sum_{k=0}^{n-1} \rho^k L_{n-k}((1-\varepsilon)v) \right) L_Q(v) + \rho^n L_0(v).
\]
Let us notice that we have in fact more: for the same \( \varepsilon \), and for any \( \tilde{v} \leq v \), \( \tilde{\rho} := \mathbb{E}(e^{\tilde{v}Q} 1_{\{1 - \varepsilon < M \leq 1\}}) < \rho \), hence, by the same method as before,
\[
L_n(\tilde{v}) \leq \left( \sum_{k=0}^{n-1} \rho^k L_{n-k}((1-\varepsilon)\tilde{v}) \right) L_Q(\tilde{v}) + \rho^n L_0(\tilde{v}). \tag{4}
\]
Let us define \( \overline{L} = \sup_{n \geq 0} L_n \). Taking the supremum over \( n \) in (4), one gets for any \( \tilde{v} \leq v \)
\[
\overline{L}(\tilde{v}) \leq \frac{1}{1 - \rho} \overline{L}((1-\varepsilon)\tilde{v}) L_Q(\tilde{v}) + L_0(\tilde{v}). \tag{5}
\]
There is \( k \in \mathbb{N} \) such that \( (1-\varepsilon)^k v < v_0 \), hence \( \overline{L}((1-\varepsilon)^k v) < +\infty \). Applying \( k \) times estimate (5), one then obtains immediately that \( \overline{L}(v) < +\infty \), which achieves the first part of the proof.

On the other hand, if \( v > v_Q \), \( R_1 \geq Q_0 \) immediately implies \( L_1(v) = +\infty \); if \( v > v_1 \), \( \rho_0 := \mathbb{E}(e^{vQ} 1_{\{M = 1\}}) > 1 \) (except in a trivial case, left to the reader) and, for all \( n \geq 0 \),
\[
L_{n+1}(v) \geq \rho_0 L_n(v),
\]
implying that \( \overline{L}(v) = +\infty \). \( \square \)

3 Some extensions and perspectives

What happens if the random variables \( (M_n, Q_n)_{n \geq 0} \) are no longer independent? We provide here a partial result under a Markovian assumption when the contractive term \( M \) is less than 1.

Let us introduce \( X = (X_n)_{n \geq 0} \) an irreducible recurrent Markov process with finite space \( E \) and \( ((M_n(x), Q_n(x))_{x \in E})_{n \geq 0} \) a sequence of i.i.d. random vectors supposed to be independent of \( X \). We assume that, for all \( x \in E \),
\[
\mathbb{P}(0 \leq M(x) < 1) = 1,
\]
but we do not assume in the sequel that \( Q \) is non negative. The sequence \( (R_n)_{n \geq 0} \) is defined by
\[
R_{n+1} = M_n(X_n)R_n + Q_n(X_n),
\]
\( R_0 \) being arbitrary (with all exponential moments). Notice that the process \( (X_n, R_n)_{n \geq 0} \) is a Markov process whereas \( (R_n)_{n \geq 0} \) is not (in general).
Proposition 3.1. Introduce $\underline{v} = \inf_{x \in E} v|Q(x)|$, with $v|Q(x)|$ defined as in (6). For any $v < \underline{v}$,
\[
\sup_{n \geq 0} \mathbb{E}\left(e^{v|\hat{R}_n|}\right) < +\infty.
\]
Moreover, if $v > \underline{v}$, then this supremum is infinite.

Proof. Let us introduce $\overline{M}_n = \max_{x \in E} M_n(x)$ and $\overline{Q}_n = \max_{x \in E} |Q_n(x)|$. The random variables $((\overline{M}_n, \overline{Q}_n))_{n \geq 0}$ are i.i.d. Define the sequence $(\overline{R}_n)_{n \geq 0}$ by
\[
\overline{R}_0 = |R_0| \quad \text{and} \quad \overline{R}_{n+1} = \overline{M}_n \overline{R}_n + \overline{Q}_n \quad \text{for } n \geq 1.
\]
Obviously, $|R_n| \leq \overline{R}_n$ for all $n \geq 0$. Thus it is sufficient to study the Laplace transforms of $(\overline{R}_n)_{n \geq 0}$. On the other hand, Theorem 2.1 ensures that $(\mathbb{E}(e^{v|R_n|}))_{n}$ is uniformly bounded as soon as $v < \overline{v}_1 = \min(\overline{v}_1, v_\overline{\Pi})$ with $\overline{v}_1 = \sup\{v \geq 0 : \mathbb{E}(e^{v|Q_1|/M_1}) < 1\} > 0$. In our case, $\overline{v}_1$ is infinite since $\mathbb{P}(\overline{M} < 1) = 1$. At last, for $v \geq 0$,
\[
\sup_{x \in E} \mathbb{E}(e^{v|Q(x)|}) \leq \mathbb{E}(e^{vQ}) = \mathbb{E}\left(\sup_{x \in E} e^{v|Q(x)|}\right) \leq \sum_{x \in E} \mathbb{E}(e^{v|Q(x)|}).
\]
Thus $\underline{v}_\overline{Q} = \inf_{x \in E} v|Q(x)|$.

On the other hand, choose $v > \underline{v}$. There exists $x_0 \in E$ such that $\mathbb{E}(e^{v|Q(x_0)|})$ is infinite. Then, for any $n \geq 0$,
\[
\mathbb{E}(e^{v|\hat{R}_{n+1}|}) \geq \mathbb{E}(e^{v|R_{n+1}|}1_{\{X_n = x_0\}}) \geq \mathbb{E}(e^{-v|\hat{R}_{n}|}e^{v|Q_n(x_0)|}1_{\{X_n = x_0\}}) \geq \mathbb{E}(1_{\{X_n = x_0\}}e^{-v|R_n|}\mathbb{E}(e^{v|Q_n(x_0)|})).
\]
The recurrence of $X$ ensures that $\{n \geq 0, \mathbb{E}(e^{v|\hat{R}_n|}) = +\infty\}$ is infinite. 

Remark 3.2. In [2], we use the previous estimates to improve the results of [3, 4] on the tails of the invariant measure of a diffusion process with Markov switching.

References


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Jean-Baptiste Bardet e-mail: jean-baptiste.bardet(AT)univ-rouen.fr

UMR 6085 CNRS Laboratoire de Mathématiques Raphaël Salem (LMRS)
Université de Rouen, Avenue de l’Université, BP 12, F-76801 Saint Etienne du Rouvray

Hélène Guérin, e-mail: helene.guerin(AT)univ-rennes1.fr

UMR 6625 CNRS Institut de Recherche Mathématique de Rennes (IRMAR)
Université de Rennes I, Campus de Beaulieu, F-35042 Rennes Cedex, France.

Florent Malrieu, corresponding author, e-mail: florent.malrieu(AT)univ-rennes1.fr

UMR 6625 CNRS Institut de Recherche Mathématique de Rennes (IRMAR)
Université de Rennes I, Campus de Beaulieu, F-35042 Rennes Cedex, France.