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On the Laplace transform of perpetuities with thin tails

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Abstract

We consider the random variables \( R \) which are solutions of the distributional equation \( R \overset{d}{=} MR + Q \), where \((Q, M)\) is independent of \( R \) and \(|M| \leq 1\). Goldie and Grübel showed that the tails of \( R \) are no heavier than exponential. Alsmeyer and al provide a complete description of the domain of the Laplace transform of \( R \). We present here a simple proof in a particular case and an extension to the Markovian case.

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1 Introduction

We define on some probability space \((\Omega, \mathcal{A}, \mathbb{P})\) a couple of random variables \((M, Q)\), a sequence \((M_n, Q_n)_{n \geq 0}\) of independent and identically distributed random vectors with the same law as \((M, Q)\), and \(R_0\) a random variable independent of the sequence \((M_n, Q_n)_{n \geq 0}\). Define the sequence \((R_n)_{n \geq 0}\) by

\[
R_{n+1} = M_n R_n + Q_n,
\]

for any \(n \geq 0\). This sequence has been extensively studied in the last decades. Under weak assumptions (see [8]) which are obviously fulfilled in our setting, it can be shown that the sequence \((R_n)_{n \geq 0}\) converges almost surely to a random variable \(R\) such that

\[
R \overset{d}{=} MR + Q, \tag{2}
\]

where \(R\) is independent of \((M, Q)\).

In [7], Kesten established that \(R\) is in general heavy-tailed \((i.e. \ not \ all \ the \ moments \ of \ R \ are \ finite)\) even if \(Q\) is light-tailed as soon as \(|M|\) can be greater than 1. Nevertheless, Goldie and Grübel [4] have shown that \(R\) can have some exponential moments if \(|M| \leq 1\). In particular, if \(Q\) and \(M\) are nonnegative the following result holds.

**Theorem 1.1** (Goldie, Grübel [4]). *Assume that*

\[
\mathbb{P}(Q \geq 0, 0 \leq M \leq 1) = 1, \quad \mathbb{P}(M < 1) > 0
\]

*and that there is \(v_Q > 0\) (possibly infinite) such that*

\[
\mathbb{E}(e^{vQ}) \begin{cases} < +\infty & \text{if } v < v_Q, \\ = +\infty & \text{if } v > v_Q. \end{cases} \tag{3}
\]

*Then, the Laplace transform \(v \mapsto \mathbb{E}(e^{vR})\) of the solution \(R\) of \((2)\) is finite on the set \((-\infty, v_{GG})\) with \(v_{GG} = v_Q \wedge \sup \{v \geq 0, \mathbb{E}(e^{vM}) < 1\}\).*
In fact, the domain of the Laplace transform of $R$ is larger than $(-\infty, v_{GG})$. In [1], a full description of this domain is established. Let us provide a simple proof under the assumptions of Theorem 1.1.

2 The main result

**Theorem 2.1.** Under the assumptions of Theorem 1.1, assuming furthermore that $R_0$ is non-negative and has all its exponential moments finite, then

$$
\sup_{n \geq 0} \mathbb{E}(e^{vR_n}) < +\infty \quad \text{and} \quad \mathbb{E}(e^{vR}) < +\infty
$$

for any $v < v_c$, where

$$
v_c = v_Q \wedge \sup\{v \geq 0, \mathbb{E}(e^{vQ1_{\{M=1\}}}) < 1\}.
$$

Moreover, for any $v > v_c$, $\sup_{n \geq 0} \mathbb{E}(e^{vR_n}) = +\infty$ and $\mathbb{E}(e^{vR}) = +\infty$.

For other recent generalizations of [4], the interested reader is referred to [6], where the authors give sharper results than ours on the tails for some specific examples.

**Proof of Theorem 2.1.** Let us start this section with the main lines of the proof of Theorem 1.1 of Goldie and Grubel [4]. For $\rho > 0$, let $M_\rho$ be the set of probability measures on $\mathbb{R}_+$ with finite exponential moment of order $\rho$, and $d_\rho$ a distance defined on $M_\rho$ by: for $\mu, \nu \in M_\rho$,

$$
d_\rho(\mu, \nu) = \int_0^\infty e^{\rho u} |\mu[u, \infty) - \nu[u, \infty)| \, du.
$$

Define the application $T$ on $M_\rho$ as follows: for $X$ with law $\mu \in M_\rho$, $T\mu$ is the law of $Q + M X$ with $(M, Q)$ independent of $X$. It is shown in [4] that,

$$
d_\rho(T\mu, T\nu) \leq \mathbb{E}(e^{\rho Q} M) d_\rho(\mu, \nu).
$$

Since

$$
\mathbb{E}(e^{vX}) = v \int_0^\infty e^{vu} \mathbb{P}(X \geq u) \, du,
$$

one can show that, for any $n \geq 0$ and $v < \min(v_0, v_Q)$ with $v_0 = \sup\{v \geq 0, \mathbb{E}(e^{vQ} M) < 1\}$,

$$
\mathbb{E}(e^{vR_n}) \leq v \frac{1 - \mathbb{E}(e^{vQ} M)^n}{1 - \mathbb{E}(e^{vQ} M)} d_v(T\mu_0, \mu_0) + \mathbb{E}(e^{vR_0}).
$$

In others words, Goldie and Grubel [4] established that for any $v < \min(v_0, v_Q)$, $(\mathbb{E}(e^{vR_n}))_n$ is uniformly bounded. This estimate can be extended to a larger domain.

Let us define $v_1 = \sup\{v \geq 0, \mathbb{E}(e^{vQ1_{\{M=1\}}}) < 1\}$ and $v_c = \min(v_1, v_Q)$. Let us fix $v < v_c$ and choose $\varepsilon > 0$ such that

$$
\rho := \mathbb{E}(e^{vQ1_{\{1-\varepsilon < M \leq 1\}}}) < 1.
$$
Then we get, for any $n \geq 0$,
\[
L_{n+1}(v) := \mathbb{E}(e^{vR_{n+1}}) = \mathbb{E}(e^{v(M_nR_n + Q_n)}) \\
\leq \mathbb{E}(e^{v(1-\varepsilon)R_n + Q_n})1_{\{M_n \leq 1-\varepsilon\}} + \mathbb{E}(e^{vR_n + Q_n})1_{\{1-\varepsilon < M_n \leq 1\}} \\
\leq L_n((1-\varepsilon)v)L_Q(v) + \rho L_n(v)
\]
where $L_Q(v) = \mathbb{E}(e^{vQ})$. By iteration of this estimate, one gets for any $n \geq 0$
\[
L_n(v) \leq \left( \sum_{k=0}^{n-1} \rho^k L_{n-k}((1-\varepsilon)v) \right) L_Q(v) + \rho^n L_0(v).
\]
Let us notice that we have in fact more: for the same $\varepsilon$, and for any $\tilde{v} \leq v$, $\tilde{\rho} := \mathbb{E}(e^{\varepsilon Q}1_{\{1-\varepsilon < M \leq 1\}}) < \rho$, hence, by the same method as before,
\[
L_n(\tilde{v}) \leq \left( \sum_{k=0}^{n-1} \rho^k L_{n-k}((1-\varepsilon)\tilde{v}) \right) L_Q(\tilde{v}) + \rho^n L_0(\tilde{v}).
\]
Let us define $\overline{T} = \sup_{n \geq 0} L_n$. Taking the supremum over $n$ in (4), one gets for any $\tilde{v} \leq v$
\[
\overline{T}(\tilde{v}) \leq \frac{1}{1-\rho} \overline{T}((1-\varepsilon)\tilde{v})L_Q(\tilde{v}) + L_0(\tilde{v}).
\]
There is $k \in \mathbb{N}$ such that $(1-\varepsilon)^k v < v_0$, hence $\overline{T}((1-\varepsilon)^k v) < +\infty$. Applying $k$ times estimate (4), one then obtains immediately that $\overline{T}(v) < +\infty$, which achieves the first part of the proof.

On the other hand, if $v > v_Q$, $R_1 \geq Q_0$ immediatly implies $L_1(v) = +\infty$; if $v > v_1$, $\rho_0 := \mathbb{E}(e^{vQ}1_{\{M=1\}}) > 1$ (except in a trivial case, left to the reader) and, for all $n \geq 0$,
\[
L_{n+1}(v) \geq \rho_0 L_n(v),
\]
implying that $\overline{T}(v) = +\infty$.

\section{Some extensions and perspectives}

What happens if the random variables $(M_n, Q_n)_{n \geq 0}$ are no longer independent? We provide here a partial result under a Markovian assumption when the contractive term $M$ is less than 1.

Let us introduce $X = (X_n)_{n \geq 0}$ an irreducible recurrent Markov process with finite space $E$ and $((M_n(x), Q_n(x))_{x \in E})_{n \geq 0}$ a sequence of i.i.d. random vectors supposed to be independent of $X$. We assume that, for all $x \in E$,
\[
\mathbb{P}(0 \leq M(x) < 1) = 1,
\]
but we do not assume in the sequel that $Q$ is non negative. The sequence $(R_n)_{n \geq 0}$ is defined by
\[
R_{n+1} = M_n(X_n)R_n + Q_n(X_n),
\]
$R_0$ being arbitrary (with all exponential moments). Notice that the process $(X_n, R_n)_{n \geq 0}$ is a Markov process whereas $(R_n)_{n \geq 0}$ is not (in general).
Proposition 3.1. Introduce \( v = \inf_{x \in E} v_{|Q(x)|} \), with \( v_{|Q(x)|} \) defined as in (5). For any \( v < v^* \),
\[
\sup_{n \geq 0} \mathbb{E} \left( e^{v|R_n|} \right) < +\infty.
\]
Moreover, if \( v > v^* \), then this supremum is infinite.

Proof. Let us introduce \( M_n = \max_{x \in E} M_n(x) \) and \( Q_n = \max_{x \in E} |Q_n(x)| \). The random variables \( (\langle M_n, Q_n \rangle)_{n \geq 0} \) are i.i.d. Define the sequence \( (\mathcal{R}_n)_{n \geq 0} \) by
\[
\mathcal{R}_0 = |R_0| \quad \text{and} \quad \mathcal{R}_{n+1} = M_n \mathcal{R}_n + Q_n \quad \text{for } n \geq 1.
\]
Obviously, \( |R_n| \leq \mathcal{R}_n \) for all \( n \geq 0 \). Thus it is sufficient to study the Laplace transforms of \( (\mathcal{R}_n)_{n \geq 0} \). On the other hand, Theorem [21] ensures that \( \mathbb{E} \left( e^{v\mathcal{R}_n} \right) \) is uniformly bounded as soon as \( v < v_\infty = \min(v_1, v^*_Q) \) with \( v_1 = \sup\{v \geq 0 : \mathbb{E}(e^{vQ_1}\mathcal{M}_1) < 1\} \). In our case, \( v_1 \) is infinite since \( \mathbb{P}(\mathcal{M} < 1) = 1 \). At last, for \( v \geq 0 \),
\[
\sup_{x \in E} \mathbb{E} \left( e^{v|Q(x)|} \right) \leq \mathbb{E} \left( e^{vQ} \right) = \mathbb{E} \left( \sup_{x \in E} e^{v|Q(x)|} \right) \leq \sum_{x \in E} \mathbb{E} \left( e^{v|Q(x)|} \right).
\]
Thus \( v^*_Q = \inf_{x \in E} v_{|Q(x)|} \).

On the other hand, choose \( v > v^* \). There exists \( x_0 \in E \) such that \( \mathbb{E}(e^{v|Q(x_0)|}) \) is infinite. Then, for any \( n \geq 0 \),
\[
\mathbb{E} \left( e^{v|R_{n+1}|} \right) \geq \mathbb{E} \left( e^{v|R_n|} 1_{\{X_n = x_0\}} \right) \geq \mathbb{E} \left( e^{-v|R_n|} e^{v|Q_n(x_0)|} 1_{\{X_n = x_0\}} \right) \geq \mathbb{E} \left( 1_{\{X_n = x_0\}} e^{-v|R_n|} \right) \mathbb{E} \left( e^{v|Q_n(x_0)|} \right).
\]

The recurrence of \( X \) ensures that \( \{n \geq 0, \mathbb{E}(e^{v|R_n|}) = +\infty\} \) is infinite. \( \square \)

Remark 3.2. In [22], we use the previous estimates to improve the results of [4] on the tails of the invariant measure of a diffusion process with Markov switching.

References


