Towards a unified logical framework of fuzzy implications to compare fuzzy sets

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Abstract— *In fuzzy set theory, comparison of fuzzy sets plays an important role. Among the several ways to compare fuzzy sets, we address the logical theoretic approach using fuzzy implications. We propose a general framework allowing to generate many measures of comparison: inclusion, similarity and distance, and study their properties. Since the literature on the use fuzzy implications for defining such measures is abundant, we also attempt to relate this work to existing research.*

Keywords— aggregation operators, distance, fuzzy implication, inclusion, similarity

1 Introduction

In fuzzy set theory, inclusion (or subsethood) and similarity measures between fuzzy sets are basic concepts. Among the different approaches to compare fuzzy sets, one finds three categories. The first one is based on measuring a distance between two real functions and do not refer to a specific interpretation, e.g. the Minkowski r-distances. The second category involves set-theoretic operations for fuzzy sets (fuzzy intersection, union, cardinality) [1, 2]. This paper addresses the third category which relies on logical theory, mainly approached by using fuzzy implications. Following the early paper by Bandler and Kohout [3] introducing the use of implications to measure inclusion of fuzzy sets, many authors have proposed new measures satisfying some specific axioms [4, 5, 6].

This paper will be organized as follows. We first recall some basic definitions and theorems on aggregation operators, triangular norms and fuzzy implications that will be used in the sequel in section 2. Next, in section 3 we propose a general framework of comparison measures of fuzzy sets (i.e. inclusion, similarity and distance) based on logical considerations, i.e. using fuzzy implications. A general description of the overall aggregation method is also provided, characterizing each measure belonging to this category. In section 4, we present the existing approaches that could be linked to this general formulation. We finally conclude and mention some application domains we have in mind in section 5.

2 Preliminaries and notations

Aggregating numbers plays an important role in decision-making systems. Values to be aggregated are generally defined on a finite real interval or on ordinal scales. In this paper, we assume with no loss of generality that they come from the unit interval. If not, a simple transformation can be found to make this assumption true. Given n numbers, an aggregation operator is a mapping $\mathcal{A} : [0,1]^n \to [0,1]$, satisfying boundary conditions

$$\mathcal{A}(0,\cdots,0) = 0 \quad \text{and} \quad \mathcal{A}(1,\cdots,1) = 1. \tag{1}$$

and monotonocity

$$\forall n \in \mathbb{N}, x_1 \leq y_1, \cdots, x_n \leq y_n \quad \text{implies} \quad \mathcal{A}(x_1,\cdots,x_n) \leq \mathcal{A}(y_1,\cdots,y_n). \tag{2}$$

Adding properties like idempotency, continuity, associativity lead to others definitions. In the literature, one finds many aggregation operators, e.g.: triangular norms (t-norms for short), OWA (*Ordered Weighted Averaging*) operators, γ-operators, or fuzzy integrals. They belong to several categories, depending on the way the values are aggregated: conjunctives, disjunctives, compensatory, and weighted operators. Briefly, an aggregation operator is said to be *conjunctive* if its output value is lower than the minimum of the input values, *compensatory* if the output value lies between the minimum and the maximum input value, and *disjunctive* if its output is greater than the maximum value of the inputs, refer to [7] for a large survey on aggregation operators.

Theorem 1. *If $\mathcal{A}$ is a strictly monotonic compensatory aggregation operator, then*

$$\mathcal{A}(x_1,\ldots,x_n) = 1 \Leftrightarrow x_1 = \cdots = x_n = 1 \tag{3}$$

and

$$\mathcal{A}(x_1,\ldots,x_n) = 0 \Leftrightarrow x_1 = \cdots = x_n = 0 \tag{4}$$

Proof.

(3) ($\Leftarrow$) obvious with Eq. (1).
($\Rightarrow$) by contraposition. We show that $\exists i$ such that $x_i < 1$ implies $\mathcal{A}(x_1,\cdots,x_n) < 1$. If $x_i < 1$ for some i, then, since $x_j \leq 1$ for all j, by strict monotonicity of $\mathcal{A}$, we get $\mathcal{A}(x_1,\cdots,x_n) < \mathcal{A}(1,\cdots,1) = 1$, which ends the contraposition.

(4) ($\Leftarrow$) obvious with Eq. (1).
($\Rightarrow$) by contraposition. We show that $\exists i$ such that $x_i > 0$ implies $\mathcal{A}(x_1,\cdots,x_n) > 0$. If $x_i > 0$ for some i, then, since $x_j \geq 0$ for all j, by strict monotonicity of $\mathcal{A}$, we get $\mathcal{A}(x_1,\cdots,x_n) > \mathcal{A}(0,\cdots,0) = 0$, which ends the contraposition.

□

Example 1. *The family of mean operators are strictly monotonic compensatory aggregation operators.*

Remark 1. *From Theorem 1, we immediately see that a disjunctive aggregation operator cannot be strictly monotonic.* $\forall j\ x_j \leq 1$*, and* $\exists i,\ x_i < 1$*,* $\exists k,\ x_k = 1$*. If* $\mathcal{A}$ *is strictly monotonic,* $\mathcal{A}(x_1, \cdots, x_n) < \mathcal{A}(1, \cdots, 1) = 1$*. Since* $\mathcal{A}$ *is disjunctive, i.e.* $\mathcal{A}(x_1, \cdots, x_n) \geq \max(x_1, \cdots, x_n)$*, we have* $1 = \max(x_1, \cdots, x_n) \leq \mathcal{A}(x_1, \cdots, x_n) < 1$*, which is a contradiction.*

A t-norm is an increasing, associative and commutative mapping $\top : [0,1]^2 \rightarrow [0,1]$ satisfying the boundary condition $\top(x, 1) = x$ for all $x \in [0,1]$. The most popular continuous t-norms are:

- Standard: $\top_S(x, y) = \min(x, y)$
- Algebraic (Product): $\top_A(x, y) = x\,y$
- Łukasiewicz: $\top_L(x, y) = \max(x + y - 1, 0)$

Alternatively, the dual operators with respect to a strict negation are called triangular conorms (t-conorms for short). A t-conorm is an increasing, associative and commutative mapping $\perp : [0,1]^2 \rightarrow [0,1]$ satisfying the boundary condition $\perp(x, 0) = x$ for all $x \in [0,1]$. The most popular continuous t-conorms are:

- Standard: $\perp_S(x, y) = \max(x, y)$
- Algebraic (Product): $\perp_A(x, y) = x + y - x\,y$
- Łukasiewicz: $\perp_L(x, y) = \min(x + y, 1)$

Various parametrical families have been introduced, e.g. the Hamacher family defined by: given $\gamma \in [0, +\infty[$,

- $x \top_H y = \frac{x\,y}{\gamma + (1-\gamma)\,(x+y-x\,y)}$
- $x \perp_H y = \frac{x+y-x\,y-(1-\gamma)\,x\,y}{1-(1-\gamma)\,x\,y}$.

Such families result in an infinite number of t-norm couples, including non parametrical ones e.g. the Algebraic couple for $\gamma = 1$.
A general problem in fuzzy logic is to handle conditional statements *if* x*, then* y where x and y are fuzzy predicates. A widely used method consists in managing them by mappings $I : [0,1] \times [0,1] \rightarrow [0,1]$ such that the truth value of I depends on the initial propositions x and y. We generally speak about an *implication function* if I is non-increasing in the first variable, non-decreasing in the second variable and $I(0,0) = I(1,1) = 1$, and $I(1,0) = 0$, see [8] for a survey on fuzzy implication functions and [9] for a large overview on the use of parametrical implications in fuzzy inference systems. The four most usual implications are:

1. *S-implications*, defined by:
$$I_\perp(x, y) = \perp(x^c, y) \tag{5}$$
where $(.)^c$ is the usual complementation $x^c = 1-x$. This implication is an immediate generalization of the usual boolean implication $x \rightarrow y \equiv x^c \vee y$.

2. *R-(for Residual) implications*, defined by:
$$I_\top(x, y) = \sup_t \{t \in [0,1] | \top(x, t) \leq y\}. \tag{6}$$
Note that if $\top$ is a left-continuous t-norm, the *supremum* operation can be substituted by the *maximum* one.

3. *QL-(for Quantum mechanic Logic) implications*, defined by:
$$I_{QL}(x, y) = \perp(x^c, \top(x, y)). \tag{7}$$

4. *D-implications*, defined by:
$$I_D(x, y) = \perp(\top(x^c, y^c), y) \tag{8}$$
which are the contraposition of QL-implications with respect to the complementation.

In the sequel, we will denote by:

- $X = \{x_1, \cdots, x_n\}$ the (supposed finite) universe of discourse,
- $\mathcal{C}(X)$ and $\mathcal{F}(X)$ the sets of all crisps and fuzzy sets in X, respectively,
- $f_A(x)$, $\forall x \in X$, the membership function of a fuzzy set A over X,
- $[\frac{1}{2}]$ the constant fuzzy set defined by $[\frac{1}{2}](x) = \frac{1}{2}$ for any $x \in X$.

3 General framework

There are several ways to compare fuzzy values or fuzzy quantities. The first one is based on a broad class of measures of equality based on a distance measure which is specified for membership functions of fuzzy sets. This approach takes its roots from studies on how to measure the distance between two real functions and do not refer to any specific interpretation. The general form of a Minkowski r-metric is usually taken and leads to well known distance functions (Hamming, Euclidean, Chebyshev):

$$d_r(A, B) = \left(\sum_{x \in X} |f_A(x) - f_B(x)|^r \right)^{1/r} \tag{9}$$

A second way to compare fuzzy values comes from some basic set-theoretic considerations where union, intersection and complementation are defined for fuzzy sets. Cardinal and possibility based measures belong to this category. In this paper, we will focus our attention on the third way: the logical framework. We present the design a unified logical framework to compare fuzzy sets, which includes usual measures depending the aggregation operators we use. The use of parametrical t-norms and t-conorms to define the implications will enable to obtain parametrical measures of comparison, where the parameters can be set according to user needs.

3.1 Inclusion measures

An inclusion measure is a relation between two fuzzy sets which indicates to which extent one fuzzy set is contained in another one. Since its original definition by $A \subset B$ iff $f_A(x) \leq f_B(x)$, for all x in X, which was crisp assessment, Bandler and Kohout enlarged this point of view by giving a degree of subsetness [3], more in the spirit of the fuzzy theory. Inclusion measures are generally defined by a set of axioms [4] and by using fuzzy implication operators [3, 10].

Definition 1. *A mapping* $\mathcal{I} : \mathcal{F}(X) \times \mathcal{F}(X) \rightarrow [0,1]$ *is called an inclusion (or subsethood) measure if it satisfies*

(P1) $\mathcal{I}(A,B)=1$ *iff* $A\subseteq B$, $\quad \forall A,B\in\mathcal{F}(X)$.

(P2) *if* $[\frac{1}{2}]\subseteq A$, *then* $\mathcal{I}(A,A^c)=0$ *iff* $A=X$.

(P3) $\forall A,B,C\in\mathcal{F}(X)$, *if* $A\subseteq B\subseteq C$, *then* $\mathcal{I}(C,A)\leq\mathcal{I}(B,A)$, *and if* $A\subseteq B$, $\mathcal{I}(C,A)\leq\mathcal{I}(C,B)$.

Note that if a measure $\mathcal{I}$ satisfies only the two last properties, it is called a weak inclusion measure.

Theorem 2. *Let $I_\top$ be a residual implication function. Given X and arbitrary $A,B\in\mathcal{F}(X)$, let*

$$\mathcal{I}(A,B)=\mathop{\mathcal{A}}_{i=1}^{n} I_\top\left(f_A(x_i),f_B(x_i)\right) \tag{10}$$

for all x_i in X, where $\mathcal{A}$ is a conjunctive or a strictly monotonic compensatory aggregation operator satisfying Eqs.(1) and (2). Then $\mathcal{I}$ is an inclusion measure.

Proof.

(P1) ($\Leftarrow$) if $A\subseteq B$, then $f_A(x_i)\leq f_B(x_i)$ for all $x_i\in X$. Since if $x\leq y$ then $I_\top(x,y)=1$, we have $I_\top(f_A(x_i),f_B(x_i))=1$ for all $x_i\in X$, which gives $\mathcal{I}(A,B)=1$ by boundary condition on $\mathcal{A}$.
($\Rightarrow$) two cases are considered
- if $\mathcal{A}$ is disjunctive, then we obviously have $f_A(x_i)\leq f_B(x_i)$ for all $x_i\in X$, since the minimum value of all implications on X is 1
- if $\mathcal{A}$ is a strictly monotonic compensatory aggregation operator, then by using Theorem 1, we have $I_\top\left(f_A(x_i),f_B(x_i)\right)=1$ for all $x_i\in X$, giving $A\subseteq B$ by the confinement principle: $x\leq y$ *iff* $I(x,y)=1$.

(P2) ($\Leftarrow$) if $A=X$, then $I_\top(f_X(x_i),\emptyset(x_i))=0$ for all $x_i\in X$. By boundary condition on $\mathcal{A}$, it follows that $\mathcal{I}(A,A^c)=0$.
($\Rightarrow$) if $\mathcal{I}(A,A^c)=0$, then $I_\top(f_A(x_i),1-f_A(x_i))=0$ for all $x_i\in X$ by Theorem 1. Assuming that $A\neq X$, then $\exists i$, x_i such that $\frac{1}{2}\leq f_A(x_i)<1$, i.e. $0<1-f_A(x_i)\leq\frac{1}{2}$.
By non-increasingness in the first variable of $I_\top$, we have $I_\top(f_A(x_i),1-f_A(x_i))\geq I_\top(1,1-f_A(x_i))=1-f_A(x_i)\neq 0$, since $I_\top$ satisfies the border principle: $I(1,x)=x,\forall x\in[0,1]$. This is a contradiction with $I_\top(f_A(x_i),1-f_A(x_i))=0$, so that $A=X$.

(P3) if $A\subseteq B\subseteq C$, $f_A(x_i)\leq f_B(x_i)\leq f_C(x_i)$, for all $x_i\in X$. By non-increasingness in the first variable and non-decreasingness in the second variable, it follows that $I_\top(f_C(x_i),f_A(x_i)\leq I_\top(f_B(x_i),f_A(x_i))$ and $I_\top(f_C(x_i),f_A(x_i))\leq I_\top(f_C(x_i),f_B(x_i))$ for all i. By monotonicity of $\mathcal{A}$, we have $\mathcal{I}(C,A)\leq\mathcal{I}(B,A)$ and $\mathcal{I}(C,A)\leq\mathcal{I}(C,B)$, which concludes the proof.

□

A necessary condition to define a strong inclusion measure $\mathcal{I}$ is that the implication I holds the confinement principle and the border principle. It is easy to show that the four usual implications $I_\top$, $I_\perp$, I_{QL} and I_D satisfy the latter, but only $I_\top$ satisfies the former. Therefore S, QL and $D-$implications define weak inclusion measures while $R-$implications define strong inclusion measures provided the operator $\mathcal{A}$ is not disjunctive, see Remark 1.

3.2 Similarity and distance measures

Definition 2. *A mapping $\mathcal{S}:\mathcal{F}(X)\times\mathcal{F}(X)\to[0,1]$ is called a similarity measure if it satisfies*

(P1) $\mathcal{S}(A,B)=\mathcal{S}(B,A)$, $\quad\forall A,B\in\mathcal{F}(X)$.

(P2) $\mathcal{S}(A,A)=1$, $\forall A\in\mathcal{F}(X)$.

(P3) $\mathcal{S}(D,D^c)=0$, $\quad\forall D\in\mathcal{C}(X)$.

(P4) $\forall A,B,C\in\mathcal{F}(X)$, *if* $A\subseteq B\subseteq C$, *then* $\mathcal{S}(A,C)\leq\mathcal{S}(A,B)\wedge\mathcal{S}(B,C)$
or, equivalently
$\forall A,B,C,D\in\mathcal{F}(X)$, *if* $A\subseteq B\subseteq C\subseteq D$, *then* $\mathcal{S}(A,D)\leq\mathcal{S}(B,C)$

However, the symmetry property (P1) is still subject to experimental investigation: if $\mathcal{S}(x,y)$ is the answer to the question *how is x similar to y?*, then, when making comparisons, subjects focus more on the feature x than on y. This corresponds to the notion of saliency [1] of x and y: if y is more salient than x, then x is more similar to y than vice versa, which is experimentally confirmed. Accordingly to [11], for a certain element of the universe of discourse X, a degree of equality of two fuzzy elements x and y can be defined by using implications as follows:

$$(x\equiv y)=\frac{1}{2}\big((x\to y)\wedge(y\to x)+(x^c\to y^c)\wedge(y^c\to x^c)\big) \tag{11}$$

where $\wedge$ stands for minimum, $\to$ is an R-implication.
Then applying Eq. (6) with $x\geq y$, gives

$$(x\equiv y)=\frac{1}{2}\big((x\to y)+(y^c\to x^c)\big) \tag{12}$$

since 1 is the neutral element of t-norms.
Furthermore, due to the opposition law: $I(x,y)=I(y^c,x^c)$, we obtain

$$(x\equiv y)=(x\to y) \tag{13}$$

A convenient way to define a similarity measure is to quantify to which extent two fuzzy membership degrees are similar, so that it is closely related to the problem of matching fuzzy quantities, or fuzzy sets similarity. So we propose to use fuzzy implication functions as similarity measures by the following theorem:

Theorem 3. *Let $I_\top$ be a residual implication function. For arbitrary $A,B\in\mathcal{F}(X)$, let*

$$\mathcal{S}(A,B)=\mathop{\mathcal{A}}_{i=1}^{n} I_\top\left(f_{(1)}(x_i),f_{(2)}(x_i)\right) \tag{14}$$

for all x_i in X, where $f_{(.)}$ is a permutation of f_A and f_B such that $f_{(1)}(x_i)=(f_A\cup f_B)(x_i)$, $f_{(2)}(x_i)=(f_A\cap f_B)(x_i)$, and $\mathcal{A}$ an aggregation operator satisfying Eqs. (1) and (2). Then $\mathcal{S}$ is a similarity measure.

Proof.

(P1) we have $I_\top(x,x)=1$, for any $x\in[0,1]$. By boundary conditions on $\mathcal{A}$, see Eq. (1), we obtain $\mathcal{S}(A,A)=1$.

(P2) by commutativity of union and intersection of fuzzy sets, we have

$$\begin{aligned}\mathcal{S}(A,B) &= \mathop{\mathcal{A}}_{i=1}^{n} I_\top\left((f_A \cup f_B)(x_i), (f_A \cap f_B)(x_i)\right)\\ &= \mathop{\mathcal{A}}_{i=1}^{n} I_\top\left((f_B \cup f_A)(x_i), (f_B \cap f_A)(x_i)\right)\\ &= \mathcal{S}(B,A)\end{aligned}$$

(P3) by definition, $I(1,0) = 0$. By boundary conditions on $\mathcal{A}$, see Eq. (1), we obtain $\mathcal{S}(D, D^c) = 0$.

(P4) since $A \subseteq B \subseteq C \subseteq D$, we have for all $x_i \in X$

$$f_D(x_i) \geq f_C(x_i) \tag{15}$$
$$f_B(x_i) \geq f_A(x_i) \tag{16}$$

By non-increasingness in the first variable and non-decreasingness in the second variable of $I_\top$, we obtain for all $x_i \in X$

$$I_\top(f_D(x_i), f_A(x_i)) \leq I_\top(f_C(x_i), f_A(x_i)) \quad \text{by Eq. (15)}$$

$$I_\top(f_C(x_i), f_A(x_i)) \leq I_\top(f_C(x_i), f_B(x_i)) \quad \text{by Eq. (16)}$$

Last, monotonicity of $\mathcal{A}$, see Eq. (2), gives $\mathcal{S}(A, D) \leq \mathcal{S}(B, C)$ which concludes the proof.

□

In contrast to inclusion measures, no restriction is imposed to the $\mathcal{A}$ operator, it can be freely chosen provided Eqs.(1) and (2) are satisfied. On another hand, here again, the confinement principle is necessary to obtain property (P1) of similarity measures. Since $\mathcal{S}(A, B)$ is reflexive and symmetrical, i.e. $\mathcal{S}(A, A) = 1$ and $\mathcal{S}(A, B) = \mathcal{S}(B, A)$ hold for any $A, B \in \mathcal{F}(X)$, $\mathcal{S}$ is a proximity relation on $\mathcal{F}(X)$.

Definition 3. *A mapping $\mathcal{D} : \mathcal{F}(X)\times\mathcal{F}(X) \to [0, 1]$ is called a distance measure if $\mathcal{D}$ has the following properties*

(P1) $\mathcal{D}(A, B) = \mathcal{D}(B, A), \quad \forall A, B \in \mathcal{F}(X)$.

(P2) $\mathcal{D}(A, A) = 0$, $\forall A \in \mathcal{F}(X)$.

(P3) $\mathcal{D}(D, D^c) = 1, \quad \forall D \in \mathcal{C}(X)$.

(P4) $\forall A, B, C \in \mathcal{F}(X)$, *if* $A \subseteq B \subseteq C$, *then* $\mathcal{D}(A, B) \leq \mathcal{D}(A, C)$ *and* $\mathcal{D}(B, C) \leq \mathcal{D}(A, C)$

Proposition 1. *For arbitrary $A, B \in \mathcal{F}(X)$, and $\mathcal{S}(A, B)$ a similarity measure defined by Eq. (14), then the mapping $\mathcal{D} : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, 1]$ defined by $\mathcal{D}(A, B) = \mathcal{S}^c(A, B)$ is a distance measure between A and B.*

3.3 Examples

Some examples of inclusion and similarity measures of the literature, and the new Hamacher inclusion and similarity measures, all obtained through the proposed general logical framework are given in Table 1. Moreover, this table shows how it is easy to check wether an inclusion measure is strong (non disjunctive $\mathcal{A}$ and $I_\top$) or weak (non disjunctive $\mathcal{A}$), see Definition 1, as well as the proper definition of a similarity measure (any $\mathcal{A}$ and $I_\top$), see Theorem 3.

For illustration purpose, Fig. 1 shows some examples of fuzzy similarity measures as well as the influence of the parameter γ for Hamacher residual implications, which are defined by:

$$I_{\top_{H_\gamma}}(x, y) = \begin{cases} 1 & \text{if } y \geq x \\ \frac{y(\gamma+x-\gamma x)}{y(\gamma+x-\gamma x)+x-y} & \text{if } y \leq x \end{cases} \tag{17}$$

where $\gamma \in [0, +\infty[$. The different plots show the similarity value of a given fuzzy set $A = \{0.4/x_1, 0.3/x_2\}$ to all the possible two-dimensional fuzzy sets B for various $I_\top$. As one could expect, the closer to x_1 or x_2 the higher the similarity. One can also note that different $I_\top$ lead to different shapes for isosurfaces of the similarity.

4 Related works

Let us relate, as far as possible, the proposed framework to some works issued from the abundant literature on the use of fuzzy implications for defining inclusion measures. In their seminal paper [3], Bandler and Kohout propose to use implication functions in order to quantify the inclusion of each element in another, and then aggregate these individual measures by a conjunctive operator: the minimum. Some years later, Hirota and Pedrycz propose to use implications for matching fuzzy quantities [11]. They aggregate the different implication truth values over X by a Choquet integral, computing the fuzzy measure with the help of a family of fuzzy sets taken as fuzzy prototypes. They also propose an entropy measure based on this matching measure, which gives an *impression concerning the uncertainty of matching*. In [14], Kosko criticizes the original definition of fuzzy set containment: B contains A if and only if $f_A(x) \leq f_B(x)$ for all x in X by Zadeh, pointing out that *if this inequality holds for all but just a few x, we can still consider A to be a subset of B to some degree*. So he proposes a second definition based on fuzzy intersection of two fuzzy sets which can be seen as the conditional probability $P(B|A)$ under certain circumstances. Furthermore, he defines the fuzziness of a fuzzy set A as the inclusion measure of $A \cup A^c$ in $A \cap A^c$, which satisfies the axioms of fuzzy entropy of De Luca and Termini [15]. Inclusion and similarity measures from a general set-theoretic point of view, coming from the proposition of Tversky [1], are described by Bouchon-Meunier *et al.* in [2]. The authors introduce a general framework for similitude, satisfiability and inclusion measures, also study the aggregation of measures of comparison, but take only two examples for the aggregation operator $\mathcal{A}$: a t-norm and the OWA operator. In this paper, we propose a study of the properties of $\mathcal{A}$ in its more general meaning. The same remark applies to the work by Young [4] proposing an axiomatization of inclusion measures and their connection to implication operators since she restricts herself to both the minimum and the arithmetical mean for $\mathcal{A}$. Furthermore, this work do not give necessary conditions on implications for the definition of strong or weak inclusion measure whereas our's does (Definition 1). Wang [13] presents two similarity measures which can be viewed as particular cases of the framework we propose. He enlarges his definition to the similarity of fuzzy elements belonging to various fuzzy sets. Again, the framework we propose allows to obtain a similarity between fuzzy elements since we compute implications for each element. Starting from Kosko and Young observations, Botana [16] presents a set of new measures derived

Table 1: Inclusion and similarity measures of the literature, and the new Hamacher inclusion and similarity measures, all obtained through the proposed general logical framework.

Inclusion Measure $\mathcal{I}$	**Aggregation Operator** $\mathcal{A}$	**Implication** I
$\mathcal{I}(A,B) = \min_{x\in X}(\min(1, 1 - f_A(x) + f_B(x)))$ as defined in [12]	min	$I_{\top_L}$
$\mathcal{I}(A,B) = \frac{1}{n}\sum_{x\in X} \min(1, 1 - f_A(x) + f_B(x))$	arithmetical mean	$I_{\top_L}$
$\mathcal{I}(A,B) = \frac{1}{n}\sum_{x\in X} \max(1 - f_A(x), f_B(x))$	arithmetical mean	$I_{\perp_S}$
$\mathcal{I}(A,B) = \frac{1}{n}\sum_{x\in X} 1 - f_A(x) + f_A(x)\, f_B(x)$ as defined in [4]	arithmetical mean	$I_{\perp_A}$
$\mathcal{I}(A,B) = \frac{1}{n}\sum_{x\in X} \frac{f_B(x)(\gamma + f_A(x) - \gamma f_A(x))}{f_B(x)(\gamma + f_A(x) - \gamma f_A(x)) + f_A(x) - f_B(x)}$	arithmetical mean	$I_{\top_{H_\gamma}}$
Similarity Measure $\mathcal{S}$	**Aggregation Operator** $\mathcal{A}$	**Implication** I
$\mathcal{S}(A,B) = \frac{1}{n}\sum_{x\in X} \frac{\min(f_A(x), f_B(x))}{\max(f_A(x), f_B(x))}$ as defined in [13]	arithmetical mean	$I_{\top_A}$
$\mathcal{S}(A,B) = \frac{1}{n}\sum_{x\in X} 1 - \lvert f_A(x) - f_B(x)\rvert$ as defined in [13]	arithmetical mean	$I_{\top_L}$
$\mathcal{S}(A,B) = \max_{x\in X} \min(f_A(x), f_B(x))$	max	$I_{\top_S}$
$\mathcal{S}(A,B) = 1 - \max_{x\in X} \lvert f_A(x) - f_B(x)\rvert$	min	$I_{\top_L}$
$\mathcal{S}(A,B) = \frac{1}{n}\sum_{x\in X} \frac{f_{(2)}(x)(\gamma + f_{(1)}(x) - \gamma f_{(1)}(x))}{f_{(2)}(x)(\gamma + f_{(1)}(x) - \gamma f_{(1)}(x)) + f_{(1)}(x) - f_{(2)}(x)}$	arithmetical mean	$I_{\top_{H_\gamma}}$

from fuzzy implications. He studies whether the introduced inclusion measures satisfy Young's axioms [4] and/or those in [12] when using Wu, Goguen, modified Goguen, Gödel and Standard strict implications. He also gives the formulation of the corresponding entropy in the sense of [15, 14]. Another approach to aggregation operators is proposed in [17] consisting in combining implication truth values through respectively disjunctive and conjunctive functions for an optimistic and pessimistic aggregation. Fan et al. discuss the links between inclusion, entropy and fuzzy implications, and propose some new axioms for these measures [5]. Burillo et al. present a family of implication operators derived from the Łukasiewicz implication in order to define a family of inclusion grade operators [6] using the minimum operator for $\mathcal{A}$. In [18], Kehagias and Konstantinidou introduce L-fuzzy valued inclusion, similarity and distance measures, i.e. mappings $\mathcal{I}, \mathcal{S}, \mathcal{D} : \mathcal{F}(X) \times \mathcal{F}(X) \to [0,1]^n$, but restrict the output to crisp values. As pointed out by the authors, the vector output can lead to a difficult interpretation of the result. Furthermore, as vectors are partially ordered they are even harder to compare, they require a new measure to compare outputs. The framework we propose can provide a vector output since an implication on each element of X is computed. By contrast to [18], it would be a *fuzzy L-fuzzy valued* measure. More recently [19, 20], a distance between fuzzy operators, fuzzy implication functions in particular, is proposed. It leads to normalized tensor-norms which allow to define a similarity of fuzzy sets, and present an heuristic to choose the most suitable implication function to a fuzzy inference system. Zhang and Zhang propose an hybrid inclusion measure in [10] and use it to define similarity and distance measures of fuzzy sets. They restrict themselves to the weighted mean for $\mathcal{A}$, so it can be viewed as a special case of the work we propose, but contrarily to Young the involved implications satisfy the confinement and the border principle. Let us finally mention the work by Fono et al. [21] where as many difference operations as many implications are used to define measures of comparison of fuzzy sets in the set-theoretic framework, but we remind that such measures are out of the scope of this paper.

5 Conclusion and perspectives

In this paper, we propose a unified logical framework to compare fuzzy sets as well as fuzzy elements. Within this framework, new measures of inclusion, similarity and distance can be easily derived. These measures depend on any fuzzy implication I, provided it satisfies the necessary conditions we give (the border and the confinement principles) and any aggregation operator $\mathcal{A}$ (for similarity and distance) or any conjunctive and strictly monotonic compensatory aggregation operator $\mathcal{A}$ (for inclusion). Choosing specific $(\mathcal{A}, I)$ enables to retrieve most of the measures of the literature.

We hope that results of this work would be of great help to set comparison functions in many fields: fuzzy mathematical morphology [6], cluster validity [5], as well as other domains e.g. image retrieval or feature selection.

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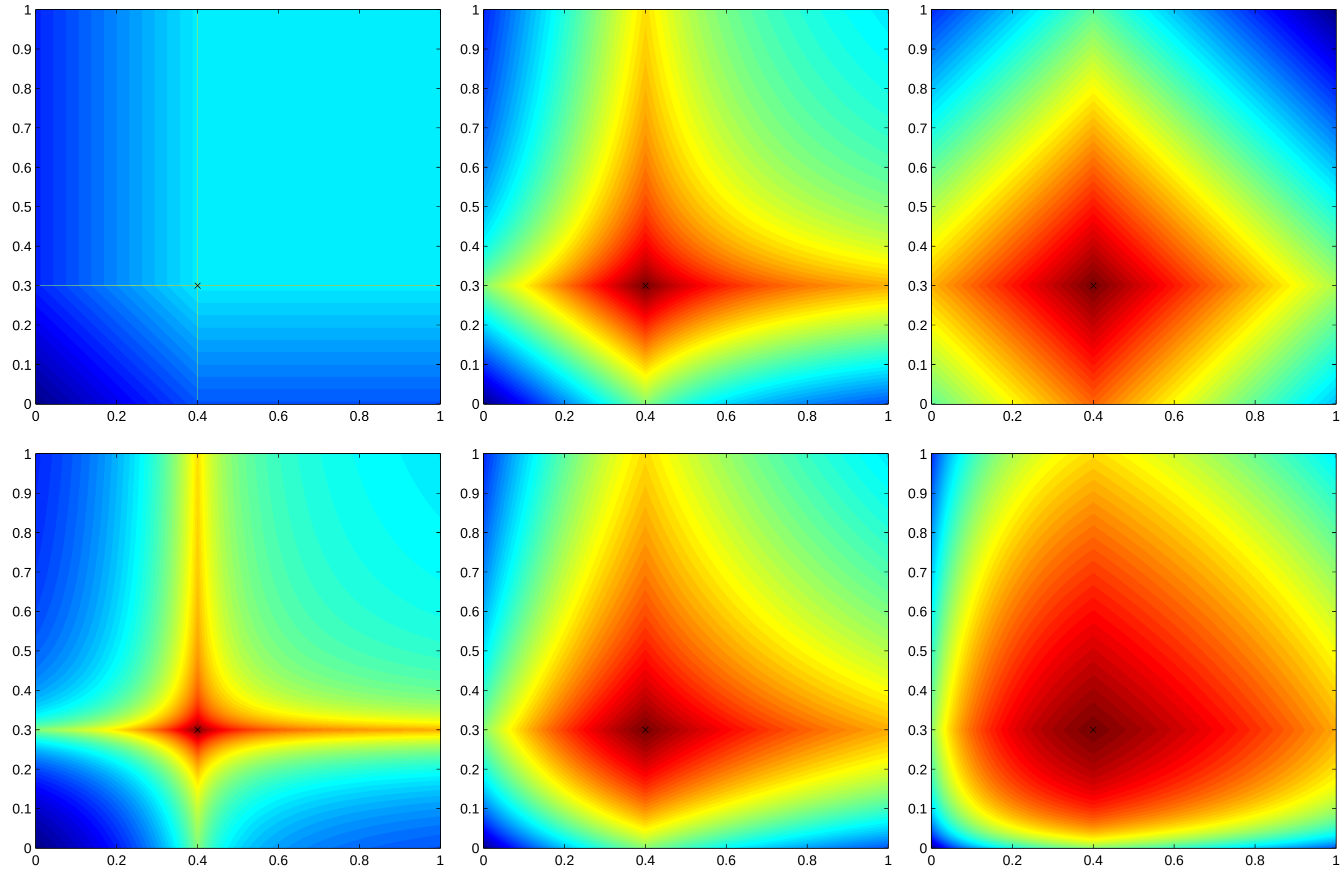

Figure 1: Examples of similarity measures of $A = \{0.4/x_1, 0.3/x_2\}$ and all fuzzy sets $\mathcal{F}(X)$, $n = 2$ where high and low values correspond to red and blue colors respectively. First row: $\mathcal{A}$ is the arithmetical mean, and we use $I_{\top_S}$, $I_{\top_A}$ and $I_{\top_L}$, respectively from left to right. Second row: $\mathcal{A}$ is the arithmetical mean and we used the Hamacher implication $I_{\top_{H_\gamma}}$, where $\gamma = 0, 2, 5$ from left to right.

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