On the structure of the category O for W-Algebras

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ON THE STRUCTURE OF THE CATEGORY $\mathcal{O}$ FOR W-ALGEBRAS

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Abstract. A W-algebra (of finite type) $\mathcal{W}$ is a certain associative algebra associated with a semisimple Lie algebra, say $\mathfrak{g}$, and its nilpotent element, say e. The goal of this paper is to study the category $\mathcal{O}$ for $\mathcal{W}$ introduced by Brundan, Goodwin and Kleshchev. We establish an equivalence of this category with a certain category of $\mathfrak{g}$-modules. In the case when e is of principal Levi type (this is always so when $\mathfrak{g}$ is of type A) the category of $\mathfrak{g}$-modules in interest is the category of generalized Whittaker modules introduced by McDowell, and studied by Milicic-Soergel and Backelin.

1. Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field $\mathbb{K}$ of characteristic zero. Choose a nilpotent element $e \in \mathfrak{g}$. Associated to the pair $(\mathfrak{g}, e)$ is a certain associative algebra $\mathcal{W}$, which is closely related to the universal enveloping algebra $U(\mathfrak{g})$. It was studied extensively during the last decade starting from Premet's paper [Pr1], see also [BGK],[BK1],[BK2],[GG],[Gi],[Lo1],[Lo2], [Pr2]-[Pr4]. Definitions of a W-algebra due to Premet, [Pr1], and the author, [Lo1], are recalled in Section 2.

In the representation theory of $U(\mathfrak{g})$ a crucial role is played by the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of $U(\mathfrak{g})$-modules. In particular, all finite dimensional $U(\mathfrak{g})$-modules and all Verma modules belong to $\mathcal{O}$. In [BGK] Brundan, Goodwin and Kleshchev introduced the notion of the category $\mathcal{O}$ for $\mathcal{W}$. This category also contains all finite dimensional $\mathcal{W}$-modules as well as analogs of Verma modules. See Section 3 for definitions.

The BGK category $\mathcal{O}$ is not always very useful. For example, for a *distinguished* nilpotent element $e \in \mathfrak{g}$ (i.e., such that the centralizer $\mathfrak{z}_\mathfrak{g}(e)$ contains no nonzero semisimple elements) $\mathcal{O}$ consists precisely of finite dimensional modules. The other extreme is the case when e is of principal Levi type. This means that there is a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ such that e is a principal nilpotent element in $\mathfrak{l}$. Here the BGK category $\mathcal{O}$ looks quite similar to the BGG one.

In [BGK], Conjecture 5.3, the authors conjectured that for e of principal Levi type there exists a category equivalence between their category $\mathcal{O}$ and a certain category of *generalized Whittaker modules* introduced by McDowell, [McD], and studied by Milicic and Soergel, [MS], and Backelin, [Ba]. We postpone the description of this category until Section 4. The main result of this paper, Theorem 4.1, gives a proof of that conjecture.

Let us describe the content of this paper. In Section 2 we recall the definition of W-algebras and the basic theorem of our paper [Lo1], the so called decomposition theorem. In Section 3 the notion of the category $\mathcal{O}$ for a W-algebra is recalled. In Section 4 we introduce the category of generalized Whittaker modules. Special cases of this category

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are, firstly, Skryabin's category of Whittaker modules (or, more precisely, the full subcategory there consisting of all finitely generated modules) and, secondly, the categories studied in [McD],[MS],[Ba]. Then we state the category equivalence theorem 4.1 generalizing the Skryabin equivalence theorem from the appendix to [Pr1] and proving the conjecture of Brundan, Goodwin and Kleshchev. The proof of Theorem 4.1 is given in Section 5. Essentially, we generalize the proof of the Skryabin equivalence theorem given in [Lo1], Subsection 3.3, checking that certain topological algebras are isomorphic.

Finally, in Section 6 we will describe some applications of our results.

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2. W-algebras

Throughout the paper everything is defined over an algebraically closed field $\mathbb{K}$ of characteristic 0.

Let G be a reductive algebraic group, $\mathfrak{g}$ its Lie algebra, and $\mathcal{U}$ the universal enveloping algebra of $\mathfrak{g}$. Fix a nonzero nilpotent element $e \in \mathfrak{g}$. Choose an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$ and set $Q := Z_G(e, h, f)$. Denote by T a maximal torus of Q. Also introduce a grading on $\mathfrak{g}$ by eigenvalues of $\operatorname{ad} h$: $\mathfrak{g} := \bigoplus \mathfrak{g}(i), \mathfrak{g}(i) := \{\xi \in \mathfrak{g} | [h, \xi] = i\xi\}$. Consider the one-parameter subgroup $\gamma : \mathbb{K}^\times \to G$ with $\frac{d}{dt}|_{t=0}\gamma = h$. Choose a G-invariant symmetric form $(\cdot, \cdot)$ on $\mathfrak{g}$, whose restriction to any algebraic reductive subalgebra is non-degenerate. This form allows one to identify $\mathfrak{g} \cong \mathfrak{g}^*$. Let $\chi = (e, \cdot)$ be the element of $\mathfrak{g}^*$ corresponding to e.

Equip the space $\mathfrak{g}(-1)$ with a symplectic form ω_χ as follows: $\omega_\chi(\xi, \eta) = \langle \chi, [\xi, \eta] \rangle$. Fix a lagrangian subspace $l \subset \mathfrak{g}(-1)$ and define the subalgebra $\mathfrak{m} := l \oplus \bigoplus_{i \leqslant -2} \mathfrak{g}(i) \subset \mathfrak{g}$. According to Premet, [Pr1], a W-algebra $\mathcal{W}$ associated with e is, by definition, $(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\operatorname{ad}\mathfrak{m}}$, where $\mathfrak{m}_\chi := \{\xi - \langle \chi, \xi \rangle, \xi \in \mathfrak{m}\}$. As Gan and Ginzburg checked in [GG], $\mathcal{W}$ does not depend on the choice of l up to some natural isomorphism. Thus we can choose a T-stable lagrangian subspace $l \subset \mathfrak{g}(-1)$ so we get an action of T on $\mathcal{W}$. Note that the image of $\mathfrak{t}$ in $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi$ consists of $\operatorname{ad}\mathfrak{m}$-invariants, for $\mathfrak{m}$ is $\mathfrak{t}$-stable and χ is annihilated by $\mathfrak{t}$. So we get an embedding $\mathfrak{t} \hookrightarrow \mathcal{W}$. It is compatible with the action of T in the sense that the differential of the T-action coincides with the adjoint action of $\mathfrak{t} \subset \mathcal{W}$. In fact, from the construction in [GG] it follows that Q acts on $\mathcal{W}$ by algebra automorphisms, see [Pr2], Subsection 2.2, for details.

One important feature of $\mathcal{W}$ is that the category $\mathcal{W}$-Mod of (left) $\mathcal{W}$-modules is equivalent to a certain full subcategory in $\mathcal{U}$-Mod to be described now. We say that a left $\mathcal{U}$-module M is a *Whittaker* module if $\mathfrak{m}_\chi$ acts on M by locally nilpotent endomorphisms. In this case $M^{\mathfrak{m}_\chi} = \{m \in M | \xi m = \langle \chi, \xi \rangle m, \forall \xi \in \mathfrak{m}\}$ is a $\mathcal{W}$-module. As Skryabin proved in the appendix to [Pr1], the functor $M \mapsto M^{\mathfrak{m}_\chi}$ is an equivalence between the category of Whittaker $\mathcal{U}$-modules and $\mathcal{W}$-Mod. A quasiinverse equivalence is given by $N \mapsto \mathcal{S}(N) := (\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi) \otimes_{\mathcal{W}} N$, where $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi$ is equipped with a natural structure of a $\mathcal{U}$-$\mathcal{W}$-bimodule.

Note also that the center of $\mathcal{W}$ can be identified with the center $\mathcal{Z}$ of $\mathcal{U}$, as follows. It is clear that $\mathcal{Z} \subset \mathcal{U}^{\operatorname{ad}\mathfrak{m}}$ whence we have a homomorphism $\mathcal{Z} \to \mathcal{W}$. This homomorphism is injective and its image coincides with the center of $\mathcal{W}$, see [Pr2], the footnote to the Question 5.1.

An alternative description of $\mathcal{W}$ was given in [Lo1]. Define the Slodowy slice $S := e + \mathfrak{z}_{\mathfrak{g}}(f)$. It will be convenient for us to consider S as a subvariety in $\mathfrak{g}^*$. Define the *Kazhdan* action of $\mathbb{K}^\times$ on $\mathfrak{g}^*$ by $t.\alpha = t^{-2}\gamma(t)\alpha$. This action preserves S and, moreover, $\lim_{t\to\infty} t.s = \chi$ for all $s \in S$. Also note that Q acts on S in a natural way.

Set $V := [\mathfrak{g}, f]$. Equip V with the symplectic form $\omega(\xi,\eta) = \langle \chi, [\xi,\eta]\rangle$, the action of $\mathbb{K}^\times, t.v = \gamma(t)^{-1}v$, and the natural action of Q.

Now let Y be a smooth affine variety equipped with commuting actions of a reductive group Q and of the one-dimensional torus $\mathbb{K}^\times$. For instance, one can take $Y = \mathfrak{g}^*, S, V^*$ equipped with the natural actions of $Q = Z_G(e,h,f)$ and the Kazhdan actions of $\mathbb{K}^\times$. Note that the grading on $\mathbb{K}[S]$ induced by the Kazhdan action is positive.

As follows from the explanation in [Lo2], Subsection 2.1, for $Y = \mathfrak{g}^*, V^*, S$ there are certain *star-products* $* : \mathbb{K}[Y] \otimes \mathbb{K}[Y] \to \mathbb{K}[Y][\hbar], f * g = \sum_{i=0}^\infty D_i(f,g)\hbar^{2i}$, satisfying the following conditions.

(1) $*$ is associative, that is, a natural extension of $*$ to $\mathbb{K}[Y][\hbar]$ turns $\mathbb{K}[Y][\hbar]$ into an associative $\mathbb{K}[\hbar]$-algebra, and 1 is a unit for this product.
(2) $D_0(f,g) = fg$ for all $f, g \in \mathbb{K}[Y]$.
(3) $D_i(\cdot,\cdot)$ is a bidifferential operator of order at most i in each variable.
(4) $*$ is a Q-equivariant map $\mathbb{K}[Y] \otimes \mathbb{K}[Y] \to \mathbb{K}[Y][\hbar]$.
(5) $*$ is homogeneous with respect to $\mathbb{K}^\times$. This, by definition, means that the degree of D_i is $-2i$ for all i.
(6) There is a Q-equivariant map $\mathfrak{q} \to \mathbb{K}[Y][\hbar], \xi \mapsto \widehat{H}_\xi$, such that $\hbar^{-2}[\widehat{H}_\xi, \bullet]$ coincides with the image of ξ under the differential of the Q-action on $\mathbb{K}[Y][\hbar]$.

This construction allows one to equip $\mathbb{K}[\mathfrak{g}^*], \mathbb{K}[V^*], \mathbb{K}[S]$ with new associative products $*_1$ defined by $f *_1 g = \sum_{i=0}^\infty D_i(f,g)$. The algebras $\mathbb{K}[\mathfrak{g}^*], \mathbb{K}[V^*], \mathbb{K}[S]$ with these new products are T (and, in fact, Q)-equivariantly isomorphic to $\mathcal{U}$, the Weyl algebra $\mathbf{A}_V$ of the vector space V, and the W-algebra $\mathcal{W}$, respectively.

We finish this section by recalling a decomposition result from [Lo1], which plays a crucial role in our construction.

Recall that if X is an affine algebraic variety and x a point of X we can consider the completion $\mathbb{K}[X]^\wedge_x := \varprojlim_k \mathbb{K}[X]/\mathbb{K}[X]\mathfrak{m}_x^k$, where $\mathfrak{m}_x$ denotes the maximal ideal corresponding to x. If X is an affine space, then taking x for the origin and choosing a basis in X, we can identify $\mathbb{K}[X]^\wedge_x$ with the algebra of formal power series. The algebra $\mathbb{K}[X]^\wedge_x$ is equipped with the topology of the inverse image. If $D : \mathbb{K}[X] \otimes \mathbb{K}[X] \to \mathbb{K}[X]$ is a bidifferential operator, then it can be uniquely extended to a continuous bidifferential operator $\mathbb{K}[X]^\wedge_x \otimes \mathbb{K}[X]^\wedge_x \to \mathbb{K}[X]^\wedge_x$.

Since our star-products satisfy (3), we can extend them to the completions $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi, \mathbb{K}[V^*]^\wedge_0$, $\mathbb{K}[S]^\wedge_\chi$. So we get new algebra structures on $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]], \mathbb{K}[V^*]^\wedge_0[[\hbar]], \mathbb{K}[S]^\wedge_\chi[[\hbar]]$. These algebras have unique maximal ideals, for instance, the maximal ideal $\widetilde{\mathfrak{m}} \subset \mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]]$ is the inverse image of the maximal ideal in $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi$. The algebra $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi$ is complete in the $\widetilde{\mathfrak{m}}$-adic topology. The similar claims hold for the other two algebras.

Consider the algebra $\mathbb{K}[S]^\wedge_\chi[[\hbar]] \otimes_{\mathbb{K}[[\hbar]]} \mathbb{K}[V^*]^\wedge_0[[\hbar]]$ and let $\widetilde{\mathfrak{m}}$ denote its maximal ideal corresponding to the point $(\chi, 0)$. Note that the algebra is not complete in the $\widetilde{\mathfrak{m}}$-adic topology. Taking the completion, we get the *completed tensor product*, which we denote by $\mathbb{K}[S]^\wedge_\chi[[\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathbb{K}[V^*]^\wedge_0[[\hbar]]$. As a vector space, the last algebra is just $\mathbb{K}[S \times V^*]^\wedge_{(\chi,0)}[[\hbar]]$.

Finally, note that there is a natural identification $\varphi : \mathfrak{z}_{\mathfrak{g}}(e) \oplus V \to \mathfrak{g}, (\xi,\eta) \mapsto \xi + \eta$.

The first two assertions of the following Proposition follow from [Lo1], Theorem 3.1.3, and the third follows from [Lo2], Theorem 2.3.1, for semisimple G and from Remark 2.3.2 for a general reductive group G.

Proposition 2.1. *There is a $Q \times \mathbb{K}^\times$-equivariant isomorphism*

$$\Phi_\hbar : \mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]] \to \mathbb{K}[S]^\wedge_\chi[[\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathbb{K}[V^*]^\wedge_0[[\hbar]]$$

of topological $\mathbb{K}[[\hbar]]$-algebras satisfying the following conditions:

1. $\Phi_\hbar(\sum_{i=0}^\infty f_i\hbar^{2i})$ *contains only even powers of $\hbar$.*
2. *The map between cotangent spaces $d_0(\Phi_\hbar)^* : \mathfrak{z}_\mathfrak{g}(e)\oplus V \to \mathfrak{g}$ induced by $\Phi_\hbar$ coincides with φ.*
3. *Let ι_1, ι_2 denote the embeddings of $\mathfrak{q}$ into $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]], \mathbb{K}[S]^\wedge_\chi[[\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathbb{K}[V^*]^\wedge_0[[\hbar]]$. Then $\Phi \circ \iota_1 = \iota_2$.*

This proposition allows to define a map from the set of two-sided ideals of $\mathcal{W}$ to the analogous set for $\mathcal{U}$. Namely, take a two-sided ideal $\mathcal{I} \subset \mathcal{W}$. As we noted in [Lo1], Subsection 3.2, there is a unique ideal $\mathcal{I}_\hbar \subset \mathbb{K}[S][\hbar]$ such that $\mathcal{I} = \mathcal{I}_\hbar/(\hbar - 1)$ and $\mathcal{I}_\hbar$ is *$\hbar$-saturated*, i.e., $\mathcal{I}_\hbar \cap \hbar\mathbb{K}[S][\hbar] = \hbar\mathcal{I}_\hbar$. Let $\mathcal{I}^\wedge_\hbar$ denote the closure of $\mathcal{I}_\hbar$ in $\mathbb{K}[S]^\wedge_\chi[[\hbar]]$. Let $\mathcal{J}_\hbar$ denote the intersection of $\Phi_\hbar^{-1}(\mathcal{I}^\wedge_\hbar\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathbb{K}[V^*]^\wedge_0[[\hbar]])$ with $\mathbb{K}[\mathfrak{g}^*][\hbar]$. Finally, set $\mathcal{I}^\dagger := \mathcal{J}_\hbar/(\hbar - 1) \subset \mathcal{U}$. By [Lo1], Proposition 3.4.1 and Theorem 1.2.2(ii), $\operatorname{Ann}_\mathcal{U}(\mathcal{S}(N)) = \operatorname{Ann}_\mathcal{W}(N)^\dagger$ for any $\mathcal{W}$-module N.

3. Category $\mathcal{O}$ for a W-algebra

Recall that we have an embedding $\mathfrak{t} \hookrightarrow \mathcal{W}$. Also we have a natural embedding of the cocharacter lattice $\mathfrak{X}^*(T) := \operatorname{Hom}(\mathbb{K}^\times, T)$ of T into $\mathfrak{t}$. Choose an element $\theta \in \mathfrak{X}^*(T) \subset \mathfrak{t}$. Let L stand for the centralizer of θ in G, this is a Levi subgroup of G. By $\mathfrak{l}$ we denote the Lie algebra of L, clearly, $e, h, f \in \mathfrak{l}$. Let T_0 denote the unit component of $Z(L) \cap T$ and $\mathfrak{t}_0$ be the Lie algebra of T_0. Note that $\theta \in \mathfrak{t}_0$.

The algebra $\mathcal{W}$ decomposes into the direct sum of weight spaces with respect to $\operatorname{ad}\theta$, $\mathcal{W} = \sum_{\alpha\in\mathbb{Z}} \mathcal{W}_\alpha$. Set

$$\mathcal{W}_{\geqslant 0} := \bigoplus_{\alpha\geqslant 0}\mathcal{W}_\alpha, \mathcal{W}_{>0} := \bigoplus_{\alpha>0}\mathcal{W}_\alpha, \mathcal{W}^+_{\geqslant 0} := \mathcal{W}_{\geqslant 0} \cap \mathcal{W}\mathcal{W}_{>0}. \tag{3.1}$$

It is clear that $\mathcal{W}_{\geqslant 0}$ is a subalgebra of $\mathcal{W}$, while $\mathcal{W}_{>0}, \mathcal{W}^+_{\geqslant 0}$ are two-sided ideals in $\mathcal{W}_{\geqslant 0}$.

Let us make a remark on the choice of generators in the left $\mathcal{W}$-ideal $\mathcal{W}\mathcal{W}_{>0}$. Now we consider $\mathcal{W}$ as the algebra $\mathbb{K}[S]$ with the modified multiplication. We have an embedding $\mathfrak{z}_\mathfrak{g}(e) \hookrightarrow \mathbb{K}[S]$ and $\mathfrak{z}_\mathfrak{g}(e)$ generates $\mathbb{K}[S]$ (for both multiplications). It is easy to see that $\mathfrak{z}_\mathfrak{g}(e)_{>0}$ generates $\mathcal{W}\mathcal{W}_{>0}$. Let $f_1, \ldots, f_n$ denote a homogeneous (with respect to $\operatorname{ad}\theta$) basis in $\mathfrak{z}_\mathfrak{g}(e)_{>0}$.

Set $\mathcal{W}^0 := \mathcal{W}_{\geqslant 0}/\mathcal{W}^+_{\geqslant 0}$.

Remark 3.1. In [BGK] Brundan, Kleshchev and Goodwin constructed an isomorphism between $\mathcal{W}^0$ and the W-algebra $\underline{\mathcal{W}}$ constructed for the pair $(\mathfrak{l}, e)$. We will also show that there is an isomorphism between the two algebras in the course of the proof of Theorem 4.1. See also Remark 5.5.

We now proceed to the definition of full subcategories $\widetilde{\mathcal{O}}(\theta), \widetilde{\mathcal{O}}^{\mathfrak{t}_0}(\theta), \mathcal{O}(\theta), \mathcal{O}^{\mathfrak{t}_0}(\theta)$ in the category $\mathcal{W}$-Mod of left $\mathcal{W}$-modules.

First, we say that a $\mathcal{W}$-module M belongs to $\widetilde{\mathcal{O}}(\theta)$ if M is finitely generated and the following condition holds:

(*) for any $m \in M$ there exists $\alpha \in \mathbb{Z}$ such that $\mathcal{W}_\beta m = 0$ for any $\beta \geqslant \alpha$.

Clearly, $\widetilde{\mathcal{O}}(\theta)$ is an abelian subcategory in the category $\mathcal{W}$-Mod. By definition, $\widetilde{\mathcal{O}}^{\mathfrak{t}_0}(\theta)$ consists of all modules in $\widetilde{\mathcal{O}}(\theta)$, where the action of $\mathfrak{t}_0$ is diagonalizable.

For example, take a finitely generated $\mathcal{W}^0$-module V. Set $\mathcal{M}^\theta(V) := \mathcal{W} \otimes_{\mathcal{W}_{\geqslant 0}} V$, where $\mathcal{W}_{\geqslant 0}$ acts on V via an epimorphism $\mathcal{W}_{\geqslant 0} \twoheadrightarrow \mathcal{W}^0$. The module $\mathcal{M}^\theta(V)$ lies in $\widetilde{\mathcal{O}}(\theta)$. It belongs to $\widetilde{\mathcal{O}}^{\mathfrak{t}_0}(\theta)$ if and only if the action of $\mathfrak{t}_0$ on V is diagonalizable.

Lemma 3.2. *For a finitely generated $\mathcal{W}$-module M the condition $(*)$ is equivalent to either of the following two conditions:*

($*'$) *$\mathcal{W}_{>0}$ acts on M by locally nilpotent endomorphisms.*

($*''$) *The elements $f_i \in \mathcal{W}_{>0}$ act on M by locally nilpotent endomorphisms.*

Proof. Clearly, $(*) \Rightarrow (*') \Rightarrow (*'')$. Let us prove the implication $(*'') \Rightarrow (*)$. Let $m_1, \ldots, m_r$ generate M. Let N be such that $f_i^N m_j = 0$ for all i, j. Let $\mathcal{I}$ denote the intersection of $\mathcal{W}_{\geqslant 0}$ with the left ideal in $\mathcal{W}$ generated by f_i^N. It is easy to see that $\operatorname{gr}(\mathcal{I})$ contains $\mathbb{K}[S]_{>\beta}$ for sufficiently large β. Since both $\mathcal{I}$ and the filtration are $\operatorname{ad}\theta$-stable, we see that $\mathcal{W}_{>\beta} \subset \mathcal{I}$. So $\mathcal{W}_{>\beta}$ annihilates $m_1, \ldots, m_r$. This condition implies (*). □

To define the two remaining subcategories we need a certain functor $\widetilde{\mathcal{O}}(\theta) \to \mathcal{W}^0$-Mod. This is the functor of taking $\mathcal{W}_{>0}$-invariants. More precisely, set

$$\mathcal{F}(M) = M^{\mathcal{W}_{>0}} := \{m \in M | w.m = 0, \forall w \in \mathcal{W}_{>0}\}.$$

Note that the functor $\mathcal{M}^\theta : \mathcal{W}^0\text{-Mod} \to \widetilde{\mathcal{O}}(\theta)$ is left adjoint to $\mathcal{F} : \widetilde{\mathcal{O}}^\theta \to \mathcal{W}^0\text{-Mod}$. Indeed,

$$\operatorname{Hom}_{\mathcal{W}}(\mathcal{W} \otimes_{\mathcal{W}_{\geqslant 0}} N, M) = \operatorname{Hom}_{\mathcal{W}_{\geqslant 0}}(N, M) = \operatorname{Hom}_{\mathcal{W}^0}(N, \mathcal{F}(M)), N \in \mathcal{W}^0\text{-Mod}, M \in \widetilde{\mathcal{O}}(\theta).$$

By definition, $\mathcal{O}(\theta)$ (resp., $\mathcal{O}^{\mathfrak{t}_0}(\theta)$) consists of all modules M in $\widetilde{\mathcal{O}}(\theta)$ (resp., $\widetilde{\mathcal{O}}^{\mathfrak{t}_0}(\theta)$) with $\dim \mathcal{F}(M) < \infty$. Note that all finite dimensional $\mathcal{W}$-modules lie in $\mathcal{O}(\theta)$.

Let us state some results describing the properties of our four categories.

Proposition 3.3. (1) *The action of $\mathfrak{t}_0$ on any module from $\mathcal{O}(\theta)$ is locally finite.*

(2) *Any module in $\mathcal{O}(\theta)$ contains a submodule from $\mathcal{O}^{\mathfrak{t}_0}(\theta)$.*

Proof. The subspace $\mathcal{F}(M) \subset M$ is finite dimensional. Let $\mathcal{F}(M)_{diag}$ be the sum of all weight subspaces for $\mathfrak{t}_0$ in $\mathcal{F}(M)$. Then $\mathcal{F}(M)_{diag}$ is a $\mathcal{W}^0$-submodule in $\mathcal{F}(M)$. Let M_0 be the image of $\mathcal{M}^\theta(\mathcal{F}(M)_{diag})$ in M under the natural homomorphism. Then the action of $\mathfrak{t}_0$ on M_0 is diagonalizable. Hence the second assertion.

Since M is a Noetherian $\mathcal{W}$-module, we see that there is a filtration $M_0 \subset M_1 \subset \ldots \subset M_k = M$ such that the action of $\mathfrak{t}_0$ on every quotient M_i/M_{i-1} is locally finite. Assertion 1 follows. □

Proposition 3.4. *Let $M \in \widetilde{\mathcal{O}}(\theta)$. Then the following conditions are equivalent:*

(1) *$M \in \mathcal{O}(\theta)$.*

(2) *For any $\alpha \in \mathbb{K}$ the root subspace $M^\alpha := \bigcup_i \ker(\theta - \alpha)^i$ is finite dimensional.*

(3) *For any $\widetilde{\alpha} \in \mathfrak{t}_0^*$ the root subspace $M^{(\widetilde{\alpha})} := \bigcup_i \left(\bigcap_{\xi \in \mathfrak{t}_0} \ker(\xi - \langle\widetilde{\alpha}, \xi\rangle)^i\right)$ is finite dimensional.*

In the proof of Proposition 3.4 we will need the following simple lemma.

Lemma 3.5. *Define the partial order on $\mathbb{K}$ as follows: $x \preceq y$ if $y-x$ is a nonnegative integer. We write $x \prec y$ if $x \preceq y, x \neq y$. For any $M \in \widetilde{\mathcal{O}}(\theta)$ there exist $\alpha_1, \ldots, \alpha_r \in \mathbb{K}$ such that for any eigenvalue β of θ on M there exists i with $\beta \preceq \alpha_i$.*

Proof. Take finite number of homogeneous generators of M and use the condition (*). □

Proposition 3.4 and Lemma 3.5 show that for $\mathfrak{t} = \mathfrak{t}_0$ the category $\mathcal{O}^{\mathfrak{t}}(\theta)$ consists of the same modules as the category $\mathcal{O}(e)$ studied in [BGK]: for example, the implication (1)⇒(3) shows that any module from $\mathcal{O}^{\mathfrak{t}}(\theta)$ lies in $\mathcal{O}(e)$. Recall that in [BGK] the category $\mathcal{O}(e)$ was defined as the full subcategory in $\mathcal{W}$-Mod consisting of all modules, where $\mathfrak{t}$ acts diagonalizably, all weight subspaces are finite dimensional, and the set of weights is bounded from above. Their notion of being "bounded from above" is equivalent to that mentioned in Lemma 3.5 although is stated in a different way.

Proof of Proposition 3.4. Let us check (1) ⇒ (2). Assume the converse. Choose maximal (w.r.t $\preceq$) $\alpha \in \mathbb{K}$ such that $\dim M^\alpha = \infty$. Let $f_1, \ldots, f_n$ be such as above. Then, by the choice of α, we see that $\dim f_i M^\alpha < \infty$ for any i. We see that $f_i M^\alpha \subset \bigoplus_{\alpha \prec \beta} M^\beta$ for any i. It follows that the intersection of the kernels of f_i in M^α is infinite dimensional. But this intersection coincides with $M^\alpha \cap \mathcal{F}(M)$. Contradiction.

The implication (2) ⇒ (3) follows from $M^{(\widetilde{\alpha})} \subset M^{\langle \widetilde{\alpha}, \theta \rangle}$. So it remains to check that (3) ⇒ (1).

For any $\widetilde{\alpha} \in \mathfrak{t}_0^*$ the weight subspace

$$\mathcal{F}(M)_{\widetilde{\alpha}} := \{v \in \mathcal{F}(M) | \xi.v = \langle \widetilde{\alpha}, \xi \rangle v, \forall \xi \in \mathfrak{t}_0\}$$

is a finite dimensional $\mathcal{W}^0$-submodule. Choose $\widetilde{\alpha}$ such that $\mathcal{F}(M)_{\widetilde{\alpha}} \neq \{0\}$. Let M_0 denote the submodule in M generated by $\mathcal{F}(M)_{\widetilde{\alpha}}$. Clearly, M_0 is isomorphic to the quotient of $\mathcal{M}^\theta(\mathcal{F}(M)_{\widetilde{\alpha}})$. If M is irreducible, then $M = M_0$. In particular, $\langle \widetilde{\beta}, \theta \rangle \prec \langle \widetilde{\alpha}, \theta \rangle$ for any $\mathfrak{t}_0$-weight $\widetilde{\beta}$ of $\mathcal{F}(M)$. It follows that $M \in \mathcal{O}(\theta)$. In general, since $\mathcal{F}$ is a left exact functor, it remains to check that M has finite length.

As Brundan, Goodwin and Kleshchev proved in [BGK], Corollary 4.11, the module $\mathcal{M}^\theta(V)$ has finite length provided V is finite dimensional and irreducible. Actually, they considered the case when $\mathfrak{t} = \mathfrak{t}_0$ but their proof extends to the general case directly. Since $\mathcal{M}^\theta$ is a right exact functor, we see that $\mathcal{M}^\theta(\mathcal{F}(M)_{\widetilde{\alpha}})$ has finite length. Thus M_0 has finite length. Finally, since M is Noetherian, we see that M has finite length. □

From this proposition and its proof we deduce the following

Corollary 3.6. *The subcategory $\mathcal{O}(\theta)$ is a Serre subcategory in $\widetilde{\mathcal{O}}(\theta)$ (i.e., it is closed with respect to taking subquotients and extensions) and any module in $\mathcal{O}(\theta)$ has finite length. Furthermore, $\mathcal{O}(\theta)$ contains all Verma modules $\mathcal{M}^\theta(V)$ for finite dimensional V.*

As Brundan, Goodwin and Kleshchev noticed in [BGK] (in the case $\mathfrak{t} = \mathfrak{t}_0$, the general case is completely analogous), the following statement holds.

Proposition 3.7. *Let V be an irreducible finite dimensional $\mathcal{W}^0$-module. There is a unique simple quotient $L^\theta(V)$ of $\mathcal{M}^\theta(V)$ and any simple module in $\mathcal{O}(\theta)$ is isomorphic to some $L^\theta(V)$.*

Remark 3.8. Note that $\widetilde{\mathcal{O}}(\theta)$ depends on the choice of θ. However, it follows from Lemma 3.2 that two categories $\widetilde{\mathcal{O}}(\theta_1), \widetilde{\mathcal{O}}(\theta_2)$ consist of the same modules provided the spaces $\mathfrak{z}_{\mathfrak{g}}(e)_{>0}$ constructed for θ_1, θ_2 coincide. So we get only finitely many different categories $\mathcal{O}(\theta)$.

4. Generalized Whittaker modules

In this section we introduce a certain category of $\mathcal{U}$-modules generalizing Whittaker modules mentioned in Section 2. Also we state our main result here.

Recall that we have fixed $\theta \in \mathfrak{X}^*(T) \cap \mathfrak{t}_0$. Let $\mathfrak{g}_{>0}$ denote the sum of all eigenspaces for $\operatorname{ad}\theta$ with positive eigenvalues. Then $\mathfrak{g}_{\geqslant 0} := \mathfrak{l} \oplus \mathfrak{g}_{>0}$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_{>0}$ is its nilpotent radical.

Recall the grading $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ introduced in the beginning of Section 2. For $\mathfrak{l}(i) := \mathfrak{l} \cap \mathfrak{g}(i)$ we have $\mathfrak{l} = \bigoplus_i \mathfrak{l}(i)$. By analogy with the subalgebra $\mathfrak{m} \subset \mathfrak{g}$ define a subalgebra $\underline{\mathfrak{m}} \subset \mathfrak{l}$ and its shift $\underline{\mathfrak{m}}_\chi$ so that we can define the W-algebra $\underline{\mathcal{W}} = (\underline{\mathcal{U}}/(\underline{\mathcal{U}}\underline{\mathfrak{m}}_\chi))^{\operatorname{ad}\underline{\mathfrak{m}}}$, where $\underline{\mathcal{U}} := U(\mathfrak{l})$. Finally, set $\widetilde{\mathfrak{m}} = \underline{\mathfrak{m}} \oplus \mathfrak{g}_{>0}, \widetilde{\mathfrak{m}}_\chi := \{\xi - \langle \xi, \chi \rangle, \xi \in \widetilde{\mathfrak{m}}\}$.

Let M be a finitely generated left $\mathcal{U}$-module. We say that M is a *generalized Whittaker module* for (e, θ) if $\widetilde{\mathfrak{m}}_\chi$ acts on M by locally nilpotent endomorphisms. The full subcategory of $\mathcal{U}$-Mod consisting of all generalized Whittaker modules for (e, θ) will be denoted by $\widetilde{\mathrm{Wh}}(e, \theta)$. By $\widetilde{\mathrm{Wh}}^{\mathfrak{t}_0}(e, \theta)$ we denote the full subcategory in $\widetilde{\mathrm{Wh}}(e, \theta)$ consisting of all modules with diagonalizable action of $\mathfrak{t}_0$.

For example, let V be a $\underline{\mathcal{W}}$-module. Set

$$\mathcal{M}^{e,\theta}(V) = \mathcal{U} \otimes_{U(\mathfrak{g}_{\geqslant 0})} \mathcal{S}_{\mathfrak{l}}(V),$$

where $\mathcal{S}_{\mathfrak{l}}(V) = (\underline{\mathcal{U}}/\underline{\mathcal{U}}\underline{\mathfrak{m}}_\chi) \otimes_{\underline{\mathcal{W}}} V$. This module always lies in $\widetilde{\mathrm{Wh}}(e, \theta)$. It lies in $\widetilde{\mathrm{Wh}}^{\mathfrak{t}_0}(e, \theta)$ if and only if the action of $\mathfrak{t}_0$ on V is diagonalizable.

Let us construct a functor from $\widetilde{\mathrm{Wh}}(e, \theta)$ to the category $\underline{\mathcal{W}}$-Mod. Set $\mathcal{G}(M) := M^{\widetilde{\mathfrak{m}}_\chi}$. The algebra $\underline{\mathcal{U}}$ acts naturally on $M^{\mathfrak{g}_{>0}}$. Since $\mathcal{G}(M) = (M^{\mathfrak{g}_{>0}})^{\underline{\mathfrak{m}}_\chi}$, there is a natural action of $\underline{\mathcal{W}}$ on $\mathcal{G}(M)$. It is clear that $\mathcal{G}(M) \neq \{0\}$ provided $M \neq \{0\}$. We say that $M \in \widetilde{\mathrm{Wh}}(e, \theta)$ is of *finite type* if $\dim \mathcal{G}(M) < \infty$. The category of all finite type modules is denoted by $\mathrm{Wh}(e, \theta)$. Finally, set $\mathrm{Wh}^{\mathfrak{t}_0}(e, \theta) = \mathrm{Wh}(e, \theta) \cap \widetilde{\mathrm{Wh}}^{\mathfrak{t}_0}(e, \theta)$.

Note also that, analogously to the previous section, the functor $\mathcal{M}^{e,\theta} : \underline{\mathcal{W}}\text{-}\mathrm{Mod} \to \widetilde{\mathrm{Wh}}(e, \theta)$ is left adjoint to $\mathcal{G}$.

The following theorem is the main result of the paper.

Theorem 4.1. *There is an equivalence $\mathcal{K} : \widetilde{\mathrm{Wh}}(e, \theta) \to \widetilde{\mathcal{O}}(\theta)$ of abelian categories and an isomorphism $\Psi : \underline{\mathcal{W}} \to \mathcal{W}^0$ satisfying the following conditions:*

1. $\mathrm{Ann}_{\mathcal{W}}(\mathcal{K}(M))^\dagger = \mathrm{Ann}_{\mathcal{U}}(M)$ *for any* $M \in \widetilde{\mathrm{Wh}}(e, \theta)$.
2. $\mathcal{K}$ *maps* $\widetilde{\mathrm{Wh}}^{\mathfrak{t}_0}(e, \theta)$ *to* $\widetilde{\mathcal{O}}^{\mathfrak{t}_0}(\theta)$, *and* $\mathrm{Wh}(e, \theta)$ *to* $\mathcal{O}(\theta)$.
3. *The functors* $\Psi^* \circ \mathcal{F} \circ \mathcal{K}$ *and* $\mathcal{G}$ *from* $\widetilde{\mathrm{Wh}}(e, \theta)$ *to* $\underline{\mathcal{W}}$-Mod *are isomorphic. Here* Ψ^* *denotes the pull-back functor between the categories of modules induced by* Ψ.
4. *The functors* $\mathcal{K} \circ \mathcal{M}^{e,\theta}, \mathcal{M}^\theta \circ \Psi^{-1*}$ *from* $\underline{\mathcal{W}}$-Mod *to* $\widetilde{\mathcal{O}}(\theta)$ *are isomorphic.*

In particular, $\mathcal{K}$ induces an equivalence of abelian categories $\mathrm{Wh}(e, \theta) \to \mathcal{O}(\theta)$. Moreover, for any irreducible finite dimensional $\underline{\mathcal{W}}$-module V the $\mathcal{U}$-module $\mathcal{M}^{e,\theta}(V)$ has a unique simple quotient $L^{e,\theta}(V)$. The last claim follows either from the theorem above or can be proved in the same way as an analogous statement in [MS], Proposition 2.1.

Now we consider an important special case. Till the end of the section we assume that e is regular in $\mathfrak{l}$. The results below in this section will not be used in the proof of Theorem 4.1.

The following (quite standard) proposition shows that the category $\mathrm{Wh}(e,\theta)$ coincides with the category considered in [MS],[Ba].

Proposition 4.2. *Let $M \in \widetilde{\mathrm{Wh}}(e,\theta)$, where e is regular in $\mathfrak{l}$. Then the following two conditions are equivalent:*

1. $\dim \mathcal{G}(M) < \infty$.
2. *The action of the center $\mathcal{Z}$ of $\mathcal{U}$ on M is locally finite.*

Proof. (1) $\Rightarrow$ (2). Thanks to Theorem 4.1 and Corollary 3.6, we may assume that M is irreducible. Thence the natural homomorphism $\mathcal{M}^{e,\theta}(\mathcal{G}(M)) \to M$ is surjective. Note that $\mathcal{G}(M)$ is $\mathcal{Z}$-stable. Since $\dim \mathcal{G}(M) < \infty$, the $\mathcal{Z}$-action on $\mathcal{G}(M)$ is locally finite. But $\mathcal{G}(M)$ generates M hence the $\mathcal{Z}$-action on M is locally finite.

(2) $\Rightarrow$ (1). According to [MS], Theorem 2.6, M has finite length. So we may assume that M is irreducible. Choose a eigenvector $u \in M^{\mathfrak{g}_{>0}}$ for θ with eigenvalue α. All eigenvalues of θ in $\mathcal{U}v$ are $\preceq \alpha$ and if $u \in \mathcal{U}v$ has eigenvalue α, then $u \in \underline{\mathcal{U}}v$. So $M^{\mathfrak{g}_{>0}}$ is an irreducible $\underline{\mathcal{U}}$-module. By the Skryabin equivalence theorem (which in this case was proved already by Kostant), $\mathcal{G}(M)$ is an irreducible $\underline{\mathcal{W}}$-module. The algebra $\underline{\mathcal{W}}$ is identified with the center $\underline{\mathcal{Z}}$ of $\underline{\mathcal{U}}$. It is known that $\underline{\mathcal{Z}}$ is finite over $\mathcal{Z}$. Since the $\mathcal{Z}$-action on M is locally finite, we see that the $\underline{\mathcal{W}}$-action on $\mathcal{G}(M)$ is locally finite. Being both locally finite and irreducible, the $\underline{\mathcal{W}}$-module $\mathcal{G}(M)$ is finite dimensional. $\square$

Clearly, $\widetilde{\mathfrak{m}}$ is a maximal subalgebra of $\mathfrak{g}$ consisting of nilpotent elements. Let $\mathfrak{b} := \mathfrak{n}_{\mathfrak{g}}(\widetilde{\mathfrak{m}})$ be the corresponding Borel subalgebra, $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra. Let Δ, Δ_+, Π be, respectively, the root system and the sets of positive roots and of simple roots corresponding to $(\mathfrak{b},\mathfrak{h})$. Let $\Pi' \subset \Pi$ consist of all simple roots α such that $\chi|_{\mathfrak{g}_\alpha} \neq 0$. Then Π' is the system of simple roots in $\mathfrak{l}$. Conjugating χ by an element of $B \cap L$, where B is the Borel subgroup of G corresponding to $\mathfrak{b}$, if necessary, we may assume that $\chi|_{\mathfrak{g}_\alpha} = 0$ for $\alpha \in \Delta_+ \setminus \Pi'$. Finally, let $W' \subset W$ be the Weyl group of $\mathfrak{l}$. As Kostant proved in [Ko], $\underline{\mathcal{W}}$ is identified with the center of $\underline{\mathcal{U}}$. So any irreducible $\underline{\mathcal{W}}$-module is one-dimensional. These irreducible modules are parametrized by W'-orbits in $\mathfrak{h}^*$ for the action given by $w \cdot \lambda = w(\lambda+\rho) - \rho$, where, as usual, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$. So we get "Verma modules" $\mathcal{M}^{e,\theta}(\lambda)$ with $\mathcal{M}^{e,\theta}(\lambda_1) = \mathcal{M}^{e,\theta}(\lambda_2)$ if and only if λ_1, λ_2 are W'-conjugate, and their simple quotients $L^{e,\theta}(\lambda)$. As Milicic and Soergel checked in [MS], Proposition 2.1, Theorem 2.6, any simple module in $\widetilde{\mathrm{Wh}}(e,\theta)$ is isomorphic to $L^{e,\theta}(\lambda)$, and $L^{e,\theta}(\lambda_1) \cong L^{e,\theta}(\lambda_2)$ if and only if λ_1, λ_2 are W'-conjugate.

Since any module in $\widetilde{\mathrm{Wh}}(e,\theta)$ (in particular, $\mathcal{M}^{e,\theta}(\lambda)$) has finite length, the multiplicity $[\mathcal{M}^{e,\theta}(\lambda) : L^{e,\theta}(\mu)]$ is defined. Theorem 6.2 in [Ba] reduces the computation of this multiplicity to a similar problem in the usual BGG category $\mathcal{O}$. Let $M(\lambda), L(\lambda)$ denote the Verma module and the irreducible module with highest weight $\lambda \in \mathfrak{h}^*$.

Theorem 4.3 (Backelin)**.** *Let $\lambda, \mu \in \mathfrak{h}^*$. If*

1. *μ and λ are W-conjugate (with respect to the $\cdot$-action),*
2. *and there is $w \in W'$ such that $w\cdot\mu$ is antidominant for $\mathfrak{l}$ (i.e., $\langle w\cdot\mu+\rho, \alpha^\vee\rangle \notin \mathbb{Z}_{>0}$ for any $\alpha \in \Delta_+$ with $\langle \alpha, \theta\rangle = 0$), and $\lambda - w\cdot\mu \in \mathrm{Span}_{\mathbb{Z}_{\geqslant 0}}(\Delta^+)$,*

then $[\mathcal{M}^{e,\theta}(\lambda) : L^{e,\theta}(\mu)] = [M(\lambda) : L(w\cdot\mu)]$. Otherwise, $[\mathcal{M}^{e,\theta}(\lambda) : L^{e,\theta}(\mu)] = 0$.

Thanks to Theorem 4.1, Theorem 4.3 allows one to compute the decomposition numbers for the category $\mathcal{O}(\theta)$.

5. Proof of the main theorem

The proof of Theorem 4.1 is based on a construction of completions from [Lo1], Subsection 3.2. Let us recall this construction here.

Let $\mathfrak{v}$ be a finite dimensional graded vector space, $\mathfrak{v} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{v}(i)$, $\mathfrak{v} \neq \mathfrak{v}(0)$, and $A := S(\mathfrak{v})$. Suppose also that a torus T_0 acts on $\mathfrak{v}$ preserving the grading. The grading on $\mathfrak{v}$ gives rise to the grading $A = \bigoplus_{i\in\mathbb{Z}} A(i)$ whence to the action $\mathbb{K}^\times : A$. Let $* : A\otimes A \to A[\hbar^2]$ be a T_0-invariant homogeneous star-product, $f*g = \sum_{i=0}^\infty D_i(f,g)\hbar^{2i}$. Suppose $D_i : A\otimes A \to A$ is a bidifferential operator of order at most i at each variable. Then we can form the associative product $\circ : A\times A \to A, f\circ g = \sum_{i=0}^\infty D_i(f,g)$. We denote A equipped with the corresponding algebra structure by $\mathcal{A}$.

We have an action of T_0 on $\mathcal{A}$ by algebra automorphisms. Suppose that we have an embedding $\mathfrak{t}_0 \hookrightarrow \mathcal{A}$ such that the differential of the T_0-action on $\mathcal{A}$ coincides with the adjoint action of $\mathfrak{t}_0$.

For $u, v \in \mathfrak{v}(1)$ denote by $\omega_1(u,v)$ the constant term of $u\circ v - v\circ u$. Choose a T_0-invariant maximal isotropic (with respect to ω_1) subspace $\mathfrak{y} \subset \mathfrak{v}(1)$ and set $\mathfrak{m} := \mathfrak{y} \oplus \bigoplus_{i\leqslant 0} \mathfrak{v}(i)$. Further, choose a homogeneous basis $v_1,\ldots,v_n$ of $\mathfrak{v}$ such that $v_1,\ldots,v_m$ form a basis in $\mathfrak{m}$. Let d_i denote the degree of v_i. We may assume that the sequence $d_1,\ldots,d_m$ is increasing and all vectors v_i are T_0-semiinvariant.

By $A^\heartsuit$ we denote the subalgebra of the formal power series algebra $\mathbb{K}[[\mathfrak{v}^*]]$ consisting of all formal power series of the form $\sum_{i<n} f_i$ for some n, where f_i is a homogeneous power series of degree i. For any $f, g \in A^\heartsuit$ we have the well-defined element $f\circ g := \sum_{i=0}^\infty D_i(f,g) \in A^\heartsuit$. The space $A^\heartsuit$ considered as an algebra with respect to $\circ$ is denoted by $\mathcal{A}^\heartsuit$. Any element $a \in \mathcal{A}^\heartsuit$ can be written in a unique way as an infinite linear combination $\widetilde{a}$ of monomials

$$v_{i_1}\circ\ldots\circ v_{i_l} \text{ with } i_1 \geqslant i_2 \geqslant \ldots \geqslant i_l \tag{5.1}$$

such that $\sum_{j=1}^l d_{i_j} \leqslant c$, where c depends on a. Let $\mathrm{F}_c\mathcal{A}$ denote the subspace consisting of all elements, where degrees of monomials are bounded by c. Then the subspaces $\mathrm{F}_c\mathcal{A}^\heartsuit$ form a filtration of $\mathcal{A}^\heartsuit$.

Pick $\theta \in \mathfrak{X}^*(T_0) \subset \mathfrak{t}_0$. Let $\mathfrak{v}_{\geqslant 0}, \mathfrak{v}_{>0}$ denote the sums of $\operatorname{ad}\theta$-eigenspaces corresponding, respectively, to nonnegative and positive eigenvalues. Let $\mathcal{A}_{\geqslant 0}, \mathcal{A}_{>0}, \mathcal{A}^\heartsuit_{\geqslant 0}, \mathcal{A}^\heartsuit_{>0}$ be defined analogously (although the action of $\operatorname{ad}\theta$ on $\mathcal{A}^\heartsuit$ is not diagonalizable, the last definition makes sense). Suppose that the eigenvalues of $v_1,\ldots,v_n$ are decreasing, and $\mathfrak{v}_{>0} \subset \mathfrak{m} \subset \mathfrak{v}_{\geqslant 0}$. Then $\mathcal{A}^+_{\geqslant 0} := \mathcal{A}_{\geqslant 0} \cap \mathcal{A}\mathcal{A}_{>0}, \mathcal{A}^{\heartsuit+}_{\geqslant 0} := \mathcal{A}^\heartsuit_{\geqslant 0} \cap \mathcal{A}^\heartsuit\mathcal{A}^\heartsuit_{>0}$ are two-sided ideals in $\mathcal{A}_{\geqslant 0}, \mathcal{A}^\heartsuit_{\geqslant 0}$. Set $\mathcal{A}^0 := \mathcal{A}_{\geqslant 0}/\mathcal{A}^+_{\geqslant 0}, \mathcal{A}^{\heartsuit 0} := \mathcal{A}^\heartsuit_{\geqslant 0}/\mathcal{A}^{\heartsuit+}_{\geqslant 0}$. Note that there is a natural inclusion $\mathcal{A}^0 \hookrightarrow \mathcal{A}^{\heartsuit 0}$.

Clearly, an element of $\mathcal{A}^0$ (resp., of $\mathcal{A}^{\heartsuit 0}$) may be thought as a finite (resp., infinite with finiteness condition stated after (5.1)) sum of monomials in $v_i \in \mathfrak{v}_0$. Also let us note that $\mathcal{A}^0$ is obtained from $S(\mathfrak{v}_0)$ in the same way as $\mathcal{A}$ is obtained from $S(\mathfrak{v})$ (i.e., using a star-product with properties listed in the beginning of the section). So we can construct the algebra $\mathcal{A}^{0\heartsuit}$. However, it is clear that $\mathcal{A}^{0\heartsuit} = \mathcal{A}^{\heartsuit 0}$.

Consider the space $\mathcal{A}^\wedge := \varprojlim \mathcal{A}/\mathcal{A}\mathfrak{m}^k$. It follows from [Lo1] (Lemma 3.2.8 and the discussion before it) that $\mathcal{A}^\wedge$ has a natural structure of a topological algebra such that the natural map $\mathcal{A} \to \mathcal{A}^\wedge$ is an algebra homomorphism. Moreover, this map is injective and extends to an injective homomorphism of algebras $\mathcal{A}^\heartsuit \to \mathcal{A}^\wedge$. The algebra $\mathcal{A}^\wedge$ consists of all infinite sums of monomials (5.1) satisfying the following condition:

for any given $j \geqslant 0$ there are only finitely many monomials with nonzero coefficients and $v_{i_{l-j}} \notin \mathfrak{m}$.

Furthermore, we can compare algebras of the form $\mathcal{A}^\wedge$ for two different star-products. The following result follows from [Lo1], Lemmas 3.2.8,3.2.9.

Proposition 5.1. *Let $\mathfrak{v}, A$ be as in the beginning of this section and $*, *'$ be two $*$-products on A satisfying the above conditions. So we get new products $\circ, \circ'$ on A, the corresponding algebras will be denoted by $\mathcal{A}, \mathcal{A}'$. Suppose there is a T_0-stable subspace $\mathfrak{y} \subset \mathfrak{v}(1)$ that is maximal isotropic for both skew-symmetric forms. Suppose, further, that any element in A can be represented as a finite sum of monomials (5.1) and also as a finite sum of analogous monomials for $\circ'$. Finally, suppose there is a homogeneous T_0-equivariant isomorphism $\Phi : \mathcal{A}^\heartsuit \to \mathcal{A}'^\heartsuit$ such that $\Phi(v_i) - v_i \in \mathrm{F}_{d_i-2}\mathcal{A} + (\mathrm{F}_{d_i}\mathcal{A} \cap \mathfrak{v}^2 A)$. Then Φ extends uniquely to a topological algebra isomorphism $\Phi : \mathcal{A}^\wedge \to \mathcal{A}'^\wedge$ with $\Phi(\mathcal{A}^\wedge\mathfrak{m}) = \mathcal{A}'^\wedge\mathfrak{m}$.*

Clearly, Φ induces an isomorphism $\mathcal{A}^{0\heartsuit} = \mathcal{A}^{\heartsuit 0} \to \mathcal{A}'^{\heartsuit 0} = \mathcal{A}'^{0\heartsuit}$, which is denoted by Φ^0. This isomorphism is extended to an isomorphism $\Phi^0 : \mathcal{A}^{0\wedge} \to \mathcal{A}'^{0\wedge}$, where $\mathcal{A}^{0\wedge} := \varprojlim \mathcal{A}^0/\mathcal{A}^0\mathfrak{m}_0^k$, $\mathfrak{m}_0 := \mathfrak{m} \cap \mathfrak{v}_0$ and $\mathcal{A}'^{0\wedge}$ is defined analogously. Again, we have $\Phi^0(\mathcal{A}^{0\wedge}\mathfrak{m}_0) = \mathcal{A}'^{0\wedge}\mathfrak{m}_0$.

Now we will discuss a certain category of $\mathcal{A}^\wedge$-modules. Namely, we consider topological $\mathcal{A}^\wedge$-modules M equipped with discrete topology. This means that any vector in M is annihilated by some neighborhood of zero, i.e., by some $\mathcal{A}^\wedge\mathfrak{m}^k$. So this category is the same as the category of $\mathcal{A}$-modules, where $\mathfrak{m}$ acts by locally nilpotent endomorphisms. This category is denoted by $\widetilde{\mathrm{Wh}}(A, \mathfrak{m})$. Also we need its subcategory $\widetilde{\mathrm{Wh}}^{\mathfrak{t}_0}(A, \mathfrak{m})$ consisting of all modules with diagonalizable action of $\mathfrak{t}_0$.

In particular, we have the following straightforward corollary of Proposition 5.1.

Corollary 5.2. *Preserve the assumptions of Proposition 5.1. Then there are equivalences $\Phi_* : \widetilde{\mathrm{Wh}}(\mathcal{A}, \mathfrak{m}) \to \widetilde{\mathrm{Wh}}(\mathcal{A}', \mathfrak{m}), \widetilde{\mathrm{Wh}}^{\mathfrak{t}_0}(\mathcal{A}, \mathfrak{m}) \to \widetilde{\mathrm{Wh}}^{\mathfrak{t}_0}(\mathcal{A}', \mathfrak{m}), \Phi^0_* : \widetilde{\mathrm{Wh}}(\mathcal{A}^0, \mathfrak{m}_0) \to \widetilde{\mathrm{Wh}}(\mathcal{A}'^0, \mathfrak{m}_0)$ induced by Φ and Φ^0. Moreover,*

$$\Phi_*(M^{\mathfrak{m}}) = \Phi_*(M)^{\mathfrak{m}} \tag{5.2}$$

for any $\mathcal{A}$-module M (with locally nilpotent action of $\mathfrak{m}$).

Let us specify now $\mathcal{A}, \mathcal{A}', \mathfrak{m}$.

The torus T_0 we are going to consider is the same as in Section 3. Fix $m \in \mathbb{N}, m > 2+2d$, where d denotes the maximal eigenvalue of $\operatorname{ad} h$ in $\mathfrak{g}$. Consider the diagonal embedding $\mathbb{K}^\times \hookrightarrow \mathbb{K}^\times \times T_0$, whose differential is given by $d_1(1) = (1, -m\theta)$.

Consider the vector space $\mathfrak{v} := \{\xi - \langle\chi, \xi\rangle, \xi \in \mathfrak{g}\}$. The group $\mathbb{K}^\times \times T_0$ naturally acts on this space (we consider the Kazhdan action of $\mathbb{K}^\times$ and the action of T_0 coming from the adjoint action of T_0 on $\mathfrak{g}$). So $\mathfrak{v}$ is graded, $\mathfrak{v} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{v}(i)$, the grading is induced by the diagonal action of $\mathbb{K}^\times$, i.e., $\mathfrak{v}(i) := \{\xi \in \mathfrak{g} | (h - m\theta)\xi = (i-2)\xi\}$. Put $\mathcal{A} := \mathcal{U}$. As we explained at the end of Section 2, the product in $\mathcal{U}$ has the required form. Set $\mathfrak{m} := \widetilde{\mathfrak{m}}_\chi$. This subspace satisfies the requirements above (that is, $\mathfrak{v}_{>0} \subset \mathfrak{m} \subset \mathfrak{v}_{\geqslant 0}$ and the eigenvalues of $v_1, \ldots, v_n$ are decreasing). Furthermore, $\mathfrak{v}_0 = \mathfrak{l}$ and $\mathcal{A}^0 = U(\mathfrak{l})$.

Note that $\mathfrak{v} := \mathfrak{z}_\mathfrak{g}(e) \oplus V$. So we can set $\mathcal{A}' := \mathbf{A}_V(\mathcal{W})$, where we write $\mathbf{A}_V(\mathcal{W})$ for $\mathbf{A}_V \otimes \mathcal{W}$. From the choice of m it follows that all θ-weight spaces of $\mathfrak{z}_\mathfrak{g}(e)$ with positive weights lie in $\bigoplus_{i<0} \mathfrak{v}(i)$. Moreover, note that $\widetilde{\mathfrak{m}} \cap V$ is a lagrangian subspace in V. Finally, we remark that $\mathcal{A}'_{\geqslant 0}/\mathcal{A}'^{+}_{\geqslant 0}$ is naturally identified with $\mathbf{A}_{V_0}(\mathcal{W}^0)$, where $V_0 := V \cap \mathfrak{v}_0$. Note that $\mathfrak{m}_0$ is contained in V_0 and is a lagrangian subspace there.

Lemma 5.3. *There is an isomorphism $\Phi : \mathcal{U}^\heartsuit \to \mathbf{A}_V(\mathcal{W})^\heartsuit$ satisfying the conditions of Proposition 5.1. For the extension $\Phi : \mathcal{U}^\wedge \to \mathbf{A}_V(\mathcal{W})^\wedge$ we have*

$$\Phi^{-1}(\mathbf{A}_V(\mathcal{I})^\wedge) \cap \mathcal{U} = \mathcal{I}^\dagger, \tag{5.3}$$

where $\mathcal{I} = \operatorname{Ann}_{\mathcal{W}}(M)$ for $M \in \widetilde{\mathcal{O}}(\theta)$, and $\mathbf{A}_V(\mathcal{I})^\wedge$ denotes the closure of $\mathbf{A}_V(\mathcal{I}) := \mathbf{A}_V \otimes \mathcal{I}$ in $\mathbf{A}_V(\mathcal{W})^\wedge$.

Proof. The algebras $\mathcal{A}^\heartsuit, \mathcal{A}'^\heartsuit$ are naturally identified with the quotients of $\mathbb{K}^\times$-finite parts of

$$\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]], \mathbb{K}[S]^\wedge_\chi[[\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathbb{K}[V^*]^\wedge_0[[\hbar]]$$

by $\hbar - 1$. So the isomorphism $\Phi_\hbar$ from Proposition 2.1 induces a T_0-equivariant isomorphism $\Phi : \mathcal{A}^\heartsuit \to \mathcal{A}'^\heartsuit$. From the properties of $\Phi_\hbar$ indicated in Proposition 2.1 it follows that Φ has the required properties.

By [Lo1], Lemma 3.2.5, there is a natural identification of $\mathbb{K}[S][\hbar]$ with the Rees algebra $R_\hbar(\mathcal{W})$. So we can consider $R_\hbar(\mathcal{I})$ as an ideal in the quantum algebra $\mathbb{K}[S][\hbar]$. Consider the closure $\overline{\mathcal{I}}_\hbar$ of $R_\hbar(\mathcal{I})$ in $\mathbb{K}[S]^\wedge_\chi[[\hbar]]$. The ideal

$$\overline{\mathcal{J}}_\hbar := \mathbb{K}[[V^*, \hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\overline{\mathcal{I}}_\hbar \subset \mathbb{K}[[\mathfrak{v}^*, \hbar]]$$

is closed, $\mathbb{K}^\times$-stable and $\hbar$-saturated. By [Lo1], Proposition 3.2.2, there is a unique ideal $\mathcal{I}^\ddagger \subset \mathcal{A}$ such that

$$R_\hbar(\mathcal{I}^\ddagger) = R_\hbar(\mathcal{A}) \cap \Phi_\hbar^{-1}(\mathbb{K}[[V^*, \hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\overline{\mathcal{I}}_\hbar).$$

Proposition 3.4.1 from [Lo1] asserts that $\mathcal{I}^\dagger = \mathcal{I}^\ddagger$. So we need to prove that

$$\Phi^{-1}(\mathbf{A}_V(\mathcal{I})^\wedge) \cap \mathcal{A} = \mathcal{I}^\ddagger. \tag{5.4}$$

We will prove that

$$\mathbf{A}_V(\mathcal{I})^\wedge \cap \mathbf{A}_V(\mathcal{W})^\heartsuit = \overline{\mathcal{J}}_\hbar^{fin}/(\hbar - 1), \tag{5.5}$$

where $\overline{\mathcal{J}}_\hbar^{fin}$ denotes the $\mathbb{K}^\times$-finite part of $\overline{\mathcal{J}}_\hbar$. This will imply (5.4).

To prove (5.5) we will show that both sides equal $\mathbf{A}_V(\mathcal{W})^\heartsuit\mathcal{I}$. First, let us check this for the right hand side. As we checked in [Lo1], Lemma 3.4.3, $\overline{\mathcal{J}}_\hbar$ is generated by its intersection with $\mathbb{K}[[S, \hbar]]$. Recall that $\overline{\mathcal{J}}_\hbar$ is $\mathbb{K}^\times$-stable. But the $\mathbb{K}^\times$-action we consider differ from the Kazhdan one by an action by inner automorphisms, so $\overline{\mathcal{J}}_\hbar \cap \mathbb{K}[[S, \hbar]]$ is stable w.r.t the Kazhdan action. Since the Kazhdan grading on $\mathbb{K}[S]$ is positive, we see that $\overline{\mathcal{J}}_\hbar$ (and so also $\overline{\mathcal{J}}_\hbar^{fin}$) is generated by its intersection with $\mathbb{K}[S][\hbar]$. It follows that the r.h.s of (5.5) is generated (as an ideal in $\mathbf{A}_V(\mathcal{W})^\heartsuit$) by $\mathcal{I}$.

The proof that the left hand side coincides with $\mathbf{A}_V(\mathcal{W})^\heartsuit\mathcal{I}$ boils down to the following two claims:

Claim 1. Any ideal in $\mathbf{A}_V(\mathcal{W})^\heartsuit$ is generated by its intersection with $\mathcal{W}$.

Claim 2. $\mathbf{A}_V(\mathcal{I})^\wedge \cap \mathcal{W} = \mathcal{I}$ (here the condition that $\mathcal{I}$ is the annihilator of a module from $\widetilde{\mathcal{O}}(\theta)$ is essential).

Let us prove Claim 1. Let $\mathcal{J}$ be a two-sided ideal in $\mathbf{A}_V(\mathcal{W})^\heartsuit$. Consider the corresponding ideal $R_\hbar(\mathcal{J}) \subset R_\hbar(\mathbf{A}_V(\mathcal{W})^\heartsuit) = \mathbb{K}[[\mathfrak{v}^*, \hbar]]_{\mathbb{K}^\times - fin}$ and its closure $\overline{R}_\hbar(\mathcal{J}) \subset \mathbb{K}[[\mathfrak{v}^*, \hbar]]$. Then we can repeat the argument above and obtain that $\overline{R}_\hbar(\mathcal{J})$ is generated by its intersection with $\mathbb{K}[S][\hbar]$. This yields $\mathcal{J} = \mathbf{A}_V(\mathcal{W})^\heartsuit(\mathcal{J} \cap \mathcal{W})$.

Proceed to the proof of Claim 2. We can form the algebras $\mathcal{W}^\wedge$ and $\mathbf{A}_V^\wedge$ from $\mathcal{W}$ and $\mathbf{A}_V$ using the general construction explained above, so that $\mathbf{A}_V(\mathcal{W})^\wedge$ is decomposed into the completed tensor product of $\mathbf{A}_V^\wedge$ and $\mathcal{W}^\wedge$. The ideal $\mathbf{A}_V(\mathcal{I})^\wedge$ coincides with $\mathbf{A}_V^\wedge\widehat{\otimes}\mathcal{I}^\wedge$, where $\mathcal{I}^\wedge$ is the closure of $\mathcal{I}$ in $\mathcal{W}^\wedge$. So it remains to check that $\mathcal{I}^\wedge\cap\mathcal{W}=\mathcal{I}$. Recall that $\mathcal{I}=\operatorname{Ann}_{\mathcal{W}}(M)$ for some module M from $\widetilde{\mathcal{O}}(\theta)$. Then $\mathcal{W}^\wedge$ acts on M, and $\mathcal{I}^\wedge\subset\operatorname{Ann}_{\mathcal{W}^\wedge}(M)$. It follows that $\mathcal{I}^\wedge\cap\mathcal{W}\subset\mathcal{I}$. The inverse inclusion is obvious.

□

Remark 5.4. It is not clear at the moment whether (5.3) holds without the restriction on $\mathcal{I}$. It looks plausible that any ideal is the annihilator of a module from $\widetilde{\mathcal{O}}(\theta)$. On the other hand, it may happen that the condition $\mathcal{I}^\wedge\cap\mathcal{W}=\mathcal{I}$ holds for any two-sided ideal $\mathcal{I}$, even if $\mathcal{I}$ is not the annihilator of a module from $\widetilde{\mathcal{O}}(\theta)$.

An isomorphism $\Phi: U(\mathfrak{g})^\heartsuit\rightarrow\mathbf{A}_V(\mathcal{W})^\heartsuit$ from the proof of Lemma 5.3 gives rise to an isomorphism $\Phi^0: U(\mathfrak{l})^\wedge\rightarrow\mathbf{A}_{V_0}(\mathcal{W}^0)^\wedge$ mapping $U(\mathfrak{l})^\wedge\mathfrak{m}_0$ to $\mathbf{A}_{V_0}(\mathcal{W}^0)^\wedge\mathfrak{m}_0$. This provides an isomorphism

$$\Psi:\underline{\mathcal{W}}=(U(\mathfrak{l})^\wedge/U(\mathfrak{l})^\wedge\mathfrak{m}_0)^{\mathfrak{m}_0}\rightarrow\mathbf{A}_{V_0}(\mathcal{W}^0)^\wedge/(\mathbf{A}_{V_0}(\mathcal{W}^0)^\wedge\mathfrak{m}_0)^{\mathfrak{m}_0}=\mathcal{W}^0 \tag{5.6}$$

we need in Theorem 4.1.

Before proceeding further let us make a remark on the isomorphism Ψ. We will use this remark in [Lo3].

Remark 5.5. Let us discuss a relation between Ψ and the embeddings $\mathfrak{t}_0\hookrightarrow\mathcal{W}^0,\underline{\mathcal{W}}$. It turns out that Ψ does not intertwine them but induces a shift on $\mathfrak{t}_0$.

It follows from assertion (iii) of Proposition 2.1 that the isomorphism $\Phi:\mathcal{U}^\heartsuit\rightarrow\mathbf{A}_V(\mathcal{W})^\heartsuit$ intertwines the embeddings of $\mathfrak{t}_0$. Let $\iota_{\mathfrak{g}},\iota_{\mathcal{W}},\iota_V$ denote the embeddings of $\mathfrak{t}_0$ to $\mathfrak{g},\mathcal{W},\mathbf{A}_V$, respectively. Of course, $\iota_{\mathfrak{g}}(\xi)$ is nothing else but ξ itself, and $\Phi(\iota_{\mathfrak{g}}(\xi))=\iota_{\mathcal{W}}(\xi)+\iota_V(\xi)$. Here we consider $\iota_{\mathcal{W}}(\xi),\iota_V(\xi)$ as elements of $\mathbf{A}_V(\mathcal{W})^\wedge$ via the natural embeddings $\mathcal{W},\mathbf{A}_V\hookrightarrow\mathbf{A}_V(\mathcal{W})^\wedge$. Let us describe ι_V. Let $\chi_1,\ldots,\chi_m$ denote all characters (with multiplicities) of the representation of $\mathfrak{t}_0$ in the lagrangian subspace $\mathfrak{m}\cap V=(\mathfrak{n}_+\cap V)\oplus\mathfrak{m}_0\subset V$. Let $u_1^+,\ldots,u_m^+$ be the corresponding eigenvectors and let $u_1^-,\ldots,u_m^-\in V$ be such that $\omega_V(u_i^-,u_j^+)=\delta_{ij}$ so that $\xi.u_i^-=-\langle\chi_i,\xi\rangle u_i^-$. Note that $\langle\chi_i,\theta\rangle\geqslant 0$ for all i and $\langle\chi_i,\theta\rangle=0$ if and only if $u_i^+\in\mathfrak{m}_0$. Now

$$\iota_V(\xi)=\frac{1}{2}\sum_{i=1}^m\langle\chi_i,\xi\rangle(u_i^+u_i^-+u_i^-u_i^+)=\sum_{i=1}^m\langle\chi_i,\xi\rangle u_i^-u_i^+-\frac{1}{2}\langle\sum_{i=1}^m\chi_i,\xi\rangle.$$

Let $\pi:\mathcal{W}_{\geqslant 0}\twoheadrightarrow\mathcal{W}^0$ be the natural projection. Recall that $\mathbf{A}_V(\mathcal{W})^{\heartsuit 0}$ is naturally identified with $\mathbf{A}_{V_0}(\mathcal{W}^0)$. The image of $\iota_{\mathcal{W}}(\xi)+\iota_V(\xi)$ in $\mathbf{A}_{V_0}(\mathcal{W}^0)$ coincides with

$$\pi(\iota_{\mathcal{W}}(\xi))+\sum_{i,\langle\chi_i,\theta\rangle=0}\langle\chi_i,\xi\rangle u_i^-u_i^+-\frac{1}{2}\langle\sum_{i=1}^m\chi_i,\xi\rangle. \tag{5.7}$$

Now let $\iota_{\mathfrak{l}},\iota_{\underline{\mathcal{W}}},\iota_{V_0}$ denote the embeddings of $\mathfrak{t}_0$ into $\mathfrak{l},\underline{\mathcal{W}},\mathbf{A}_{V_0}$, respectively, and let Φ^0 be an isomorphism $U(\mathfrak{l})^\wedge\rightarrow\mathbf{A}_{V_0}(\underline{\mathcal{W}})^\wedge$ similar to that from [Lo1], Theorem 1.2.1. Then, by [Lo2], Theorem 2.3.1 and Remark 2.3.2, we have $\Phi_{\mathfrak{l}}(\iota_{\mathfrak{l}}(\xi))=\iota_{\underline{\mathcal{W}}}(\xi)+\iota_{\mathbf{A}_{V_0}}(\xi)$. So

$$\Phi_{\mathfrak{l}}(\iota_{\mathfrak{l}}(\xi))=\iota_{\underline{\mathcal{W}}}(\xi)+\sum_{i,\langle\chi_i,\theta\rangle=0}\langle\chi_i,\xi\rangle u_i^-u_i^+-\frac{1}{2}\langle\sum_{i,\langle\chi_i,\theta\rangle=0}\chi_i,\xi\rangle. \tag{5.8}$$

Since Ψ is given by (5.6), from (5.7),(5.8) we see that Ψ^{-1} maps $\pi(\iota_{\mathcal{W}}(\xi))$ to

$$\iota_{\underline{\mathcal{W}}}(\xi)+\frac{1}{2}\langle\sum_{i:\langle\chi_i,\theta\rangle>0}\chi_i,\xi\rangle$$

In [BGK], Subsection 4.1, Brundan, Goodwin and Kleshchev also constructed an isomorphism $\mathcal{W}^0\rightarrow\underline{\mathcal{W}}$ (in the case when $\mathfrak{t}=\mathfrak{t}_0$). Their isomorphism sends $\pi(\iota_{\mathcal{W}}(\xi))$ to $\iota_{\underline{\mathcal{W}}}(\xi)-\langle\delta,\xi\rangle$, where δ is defined as follows. Pick a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ containing $\mathfrak{t}$ and h. Let Δ_- denote the set of all roots α with $\langle\alpha,\theta\rangle<0$. Then

$$\delta=\frac{1}{2}\sum_{\alpha\in\Delta_-,\langle\alpha,h\rangle=-1}\alpha+\sum_{\alpha\in\Delta_-,\langle\alpha,h\rangle\leqslant-2}\alpha.$$

Let us check that $\delta|_{\mathfrak{t}_0}=-\frac{1}{2}\sum_{i,\langle\chi_i,\theta\rangle>0}\chi_i$.

Since e,h are $\mathfrak{t}_0$-invariant, the representation theory of $\mathfrak{sl}_2$ implies

$$\delta|_{\mathfrak{t}_0}=\frac{1}{2}\left(\sum_{\alpha\in\Delta_-,\langle\alpha,h\rangle=-1}\alpha|_{\mathfrak{t}_0}\right)+\frac{1}{2}\left(\sum_{\alpha\in\Delta_-,\langle\alpha,h\rangle\leqslant-2}\alpha|_{\mathfrak{t}_0}+\sum_{\alpha\in\Delta_-,\langle\alpha,h\rangle\geqslant 2}\alpha|_{\mathfrak{t}_0}\right)=$$
$$\frac{1}{2}\sum_{\alpha\in\Delta_-}\alpha|_{\mathfrak{t}_0}-\frac{1}{2}\sum_{\alpha\in\Delta_-,\langle\alpha,\theta\rangle=0,1}\alpha|_{\mathfrak{t}_0}$$

The last expression is the sum of weights of $\mathfrak{t}_0$ in $\mathfrak{n}_-\cap V=[\mathfrak{n}_-,f]$. Since $\mathfrak{n}_-\cap V$ and $\mathfrak{n}_+\cap V$ are dual $\mathfrak{t}_0$-modules, we are done.

Let us complete the proof of Theorem 4.1. Note that $\widetilde{\mathrm{Wh}}(e,\theta)=\widetilde{\mathrm{Wh}}(\mathcal{A},\mathfrak{m}),\widetilde{\mathrm{Wh}}^{\mathfrak{t}_0}(e,\theta)=\widetilde{\mathrm{Wh}}^{\mathfrak{t}_0}(\mathcal{A},\mathfrak{m})$. On the other hand, let us construct an equivalence $\widetilde{\mathrm{Wh}}(\mathcal{A}',\mathfrak{m})\rightarrow\widetilde{\mathcal{O}}(\theta)$. This functor is given by $\mathcal{K}':M\mapsto M^{\mathfrak{m}\cap V},M\in\widetilde{\mathrm{Wh}}(\mathcal{A}',\mathfrak{m})$. A quasiinverse functor is given by $N\mapsto\mathbb{K}[\mathfrak{m}\cap V]\otimes N,N\in\widetilde{\mathcal{O}}(\theta)$. The claim that these two functors are quasiinverse follows from the representation theory of Heisenberg Lie algebras, see the proof of Proposition 3.3.4 in [Lo1]. It follows directly from the construction of $\mathcal{K}'$ that

$$M^{\mathfrak{m}}=(\mathcal{K}'(M))^{\mathcal{W}_{>0}}. \tag{5.9}$$

Now we set $\mathcal{K}:=\mathcal{K}'\circ\Phi_*$. Let us check that $\mathcal{K}$ has the required properties. The equality $\mathrm{Ann}_{\mathcal{W}}(\mathcal{K}(M))^\dagger=\mathrm{Ann}_{\mathcal{U}}(M)$ stems from Lemma 5.3. (5.9) and (5.2) imply assertion 3. Assertion 4 follows from assertion 3 and the adjointness of functors mentioned in Sections 3,4. The second assertion of the theorem now follows from Corollary 5.2.

6. Applications

Let us discuss some applications of Theorem 4.1. In [Lo3] we will apply it to the study of one-dimensional representations of W-algebras. More precisely, we will give a criterium for $\dim L^\theta(V)<\infty$ in terms of the annihilator of $\Psi_*(V)\in\underline{\mathcal{W}}$-Mod. In particular, this criterium will prove of Conjecture 5.2 from [BGK]. Although it is mentioned in [BGK] that their Conjecture 5.3 (which is a special case of Theorem 4.1) implies Conjecture 5.2, [BGK] contains no proof of the implication.

Then, under some conditions on e, we will get a criterium (in terms of V) for a finite dimensional module $L^\theta(V)$ to have dimension 1. More precisely, we will check that whenever $\mathfrak{q}$ is semisimple, the following conditions are equivalent provided $L^\theta(V)$ is finite dimensional:

- $\dim L^\theta(V) = 1$.
- $\dim V = 1$ and $\mathfrak{t}$ acts by 0 on V (considered as a $\mathcal{W}^0$-module).

Since the criterium for $L^\theta(V)$ to be finite dimensional is stated, in a sense, in terms of $\underline{\mathcal{W}}$, we need Remark 5.5. The condition that $\mathfrak{q}$ is semisimple is fulfilled for all so called rigid nilpotent elements in exceptional Lie algebras. Together with results of Premet, [Pr4], on "parabolic induction" for one-dimensional representations of W-algebras (also reproved in [Lo3]) this should allow one to complete the proof of Premet's conjecture, [Pr2], that any W-algebra has a one-dimensional representation.

Another application, as we learned from Jonathan Brundan, is to finite dimensional irreducible representations of Yangians. For type A, Brundan and Kleshchev identified W-algebras with quotients (truncations) of shifted Yangians. The latter generalize the usual Yangians introduced by Drinfeld, see [BK1]. Using this presentation of W-algebras, they classified, [BK2], all their irreducible finite dimensional representations (Theorem 7.9) and also all irreducible finite dimensional representations of shifted Yangians (Corollary 7.10), generalizing results of Drinfeld, [D], on the usual Yangians. The classification for W-algebras is made in terms of some Young diagrams. On the other hand, using the results announced in the previous paragraph, it is possible to describe irreducible $\underline{\mathcal{W}}$-modules V such that $\dim L^\theta(\Psi_*^{-1}(V)) < \infty$ using the classical results of Joseph, [J], on combinatorial description of primitive ideals in $U(\mathfrak{sl}_n)$ with given associated variety (which is again stated in terms of some Young diagrams). In this way one can recover Brundan-Kleshchev classification for W-algebras, see Section 5.2 of [BGK]. Then, perhaps after some work, one can recover the classification for shifted Yangians.

In the other classical types Brown identified the W-algebras for rectangular nilpotent elements e with the truncations of so called twisted Yangians, see [Br1]. Recall that a nilpotent element in a classical Lie algebra is called rectangular if all the numbers in the corresponding partition are the same. Such an element is always of principal Levi type. In [Br2] Brown used Molev's classification, [Mo], of irreducible finite dimensional representations for twisted Yangians to classify those for W-algebras (in the rectangular case). The answer is given in purely combinatorial terms. On the other hand, [BGK], Conjecture 5.2, together with results of Barbash and Vogan, [BV], should make it possible to recover Brown's classification.

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