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ROBUST IMAGE REGISTRATION BASED ON
A PARTITION OF UNITY FINITE ELEMENT METHOD

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ABSTRACT

In this paper, we present a robust, hierarchical Partition of
Unity Finite Element Method (PUFEM) to compute the transforma-
tion between two images, which is represented by a
non-rigid, locally polynomial displacement field. The parti-
tion of unity property offers an efficient optimization scheme
by breaking down the global minimization of the mismatch
energy into independent, local minimizations. Moreover, the
regularization introduced by our approach enables us to con-
trol the range of the smoothness. Our method was applied to
cardiac ultrasound image sequences to propagate the segmen-
tation of anatomical structures of interest.

Index Terms— Image registration, partition of unity, op-
tical flow, robust estimation, cardiac ultrasound

1. INTRODUCTION

Image registration is an essential task in medical image pro-
cessing, and many subsequent processes depend on it, such
as segmentation propagation, strain quantification or back-
ground subtraction. Parametric representations such as Ra-
dial Basis Functions [1], Bsplines [2, 3, 4] and other Finite
Element Methods (FEM) [5] have been widely used for image
registration. However, they use motion models with built-in
smoothness that may be difficult to adapt. In [6], we intro-
duced Partition of Unity Finite Element Method (PUFEM)
for the first time, with encouraging results on synthetically
warped images. The main advantage of this method com-
pared to the aforementioned ones is the controllability of the
range of the smoothness. Our current contributions include a
generalization of the PUFEM for registration, through the in-
roduction of a robust estimator in the matching term in order
to reduce the influence of outliers. The paper is organized as
follows. We first explain how the warping field is represented,
then we present our variational approach to register a pair of
images and our minimization strategy. Finally, we show some
results on sequences of cardiac ultrasound images.

2. REPRESENTATION OF THE WARPING FIELD

In this first section, we give an overview on the mathematical
framework of the PUFEM [7] that we use to model a vector
field $u$. Each component $u$ is a real-valued function defined
on an open bounded domain $\Omega \subset \mathbb{R}^d$. The basic idea is to
locally fit $u$ with $d$-dimensional polynomials and smoothly
blend them afterwards to obtain a regular representation. To
that end, we define a set $N$ of nodes distributed over $\Omega$. A
node $n$ is characterized by:
- a point $c^{(n)} \in \Omega$, called center of the node $n$,
- an open bounded subdomain $\Omega^{(n)} \subset \mathbb{R}^d$ containing $c^{(n)}$, called patch,
- an $\mathbb{R}$-valued function $\varphi^{(n)}$ defined on $\mathbb{R}^d$, called PU-
function, whose support is included in $\Omega^{(n)}$,
- a set $B^{(n)} = \{ p^{(n)}_r | r \in R^{(n)} \}$ of functions from $\Omega^{(n)}$ to $\mathbb{R}$,
called the local basis at node $n$, with $R^{(n)}$ being the set of
indices for the local basis.

We choose the $p^{(n)}_r$ to be monomials of all degrees up to $q$, with $c^{(n)}$ as origin, so that $u$ is locally modelled at node $n$ by a polynomial:

$$u^{(n)} = \sum_{r \in R^{(n)}} \alpha_r^{(n)} p_r^{(n)}$$

(1)

where the $\alpha_r^{(n)}$ are real coefficients.

We assume the families $(\Omega^{(n)})_{n \in \mathbb{N}}$ and $(\varphi^{(n)})_{n \in \mathbb{N}}$ to fulfill the Partition of Unity conditions i. e.:

$$\Omega \subset \bigcup_{n \in \mathbb{N}} \Omega^{(n)}$$

(2)

$$\forall x \in \Omega^{(n)} \sum_{n \in \mathbb{N}} \varphi^{(n)}(x) = 1$$

(3)

The global representation is then constructed by blending the $u^{(n)}$ with the PU-functions:

$$u = \sum_{n \in \mathbb{N}} \varphi^{(n)} u^{(n)}$$

(4)

Unlike more conventional FEMs, PUFEMs allow a patch to overlap its neighbours. For the sake of computational effi-
ciency, our nodes are distributed over a regular rectangular

array with an inter-node spacing $h_i$ along the $i^{th}$ coordinate axis ($i = 1, \ldots, d$). Each patch $\Omega^{(n)}$ is an $h_1 \times \cdots \times h_d$ cube centered on $\mathbf{c}^{(n)}$. This configuration is illustrated on Fig. 1.a. The $\varphi^{(n)}$ are separable piecewise polynomials, equal to 1 at $\mathbf{c}^{(n)}$ and decreasing with the distance to $\mathbf{c}^{(n)}$ (cf. Fig. 1.b).

**Notations:** in the following, $u_i$ denotes the $i^{th}$ component of $\mathbf{u}$.

![Fig. 1. Example of Partition of Unity configuration in 2D.](image)

Fig. 1. Example of Partition of Unity configuration in 2D.

The displacement field $\mathbf{u}$ and $a^{(n)}_{i,r}$ the coefficients representing $u_i$; $a^{(n)}$ is the vector obtained by piling up the $a^{(n)}_{i,r}$ at node $n$ and $\mathbf{a}$ by piling up the $a^{(n)}$. Similarly for a second field $\mathbf{v}$, we adopt the notations $v_i$, $b^{(n)}_{i,r}$, $\mathbf{b}^{(n)}$ and $\mathbf{b}$.

According to (1) and (4), $\mathbf{u}$ is as regular as the PU-functions per se. However, we want to impose a controllable, “long range” regularization, or rather, globality. To this end, we introduce the notion of Sobolev non-conformity between two neighbouring nodes $m$ and $n$ through the energy:

$$S_{\kappa}^{(m,n)}(\mathbf{u}) = \sum_{|\alpha| \leq \kappa} \int_{\Omega^{(m,n)}} \varphi^{(m)}(\mathbf{x}) \left\| D^\alpha \mathbf{u}^{(m)} - D^\alpha \mathbf{u}^{(n)} \right\|^2$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $D^\alpha$ is the partial derivative operator in the standard multi-index notations. This local energy has an intuitive interpretation: it penalizes the global field $\mathbf{u}$ if its local representations at nodes $m$ and $n$ and their derivatives up to order $\kappa$ differ in the overlapping region $\Omega^{(m,n)}$. The global conformity energy is then defined by:

$$S_{\kappa}(\mathbf{u}) = \sum_{n \in N} \sum_{m \in \mathcal{V}(n)} S_{\kappa}^{(m,n)}(\mathbf{u})$$

where $\mathcal{V}(n)$ is the set of neighbours of node $n$ in 4-connexity. This inter-node conformity constraint is a key feature of our method. This energy is zero when all the local representations are equal, i.e. when $u$ is globally polynomial. Thus, in the case of local affine bases, global translation, rotation, scaling and shearing are not penalized.

### 3. ROBUST REGISTRATION FORMULATION

A reference image $R$ and a template $T$ are registered by minimizing the mismatch energy:

$$\mathcal{M}(\mathbf{u}) = \int_{\Omega} \rho \left( T \circ (\mathbf{id} + \mathbf{u}) - R \right)$$

w.r.t. $\mathbf{u}$. $\rho$ is a convex M-estimator in the form $\rho(s) = \Psi(s^2)$, which has the effect of reducing the influence of outliers (see [8, 9, 10]). For example, we choose $\Psi(s^2) = \sqrt{s^2 + \epsilon^2}$. This energy is difficult to minimize as such and we need to remove the nonlinearity due to $T$ and $\rho$.

We first decompose the optimization into optical flow steps. Considering one iteration, let $\mathbf{u}$ be the current unknown, $\mathbf{v}$ the solution of the previous iteration, and $\delta \mathbf{u}$ the unknown increment so that $\mathbf{u} = \mathbf{v} + \delta \mathbf{u}$. In order to remove the nonlinearity in the argument of $\rho$ in (6), we use a first order Taylor expansion of $T$:

$$T \circ (\mathbf{id} + \mathbf{u}) \approx T \circ (\mathbf{id} + \mathbf{v}) + \nabla T \circ (\mathbf{id} + \mathbf{v}) \cdot \delta \mathbf{u}$$

The framework presented in section 2 enables us to derive a simple scheme by approximating the problem by a set of independent subproblems, each confined to a node. By expanding $\delta \mathbf{u}$ as in (4) and using property (3), $\mathcal{M}(\mathbf{v} + \delta \mathbf{u})$ is approximated by:

$$\int_{\Omega} \rho \left( \sum_{n \in N} \varphi^{(n)} \left[ T \circ (\mathbf{id} + \mathbf{v}) - R + \nabla T \circ (\mathbf{id} + \mathbf{v}) \cdot \delta \mathbf{u}^{(n)} \right] \right) \leq \mathcal{M}_{\mathbf{v}}^{(n)}(\delta \mathbf{u}^{(n)})$$

where $\rho$ being convex, we can apply Jensen’s inequality [11] to get:

$$\int_{\Omega} \rho \left( \sum_{n \in N} \varphi^{(n)} \xi^{(n)} \right) \leq \sum_{n \in N} \int_{\Omega} \varphi^{(n)} \rho(\xi^{(n)})$$

This provides an upper bound for $\mathcal{M}(\mathbf{v} + \delta \mathbf{u})$ which we minimize instead: if all the local energies $\mathcal{M}_{\mathbf{v}}^{(n)}(\delta \mathbf{u}^{(n)})$ can be made small, then we are sure that $\mathcal{M}(\mathbf{v} + \delta \mathbf{u})$ will be small. This upper bound is interesting because it removes the dependencies between the nodes.

However the nonlinearity due to $\rho$ still remains. It can be shown that each $\mathcal{M}_{\mathbf{v}}^{(n)}(\delta \mathbf{u}^{(n)})$ can be approximated by a quadratic form of the $\delta a^{(n)}_{i,r}$:

$$\mathcal{M}_{\mathbf{v}}^{(n)}(\delta \mathbf{u}^{(n)}) \approx (\delta a^{(n)}_{i,r})^T M^{(n)}(\delta a^{(n)}) + 2 (g^{(n)}_{i,r})^T \cdot \delta a^{(n)} + C$$

where $C$ is a constant. $M^{(n)}$ is a positive semidefinite, symmetric matrix and $g^{(n)}_{i,r}$ a vector; their entries are defined by:

$$m^{(n)}_{i,r,s} = \int_{\Omega^{(n)}} \varphi^{(n)} \Psi' \left( (T_v - R)^2 \right) \partial_i T_r \partial_j T_r p^{(n)}_{s}$$

$$g^{(n)}_{i,r} = \int_{\Omega^{(n)}} \varphi^{(n)} \Psi' \left( (T_v - R)^3 \right) (T_v - R) \partial_i T_r p^{(n)}_{s}$$

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where $T_v \triangleq T \circ (\text{id} + v)$ and $\partial_i T_v \triangleq \partial_i T \circ (\text{id} + v)$. The above inequality and the latter quadratic form are generalizations of the results established in [6].

Thus, adding the inter-node conformity constraint (5), the total energy we consider at each iteration is:

$$
\tilde{E}(\mathbf{a}) = \sum_{n \in \mathcal{N}} \left( \mathbf{a}^{(n)} - \mathbf{b}^{(n)} \right)^T \mathbf{M}^{(n)} \left( \mathbf{a}^{(n)} - \mathbf{b}^{(n)} \right) + 2 \sum_{n \in \mathcal{N}} (\mathbf{g}^{(n)})^T \left( \mathbf{a}^{(n)} - \mathbf{b}^{(n)} \right) + \gamma \mathcal{S}_c(\mathbf{a})
$$

(7)

with $\gamma$ being a parameter that controls the globality of $\mathbf{u}$. For example, if all the local representations are affine, then increasing $\gamma$ forces $\mathbf{u}$ towards a global affine displacement field.

4. MINIMIZATION

The conformity term $\mathcal{S}_c(\mathbf{a})$ is a quadratic function of the coefficients $a_{ijr}$, and hence so is $\tilde{E}(\mathbf{a})$. Minimizing (7) is then equivalent to solving a linear system of size $\sum_n \text{Card}\mathcal{R}^{(n)}$. This system is sparse since:

- the approximated matching term (i.e., the upper bound derived above) does not relate coefficients from different nodes,
- the conformity (5) only relates nearest neighbour nodes.

We use conjugate gradient descent which is well-suited for solving sparse linear systems [12].

The scheme we have just described is integrated in a hierarchical environment in order to avoid local minima. We define a dyadic pyramid of PU-configurations. Once the solution has been computed at one level, it is projected on the basis of the next finer level to provide an initialization (see details in [6]). This global-to-local strategy is coupled with a multiscale approach: starting with a low-resolution image (high scale) at the coarsest level, we decrease the scale as the node array is refined.

5. RESULTS

We first show results on a pair of synthetic binary images to demonstrate how the globality can be controlled through the parameter $\gamma$ (cf. Fig. 2). The local bases are chosen affine. The transformation is globally affine when $\gamma$ is large (second image), and adapts locally as $\gamma$ decreases.

We applied our method to sequences of cardiac ultrasound 2D images, in order to track the endocardium of the left ventricle. To this end, we propagated initial manual segmentations by successively applying the displacement fields computed from each pair of consecutive frames. We used a 4-level pyramid starting from $5 \times 5$ nodes for the coarsest level to $33 \times 33$ nodes for the finest, the image being of size $608 \times 428$ pixels. The local bases are chosen quadratic.

![Fig. 2. The template (upper left) has been registered to the reference (a,b,c) with decreasing values of the parameter $\gamma$: (a) 50, (b) 5 and (c) 0.1. The first line shows the template’s contours superimposed on the reference image. The second line displays the deformations of an initially regular grid.](image)

![Fig. 3 displays results on a long-axis view sequence. Notice that the valve has an out-of-plane motion between the last two frames, but does not disturb the propagation. Fig. 4 displays results on a more difficult case: the sequence is low-contrasted and exhibits out-of-plane motion.](image)

6. SUMMARY AND CONCLUSION

We have proposed a generalization of the hierarchical Partition of Unity Finite Element Method presented in [6] to solve a variational approach to image registration. This new method offers several features to improve the robustness to outliers: the introduction of an M-estimator in the matching term is a well-known technique and can be easily integrated in the optimization scheme; the global-to-local strategy and the inter-node conformity constraint enable us to control the globality of the field, as opposed to existing parametric methods. In order to demonstrate the performance of our algorithm, we have applied our method to a sequence of cardiac ultrasound images to track the myocardium of the left ventricle.

In future works, we shall consider cases where abrupt spatial variations in the displacement field occur. The globality constraint may prevent the algorithm from correctly capturing the motion field. To deal with this issue, we shall introduce M-estimator techniques in the conformity penalty term.
Fig. 3. Propagation of a segmentation of the left ventricle’s endocardium in a long-axis view sequence. The initial contour was manually drawn (top left image). The second column shows the propagation of a grid, in order to visualize the deformation field.

7. REFERENCES


