Stein block thresholding for wavelet-based image deconvolution

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Abstract: In this paper, we propose a fast image deconvolution algorithm that combines adaptive block thresholding and Vaguelet-Wavelet Decomposition. The approach consists in first denoising the observed image using a wavelet-domain Stein block thresholding, and then inverting the convolution operator in the Fourier domain. Our main theoretical result investigates the minimax rates over Besov smoothness spaces, and shows that our block estimator can achieve the optimal minimax rate, or is at least nearly-minimax in the least favorable situation. The resulting algorithm is simple to implement and fast. Its computational complexity is dominated by that of the FFT in the Fourier-domain inversion step. We report a simulation study to support our theoretical findings. The practical performance of our block vaguelet-wavelet deconvolution compares very favorably to existing competitors on a large set of test images.

Keywords and phrases: Image deconvolution, Block thresholding, Wavelets, Minimax, LaTeX.

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1. Introduction

In this paper, we consider the two-dimensional convolution model with Gaussian white noise $\sim N(0, \sigma^2)$. We observe the stochastic process $Y(\cdot)$ where

$$dY(x) = T(f)(x)dx + \sigma dW(x),$$

$x \in [0, 1]^2$, $W(\cdot)$ is a (non-observed) white Gaussian noise, $T(f)(x) = (f * g)(x)$ is the two-dimensional convolution operator on $[0, 1]^2$, $g$ is a known kernel (called also point spread function PSF), both $f$ and $g$ are one-periodic functions belonging to $L^2([0, 1]^2)$. In the sequel, the Fourier transform of a function $h$ will be denoted $\mathcal{F}(h)(l) = \int_{[0,1]^2} h(x)e^{-2\pi i <l,x>}dx$. The observation model (1.1) illustrates the action of a linear time-invariant system on an input image $f$ when the data are corrupted with additional noise. The deconvolution is to estimate $f$ from $Y$ which is a longstanding inverse problem in image processing.

There is an extensive statistical literature on wavelet-based deconvolution problems. For obvious space limitations, we only focus on some of them. In 1D, Donoho in [8] gave the first discussion of wavelet thresholding in linear inverse problems and introduced the Wavelet-Vaguelet Decomposition (WVD). The WaveD algorithm of [11] is an adaptation of WVD to the one dimensional deconvolution problem. Abramovich and Silverman in [1] proposed another procedure; the Vaguelet-Wavelet Decomposition (VWD). The original estimator based on VWD is defined with standard term-by-term thresholding rules. It has been improved by [2] using a Stein block thresholding rule. As for VWD, the original WaveD procedure based on term-by-term thresholding has been recently improved by [4] using again block thresholding.

In 2D, the WVD approach was refined in [12] who proposed a mirror wavelet basis adapted to capture the singularity of the spectrum of the inverse of $h$. The authors in [14] advocated a hybrid approach known as ForWarD. In [7], the authors proposed an adaptive wavelet estimator based on two-dimensional version of the WaveD algorithm of [11] which enjoys good numerical performance. Deconvolution methods based on variational or bayesian formulations with sparsity-promoting regularization over wavelet coefficients have been recently proposed; see e.g. [10, 6, 9, 3] and others. These algorithms are based on iterative thresholding. A few papers have also focused on deconvolution by using the SURE principle to minimize an unbiased estimate of the MSE of deconvolution estimators operating by thresholding in orthogonal or redundant wavelet bases [16, 15].

However, so far, these wavelet deconvolution algorithms were based on term-by-term thresholding which under-performs for many images. The drawback of individual cannot be circumvented by fine-tuning the regularization/threshold
parameter. All these reasons motivated us to develop an adaptive estimator of $f$ based on combining two-dimensional Stein block thresholding and VWD. The approach consists in first denoising the observed image using a wavelet-domain block thresholding, and then inverting the convolution operator in the Fourier domain. It can be viewed as a multi-dimensional version of the procedure developed by [2]. From a theoretical point of view, taking the minimax approach over the Besov balls $B^s_{p,q}(M)$ (to be defined in Section 2) and under the $L_2$ risk, we prove that our estimator achieves near optimal rates of convergence. These rates are for instance better than those attained by the two-dimensional WaveD of [7]. This result can be viewed as a 2D extension of the one of [2]. It is a consequence of a general theorem established by [5] on the minimax performances of multi-dimensional block thresholding. From a practical point of view, our algorithm is very simple to implement and runs very fast. Its performances compare very favorably to alternative deconvolution algorithms such as [10, 6, 14, 7] over a large set of test images.

The paper is organized as follows. Section 2 briefly reviews wavelets and Besov balls. Section 3 describes the block thresholding-based deconvolution estimator. The minimax performances of this estimator are investigated in Section 4. Section 5 contains experimental results. Section 6 gives a conclusion. The proofs are postponed in Section 7.

2. Wavelets and Besov balls

We consider an orthonormal wavelet basis generated by dilations and translations of a "father" Meyer-type wavelet $\phi$ and a "mother" Meyer-type wavelet $\psi$. The main features of such wavelets are:

1. they are bandlimited, i.e. the Fourier transforms of $\phi$ and $\psi$ have compact supports respectively included in $[-4\pi 3^{-1}, 4\pi 3^{-1}]$ and $[-2\pi 3^{-1}, -2\pi 3^{-1}] \cup [2\pi 3^{-1}, 8\pi 3^{-1}]$.
2. for any frequency in $[-2\pi, -\pi] \cup [\pi, 2\pi]$, there exists a constant $c > 0$ such that the magnitude of the Fourier transform of $\psi$ is lower bounded by $c$.
3. the functions $(\phi, \psi)$ are $C^\infty$ as their Fourier transforms have a compact support, and $\psi$ has an infinite number of vanishing moments as its Fourier transform vanishes in a neighborhood of the origin:

$$\int_{-\infty}^{\infty} t^u \psi(t) dt = 0, \quad \forall u \in \mathbb{N}.$$ 

If the Fourier transforms of $\phi$ and $\psi$ are also in $C^r$ for a chosen $r \in \mathbb{N}$, then for it can be easily shown that $\phi$ and $\psi$ decay as

$$|\phi(t)| = O \left((1 + |t|)^{-r-1}\right), \quad |\psi(t)| = O \left((1 + |t|)^{-r-1}\right),$$

meaning that $\phi$ and $\psi$ are not very well localized in time. This is why a Meyer wavelet transform is generally implemented in the Fourier domain.
For the purpose of this paper, we use the periodized Meyer wavelet bases on the unit interval. For any \( t \in [0, 1] \), any integer \( j \) and any \( k \in \{0, \ldots, 2^j - 1\} \), let
\[
\phi_{j,k}(t) = 2^{j/2} \phi(2^j t - k), \quad \psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k)
\]
be the elements of the wavelet basis, and
\[
\phi_{j,k}^{\per}(t) = \sum_{l \in \mathbb{Z}} \phi_{j,k}(t - l), \quad \psi_{j,k}^{\per}(t) = \sum_{l \in \mathbb{Z}} \psi_{j,k}(t - l),
\]
their periodized versions. There exists an integer \( j_* \) such that the collection \( \{\phi_{j,k}^{\per}; k = 0, \ldots, 2^{j_*} - 1; \psi_{j,k}^{\per}; \ j = j_*, \ldots, \infty, \ k = 0, \ldots, 2^j - 1\} \) forms an orthonormal basis of \( L^2_{\per}([0, 1]) \). In what follows, the superscript "\( \per \)" will be dropped to lighten the notation.

For the purposes of this paper, we consider the tensor product wavelet basis on \( L_2([0, 1]^2) \). Let us briefly recall the construction of such a basis (see, for instance, [13]). Let us define the tensor-product wavelets \( \Phi, \Psi^1, \Psi^2 \) and \( \Psi^3 \) as
\[
\Phi(x) = \phi(x)\phi(y), \quad \Psi^1(x) = \psi(x)\phi(y), \quad \Psi^2(x) = \phi(x)\psi(y), \quad \Psi^3(x) = \psi(x)\psi(y),
\]
for any \( \forall x = (x, y) \in [0, 1]^2 \). For any orientation \( i \in \{1, 2, 3\} \), scale \( j \geq 0 \) and spatial location \( k = (k_1, k_2) \in D_j = \{0, \ldots, 2^j - 1\}^2 \), we define the translated and scaled versions \( \Phi_{j,i}(x) = 2^j \Phi(2^j x - k_1, 2^j y - k_2) \) and \( \Psi_{j,i}(x) = 2^i \Psi^i(2^j x - k_1, 2^j y - k_2) \).

Any function \( f \in L_2([0, 1]^2) \) can be expanded into a wavelet series
\[
f(x) = \sum_{k \in D_{j_0}} \alpha_{j_0,k} \Phi_{j_0,k}(x) + \sum_{i=1}^3 \sum_{j \geq j_0} \sum_{k \in D_j} \beta_{j,i,k} \Psi_{j,i,k}(x),
\]
where \( \alpha_{j,k} = \int_{[0,1]^2} f(x) \Phi_{j,k}(x) \, dx \) and \( \beta_{j,i,k} = \int_{[0,1]^2} f(x) \Psi_{j,i,k}(x) \, dx \) are the wavelet coefficients of \( f \).

We say that a function \( f \) in \( L_2([0, 1]^2) \) belongs to the bi-dimensional (isotropic) Besov ball \( \mathcal{B}^s_{p,q}(M) \) if, and only if, \( \int_{[0,1]^2} f^2(x) \, dx \leq M \) and there exists a constant \( M_\alpha \), depending on \( M \), such that the wavelet coefficients of \( f \) satisfy
\[
\left( \sum_{i=1}^3 \sum_{j \geq 0} \right) 2^{(j+1-2/p)q} \left( \sum_{k \in D_j} |\beta_{j,i,k}|^p \right)^{1/p} \leq M_\alpha,
\]
with a smoothness parameter \( s > 0 \) and the norm parameters: \( 0 < p \leq \infty \) and \( 0 < q \leq \infty \). Such Besov spaces contain both smooth images and those with sharp edges.
3. The deconvolution block estimator

3.1. Smoothness of the kernel $g$

For the theoretical study, the following assumption on $g$ will be essential. It is similar to the one employed in [8, 7, 14, 2]. We suppose that there exist four constants, $c > 0$, $C > 0$, $\delta_1 > 1/2$ and $\delta_2 > 1/2$, such that, for any $l = (l_1, l_2) \in \mathbb{Z}^2$, the Fourier transform of $g$ satisfies

$$c(1 + |l_1|^{\delta_1})^{-1}(1 + |l_2|^{\delta_2})^{-1} \leq |\mathcal{F}(g)(l)| \leq C(1 + |l_1|^{\delta_1})^{-1}(1 + |l_2|^{\delta_2})^{-1}. \quad (3.1)$$

In words, this means that the Fourier transform of the blurring PSF decays in a polynomial fashion within its bandwidth. For example, it is easy to check that the square integrable one-periodic function $g$ defined by $g(x, y) = h(x)h(y)$ where $h(x) = \sum_{m \in \mathbb{Z}} e^{-|x+m|}$, $x \in [0, 1]$, satisfies (3.1). Indeed, for any $l \in \mathbb{Z}$, we have $\mathcal{F}(h)(l) = 2(1 + 4\pi^2 l^2)^{-1}$. Hence, for any $l = (l_1, l_2) \in \mathbb{Z}^2$, $\mathcal{F}(g)(l) = \mathcal{F}(h)(l_1)\mathcal{F}(h)(l_2)$ satisfies (3.1) with $c = 4(1 + 4\pi^2)^{-2}$, $C = (2\pi)^{-2}$ and $\delta_1 = \delta_2 = 2$. This assumption goes by the name of ordinary smooth case.

3.2. Vaguelet-Wavelet decomposition

Although the VWD is valid for more general operators $T$, we here restrict our description to the case of convolution where the VWD takes a simple form.

Proposition 3.1. For any $i \in \{1, 2, 3\}$, $j \geq j_0$ and $k \in D_j$, set

$$w_{j, i, k}(x) = 2^{-j(\delta_1 + \delta_2)}T^{-1}(\psi_{j, i, k})(x), \quad x \in [0, 1]^2,$$

where, for any $h \in L_2([0, 1]^2)$,

$$T^{-1}(h)(x) = \mathcal{F}^{-1}\left(\mathcal{F}(h)(\cdot)/\mathcal{F}(g)(\cdot)\right)(x) = \int_{[0,1]^2} (\mathcal{F}(h)(l)/\mathcal{F}(g)(l)) e^{2\pi i <l, x>} dl. \quad (3.2)$$

Then, under assumption (3.1), there exist two constants $c > 0$ and $C > 0$ such that

$$c \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j, i, k}^2 \leq \int_{[0,1]^2} \left(\sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j, i, k} \omega_{j, i, k}(x)\right)^2 dx \leq C \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j, i, k}^2.$$

for every sequence $(a_{j, i, k})$. 


Thanks to Proposition 3.1, under assumption (3.1), any function $f \in \mathbb{L}_2([0,1]^2)$ can be expanded into a vaguelet-wavelet series

$$f(x) = \sum_{k \in D_{j_0}} \vartheta_{j_0,k} w_{j_0,k}(x) + \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} \theta_{j,i,k} w_{j,i,k}(x),$$

$x \in [0,1]^2$, where $\vartheta_{j_0,k} = 2^{j_0(\delta_1 + \delta_2)} \int_{[0,1]^2} T(f)(x) \Phi_{j_0,k}(x) dx,$

$$\theta_{j,i,k} = 2^{j(\delta_1 + \delta_2)} \int_{[0,1]^2} T(f)(x) \Psi_{j,i,k}(x) dx,$$

$$w_{j_0,k}(x) = 2^{-j_0(\delta_1 + \delta_2)} T^{-1}(\Phi_{j_0,k})(x), \quad w_{j,i,k}(x) = 2^{-j(\delta_1 + \delta_2)} T^{-1}(\Psi_{j,i,k})(x).$$

For any $h \in \mathbb{L}_2([0,1]^2)$, $T^{-1}(h)(x)$ is defined by (3.2). Further details on vaguelet-wavelet series can be found in [1].

3.3. Gaussian sequence model

The first step to estimate $f$ consists in estimating the unknown wavelet coefficients of $T(f)$: $(\vartheta_{j_0,k})_k$ and $(\theta_{j,i,k})_{j,i,k}$ from the observation $Y$ in (1.1). It follows from (1.1) that

$$y_{j,i,k} = \theta_{j,i,k} + \sigma z_{j,i,k}, \quad (3.3)$$

where

$$y_{j,i,k} = 2^{j(\delta_1 + \delta_2)} \int_{[0,1]^2} \Psi_{j,i,k}(x) dY(x), \quad z_{j,i,k} = 2^{j(\delta_1 + \delta_2)} \int_{[0,1]^2} \Psi_{j,i,k}(x) dW(x).$$

Thanks to the orthonormality of the wavelet basis, the random variables $(z_{j,i,k})_{j,i,k}$ are Gaussian i.i.d. with mean 0 and variance $2^{2j(\delta_1 + \delta_2)}$.

3.4. Two-dimensional block thresholding estimator

Let the observed image be defined on a $n \times n$ discrete grid of equally-spaced pixels $\{Y(i/n, j/n); (i,j) \in \{1, \ldots, n\}^2\}$. Let $L = [(2 \log(n))^{1/2}]$ be the block length, $j_0 = [\log_2 L]$ is the coarsest decomposition scale, and $J_s = [(1/(\delta_1 + \delta_2)) \log_2(n)]$. Consider the sequence model (3.3) with $\sigma = 1/n$. For any $k \in D_{j_0}$, we set

$$\widehat{\vartheta}_{j_0,k} = 2^{j_0(\delta_1 + \delta_2)} \int_{[0,1]^2} \Phi_{j_0,k}(x) dY(x).$$

For any $j \in \{j_0, \ldots, J_s\}$, let $A_j = \{1, \ldots, [2^jL^{-1}]\}^2$ be the set indexing the blocks at scale $j$, and for each block index $K = (K_1, K_2) \in A_j$,

$$U_{j,K} = \{k \in D_j; (K_1 - 1)L \leq k_1 \leq K_1 L - 1, \ (K_2 - 1)L \leq k_2 \leq K_2 L - 1\}$$

is the set indexing the positions of coefficients within the $K$th block $U_{j,K}$.

For any $k \in U_{j,K}$, $K \in A_j$ and $i \in \{1, 2, 3\}$, we estimate the wavelet coefficients $\theta_{j,i,k}$ of $T(f)$ from $y_{j,i,k}$ in (3.3) as
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\[ \hat{\theta}_{j,i,k} = y_{j,i,k} \text{ if } j \in \{0, \ldots, j_0 - 1\}; \]

\[ \hat{\theta}_{j,i,k} = y_{j,i,k} \left( 1 - \frac{\lambda_* \sigma^2 2^j (\delta_1 + \delta_2)}{L^2} \sum_{k \in U_{j,k}} y_{j,i,k}^2 \right) \text{ if } j \in \{j_0, \ldots, J_*\}; \]

\[ \hat{\theta}_{j,i,k} = 0 \text{ if } j > J_* . \]

where \((a)_+ = \max(a, 0)\), and \(\lambda_*\) is the root of \(x - \log x = 3\), i.e. \(\lambda_* = 4.50524\ldots\).

To estimate \(f\), we reconstruct it from these block-thresholded coefficients as

\[ \hat{f}(x) = \sum_{k \in D_{j_0}} \hat{\vartheta}_{j_0,k} \omega_{j_0,k}(x) + \sum_{i=1}^{J_*} \sum_{j=j_0}^{J_*} \sum_{K \in A_j} \sum_{k \in U_{j,k}} \hat{\theta}_{j,i,k} w_{j,i,k}(x), \quad (3.4) \]

\(x \in [0,1]^2\).

4. Optimality result

Theorem 4.1 below investigates the minimax rates of convergence attained by \(\hat{f}\) over \(B_{p,q}^s(M)\) under the \(L_2\) risk.

**Theorem 4.1.** Consider the model (1.1). Let \(\hat{f}\) be the estimator defined by (3.4). Then there exists a constant \(C > 0\) such that

\[ \sup_{f \in B_{p,q}^s(M)} \mathbb{E} \left( \int_{[0,1]^2} \left( \hat{f}(x) - f(x) \right)^2 dx \right) \leq Cv_\sigma, \]

where

\[ v_\sigma = \begin{cases} \sigma^{2s/(s+\delta_1+\delta_2+1)}, & \text{for } 2 \leq p, \\ (\sigma |\log(\sigma)|)^{2s/(s+\delta_1+\delta_2+1)}, & \text{for } p < 2, \ sp > c, \end{cases} \quad (4.1) \]

\(c = 2 \vee (2-p)(\delta_1 + \delta_2 + 1)\).

Using lower bound techniques, one can prove that \(v_\sigma\) is optimal except in the cases \(p < 2\) where there is an extra logarithmic term. It can also be shown that \(v_\sigma\) is better than the one achieved by the conventional term-by-term thresholding estimators (WaveD [7], etc). The main difference is for the case \(p \geq 2\) where there is no extra logarithmic term.

5. Experimental results

The proposed block VWD deconvolution method has been compared to three deconvolution methods from the literature: ForWarD [14], wavelet-domain iterative soft-thresholding (IST) with 100 iterations [10, 6], and WaveD [7]. For fair comparison, the regularization parameter of the IST method was tweaked manually to reach its best performance. For reliable comparison, we applied the
deconvolution algorithms to six standard grayscale images of size $512 \times 512$ (Barbara, Lena, Boat) and $256 \times 256$ (Cameraman, House, Peppers). The blurred images were corrupted by a zero-mean white Gaussian noise such that the blurred signal-to-noise ratio $(\text{BSNR} = 10 \log_{10} \left( \frac{\|f \ast g\|_\infty}{\sigma^2} \right))$ ranged from 10 to 40 dB. At each combination of test image and noise level, ten noisy versions were generated and each deconvolution algorithm was applied to each noisy realization. The output SNR improvement (ISNR) was averaged over the ten replications. The results are shown in Table 1 where the PSF was $g(i, j) = e^{-|i/n|^{0.5} + |j/n|^{0.5}}$. Each table corresponds to the ISNR as a function of BSNR for each image. These results clearly show that our approach compares very favorably to ForWarD and iterative thresholding. It is even able to outperform them particularly at low BSNR, while having substantially less computational cost as reported in Table 2. These quantitative results are confirmed by visual inspection of Fig. 1 and Fig. 2 which display the results on Barbara and Boat for BSNR=30dB. Again, owing to block thresholding, our VWD deconvolution is able to recover many details (e.g. textured areas on Barbara trousers) better that the other methods.

Following the philosophy of reproducible research, a toolbox is made available freely for download at the address

\begin{verbatim}http://www.greyc.ensicaen.fr/~jfadili/software.html\end{verbatim}

This toolbox is a collection of Matlab functions, scripts and datasets for image block denoising. It requires at least WaveLab 8.02 [17] to run properly. The toolbox implements the proposed block denoising procedure with several transforms and contains all scripts to reproduce the figures and tables reported in this paper.

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\textbf{Barbara $512 \times 512$} & \multicolumn{5}{c|}{\textbf{Lenna $512 \times 512$}} \\
\hline
\textbf{BSNR (dB)} & 10 & 15 & 20 & 25 & 30 & 40 & 10 & 15 & 20 & 25 & 30 & 40 \\
\hline
Ours & 2.93 & 2.56 & 2.90 & 3.66 & 5.68 & 11.11 & 5.08 & 5.13 & 6.15 & 7.81 & 8.63 & 12.82 \\
ForWarD [14] & 0.00 & 0.00 & 2.91 & 3.58 & 4.92 & 10.24 & 0.17 & 4.29 & 6.16 & 7.46 & 8.76 & 12.18 \\
IST [10, 6] & 2.87 & 2.54 & 2.94 & 5.26 & 4.33 & 7.69 & 4.82 & 4.60 & 6.32 & 7.62 & 9.08 & 12.66 \\
\hline
\textbf{Boat $512 \times 512$} & \multicolumn{5}{c|}{\textbf{Cameraman $256 \times 256$}} \\
\hline
\textbf{BSNR (dB)} & 10 & 15 & 20 & 25 & 30 & 40 & 10 & 15 & 20 & 25 & 30 & 40 \\
\hline
Ours & 3.69 & 3.87 & 5.08 & 6.71 & 7.66 & 11.98 & 3.47 & 3.53 & 4.79 & 5.89 & 7.84 & 11.40 \\
ForWarD [14] & 1.00 & 3.70 & 5.17 & 6.38 & 7.75 & 11.42 & 0.00 & 2.16 & 4.32 & 6.29 & 8.16 & 12.40 \\
IST [10, 6] & 3.30 & 3.31 & 5.26 & 6.52 & 8.32 & 11.89 & 3.07 & 3.18 & 4.77 & 5.86 & 7.86 & 11.75 \\
\hline
\textbf{House $256 \times 256$} & \multicolumn{5}{c|}{\textbf{Peppers $256 \times 256$}} \\
\hline
\textbf{BSNR (dB)} & 10 & 15 & 20 & 25 & 30 & 40 & 10 & 15 & 20 & 25 & 30 & 40 \\
\hline
IST [10, 6] & 5.21 & 5.67 & 7.73 & 9.25 & 10.74 & 14.58 & 3.69 & 4.28 & 7.07 & 8.81 & 11.01 & 15.01 \\
\hline
\end{tabular}
\caption{Comparison of average ISNR in dB over ten realizations for various images.}
\end{table}
Algorithm & Ours & ForWarD [14] & IST [10, 6] & WaveD [7] \\ 512 × 512 & 0.62 & 5.2 & 119 & 4.08 \\ 256 × 256 & 0.15 & 1.05 & 29 & 0.76 \\

Table 2  
Average execution times (seconds) over then replications for 512 × 512 and 256 × 256 images. The algorithms were run under Matlab with an 2.53GHz Intel Core Duo CPU, 4Gb RAM.

6. Conclusion

In this paper, a wavelet-based deconvolution algorithm was presented. It combines the benefits of block-thresholding with vaguelet-wavelet decomposition. Its theoretical and practical performances were established. Although we focused on convolution, the approach can handle other operators $T(f)$. A possible perspective of the present work that we are currently investigating is the theoretical properties of the procedure when other transforms than wavelets (curvelets for instance) are used.

7. Proofs

Proof of Proposition 3.1. For every sequence $(a_{j,i,k})$, the Parseval-Plancherel theorem implies that

$$
\int_{[0,1]^2} \left( \sum_{i=1}^3 \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k} \omega_{j,i,k}(x) \right)^2 d \mathbf{x} \\
= 2^{-2j(\delta_1 + \delta_2)} \int_{[0,1]^2} \left( \mathcal{F}^{-1} \left( \sum_{i=1}^3 \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k} \mathcal{F}(\Psi_{j,i,k})(\mathbf{x}) / \mathcal{F}(g)(\mathbf{x}) \right) \right)^2 d \mathbf{x} \\
= 2^{-2j(\delta_1 + \delta_2)} \int_{[0,1]^2} \left( \sum_{i=1}^3 \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k} \mathcal{F}(\Psi_{j,i,k})(\mathbf{x}) / \mathcal{F}(g)(\mathbf{x}) \right)^2 d \mathbf{x} \\
= 2^{-2j(\delta_1 + \delta_2)} \int_{[0,1]^2} \left( 1 / |\mathcal{F}(g)(\mathbf{x})|^2 \right) \left( \mathcal{F} \left( \sum_{i=1}^3 \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k} \Psi_{j,i,k} \right)(\mathbf{x}) \right)^2 d \mathbf{x}.
$$

(7.1)

Let $C_{j,i} = \text{supp}(\Psi_{j,i,k})$. Then by definition of the wavelet basis and under assumption (3.1), there exist two constants $c > 0$ and $C > 0$ such that

$$
c^{2j(\delta_1 + \delta_2)} \leq \inf_{\mathbf{x} \in C_{j,i}} (1 / |\mathcal{F}(g)(\mathbf{x})|^2) \leq \sup_{\mathbf{x} \in C_{j,i}} (1 / |\mathcal{F}(g)(\mathbf{x})|^2) \leq C^{2j(\delta_1 + \delta_2)}. \quad (7.2)
$$
Using again the Plancherel-Parseval theorem, we obtain
\[
\int_{[0,1]^2} \left( \mathcal{F} \left( \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k} \Psi_{j,i,k} \right) \right)^2 \, dx
\]
\[= \int_{[0,1]^2} \left( \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k} \Psi_{j,i,k}(x) \right)^2 \, dx = \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k}^2. \tag{7.3}
\]

Putting (7.1), (7.2) and (7.3) together, there exist two constants \(c > 0\) and \(C > 0\) such that
\[
c \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k}^2 \leq \int_{[0,1]^2} \left( \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k} \omega_{j,i,k} \right)^2 \, dx \leq C \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k}^2.
\]

Proof of Theorem 4.1. Using the Gaussian sequence (3.3), Theorem 4.1 can be proved by applying [5, Theorem 3.1]: it is enough to prove the existence of two constants \(Q_3 > 0\) and \(Q_4 > 0\) (independent of \(n\)) such that the conditions (i) and (ii) below are satisfied:

(i) \[\sup_{j \in \{0, \ldots, J_\ast\}} \sup_{i \in \{1, 2, 3\}} \sup_{k \in D_j} 2^{-4(\delta_1 + \delta_2)j} \mathbb{E} \left( \sum_{i=1}^{3} \sum_{j \geq j_0} \sum_{k \in D_j} a_{j,i,k}^2 \right) \leq Q_3.\]

(ii) for any \(a = (a_k)_{k \in D_j}\) such that \(\sup_{j \in \{0, \ldots, J_\ast\}} \sup_{K \in \mathcal{A}_j} \sum_{k \in U_{j,K}} a_k^2 \leq 1\), we have
\[\sup_{j \in \{0, \ldots, J_\ast\}} \sup_{i \in \{1, 2, 3\}} \sup_{K \in \mathcal{A}_j} 2^{-2(\delta_1 + \delta_2)j} \mathbb{E} \left( \sum_{k \in U_{j,K}} a_k \omega_{j,i,k} \right)^2 \leq Q_4.
\]

Since the random variables \((z_{j,i,k})_{j,i,k}\) are Gaussian i.i.d. with mean 0 and variance \(2^{2j(\delta_1 + \delta_2)}\), these bounds are obvious.

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References


