



**HAL**  
open science

## Jarnik's convex lattice $n$ -gon for non-symmetric norms

Imre Barany, Nathanael Enriquez

► **To cite this version:**

Imre Barany, Nathanael Enriquez. Jarnik's convex lattice  $n$ -gon for non-symmetric norms. 2009.  
hal-00434287

**HAL Id: hal-00434287**

**<https://hal.science/hal-00434287>**

Preprint submitted on 22 Nov 2009

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Jarník's convex lattice $n$ -gon for non-symmetric norms

Imre Bárány, Nathanaël Enriquez

November 22, 2009

## Abstract

What is the minimum perimeter of a convex lattice  $n$ -gon? This question was answered by Jarník in 1926. We solve the same question in the case when perimeter is measured by a (not necessarily symmetric) norm.

**Keywords :** convex lattice polygon, isoperimetric problem, variational problem.

**2000 Mathematics Subject Classification :** 52B60, 52C05, 49Kxx.

## 1 Introduction

What is the minimal perimeter  $L_n$  that a convex lattice polygon with  $n$  vertices can have? In 1926 Jarník [4] proved that  $L_n = \frac{\sqrt{6\pi}}{9}n^{3/3} + O(n^{3/4})$ . The aim of this paper is to extend this result to all, not necessarily symmetric, norms in the plane. As usual, such a norm is defined by a convex compact set  $D \subset \mathbf{R}^2$  with  $0 \in \text{int } D$ , and the norm of  $x \in \mathbf{R}^2$  is

$$\|x\| = \|x\|_D = \min\{t \geq 0 : x \in tD\}.$$

Let  $\mathbf{Z}^2$  be the lattice of integer points in  $\mathbf{R}^2$ , and write  $\mathcal{P}_n$  ( $n \geq 3$ ) for the set of all convex lattice  $n$ -gons in  $\mathbf{R}^2$ , that is,  $P \in \mathcal{P}_n$  if  $P = \text{conv}\{z_1, \dots, z_n\}$  where  $z_1, \dots, z_n \in \mathbf{Z}^2$  are the vertices, in anticlockwise order, of  $P$ . The  $D$ -perimeter of  $P$  is defined by

$$\text{Per } P = \text{Per}_D P = \sum_{i=1}^n \|z_{i+1} - z_i\|_D$$

where  $z_{n+1} = z_1$  by convention. Note that for a non-symmetric  $D$ ,  $\text{Per}_D P$  depends on the orientation of  $P$  as well. Define now

$$L_n = L_n(D) = \min\{\text{Per}_D P : P \in \mathcal{P}_n\} \quad (1.1)$$

Since  $D$  will be kept fixed throughout, we will often write  $\text{Per } P$  and  $L_n$  instead of  $\text{Per}_D P$  and  $L_n(D)$ .

In this paper we determine the asymptotic behaviour of  $L_n(D)$  for all norms. We will also show that, after suitable scaling, the minimizing polygons have a limiting shape. The same results were proved by Maria Prodromou [5] in 2005 in the case when  $D$  is symmetric, that is,  $D = -D$ . We will see that most of the difficulties in the non-symmetric case do not come up in the symmetric one.

Define  $\mathcal{F}$  as the set of all positive continuous functions  $r : [0, 2\pi] \rightarrow \mathbf{R}^+$  with  $r(0) = r(2\pi)$ . Such a function is the radial function of a *starshaped* set in  $\mathbf{R}^2$ ; such a set contains the origin in its interior and the half-line starting at the origin in direction  $u(t) = (\cos t, \sin t)$  intersects its boundary at a single point which is at distance  $r(t)$  from the origin. We write  $\mathcal{S}$  for the set of all starshaped sets in  $\mathbf{R}^2$ . Every convex compact set  $K \subset \mathbf{R}^2$  with  $0 \in \text{int } K$  is, of course, starshaped. We denote by  $\mathcal{F}^c$  the set of radial functions of all such convex compact sets.

Let  $r_0 \in \mathcal{F}^c$  be the radial function of  $D$ . The problem of determining  $L_n(D)$  is closely related to the following variational problem, to be denoted by  $VP(r_0)$ . We seek a radial function  $r \in \mathcal{F}$  that minimizes

$$\begin{aligned} & \int_0^{2\pi} r^3(t)/r_0(t) dt \\ \text{subject to} & \quad \int_0^{2\pi} r^3(t) \cos t dt = 0, \quad \int_0^{2\pi} r^3(t) \sin t dt = 0, \\ \text{and} & \quad \frac{1}{2} \int_0^{2\pi} r^2(t) dt = 1. \end{aligned} \tag{1.2}$$

Assume  $r(t)$  is the radial function of a convex (or starshaped) compact set  $K \subset \mathbf{R}^2$ . Then the first condition says that the centre of gravity,  $g(K)$ , of  $K$  is at the origin, and the second condition says that  $\text{Area } K = 1$ . We will explain later the meaning of the function to be minimized. Using the results concerning  $L_n$  we will prove the following.

**Theorem 1.1** *There is a unique solution  $r \in \mathcal{F}$  to the variational problem. It is the radial function of a convex compact set in  $\mathbf{R}^2$  defined as the only function of the form*

$$\frac{1}{r} = \frac{a}{r_0} + b \cos t + c \sin t$$

with  $a > 0$ ,  $b, c \in \mathbf{R}$ , that satisfies the constraints of  $VP(r_0)$ .

Notice that all the positive functions of the form  $\frac{a}{r_0} + b \cos t + c \sin t$  are radial functions of a convex set. Indeed, the sign of the curvature is given, in the differentiable case, by the sign of  $(\frac{1}{r})'' + \frac{1}{r}$  which happens to be equal to  $a((\frac{1}{r_0})'' + \frac{1}{r_0})$ , which is always positive because  $D$  is convex. This result can easily be extended to the non differentiable case.

We mention further that the solution to  $VP(r_0)$  is unique in a larger class than  $\mathcal{F}$ . This will be clear from the proof.

## 2 Results and notations

Assume that the vertices of a minimizer  $P_n \in \mathcal{P}_n$  are  $z_1, \dots, z_n$  in anticlockwise order (which is the orientation giving the minimal  $D$ -perimeter). Then  $E_n = \{z_2 - z_1, \dots, z_n - z_{n-1}, z_1 - z_n\}$  is the edge set of  $P_n$ . Define  $C_n = \text{conv } E_n$ . Note that the  $E_n$  determines  $P_n$  uniquely (up to translation). Even more generally, the following is true.

**Proposition 2.1** *Suppose  $V \subset \mathbf{R}^2$  is a finite set of vectors whose sum is zero. Assume further that  $u, v \in V$ ,  $u = \lambda v$  with  $\lambda > 0$  implies that  $u = v$ . Then there is a unique (up to translation) convex polygon whose edge set is equal to  $V$ .*

**Proof.** This is very simple. One has to order (cyclically) the vectors in  $V$  by increasing slope as  $v_1, \dots, v_n, v_1$ . Then the polygonal path through the points  $0, v_1, v_1 + v_2, v_1 + v_2 + v_3, \dots, v_1 + \dots + v_n = 0$  in this order is a convex polygon with edge set  $V$ . Uniqueness is clear.  $\square$

We call this construction the *increasing slope construction*. Here come our main results. We let  $\mathcal{K}$  denote the family of all convex compact sets in  $\mathbf{R}^2$  with non-empty interior. For  $K, L \in \mathcal{K}$ ,  $\text{dist}(K, L)$  denotes their Hausdorff distance.

**Theorem 2.2** *There is a unique  $C \in \mathcal{K}$  such that  $\lim \text{dist}((\text{Area } C_n)^{-1/2} C_n, C) = 0$ . Moreover,  $g(C) = 0$  and  $\lim n^{-3/2} L_n(D)$  exists and equals*

$$\alpha(D) = \frac{\pi}{\sqrt{6}} \int_C \|x\| dx.$$

We will prove the uniqueness part of Theorem 1.1 by showing that the radial function of  $C$  is the unique solution to the variational problem  $VP(r_0)$ .

**Theorem 2.3** *There is a convex set  $P \subset \mathbf{R}^2$  such that the following holds. Let  $P_n$  be an arbitrary sequence of minimizers, of  $L_n(D)$ , translated so that  $\min\{x : (x, y) \in P_n\}$  is reached at the origin. Then  $\lim \text{dist}(n^{-3/2} P_n, P) = 0$ .*

We explain in Section 10 how and why  $P$  is determined uniquely by  $C$ . Moreover, it is shown in section 11 that the round shape found for  $P$  in Jarník's case is obtained if and only if the unit ball  $D$  is given by an ellipse having a focus point at the origin.

To avoid some trivial complications in the proofs we assume that  $D$  is strictly convex. We emphasize however that the above results are valid without this extra condition. We make another simplifying assumption, namely, that

$$\text{Area } D = 1 \tag{2.1}$$

This is just a convenient scaling of the unit ball which leaves the set of minimizers, and the corresponding  $E_n, C_n$  and consequently  $C, P$  unchanged.

The strategy of proof of the key Theorem 2.2 is as follows. We put together the following ingredients :

- almost all primitive vectors of  $C_n$  belong to  $E_n$  (Section 7),
- the normalized convex hulls  $(\text{Area } C_n)^{-1/2} C_n$  are sandwiched between two fixed Euclidean balls (Section 6), so that the Blaschke selection theorem applies (Section 9),
- the radial functions of the only possible limiting points of the sequence  $(\text{Area } C_n)^{-1/2} C_n$  are solutions of  $VP(r_0)$  (Section 5). Moreover, the variational problem  $VP(r_0)$  has a unique solution (Section 8).

### 3 Auxiliary lemmas

We write  $\mathbf{P}$  for the set of primitive vectors in  $\mathbf{Z}^2$ , i.e.,  $z = (x, y) \in \mathbf{Z}^2$  ( $z \neq 0$ ) is in  $\mathbf{P}$  if  $x$  and  $y$  are relatively prime. The following two claims are very simple.

**Claim 3.1** For all  $n \geq 3$ ,  $L_n < L_{n+1}$ .

**Proof.** Let  $P_{n+1} = \text{conv}\{z_0, z_1, \dots, z_n\}$  be a minimizer for  $L_{n+1}$  and set  $P_n^* = \text{conv}\{z_1, \dots, z_n\}$ . Then  $L_n \leq \text{Per } P_n^* < L_{n+1}$ .  $\square$

**Claim 3.2**  $E_n \subset \mathbf{P}$ .

**Proof.** Assume  $P_n$  is a minimizer and the edge  $z_2 - z_1 \notin \mathbf{P}$ , say. Then the segment  $[z_1, z_2]$  contains an integer  $z \in \mathbf{Z}^2$  distinct from  $z_1, z_2$ . The convex lattice  $n$ -gon  $\text{conv}\{z_1, z, z_3, \dots, z_n\}$  has shorter  $D$ -perimeter than  $P_n$  because the triangle  $\text{conv}\{z_1, z_2, z_3\}$  contains the triangle  $\text{conv}\{z_1, z, z_3\}$  so the latter has shorter  $D$ -perimeter.  $\square$

The following lemma will be useful when proving that most points in  $C_n \cap \mathbf{P}$  belong to  $E_n$ .

**Lemma 3.3** Assume  $a, b \in E_n$  and  $a \neq \pm b$ . Let  $T$  be the parallelogram with vertices  $0, a, b, a + b$ . If  $x, y \in (T \cap \mathbf{P}) \setminus E_n$  and  $x \neq y$ , then  $x + y \notin T$ .

**Proof.** If  $x + y \in T$  were the case, then set  $E^* = E_n \cup \{x, y, z\} \setminus \{a, b\}$  where  $z = a + b - x - y$ . The increasing slope construction works now because  $\sum_{z \in E^*} z = 0$  and gives rise to convex lattice  $(n + 1)$ -gon  $P$  if there is no  $u \in E_n$  with  $u = \lambda z$  with  $\lambda > 0$ . If there is such a  $u$ , we replace  $u$  and  $z$  by  $u + z$  in  $E^*$ , and the increasing slope construction gives a convex lattice  $n$ -gon  $P$ . We claim that  $P$  has shorter  $D$ -perimeter than  $P_n$ . This clearly finishes the proof.

To prove  $\text{Per } P < \text{Per } P_n$  we have to show that  $\|x\| + \|y\| + \|z\| < \|a\| + \|b\|$ . Assume that the anticlockwise angle from  $a$  to  $b$  is smaller than  $\pi$ . Then  $x, y, z \in \text{pos}\{a, b\}$  where  $\text{pos}\{a, b\}$  is the cone hull of  $a$  and  $b$ . Order the vectors  $a, b, x, y, z$  by anticlockwise increasing slope. The outcome is  $a, x, z, y, b$  say. Then the

triangle  $\Delta = \text{conv}\{0, a, a + b\}$  contains the quadrilateral  $Q = \text{conv}\{0, x, x + z, x + y + z\}$  so the latter has shorter  $D$ -perimeter. Now  $a + b = x + y + z$  and

$$\text{Per } Q = \|x\| + \|y\| + \|z\| + \|x + y + z\| < \text{Per } \Delta = \|a\| + \|b\| + \|a + b\|,$$

and  $\text{Per } P < \text{Per } P_n$  follows.  $\square$

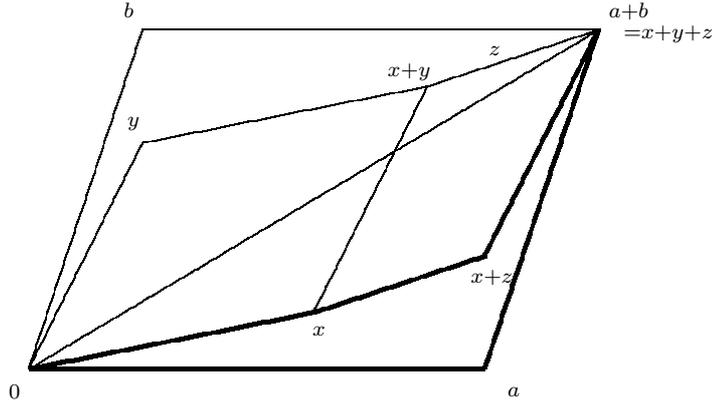


Figure 1. The proof of Lemma 3.3

We write  $B$  for the Euclidean unit ball in  $\mathbf{R}^2$  and  $|x|$  for the Euclidean norm of  $x \in \mathbf{R}^2$ . Since  $D$  is compact convex and  $0 \in \text{int } D$ , there are positive constants  $d_1, d_2$  such that  $d_1B \subset D \subset d_2B$ , or, equivalently,

$$d_1|x| \leq \|x\| \leq d_2|x|, \text{ for every } x \in D.$$

In what follows  $c, c_1, c_2, \dots$  denote positive constants independent of  $n$ . We will also use Vinogradov's convenient  $\ll$  notation:  $f(n) \ll g(n)$  means that there are positive constants  $c$  and  $n_0$  such that  $cf(n) \leq g(n)$  for all  $n \geq n_0$ . Of course, the constants do not depend on  $n$ . But they depend on  $D$ , more precisely, they depend on the constants  $d_1, d_2$ .  $f(n) \gg g(n)$  has the same meaning but with  $f(n) \geq cg(n)$ . We will also use the big Oh and little oh notation.

We need some standard estimates on the distribution of lattice points and primitive points in a convex body  $K \in \mathcal{K}$ , see [3] or [1] for a proof. Let  $L$  denote the Euclidean perimeter of  $K$ . We assume that  $L > 3$ , say, but we think of  $K$  as "large". In fact, in most applications  $L$  tends to infinity. The following estimate is simple and well-known.

$$\left| |K \cap \mathbf{Z}^2| - \text{Area } K \right| \leq 2L. \tag{3.1}$$

This implies, with the standard method using the Möbius function, that

$$\left| |K \cap \mathbf{P}| - \frac{6}{\pi^2} \text{Area } K \right| \leq 3L \log L. \tag{3.2}$$

Assume next that  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  is a 1-homogeneous function, that is,  $f(\lambda x) = \lambda f(x)$  for every  $x \in \mathbf{R}^2$  and  $\lambda \geq 0$ . Writing  $M = \max\{|f(z)| : z \in K\}$  the following estimates hold.

$$\left| \sum_{z \in K \cap \mathbf{Z}^2} f(z) - \int_K f(z) dz \right| \leq 2ML, \quad (3.3)$$

$$\left| \sum_{z \in K \cap \mathbf{P}} f(z) - \frac{6}{\pi^2} \int_K f(z) dz \right| \leq 3ML \log L. \quad (3.4)$$

The same estimates hold when  $K$  is a (non-convex but) starshaped set whose boundary consists of finitely many line segments. (Then, of course, the perimeter of  $K$  is a finite number  $L$ .) This fact will be needed in Section 5.

These estimates will be used quite often in the case when  $K = \lambda K_0$ , and  $\lambda \rightarrow \infty$  with  $K_0$  fixed. Then formulae (3.1), (3.2), (3.3), (3.4) have the following simpler form:

$$|K \cap \mathbf{Z}^2| = \lambda^2 \text{Area } K_0 (1 + O(\lambda^{-1})), \quad (3.5)$$

$$|K \cap \mathbf{P}| = \frac{6}{\pi^2} \lambda^2 \text{Area } K_0 ((1 + O(\lambda^{-1} \log \lambda))). \quad (3.6)$$

$$\sum_{z \in K \cap \mathbf{Z}^2} f(z) = \lambda^3 \int_{K_0} f(z) dz + O(\lambda^2), \quad (3.7)$$

$$\sum_{z \in K \cap \mathbf{P}} f(z) = \frac{6}{\pi^2} \lambda^3 \int_{K_0} f(z) dz + O(\lambda^2 \log \lambda). \quad (3.8)$$

The constant in the big Oh notation depends only on  $K_0$ . Here  $K_0$  is either a convex set or a starshaped set with boundary consisting of finitely many line segments.

## 4 Bounding $L_n$

In this section we give upper and lower bounds on  $L_n$ .

**Claim 4.1**  $L_n \gg n^{3/2}$ .

**Proof.** Here we use the following *density principle*. The sum of the lengths of  $n$  distinct primitive vectors is at least as large as the sum of the lengths of the  $n$  shortest (distinct) primitive vectors. We will see the same principle in action a few more times.

Let  $v_1, \dots, v_n$  be the  $n$  shortest (in  $D$ -norm) vectors in  $\mathbf{P}$  (ties broken arbitrarily). Set  $\lambda = \max\{\|v_i\| : i = 1, \dots, n\}$ . Then  $(\text{int } \lambda D) \cap \mathbf{P} \subset \{v_1, \dots, v_n\} \subset \lambda D$ . The boundary of  $\lambda D$  contains at most  $\text{Per}_B \lambda D \leq 2\pi d_2 \lambda$  lattice points. So

$|\lambda D \cap \mathbf{P}| - 2\pi d_2 \lambda \leq n \leq |\lambda D \cap \mathbf{P}|$ . Using (3.6) with  $\lambda D$  (recalling  $\text{Area } D = 1$ ) gives

$$|\lambda D \cap \mathbf{P}| = \frac{6}{\pi^2} \lambda^2 (1 + O(\lambda^{-1} \log \lambda)).$$

This shows that  $n = \frac{6}{\pi^2} \lambda^2 (1 + O(\lambda^{-1} \log \lambda))$  implying that  $\lambda = (\frac{\pi}{\sqrt{6}} + o(1)) n^{1/2}$ . Using this in (3.8) with  $\lambda D$  gives

$$\begin{aligned} L_n &\geq \sum_1^n \|v_i\| \geq \sum_{z \in \text{int}(\lambda D) \cap \mathbf{P}} \|z\| \\ &\geq \left( \frac{6}{\pi^2} - O(\lambda^{-1} \log \lambda) \right) \lambda^3 \int_D \|z\| dz \gg n^{3/2}. \end{aligned}$$

□

**Claim 4.2**  $L_n \ll n^{3/2}$ .

**Proof.** Again, let  $v_1, \dots, v_n$  be the  $n$  shortest (in  $D$ -norm) vectors in  $\mathbf{P}$  and set  $v_0 = -\sum_1^n v_i$ . By the increasing slope construction the vectors  $v_0, v_1, \dots, v_n$  form the edge set of a unique (up to translation) convex lattice  $n$ -gon or  $n+1$ -gon. We estimate its  $D$ -perimeter from above using the estimates on  $\lambda$  from the previous proof.

$$\sum_1^n \|v_i\| \leq \sum_{z \in \lambda D} \|z\| \leq \left( \frac{6}{\pi^2} + O(\lambda^{-1} \log \lambda) \right) \lambda^3 \int_D \|z\| dz \ll n^{3/2},$$

We need to estimate  $\|v_0\|$  as well.

$$\|v_0\| \ll | -v_0 | = |v_0| \ll \| -v_0 \| = \left\| \sum_1^n v_i \right\| \leq \sum \|v_i\| \ll n^{3/2}.$$

This shows that, indeed,  $L_n \ll n^{3/2}$ . □

We mention that for a symmetric norm and for even  $n$ , the  $n$  shortest vectors can be chosen in pairs  $z, -z$  which is clearly optimal for  $L_n$ . The case of odd  $n$  only causes only a minor difficulty.

**Corollary 4.3**  $\liminf n^{-3/2} L_n$  exists and equals  $\alpha = \alpha(D) > 0$ , say.

## 5 Connection between $L_n$ and $VP(r_0)$

**Lemma 5.1** Assume  $S \in \mathcal{S}$  with  $\text{Area } S = 1$ ,  $g(S) = 0$ . Then  $r \in \mathcal{F}$ , the radial function of  $S$ , is a feasible solution to  $VP(r_0)$ . Moreover, there is  $Q_n \in \mathcal{P}_n$  (for every  $n \geq 3$ ) with

$$\lim n^{-3/2} \text{Per } Q_n = \frac{\pi}{\sqrt{6}} \int_S \|z\| dz = \frac{\pi}{3\sqrt{6}} \int_0^{2\pi} \frac{r^3(t)}{r_0(t)} dt.$$

We remark that the last identity follows from a simple integral transformation.

**Proof.** Feasibility of  $r$  is evident. We want to prove that for all  $\varepsilon > 0$  (that we will suppose small enough without restricting the generality), there is  $Q_n \in \mathcal{P}_n$ , for every  $n \geq 3$ , with

$$\frac{\pi}{\sqrt{6}} \int_S \|z\| dz - \varepsilon \leq \liminf n^{-3/2} \text{Per } Q_n \leq \limsup n^{-3/2} \text{Per } Q_n \leq \frac{\pi}{\sqrt{6}} \int_S \|z\| dz + \varepsilon.$$

Since we would like to deal with sets whose perimeter can be defined and controlled, we introduce, for all  $m \geq 3$ , the  $m$ -gon approximation of  $S$ , whose vertices are  $r(\frac{2\pi k}{m})u(\frac{2\pi k}{m})$  for  $k = 0, \dots, m-1$ , recall that  $u(t) = (\cos t, \sin t)$ . The sequence  $S_m$  converges uniformly to  $S$  as  $m$  goes to infinity. Moreover, since  $g(S) = 0 \in \text{int } S$ , there are constants  $c_1, c_2 > 0$  such that, for  $m$  large enough,

$$c_1 B \subset S_m \subset c_2 B.$$

We fix now  $m$  large enough so that the above condition is satisfied, as well as

$$\begin{aligned} \left\| \frac{6}{\pi^2} \int_{S_m} z dz \right\| &< c\varepsilon \\ \left| \frac{\pi}{\sqrt{6}} \int_S \|z\| dz - \frac{\pi}{\sqrt{6}} \int_{S_m} \|z\| dz \right| &< c\varepsilon \\ |\text{Area } S_m - 1| &< c\varepsilon \end{aligned}$$

where  $c$  is a positive constant depending only on  $S$  that will be adjusted later.

Now, there is a minimal  $\lambda > 0$  (depending on  $m$ ) so that  $|\mathbf{P} \cap \lambda S_m| \geq n$ . Let  $L_m$  denote the Euclidean perimeter of  $S$ . There are at most  $\lambda L_m$  lattice points on the boundary of  $\lambda S_m$ . Then, formula (3.6) applies and shows that

$$|\mathbf{P} \cap \lambda S_m| = \left( \frac{6}{\pi^2} \text{Area } S_m + O(\lambda^{-1} \log \lambda) \right) \lambda^2,$$

implying  $\lambda = \pi \sqrt{n / (6 \text{Area } S_m)} (1 + o(1))$ .

We apply formula (3.8) to  $\lambda S$  with  $f(z) = z$ , or more precisely with  $f(z) = x$  and  $f(z) = y$  where  $z = (x, y)$  to get

$$\sum_{z \in \mathbf{P} \cap \lambda S_m} z = \frac{6}{\pi^2} \lambda^3 \int_{S_m} z dz + O(\lambda^2 \log \lambda)$$

Let  $\mathbf{P} \cap \lambda S_m = \{z_1, \dots, z_l\}$  (of course  $l \geq n$ ) and define  $z_0 = -\sum_1^l z_i$ . The previous equality implies that for  $n$  large enough  $\|z_0\| \leq 2c\varepsilon \lambda^3$ . The increasing slope construction applies to  $\{z_0, z_1, \dots, z_l\}$  and gives a convex lattice  $l$  or  $l+1$ -gon  $T^n$ . Note that  $T^n$  has a *special edge*, the one parallel to, and having the

same direction as,  $z_0$ . All other edges of  $T^n$  are short, shorter than  $c_2 d_2 \lambda \ll n^{1/2}$  in  $D$ -norm. We claim now that, for a suitable choice of  $c$  (depending only on  $S$ ), and  $\varepsilon$  small enough,

$$\frac{\pi}{\sqrt{6}} \int_S \|z\| dz - \varepsilon \leq \liminf n^{-3/2} \text{Per } T_n \leq \limsup n^{-3/2} \text{Per } T^n \leq \frac{\pi}{\sqrt{6}} \int_S \|z\| dz + \varepsilon$$

We use again (3.8) this time with  $f(z) = \|z\|$  to get

$$\sum_1^l \|z_i\| = \frac{6}{\pi^2} \lambda^3 \int_{\lambda S_m} \|z\| dz (1 + o(1)) = \frac{\pi}{\sqrt{6}} \frac{n^{3/2}}{(\text{Area } S_m)^{3/2}} \int_{S_m} \|z\| dz (1 + o(1)).$$

The claim follows since  $\text{Per } T^n$  differs from  $\sum_1^m \|z_i\|$  by  $\|z_0\| \leq 2c\varepsilon\lambda^3$ .

Finally, let  $Q_n$  be the convex hull of  $n$  consecutive vertices of  $T^n$ , including the two endpoints of the special edge. Then  $Q_n \in \mathcal{P}_n$  and  $\text{Per } Q_n \leq \text{Per } T^n$  and also,  $\text{Per } Q_n$  is at least  $\text{Per } T^n$  minus the sum of the  $D$ -length of the missing edges, which is  $\ll n$  as one can easily check. Thus  $|\text{Per } Q_n - \text{Per } T^n| \ll n$ . The requirements on the constant  $c$  are now clear.  $\square$

We mention here that Lemma 5.1 implies Claim 4.2 by simply choosing any  $S \in \mathcal{S}$  with  $g(S) = 0$  and  $\text{Area } S = 1$ , for instance the Euclidean disk centred at the origin and having area 1.

## 6 Bounding $C_n$

Our next target is to give bounds on the width and diameter of  $C_n = \text{conv } E_n$ .

**Claim 6.1** *The width of  $E_n$ ,  $w(E_n)$ , satisfies  $w(E_n) \gg n^{1/2}$ .*

**Proof.** Set  $w = w(E_n)$ . Clearly,

$$L_n = \sum_{v \in E_n} \|v\| \gg \sum_{v \in E_n} |v| \geq M_n(w),$$

where  $M_n(w)$  is the sum of the lengths of the  $n$  shortest (in Euclidean norm) distinct vectors in  $\mathbf{Z}^2$  lying in a strip of width  $w$ .

A simple yet technical computation, delayed to Appendix 1, shows that  $w \leq \gamma n^{1/2}$  (where  $\gamma \in (0, 1/2]$ ) implies  $M_n(w) \gg n^{3/2}/\gamma$ . This finishes the proof of Claim 6.1, because then  $n^{3/2} \gg L_n \gg M_n(w) \gg n^{3/2}/\gamma$  would lead to contradiction if  $\gamma$  were too small.  $\square$

**Claim 6.2** *Assume the smallest Euclidean ball centred at 0 and containing  $E_n$  is  $RB$ . Then  $R \ll n^{1/2}$ .*

**Proof.** Assume  $a$  is the farthest point (in Euclidean distance) from the origin in  $E_n$ . Then  $|a| = R$ . Claim 4.2 implies that  $|a| \leq L_n \ll n^{3/2}$ . Since  $w(E_n) \gg n^{1/2}$  by the previous claim, there is a point  $b \in E_n$  whose distance from the line  $\{x = ta : t \in \mathbf{R}\}$  is  $\geq \frac{1}{2}w(E_n) \gg n^{1/2}$ .

The perimeter of the triangle  $\Delta = \text{conv}\{0, a, b\}$  is  $|a| + |b| + |a - b| \leq 4|a|$  because  $|b| \leq |a|$  and  $|a - b| \leq |a| + |b| \leq 2|a|$ . Here  $\text{Area } \Delta = \frac{1}{2}|a|h$  where  $h$  is the corresponding height of  $\Delta$ . Since  $w(E_n) \geq n^{1/2}$ ,  $h \gg n^{1/2}$ .

Then by (3.2) for large enough  $n$ ,

$$\left| |\mathbf{P} \cap \frac{1}{2}\Delta| - \frac{6}{\pi^2} \text{Area } \frac{1}{2}\Delta \right| \leq 3 \cdot 2|a| \log 2|a| \ll h|a| \frac{\log |a|}{\sqrt{n}} \ll \text{Area } \Delta \frac{\log n}{\sqrt{n}}$$

implying that  $|\mathbf{P} \cap \frac{1}{2}\Delta| \geq \frac{1}{8} \text{Area } \Delta$ , again when  $n$  is large enough.

Assume now that  $\text{Area } \Delta > 16n$ . Then  $|\mathbf{P} \cap \frac{1}{2}\Delta| \geq 2n$ . Since  $|E_n| \leq n$ ,  $\frac{1}{2}\Delta$  contains two distinct elements  $x, y \in \mathbf{P} \setminus E_n$  and, evidently,  $x + y \in \Delta$ . Then  $x, y, x + y \in \text{conv}\{0, a, b, a + b\}$  contradicting Lemma 3.3.

Thus  $\text{Area } \Delta = \frac{1}{2}|a|h \leq 16n$ , and so  $R = |a| \ll n^{1/2}$ .  $\square$

We need one more fact about  $C_n$ :

**Claim 6.3** *Assume  $rB$  is the largest ball centred at 0 and contained in  $C_n$ . Then  $r \gg n^{1/2}$ .*

**Proof.** Let  $a$  be the nearest point to 0 on the boundary of  $C_n$ . Thus  $r = |a|$ . Define  $E^+ = E_n \cap \{x \in \mathbf{R}^2 : ax > 0\}$  and  $E^- = E_n \cap \{x \in \mathbf{R}^2 : ax < 0\}$ , and set  $f(x) = ax/|a|$  which is just the component of  $x \in \mathbf{R}^2$  in direction  $a$ . To have simpler notation we write  $f(X) = \sum_{x \in X} f(x)$  when  $X \in \mathbf{R}^2$  is a finite set. Since  $\sum_{z \in E_n} z = 0$ ,  $f(E^+) + f(E^-) = 0$  (because  $f(z) = 0$  when  $az = 0$ ). We will show, however, that  $|a| \leq \gamma n^{1/2}$ , for a suitably small  $\gamma > 0$ , implies that

$$f(E^+) + f(E^-) < 0. \quad (6.1)$$

Define  $F^+ = \{x \in \mathbf{R}^2 : 0 < f(x) \leq \gamma n^{1/2}\} \cap rB$  with  $R \ll n^{1/2}$  from Claim 6.2. The density principle tells now that  $f(E^+) \leq f(\mathbf{P} \cap F^+) \leq f(\mathbf{Z}^2 \cap F^+)$  and the last sum can be estimated as follows. Let  $Q(z)$  be the unit cube centred at  $z$ . Again,  $\text{Area } Q(z) \cap F^+ \geq 1/4$  for all  $z \in \mathbf{Z}^2 \cap F^+$ . This implies that, for large enough  $n$ ,

$$m := |\mathbf{Z}^2 \cap F^+| \leq \text{Area } F^+ / 4 \ll R|a| \ll \gamma n.$$

We use now (3.3):

$$\left| f(\mathbf{Z}^2 \cap F^+) - \int_{F^+} f(z) dz \right| \ll R|a|.$$

It is easy to see that  $\int_{F^+} f(z) dz \ll |a|^2 R$  implying that  $f(\mathbf{Z}^2 \cap F^+) \ll |a|^2 R \ll \gamma^2 n^{3/2}$ .

Define  $F^- = \{x \in \mathbf{R}^2 : 0 > f(x) \geq -\lambda\gamma n^{1/2}\} \cap RB$  where  $\lambda > 0$  is chosen so that  $F^-$  contains exactly  $n - m - k$  lattice points. Here  $k$  is the number of lattice points on the line  $ax = 0$  so  $k \leq 2R + 1 \ll n^{1/2}$ . Note that  $\lambda\gamma n^{1/2} \ll R$  since  $E_n \subset RB$  consists of exactly  $n$  vectors. Choosing  $\gamma$  small enough guarantees that  $m < 0.1n$  which, in turn, guarantees that  $\lambda > 1$  and further, that  $|F^- \cap \mathbf{Z}^2| \geq 0.8n$ . The Euclidean perimeter of  $F^-$  is at most  $4R + \lambda\gamma n^{1/2} \ll R$  and (3.1) shows that  $||F^- \cap \mathbf{Z}^2| - \text{Area } F^-| \ll R$ . Clearly  $\text{Area } F^- \ll R\lambda\gamma n^{1/2}$ , implying that

$$0.8n < |F^- \cap \mathbf{Z}^2| \leq \left(1 + O\left(\frac{1}{\lambda\gamma n^{1/2}}\right)\right) \text{Area } F^- \ll R\lambda\gamma n^{1/2} \ll \lambda\gamma n,$$

which implies  $\lambda\gamma \gg 1$ .

The density principle says now that  $f(E^-) \leq f(F^-)$  (note that  $f$  is negative on  $F^-$  and  $E^-$ ), and  $f(F^0)$  can be estimated using (3.3):

$$\left|f(F^-) - \int_{F^-} f(z)dz\right| \ll R^2 \ll n,$$

because  $\max\{|f(x)| : x \in F^-\} \leq R$ . Now  $f(z)$  is negative on  $F^-$ . It is easy to check that  $\lambda^2\gamma^2 nR \ll -\int_{F^-} f(z)dz \ll \lambda^2\gamma^2 nR$ . So we have

$$-f(F^-) \geq \int_{F^-} -f(z)dz + O(n) \gg \int_{F^-} -f(z)dz \gg \lambda^2\gamma^2 nR \gg n^{3/2}$$

This shows that (6.1) indeed holds if  $\gamma > 0$  is chosen small enough because  $0 < f(\mathbf{Z}^2 \cap F^+) \ll \gamma^2 n^{3/2}$  and  $-f(\mathbf{Z}^2 \cap F^-) \gg n^{3/2}$ .  $\square$

**Corollary 6.4** *There are positive numbers  $r$  and  $R$  (depending only on  $D$ ) such that for all  $n \geq 3$*

$$rB \subset (\text{Area } C_n)^{-1/2} C_n \subset RB.$$

## 7 Almost all primitive points of $C_n$ are in $E_n$

We begin by stating a geometric lemma which is about a special kind of approximation. The technical proof is postponed to Appendix 2.

**Lemma 7.1** *Assume  $K \in \mathcal{K}$  is a convex polygon with  $rB \subset K \subset RB$ . Then for every  $\delta \in (0, 0.02(r/R)^2]$  there are vertices  $v_1, \dots, v_m$  of  $K$  such that with  $Q = \text{conv}\{v_1, \dots, v_m\}$  the following holds:*

- $Q \subset K \subset (1 + 4R^2 r^{-2} \delta)Q$ ,
- for all  $i$ , the angle  $\angle v_i 0 v_{i+1}$  is at least  $\delta$ .

**Lemma 7.2** *For every  $\varepsilon > 0$  there is  $n_0 = n_0(\varepsilon, D)$  such that for all  $n \geq n_0$ ,  $(1 - \varepsilon)C_n \cap \mathbf{P} \subset E_n$ .*

**Proof.** Let  $r_n$ , resp.  $R_n$  be the maximal, minimal radius such that  $r_n B \subset C_n \subset R_n B$ . It follows from Claims 6.1 and 6.3 that  $R_n/r_n \leq c$  with a suitable positive constant depending only on  $D$ . Thus Lemma 7.1 can be applied with  $K = C_n$  and  $\delta = \varepsilon/(8c^2)$  (if  $\varepsilon \leq 0.02/8$  which we can clearly assume). We get a polygon  $Q = \text{conv}\{v_1, \dots, v_m\}$  satisfying  $C_n \subset (1 + \varepsilon/2)Q$ .

Assume, contrary to the statement of the lemma, that there is an  $x \in (1 - \varepsilon)C_n \cap \mathbf{P} \setminus E_n$ . One of the cones  $\text{pos}\{v_i, v_{i+1}\}$  contains  $x$ , say in the cone  $W := \text{pos}\{v_1, v_2\}$ . Define  $\Delta = \text{conv}\{0, v_1, v_2\}$ . Thus  $\Delta \subset C_n \cap W \subset (1 + \varepsilon/2)\Delta$ . As  $x \in (1 - \varepsilon)C_n \cap W$ ,  $v_1 + v_2 - x \in W \setminus (1 + \varepsilon)\Delta$ . The triangle  $\Delta^* = ((v_1 + v_2 - x) - W) \setminus (1 + \varepsilon/2)\Delta$  is disjoint from  $C_n$ . We claim that it contains a primitive point  $y$ . This will finish the proof since then  $x, y, x + y$  all lie in the parallelogram with vertices  $0, v_1, v_2, v_1 + v_2$  contradicting Lemma 3.3.

We prove the claim by using (3.2):  $\text{Area } \Delta^* \gg \varepsilon^3 n$  because its angle at  $v_1 + v_2 - x$  is at least  $\delta$ , and the neighbouring sides are of length at least  $\varepsilon|v_1|/2$  and  $\varepsilon|v_2|/2$  and  $|v_1|, |v_2| \gg n^{1/2}$ . Further, its perimeter is at most  $|v_1| + |v_2| + |v_1 - v_2| \ll n^{1/2}$ . Thus

$$\left| |\Delta^* \cap \mathbf{P}| - \frac{6}{\pi^2} \text{Area } \Delta^* \right| \ll (\log n)n^{1/2}.$$

Here  $\frac{6}{\pi^2} \text{Area } \Delta^*$  is of order  $\varepsilon^3 n$  and the error term is of order  $(\log n)n^{1/2}$ . Since  $\varepsilon$  fixed,  $\Delta^*$  contains a primitive vector if  $n$  is large enough.  $\square$

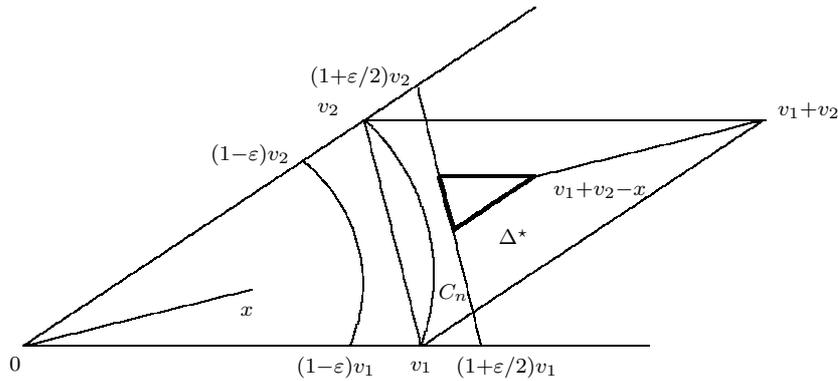


Figure 2. The proof of Lemma 7.2

## 8 Proof of Theorem 2.2

In this section we prove Theorem 2.2 apart from the uniqueness of  $C$  and  $r$  which will be shown in the next section.

The Blaschke selection theorem and Corollary 6.4 imply that every subsequence of  $(\text{Area } C_n)^{-1/2} C_n$  contains a convergent (in Hausdorff metric) subsequence. Corollary 4.3 guarantees then the existence of positive integers  $n_1 < n_2 < \dots$  such that  $\lim n_k^{-3/2} L_{n_k} = \alpha$  and  $\lim \text{dist}((\text{Area } C_{n_k})^{-1/2} C_{n_k}, C) = 0$  for some convex body  $C \in \mathcal{K}$ . Define  $\lambda_k = \sqrt{\text{Area } C_{n_k}}$  and set, for simpler writing,  $C^k = \lambda_k^{-1} C_{n_k}$ . It is evident that  $rB \subset C \subset RB$ , showing that, for every  $\delta > 0$ ,  $(1-\delta)C \subset C^k \subset (1+\delta)C$  for all large enough  $k$ . Since  $n_k = \frac{6}{\pi^2} \text{Area } C_{n_k} (1+o(1))$ ,  $\lambda_k = \frac{\pi}{\sqrt{6}} \sqrt{n_k} (1+o(1))$ .

It follows immediately that  $\text{Area } C = 1$ . We show next that  $\int_C z dz = 0$ . For this it suffices to prove that  $\int_C f(z) dz = 0$  in the case when  $f$  is the linear function  $f(z) = x$  and  $f(z) = y$  where  $z = (x, y)$ . Choose  $\varepsilon > 0$  and then, using Lemma 7.1,  $k_0$  so large that, for  $k > k_0$ ,

$$(1 - \varepsilon/2)C_{n_k} \cap \mathbf{P} \subset E_{n_k} \subset C_{n_k} \cap \mathbf{P}.$$

It follows now that there is a  $k_1$  so that for all  $k > k_1$

$$(1 - \varepsilon)\lambda_k C \cap \mathbf{P} \subset E_{n_k} \subset (1 + \varepsilon)\lambda_k C \cap \mathbf{P}. \quad (8.2)$$

Using the notation  $f(X) = \sum_{z \in X} f(z)$  when  $X \subset \mathbf{R}^2$  is finite, we have  $f(E_{n_k}) = 0$ . Next,

$$\begin{aligned} |f(\mathbf{P} \cap \lambda_k C)| &= |f(\mathbf{P} \cap \lambda_k C) - f(E_{n_k})| \\ &\leq |f(\mathbf{P} \cap [(1 + \varepsilon)\lambda_k C \setminus (1 - \varepsilon)\lambda_k C])| \\ &\ll \varepsilon \lambda_k \max\{f(z) : z \in \lambda_k C\} \ll \varepsilon n_k. \end{aligned}$$

On the other hand, by (3.8),

$$|f(\mathbf{P} \cap \lambda_k C)| = \frac{6}{\pi^2} \lambda_k^3 \int_C f(z) dz \left(1 + O(\lambda_k^{-1} \log \lambda_k)\right)$$

as one can check easily. So if  $\int_C f(z) dz \neq 0$ , then  $f(\mathbf{P} \cap \lambda_k C)$  is of order  $n_k^{3/2}$ . But as we have just shown,  $|f(\mathbf{P} \cap \lambda_k C)| \ll \varepsilon n_k$ . So indeed,  $\int_C f(z) dz = 0$ , or, in other words,  $g(C) = 0$ .

An almost identical proof, this time with the 1-homogeneous function  $f(z) = \|z\|$  gives

$$\frac{\pi}{\sqrt{6}} \int_C \|x\| dx = \alpha(D).$$

We only give a sketch: Equation (8.2) shows that

$$\left| \sum_{z \in \mathbf{P} \cap \lambda_k C} \|z\| - \sum_{z \in \mathbf{P} \cap E_{n_k}} \|z\| \right| \ll \varepsilon n_k.$$

Here  $\sum_{z \in \mathbf{P} \cap E_{n_k}} \|z\| = L_{n_k}$  and so  $\lim n_k^{-3/2} \sum_{z \in \mathbf{P} \cap \lambda_k C} \|z\| = \alpha(D)$ . The estimate (3.4) says now that

$$\left| \sum_{z \in \mathbf{P} \cap \lambda_k C} \|z\| - \frac{6}{\pi^2} \int_{\lambda_k C} \|x\| dx \right| \ll n_k \log n_k,$$

and  $\frac{\pi}{\sqrt{6}} \int_C \|x\| dx = \alpha(D)$  follows.

Lemma 5.1 applies now because  $g(C) = 0$  and  $\text{Area } C = 1$ . So there is a sequence  $Q_n \in \mathcal{P}_n$  with  $\lim n^{-3/2} \text{Per } Q_n = \alpha(D)$ . Then  $L_n \leq \text{Per } Q_n$  implies that  $\lim n^{-3/2} L_n = \alpha(D)$ .  $\square$

## 9 The variational problem

Next we turn to uniqueness. As first step we treat a special case.

**Lemma 9.1** *Let  $r_0$  be the radial function of  $D \in \mathcal{K}$  with  $g(D) = 0$ . Then  $r_0$  is the unique solution to  $VP(r_0)$ .*

**Proof.** We consider the variational problem which ignores the constraints about the center of gravity :

$$\begin{aligned} & \text{minimize } \int_0^{2\pi} r^3(t)/r_0(t) dt \\ & \text{subject to } \int_0^{2\pi} r^2(t) dt = 2 \end{aligned}$$

From Hölder's inequality :

$$\int_0^{2\pi} r^2 \leq \left( \int_0^{2\pi} \frac{r^3}{r_0} \right)^{2/3} \left( \int_0^{2\pi} r_0^2 \right)^{1/3}$$

which is an equality if and only if  $r$  and  $r_0$  are proportional. In our case  $\int_0^{2\pi} r^2 = \int_0^{2\pi} r_0^2 = 2$  and so  $r = r_0$ .  $\square$

We now use the previous lemma to treat the general case:

**Lemma 9.2** *There exists a unique solution  $r \in \mathcal{F}$  to problem  $VP(r_0)$ . This solution is equal to*

$$r = \left( \frac{a}{r_0} + b \cos t + c \sin t \right)^{-1}$$

where  $a > 0$ ,  $b, c$  are the unique real numbers which make the function  $r$  satisfy the three constraints of  $VP(r_0)$ .

**Proof.** We prove in Appendix 3 that every optimal solution  $r \in \mathcal{F}^c$  to  $VP(r_0)$  is of the form  $r(t) = (\frac{a}{r_0} + b \cos t + c \sin t)^{-1}$  with suitable constants  $a, b, c \in \mathbb{R}$ . We have shown that the radial function,  $r(t)$ , of  $C$  from Theorem 2.2 is an optimal solution to  $VP(r_0)$ . As  $C$  is convex,  $r(t)$  is equal to  $(\frac{a}{r_0} + b \cos t + c \sin t)^{-1}$ . According to the previous Lemma, the unique solution to the variational problem  $VP(r)$  is  $r$ .

Consider now another optimal solution,  $r^*$ , to  $VP(r_0)$ . It is clear that  $r^*$  is a feasible solution to  $VP(r)$  and that

$$\int_0^{2\pi} \frac{r^{*3}}{r_0} = \int_0^{2\pi} \frac{r^3}{r_0}.$$

Further,

$$a \int_0^{2\pi} \frac{r^{*3}}{r_0} = \int_0^{2\pi} r^{*3} \left( \frac{a}{r_0} + b \cos t + c \sin t \right) = \int_0^{2\pi} \frac{r^{*3}}{r},$$

and, in the same way,

$$a \int_0^{2\pi} \frac{r^3}{r_0} = \int_0^{2\pi} r^3 \left( \frac{a}{r_0} + b \cos t + c \sin t \right) = \int_0^{2\pi} \frac{r^3}{r}.$$

So  $r^*$ , too, is an optimal solution to  $VP(r)$ . By the Lemma,  $r = r^*$ , and  $a > 0$  follows as well.  $\square$

**Remark:** After reading this proof, one easily understands that  $r(t)$  is the unique solution to the variational problem in a class of functions larger than  $\mathcal{F}$ .

## 10 Proof of Theorem 2.3

This is fairly simple once we know that  $C$  is unique. Let  $u(t) = (\cos t, \sin t)$  be the unit vector in direction  $t \in [0, 2\pi]$ . When a minimizer  $P_n$  is translated as Theorem 2.3 specifies, the sum of the edges of  $P_n$  having direction between  $u(0)$  and  $u(t)$  is very close to the sum of the primitive vectors having direction between  $u(0)$  and  $u(t)$  in  $C_n$ . The latter, divided by  $n^{3/2}$  is very close to  $P(t) = \int_{C(t)} z dz$  where  $C(t)$  is the set of vectors in  $C$  with direction between  $u(0)$  and  $u(t)$ . The curve  $P(t)$  is closed (because  $g(C) = 0$ ) and convex (this has been shown in [2]), so it is the boundary of a convex set  $P$ . The simple and straightforward checking of

$$\lim \text{dist} (n^{-3/2} P_n, P) = 0$$

is left to the reader. We remark that the convexity of  $P(t)$  follows also from the fact that the boundary of  $P_n$ , after suitable rescaling, tends to  $P(t)$ .  $\square$

The same construction  $C \rightarrow P$  with  $P(t) = \int_{C(t)} z dz$  is used, with a similar purpose, in [2]. Further properties of the construction are also established there.

## 11 An example

We concentrate now on the cases when the solution is constant which correspond to the case when the limit shape of the polygon is a circle.

**Lemma 11.1** *The solution is constant if and only if  $1/r_0$  is of the form  $a + b \cos \theta + c \sin \theta$ , or, in other words, when  $r_0$  is the radial function of an ellipse having its focus point at the origin.*

**Proof.** Suppose the solution is constant, the form of  $r_0$  is then directly derived from Lemma 9.2. Conversely, if  $\frac{1}{r_0}$  is of the form  $a + b \cos \theta + c \sin \theta$ , the solution is then also of the form  $\frac{1}{r} = a' + b' \cos \theta + c' \sin \theta$ . This says that it is the radial function of an ellipse having its focus point at the origin. We conclude by observing that the only ellipses whose centre of gravity is at the same time their focus point, are circles.  $\square$

## 12 Appendix 1

**Lemma 12.1** *Let  $M_n(w)$  be the sum of the lengths of the  $n$  shortest (in Euclidean norm) distinct vectors in  $\mathbf{Z}^2$  lying in a strip of width  $w$ , centred at the origin. Suppose  $\gamma \in (0, 1/2]$ , then  $w \leq \gamma n^{1/2}$  implies  $M_n(w) \gg n^{3/2}/\gamma$ .*

**Proof.** It is clear that this set of vectors is just the set of lattice points contained in  $A := dB \cap T$  where  $T$  is a strip of width  $w$ , centred at the origin, and  $d$  is a suitable radius making  $A \cap \mathbf{Z}^2$  have exactly  $n$  elements (ties broken arbitrarily). Let  $\varphi$  denote the angle that the strip  $T$  makes with the  $x$ -axis of  $\mathbf{R}^2$ . We may assume by symmetry that  $\varphi \in [0, \pi/4]$ .

Observe first that  $d \geq \sqrt{n}/2$  since otherwise the disk  $dB$  would contain fewer than  $n$  lattice points. Let  $Q(z)$  denote the unit square centred at  $z \in \mathbf{R}^2$  and let  $\ell_k$  be the line with equation  $x = k$  ( $k$  is an integer). Clearly,  $\ell_k$  intersects  $S$  in a segment of length  $w \cos \varphi$ , and so  $\ell_k \cap \mathbf{Z}^2$  contains at least  $\lfloor w/\cos \varphi \rfloor$  and at most  $\lfloor w/\cos \varphi \rfloor + 1$  lattice points from  $S$ .

Assume first that  $w/\cos \varphi \geq 1$ . As is easy to see, Area  $A \cap Q(z)$  is at least  $1/4$  for  $z \in A \cap \mathbf{Z}^2$ . Hence, Area  $A \geq n/4$ . Since Area  $A < 2dw$ ,  $d > n/(4w)$  follows.

For simpler notation write  $u = (d \cos \varphi)/2$ . For the lines  $\ell_k$  with  $k \in [u, 2u - w/2]$ ,  $\ell_k \cap A$  contains at least  $\lfloor w/\cos \varphi \rfloor$  lattice points. Since  $w < u$ , there are at least  $\lfloor 2u - w/2 \rfloor - \lfloor u \rfloor \gg u$  such lines. All of them have distance at least  $(d - w)/2 \gg d$  from the origin. Consequently, using the bounds  $w \leq \gamma n^{1/2}$  and  $d \geq n^{1/2}/2$  generously,

$$M_n(w) \gg d \left\lfloor \frac{w}{\cos \varphi} \right\rfloor u \gg d^2 w \geq \left( \frac{n}{4w} \right)^2 w \gg \frac{1}{\gamma} n^{3/2}.$$

Assume next that  $w/\cos\varphi < 1$ . There are at most six  $z \in A \cap \mathbf{Z}^2$  such that  $Q(z)$  intersects the boundary of  $dB$ . For the other  $z \in A \cap \mathbf{Z}^2$ ,  $Q(z)$  intersects the boundary of  $A$  in one or two line segments, whose total length is between  $1/\cos\varphi$  and  $2/\cos\varphi$ . For distinct lattice points in  $A \cap \mathbf{Z}^2$  the corresponding segments do not overlap. This implies that

$$\frac{n-6}{\cos\varphi} \leq 4d \leq \frac{2(n-6)}{\cos\varphi}.$$

Each line  $\ell_k$  with  $|k| \leq 2u/3$  contains at most one lattice point from  $A$ . The remaining points from  $A \cap \mathbf{Z}^2$ , and there are at least  $n - 2\lfloor 2u/3 \rfloor - 1$  of them, are at distance  $\frac{d}{3} - 1$  from the origin. Hence, we see

$$M_n(w) \geq \left(\frac{d}{3} - 1\right) \left(n - \lfloor 2\frac{d\cos\varphi}{3} \rfloor - 1\right) \gg n^2.$$

□

## 13 Appendix 2

We start the proof of Lemma 7.1 with the following Claim.

**Claim 13.1** *Suppose  $a, b, c, d$  are vertices of  $K$  (in anticlockwise order),  $[a, b]$  and  $[c, d]$  are edges of  $K$ , and  $\angle b0c < 3\delta$ . Let  $x$  be the intersection point of the lines through  $a, b$  and  $c, d$ , and let  $y$  be the intersection point of the lines through  $0, x$  and  $a, c$ . Then  $|x - y| \leq 4\delta(R/r)^2|y|$ .*

**Proof.** The condition  $rB \subset K \subset RB$  implies that  $\beta = \angle 0xb = \angle 0ba - \angle xba > \arcsin r/R - 3\delta$  since

$$\sin \angle 0ba = \frac{d(0, \ell_{a,b})}{|b|}$$

( $\ell_{a,b}$  being the line through  $a$  and  $b$ )  $|b| < R$ ,  $d(0, \ell_{a,b}) > r$  by assumption, so that  $\sin \angle 0ba > r/R$ , see Figure 3.

Further  $\angle xyc = \angle 0xa - \angle x0b > \beta$ . The sine theorem in the triangle  $x, y, c$  shows that

$$\frac{|x - y|}{|x - c|} = \frac{\sin \angle cxy}{\sin \angle cyx},$$

and similarly, the sine theorem in the triangle  $x, 0, c$  shows that

$$\frac{|x - c|}{|x|} = \frac{\sin \angle c0x}{\sin \angle 0cx}.$$

Multiplying them gives

$$\frac{|x - y|}{|x|} = \frac{\sin \angle cxy \sin \angle c0x}{\sin \angle cyx \sin \angle 0cx} < \frac{\sin 3\delta}{(r/R) \sin \beta}.$$

Next, since  $|y| = |x| - |x - y|$ , we have

$$\frac{|x|}{|x| - |x - y|} = \frac{1}{1 - \frac{|x-y|}{|x|}} < \frac{1}{1 - \frac{\sin 3\delta}{(r/R)\sin\beta}}$$

We use this inequality next in the form

$$\frac{|x - y|}{|y|} < \frac{\sin 3\delta}{(r/R)\sin\beta} \cdot \frac{|x|}{|x| - |x - y|} < \frac{\sin 3\delta}{(r/R)\sin\beta - \sin 3\delta} < 4\delta \left(\frac{R}{r}\right)^2,$$

where we only have to check the validity of the last inequality. This is a matter of direct computation using that  $\sin\beta > \sin(\arcsin(r/R) - 3\delta) > (r/R)\cos 3\delta - \sin 3\delta$  and the assumption that  $\delta < 0.02(r/R)^2$  implying, in particular, that  $\delta < 0.02$ . What is to be checked now is that

$$\tan 3\delta \left[ 1 + 4\delta \left(\frac{R}{r}\right)^2 \left(\frac{r}{R} + 1\right) \right] \leq 4\delta.$$

Here  $\delta(R/r)^2 < 0.02$  and so the expression in the square bracket is at most 1.16 and the inequality follows. We omit the details.  $\square$

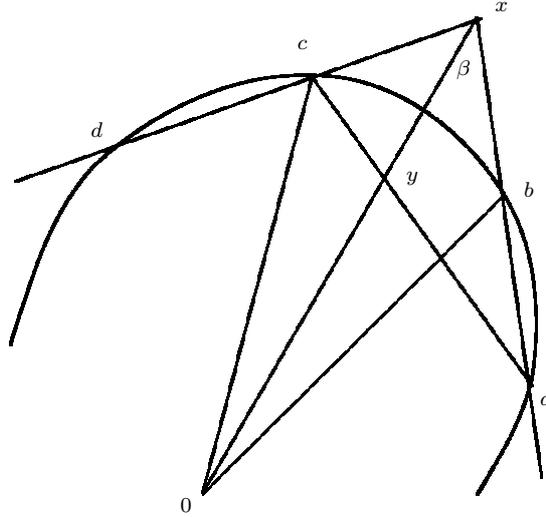


Figure 3. The proof of Claim 13.1

The **Proof** of Lemma 7.1 is an algorithm that constructs the vertex set  $V$  of  $Q$ . We start with  $V = \emptyset$ . We call the edge  $[a, b]$  of  $K$  *special* if  $\angle a0b \geq \delta$ . Let  $W$  be a cone with apex at 0 and angle  $\delta$ . It follows that if  $W$  is disjoint from all special edges, then it contains a vertex of  $K$ .

**Case 1.** Let  $[a_1, b_1], [a_2, b_2], \dots, [a_k, b_k]$  be consecutive special edges in anticlockwise order so that  $\angle b_i 0 a_{i+1} < 3\delta$  for all  $i = 1, \dots, k - 1$  (or up to  $k$  if  $\angle b_k 0 a_1 < 3\delta$ ). We call this a maximal chain of consecutive special edges if there is no special edge  $[a, b]$  with  $\angle b 0 a_1 < 3\delta$  or  $\angle b_k 0 a < 3\delta$ .

For such a maximal chain we put the vertices  $a_1, \dots, a_k, b_k$  (or  $a_1, \dots, a_k$  if  $\angle b_k 0 a_1 < 3\delta$ ) into  $V$ , and we do so for all such maximal chains.

**Case 2.** Let  $[a_1, b_1]$  and  $[a_2, b_2]$  be consecutive special edges with vertices  $a_1, b_1, a_2, b_2$  in anticlockwise order so that  $\gamma := \angle b_1 0 a_2 \geq 3\delta$ . Then we choose  $\delta' \in [\delta, 3\delta]$  so that  $\gamma/\delta'$  is an odd integer, say  $2h + 1$ . This is always possible since there is an odd integer between  $\gamma/(3\delta)$  and  $\gamma/\delta$  because their difference is  $\gamma/\delta - \gamma/(3\delta) = 2\gamma/(3\delta) \geq 2$ .

Subdivide now the cone  $\text{pos}\{b_1, a_2\}$  into  $2h + 1$  subcones, each of angle  $\delta'$  and pick a vertex  $u_1, \dots, u_h$  from every second subcone. Finally, put  $b_1, u_1, \dots, u_h, a_2$  into  $V$ .

If there are only two special edges  $[a_1, b_1]$  and  $[a_2, b_2]$ , then one has to do the same construction between edges  $[a_2, b_2]$  and  $[a_1, b_1]$  as well. If there is only one special edge, then the construction is carried out from  $b_1$  to  $a_1$  as if one had two special edges  $[a_1, b_1]$  and  $[b_1, a_1]$ .

Finally, if there are no special edges, then we chose a  $\delta' \in [\delta, 2\delta]$  so that  $2\pi/\delta'$  is an even integer,  $2h$ , say. This is evidently possible. Subdivide the plane into cones of angle  $\delta'$  (with apex at 0) and choose a vertex  $u_1, \dots, u_h$  from every second cone, and set  $V = \{u_1, \dots, u_h\}$ .

The algorithm is finished. By construction  $\angle v_i 0 v_{i+1} \geq \delta$ : for the angle at 0. Finally we check condition  $K \subset (1 + 4\delta(R/r)^2)Q$ . Let  $v_i, v_{i+1}, v_{i+2}, v_{i+3}$  be four consecutive vertices of  $Q$  in anticlockwise order. Rename these points as  $a, b, c, d$  as in the Claim. Then  $K \cap \text{pos}(b, c) \setminus Q$  is contained in the triangle  $b, c, x$  from the Claim. Now  $y \in Q$  because  $y$  lies on the segment  $[a, c]$ , and so  $x \in (1 + 4\delta(R/r)^2)Q$  according to the Claim. So the triangle  $b, c, x$  is contained  $(1 + 4\delta(R/r)^2)Q$ .  $\square$

## 14 Appendix 3

It happens that standard theorems of the Calculus of Variations (see for instance [6]) are stated in a  $C^1$  setting, and suppose also that the function  $r_0$  involved in the problem is  $C^1$ . Since these conditions are not satisfied in our problem, we have to elaborate the following statement:

**Lemma 14.1** *All the solutions  $r \in \mathcal{F}^c$  satisfying problem  $VP(r_0)$  are of the form*

$$r = \left( \frac{a}{r_0} + b \cos t + c \sin t \right)^{-1}$$

where  $a, b, c$  are real numbers which make the function  $r$  satisfy the three constraints of problem (1.2).

**Proof.** Consider  $r$  an optimal solution in  $\mathcal{F}^c$ . Let  $h$  be a function on  $[0, 2\pi]$  such that the perturbed function  $r_\varepsilon := r + \varepsilon h$  remains in  $\mathcal{F}^c$  for  $\varepsilon$  in a neighbourhood of 0 (notice that all the twice differentiable functions are convenient). This

perturbation won't be feasible in general. We want to modify it in order to make it feasible. That is what we do in the two first steps.

**Step 1.** We translate the set defined by the function  $r_\varepsilon$  in order to get a centred set defined by a new radial function  $\tilde{r}_\varepsilon$  we evaluate up to some  $o(\varepsilon)$ .

In the following, the notation  $o(\varepsilon)$  stands for some family of functions, which may be constant, indexed by  $\varepsilon$ , such that both  $\frac{o(\varepsilon)}{\varepsilon}$  converges to 0 as  $\varepsilon$  goes to 0, and  $\frac{o(\varepsilon)}{\varepsilon}$  is dominated.

The coordinates of the centre of gravity of the set defined by  $r_\varepsilon$  are

$$\left( \int_0^{2\pi} r_\varepsilon^3 \cos t dt, \int_0^{2\pi} r_\varepsilon^3 \sin t dt \right) = \varepsilon \left( \int_0^{2\pi} 3r^2 h \cos t dt, \int_0^{2\pi} 3r^2 h \sin t dt \right) + o(\varepsilon)$$

Recall that  $u(t) = (\cos t, \sin t)$ . Define the numbers  $r_h, \theta_h$  by setting  $r_h u(\theta_h) := (\int_0^{2\pi} 3r^2 h \cos t dt, \int_0^{2\pi} 3r^2 h \sin t dt)$ .

For a given  $\theta$ , the polar coordinates of  $r_\varepsilon u(\theta) - \varepsilon r_h u(\theta_h) + o(\varepsilon)$  are given by

$$\tilde{\theta}(\theta) := \theta - \varepsilon r_h \frac{\sin(\theta_h - \theta)}{r(\theta)} + o(\varepsilon)$$

$$\tilde{r}(\theta) = r_\varepsilon(\theta) - \varepsilon r_h \cos(\theta_h - \theta) + o(\varepsilon)$$

Hence,  $\tilde{r}$  can be expressed as a function of  $\tilde{\theta}$  as follows:

$$\tilde{r}(\tilde{\theta}) = r_\varepsilon \left( \tilde{\theta} + \varepsilon r_h \frac{\sin(\theta_h - \tilde{\theta})}{r(\theta)} + o(\varepsilon) \right) - \varepsilon r_h \cos(\theta_h - \tilde{\theta} + o(\varepsilon)) + o(\varepsilon)$$

Using now the almost everywhere differentiability of  $r$  (and therefore of  $r_\varepsilon$ ) which is inherited from convexity, we obtain that, almost everywhere,

$$\tilde{r}(\theta) = r(\theta) + \varepsilon \left[ h(\theta) + r_h \sin(\theta_h - \theta) \frac{r'}{r}(\theta) - r_h \cos(\theta_h - \theta) \right] + o(\varepsilon)$$

Note that the domination of  $\frac{o(\varepsilon)}{\varepsilon}$  in the last step is due to the fact that the left and right derivatives of  $r$  are bounded on  $[0, 2\pi]$ .

**Step 2.** We obtain a completely feasible function  $r^f$ , by normalizing  $\tilde{r}$  by the area of the set defined by  $\tilde{r}$ , which is the same as the area of the set defined by  $r_\varepsilon$ , since the two sets are obtained one from the other by a translation.

Define,

$$r^f(\theta) = \frac{\tilde{r}(\theta)}{(\frac{1}{2} \int_0^{2\pi} r_\varepsilon^2)^{1/2}} = \frac{\tilde{r}(\theta)}{(\frac{1}{2} \int_0^{2\pi} (r + \varepsilon h)^2)^{1/2}} = \tilde{r}(\theta) \left( 1 - \frac{\varepsilon}{2} \left( \int_0^{2\pi} r h \right) + o(\varepsilon) \right)$$

The function  $r^f(\theta)$  can be written as  $r(\theta)$  times the function

$$1 + \varepsilon \left[ \frac{h(\theta)}{r(\theta)} + r_h \sin(\theta_h - \theta) \frac{r'}{r^2}(\theta) - r_h \frac{\cos(\theta_h - \theta)}{r(\theta)} - \frac{1}{2} \left( \int_0^{2\pi} r h \right) \right] + o(\varepsilon)$$

**Step 3.** Now, we test the optimality of the function  $r$  by considering the functional applied to the feasible perturbation  $r^f$  and writing the integral  $\int_0^{2\pi} \frac{(r^f)^3}{r_0}$  as  $\int_0^{2\pi} \frac{r^3}{r_0}$  plus

$$3\varepsilon \int_0^{2\pi} \frac{r^3}{r_0} \left[ \frac{h(\theta)}{r(\theta)}(\theta) + r_h \sin(\theta_h - \theta) \frac{r'(\theta)}{r^2(\theta)}(\theta) - r_h \frac{\cos(\theta_h - \theta)}{r(\theta)} - \frac{1}{2} \left( \int_0^{2\pi} rh \right) \right] + o(\varepsilon)$$

When developing the sine and cosine in the above bracket and performing the integration on  $\theta$  (and keeping in mind that  $r_h$  and  $\theta_h$  are constants that don't depend on  $\theta$ !) we deduce that, if  $r$  is optimal, there exist real constants  $A$ ,  $B$  and  $C$  such that for all twice differentiable function  $h$ ,

$$\int_0^{2\pi} \frac{r^2 h}{r_0} + Ar_h \sin \theta_h + Br_h \cos \theta_h + C \int_0^{2\pi} rh = 0$$

Recall that  $(r_h \cos \theta_h, r_h \sin \theta_h) = (\int_0^{2\pi} 3r^2 h \cos t dt, \int_0^{2\pi} 3r^2 h \sin t dt)$ .

Therefore, for all twice differentiable functions  $h$

$$\int_0^{2\pi} hr^2 \left( \frac{1}{r_0} + 3A \cos \theta + 3B \sin \theta + \frac{C}{r} \right) = 0$$

which implies that the bracket inside the integral is 0.  $\square$

## 15 Acknowledgements

The first author was partially supported by Hungarian National Science Foundation Grants T 032452 and T 60427, and also by the Discrete and Convex Geometry project, MTKD-CT-2005-014333, of the European Community. The second author was partially supported by ANR grant MEMEMO.

## References

- [1] Bárány, I., Tokushige, N., The minimum area convex lattice  $n$ -gon, *Combinatorica*, **24** (2004), 171–185.
- [2] Bárány, I., Prodromou, M., On maximal convex lattice polygons inscribed in a plane convex set, *Israel J. Math.*, **154** (2006), 337–360.
- [3] Hardy, G. H., Wright, E. M., An introduction to the theory of numbers, Clarendon Press, Oxford, 1979.
- [4] Jarník, V., Über Gitterpunkte in konvexen Kurven. *Math. Zeit.*, **24** (1926), 500–518.

- [5] Prodromou, M., Limit shape of convex lattice polygons with minimal perimeter, *Discrete Math.*, **300** (2005), 139–151.
- [6] Sagan, H., Introduction to the calculus of variations. Corrected reprint of the 1969 original. Dover Publications, Inc., New York, 1992. xviii+449 pp.

Imre Bárány  
Rényi Institute of Mathematics,  
Hungarian Academy of Sciences  
H-1364 Budapest Pf. 127 Hungary  
barany@renyi.hu  
and  
Department of Mathematics  
University College London  
Gower Street, London, WC1E 6BT, UK

Nathanaël Enriquez  
Laboratoire Modal'X  
Université Paris-Ouest  
200 Avenue de la République, 92001 Nanterre, France  
nenriquez@u-paris10.fr  
and  
Laboratoire de Probabilité et Modèles Aléatoires  
Université Paris 6  
4 Place Jussieu, 75005 Paris, France