Self-stabilizing processes: uniqueness problem for stationary measures and convergence rate in the small noise limit

Samuel Herrmann, Julian Tugaut

Self-stabilizing processes: uniqueness problem for stationary measures and convergence rate in the small noise limit

S. Herrmann and J. Tugaut
Institut of Mathématiques Elie Cartan - UMR 7502
Nancy-Université, CNRS, INRIA
B.P. 239, 54506 Vandoeuvre-lès-Nancy Cedex, France

November 21, 2009

Abstract

In the context of self-stabilizing processes, that is processes attracted by their own law, leaving in some potential landscape, we investigate different properties of the invariant measures. The interaction between the process and its law leads to nonlinear stochastic differential equations. In [7], the authors proved that, for linear interaction and under suitable conditions, there exist some unique symmetric limit measure associated to the set of invariant measures in the small noise limit. The aim of this study is essentially to point out that this statement leads to the existence, as the noise intensity is small, of one unique symmetric invariant measure for the self-stabilizing process. Informations about the asymmetric measures shall be presented too. The main key consists in estimating the convergence rate for sequences of stationary measures using generalized Laplace's method approximations.

Key words and phrases: self-interacting diffusion; McKean-Vlasov equation, stationary measures; double well potential; perturbed dynamical system; Laplace's method; fixed point theorem; uniqueness problem.

2000 AMS subject classifications: primary 60G10; secondary: 60J60, 60H10, 41A60

1 Introduction

In the framework of nonlinear diffusions, self-stabilizing stochastic processes play a particular rule. Introduced by McKean [8] these processes attracted by their own law are solution of the so-called *McKean-Vlasov equation*:

$$dX_t = dW_t + b[X_t, u_t]\,dt, \quad X_0 = x \in \mathbb{R}, \tag{1.1}$$

where u_t is the law of X_t, $b[x,u] := \int_{\mathbb{R}} b(x,y)u(dy)$ for any probability measure u and $(W_t,\, t \geq 0)$ represents some one-dimensional Brownian motion. A solution of (1.1) is in fact a couple (X_t, u_t) such that, for any $t \geq 0$, u_t represents the distribution of the variable X_t. Such processes appear naturally in huge systems of particles in interaction by the so-called *propagation of chaos* phenomenon, see [11] for an introduction to this topic.

The common mathematical problems related to these self-stabilizing processes concern the existence and uniqueness of solutions to (1.1) and ergodicity properties like the existence and uniqueness of stationary measures, the convergence of the law of X_t to the invariant law as time elapses. A relative numerous literature, based on fixed point technics, free energy methods or logarithmic Sobolev inequalities, presents results concerning the existence and uniqueness of invariant measures and ergodic behavior. Each study deals with some particular family of interaction function b, let us present an incomplete selection of works: [1], [2], [3], [4], [9], [12], [13], [14]. In the situations described previously, the results are quite similar than those developed in the classical diffusion context even if the methods of proof are clearly different.

However the self-attraction structure of (1.1) can lead to surprising phenomena like non-uniqueness of invariant measures. The aim of this paper is namely to focus our attention to some of them. Let us introduce the process we are interesting in: the solution $(X_t,\, t \geq 0)$ of the following one-dimensional McKean-Vlasov equation:

$$dX_t = \sqrt{\varepsilon}dW_t - V'(X_t)dt - \int_{\mathbb{R}} F'(X_t - x)du_t^{\varepsilon}(x)dt, \tag{1.2}$$

where $u_t^{\varepsilon}(x)$ represents the distribution of X_t and ε is a small positive parameter. In other words the function b introduced above satisfies $b(x,y) := -V'(x) - F'(x-y)$: V is called the environment potential and F represents the interaction potential. The functions V and F are assumed to verify different conditions developed in Section 1.1 and related to [1] and [2]. Let us just note two principal properties: F is an even convex function with $F(0) = 0$ and $\lim_{x\to\infty} F(x) = +\infty$ and V is an even double-well potential whose global minima are reached for $x = -a$ and $x = a > 0$.

In some preceding paper [6], the authors pointed out, under some suitable conditions and for small noise intensity ε, that the nonlinearity of the dynamical system permits the existence of at least three invariant measures, one symmetric (due to the symmetry of F and V) and two so-called *outlying* measures which are concentrated around $-a$ or a, the bottoms of the double-well landscape V. Moreover, in the particular case of convex functions V'' and linear functions F', there exist exactly three invariant measures for ε small enough. The aim of this paper is to take the first steps in order to generalize this nice result to general interaction functions F. In particular, we shall prove three essential statements concerning the set of invariant measures whose first moments are uniformly bounded with respect to the parameter ε (the number of moments to consider will be specified in the following): if V'' and F'' are convex functions then, for ε small enough,

- There exists some unique invariant measure which converges towards δ_a in the small noise limit.

- There exists some unique invariant measure converging towards δ_{-a} in the small noise limit.

- There exists some unique *symmetric* stationary measure provided that $F''(0) \neq \sup_{x\in\mathbb{R}} -V''(x)$.

These results developed in Section 7 permit to present informations concerning the set of invariant measures for the self-stabilizing process (1.1). Indeed the authors proved in [7] that any symmetric measure converges to the discrete measure $\frac{1}{2}\,\delta_{-x_0} + \frac{1}{2}\,\delta_{x_0}$, as $\varepsilon \to 0$, where x_0 is the unique solution (see Theorem 5.4 in [7]) of the system

$$\begin{cases} V'(x_0) + \frac{1}{2}F'(2x_0) = 0, \\ V''(x_0) + \frac{1}{2}F''(0) + \frac{1}{2}F''(2x_0) \geq 0. \end{cases}$$

Let us just note that $x_0 = 0$ when $F''(0) \geq -\sup_{x\in\mathbb{R}} V''(x)$. Furthermore the existence of some family of asymmetric measures converging towards δ_a, respectively δ_{-a}, is presented in [7].

In other words, the statements proved in this paper suggest the following conjecture: under some conditions (convexity of F'' and V'' for instance), *for any $M > 0$ large enough, there exists $\varepsilon_0 > 0$ such that* (1.1) *admits exactly three invariant measures whose first moments are bounded by M for all $\varepsilon < \varepsilon_0$.*

The results announced above are proved using the convergence rate for sequences of invariant measures denoted by $(u_\varepsilon,\, \varepsilon > 0)$ and associated to some limit measure u_0. In fact, the convergence rate depends on the measure u_0 considered: the three different cases δ_{-a}, δ_a (Section 6) and $\frac{1}{2}\,\delta_{-x_0} + \frac{1}{2}\,\delta_{x_0}$ (Section 3 for $x_0 = 0$ and Section 4 for $x_0 > 0$) shall be analyzed. A particular convergence rate problem shall be analyzed in the borderline between the situations satisfying $x_0 > 0$ and those associated to $x_0 = 0$ (see Section 5). In all these situations, arguments based on Laplace's method type approximations permit to obtain some equivalence in the small noise limit of the following expression:

$$\mathcal{D}_\varepsilon := \langle f, u_\varepsilon \rangle - \langle f, u_0 \rangle \quad \text{where} \quad \langle f, u \rangle := \int_{\mathbb{R}} f(x) u(dx),$$

and f is a $\mathcal{C}^4(\mathbb{R})$-continuous function with polynomial growth.

The paper shall begin with the detailed assumptions concerning the function F and V of (1.2) and some preliminary key asymptotic results.

1.1 Main assumptions

We assume the following properties for the function V:

(V-1) *Regularity:* $V \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. $\mathcal{C}^\infty$ denotes the Banach space of infinitely bounded continuously differentiable function.

(V-2) *Symmetry:* V is an even function.

(V-3) V is a double-well potential. The equation $V'(x) = 0$ admits exactly three solutions : a, $-a$ and 0 with $a > 0$; $V''(a) > 0$ and $V''(0) < 0$. The bottoms of Wells are reached for $x = a$ and $x = -a$.

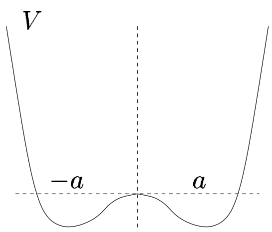

Figure 1: Potential V

(V-4) There exist two constants $C_4, C_2 > 0$ such that $\forall x \in \mathbb{R}$, $V(x) \geq C_4 x^4 - C_2 x^2$.

(V-5) $\lim_{x \to \pm\infty} V''(x) = +\infty$ and $\forall x \geq a$, $V''(x) > 0$.

(V-6) *Analyticity:* There exists an analytic function $\mathcal{V}$ such that $V(x) = \mathcal{V}(x)$ for all $x \in [-a; a]$.

(V-7) The growth of the potential V is at most polynomial: there exist $q \in \mathbb{N}^*$ and $C_q > 0$ such that $|V'(x)| \leq C_q \left(1 + x^{2q}\right)$.

(V-8) *Initialization:* $V(0) = 0$.

Typically, V is a double-well polynomial function. But our results can be applied to more general functions: regular functions with polynomial growth as $|x|$ becomes large. We introduce the parameter ϑ which plays some important role in the following:

$$\vartheta := \sup_{x \in \mathbb{R}} -V''(x). \tag{1.3}$$

Let us note that the simplest example (most famous in the literature) is $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ which bottoms are localized in -1 and 1 and with parameter $\vartheta = 1$. Let us now present the assumptions concerning the attraction function F.

(F-1) F is an even polynomial function with $F(0) = 0$. Indeed we consider some classical situation: the attraction between two points x and y only depends on the distance $F(x - y) = F(y - x)$.

(F-2) F is a convex function.

(F-3) F' is a convex function on $\mathbb{R}_+$ therefore for any $x \geq 0$ and $y \geq 0$ such that $x \geq y$ we get $F'(x) - F'(y) \geq F''(0)(x - y)$.

(F-4) The polynomial growth of the attraction function F is related to the growth condition (V-7): $|F'(x) - F'(y)| \leq C_q |x - y| (1 + |x|^{2q-2} + |y|^{2q-2})$.

Let us define the parameter $\alpha \geq 0$ which shall play some essential role in following:

$$F'(x) = \alpha x + F_0'(x) \quad \text{with } \alpha := F''(0) \geq 0. \tag{1.4}$$

2 Preliminary results

In [7], the authors considered the asymptotic behavior of invariant measures $(u_\varepsilon)_{\varepsilon>0}$ for the self-stabilizing process (1.2), as the noise intensity ε tends to 0. Some simple arguments permit to present the invariant measure in some particular exponential form: this idea was previously presented in [6]. Indeed, defining

$$W_\epsilon(x) := V(x) + F * u_\epsilon(x) - F * u_\epsilon(0), \tag{2.1}$$

the following expression holds

$$u_\varepsilon(x) = \frac{\exp[-\frac{2}{\varepsilon}W_\varepsilon(x)]}{\int_{\mathbb{R}} \exp[-\frac{2}{\varepsilon}W_\varepsilon(y)]dy}. \tag{2.2}$$

In order to study the asymptotic behavior of invariant measures, we shall use Laplace method type results and estimate $(W_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$. This section will concern these essential asymptotic analysis steps. Let us first introduce

$$\mu_k(\varepsilon) := \int_{\mathbb{R}} |x|^k u_\varepsilon(x)dx.$$

Under weak conditions on the moments, namely the family $\{\mu_{2n}(\varepsilon),\, \varepsilon > 0\}$ with $2n := \deg(F)$ is bounded (see Proposition 3.3 and Theorem 3.6 in [7]), there exist a sequence $(\varepsilon_k)_{k\geq 0}$ tending towards 0 and some regular function W_0 such that:

- $W_{\varepsilon_k}^{(j)}$ converges uniformly on each compact subset of $\mathbb{R}$ to $W_0^{(j)}$, for any $j \in \mathbb{N}$,
- the sequence $(u_{\varepsilon_k})_{k\geq 1}$ converges weakly towards some discrete probability measure given by $u_0 := \sum_{i=1}^r p_i\ \delta_{A_i}$ with $p_i > 0$ and $A_1, \ldots, A_r$ are locations of the global minimum of $W_0 := V + F * u_0 - F * u_0(0)$. Since F is an even function, we get $W_0(x) = V(x) + \sum_{j=1}^r p_j(F(x - A_j) - F(A_j))$.

We shall now use these previous results in order to obtain some estimation of the speed of convergence for the subsequence $(u_{\varepsilon_k})_{k\geq 1}$.
Attention ! In the following we shall drop the index ε_k just replaced by ε for notational simplicity but the reader has to keep in mind that both previous properties (uniform convergence of the pseudo-potential and weak onvergence of the measures) are satisfied.
First we present some special Laplace method: we define $\mathcal{A} := \{A_j\,;\, 1 \leq j \leq r\}$ the support of the limit measure u_0 and $\mathcal{B}$ the set of all locations for W_0's global minimum which don't belong to $\mathcal{A}$.

Lemma 2.1. *Let $(u_\epsilon)_{\epsilon>0}$ a sequence of stationary measures which converges weakly to u_0. We assume moreover that $\{\mu_{2n}(\varepsilon),\, \varepsilon > 0\}$ is bounded with $2n := \deg(F)$. Let $W_\epsilon := V + F * u_\epsilon - F * u_\epsilon(0)$ and $W_0 := V + F * u_0 - F * u_0(0)$. We denote by $A_1 < \cdots < A_r$ (respectively $B_1 < \cdots < B_s$ if $s > 0$) the elements of $\mathcal{A}$ (resp. $\mathcal{B}$).*

1. *Let us consider the set of intervals $(I_i)_{1\leq i\leq r+s}$ which correspond to the Voronoï cells centered in the elements of $\mathcal{D} := \mathcal{A}\cup\mathcal{B}$. If $W_0''(D) > 0$ for all $D\in\mathcal{D}$, W_ϵ reaches its global minimum at some unique location in I_i denoted by D_i^ϵ (also denoted by $A_\cdot^\epsilon$ or $B_\cdot^\epsilon$), $1\leq i\leq r+s$, which converges to $D_i\in\mathcal{D}$. Then, D_i^ϵ satisfies the following asymptotic development:*

$$D_i^\epsilon = D_i - \frac{W_\epsilon'(D_i)}{W_0''(D_i)} + o\left\{W_\epsilon'(D_i)\right\}, \quad 1\leq i\leq r+s. \tag{2.3}$$

2. *If u_ϵ is symmetric, if $u_0=\delta_0$, and if both W_0 and F'' are convex functions, then 0 is the unique location of the global minimum of W_ϵ. Furthermore, if F' is not a linear function, we get $W_\epsilon''(0)>0$.*
3. *If $W_0''(D) > 0$ for all $D\in\mathcal{D}$, then for any function $f\in\mathcal{C}^4(\mathbb{R},\mathbb{R})$ with polynomial growth, we have as $\varepsilon\to 0$:*

$$\begin{aligned}\int_{\mathbb{R}} f(t)e^{-\frac{2W_\epsilon(t)}{\epsilon}}dt &= \sum_{j=1}^{r}\sqrt{\frac{\pi\epsilon}{W_\epsilon''(A_j^\epsilon)}}\, e^{-\frac{2W_\epsilon(A_j^\epsilon)}{\epsilon}}\Big\{f(A_j^\epsilon)+\gamma_j(f)\epsilon+o(\epsilon)\Big\} \\ &+ \sum_{l=1}^{s}\sqrt{\frac{\pi\epsilon}{W_\epsilon''(B_l^\epsilon)}}\, e^{-\frac{2W_\epsilon(B_l^\epsilon)}{\epsilon}}\Big\{f(B_l)+o(1)\Big\}\end{aligned} \tag{2.4}$$

with

$$\gamma_j(f) := f(A_j)\left(\frac{5\ \mathcal{W}_{3,j}^2}{48\ \mathcal{W}_{2,j}^3} - \frac{\mathcal{W}_{4,j}}{16\ \mathcal{W}_{2,j}^2}\right) - f'(A_j)\frac{\mathcal{W}_{3,j}}{4\ \mathcal{W}_{2,j}^2} + \frac{f''(A_j)}{4\ \mathcal{W}_{2,j}}. \tag{2.5}$$

Here $\mathcal{W}_{k,j} := W_\epsilon^{(k)}(A_j^\epsilon)$.

Proof. **1.** $(W_\epsilon)_{\epsilon>0}$ satisfies the assumptions of Lemma A.4. Indeed, $W_\epsilon^{(j)}$ converges uniformly towards $W_0^{(j)}$, for $j\in\mathbb{N}$, on all compact subsets of $\mathbb{R}$, see Proposition 3.3 in [7]. Besides, since F is a even polynomial function of degree $2n$ and since the moments are bounded, $F*u_\varepsilon(x) - F*u_\varepsilon(0)\geq P(x)$ where P is a polynomial function independent of ϵ whose principal term is positive. Therefore, using (V-4), we obtain the following lower bound: $W_\epsilon(x)\geq C_4x^4 - C_2x^2 + P(x)$. The application of Lemma A.4 provides the existence of A_j^ϵ and B_l^ϵ. Let D be a location for the global minimum of W_0. If $W_0''(D)>0$, the uniform convergence of D^ϵ towards D and the convergence of W_ϵ towards W_0, on each compact set, imply $W_\epsilon''(D^\epsilon)>0$ for ϵ small enough. The asymptotic development (2.3) comes directly from Lemma A.4.
2. If $u_0=\delta_0$ then Theorem 3.6 in [7] implies that 0 is one global minimum of W_0 and by the way $W_0(x)\geq W_0(0)=0$ for all $x\in\mathbb{R}$. Furthermore, since F'' is a convex function and u_ε is symmetric and absolutely continuous with respect to the Lebesgue measure, see [6], we obtain the following lower bound

$$W_\epsilon''(x) - W_0''(x) = \int_{\mathbb{R}_+}\Big(F''(x-z)+F''(x+z)-2F''(x)\Big)u_\epsilon(z)dz \geq 0.$$

Due to the convexity of W_0, we obtain the convexity of W_ϵ: $W_\epsilon''(x) \geq 0$ for all $x \in \mathbb{R}$. Let us note that $W_\varepsilon(0) = 0$ and $W_\varepsilon'(0) = W_0'(0) = 0$. We deduce that $W_\epsilon(x) \geq W_0(x) \geq 0$ for all $x \in \mathbb{R}$.

Let us prove that the global minimum of W_ϵ is only reached at 0. If there exists some $m > 0$ such that $W_\varepsilon(m) = 0$, then due to the convexity $W_\varepsilon(x) = W_0(x) = 0$ for any $x \in [0, m]$. By definition, since $u_0 = \delta_0$, we get $W_0 = V + F$. By (V-6), we know that V is an "analytic function" on $[-a, a]$ and F is polynomial. Therefore $W_0(x) = V(x) + F(x) = 0$ and $W_0''(x) = V''(x) + F''(x) = 0$ for any $x \in [-a, a]$. This contradicts the hypotheses (V-3) and (F-2) which imply that $W''(a) > 0$. Finally we conclude that 0 is the unique location of the global minimum of W_ϵ.
Besides, if F' is not a linear function then $F''(z) > F''(0)$ for all $z \neq 0$. Consequently, $W_\epsilon''(0) - W_0''(0) = \int_{\mathbb{R}} (F''(z) - F''(0))\, u_\epsilon(z) dz > 0$ because F' is odd and convex on $\mathbb{R}_+$. Therefore, $W_\epsilon''(0) > W_0''(0) \geq 0$ and so 0 is the unique location of the global minimum.
3. As B_l^ϵ tends to B_l, we have $f(B_l^\epsilon) = f(B_l) + o(1)$ so that (2.4) is a direct consequence of Lemma A.4. □

The previous lemma is essential for the computation of the convergence's speed. In order to complete this asymptotic description, we need to estimate the behavior of $W_\epsilon(A_j^\epsilon)$ for any j as $\varepsilon \to 0$.

Proposition 2.2. *Let $\mathcal{D}$ defined in Lemma 2.1. We assume that $W_\epsilon''(D) > 0$ for all $D \in \mathcal{D} = \mathcal{A} \bigcup \mathcal{B}$. If A_j and A_k are two elements of $\mathcal{A}$ with the corresponding asymptotic weight: $u_0(A_j) = p_j$ and $u_0(A_k) = p_k$, we denote by A_j^ε and A_k^ε the corresponding arg min defined in the statement of Lemma 2.1. Then the following asymptotic development holds*

$$\lim_{\epsilon \to 0} \frac{W_\epsilon(A_j^\epsilon) - W_\epsilon(A_k^\epsilon)}{\epsilon} = -\frac{1}{4} \log\left(\frac{W_0''(A_j)}{W_0''(A_k)}\right) - \frac{1}{2} \log\left(\frac{p_j}{p_k}\right). \tag{2.6}$$

Moreover, for any $B \in \mathcal{B} \neq \emptyset$ we denote B^ε the corresponding arg min presented in Lemma 2.1 and obtain

$$\lim_{\epsilon \to 0} \frac{W_\epsilon(B^\epsilon) - W_\epsilon(A_j^\epsilon)}{\epsilon} = +\infty, \quad \textit{for all } 1 \leq j \leq r. \tag{2.7}$$

Proof. By Theorem 3.6 in [7], the limit measure u_0 is a discrete measure constructed as follows $u_0 = \sum_{j=1}^r p_j \delta_{A_j} + \sum_{l=1}^s q_l \delta_{B_l}$ where the weights are defined by

$$p_j = \lim_{\epsilon \to 0} \int_{A_j - \delta}^{A_j + \delta} u_\epsilon(x) dx \quad \text{and} \quad q_l = \lim_{\epsilon \to 0} \int_{B_l - \delta}^{B_l + \delta} u_\epsilon(x) dx, \quad 1 \leq j \leq r,\ 1 \leq l \leq s.$$

The only assumption on δ is that all the intervals $[A_\cdot - \delta, A_\cdot + \delta]$ and $[B_\cdot - \delta, B_\cdot + \delta]$ are disjoint. By definition of the set $\mathcal{A}$, $p_i \neq 0$ for all $1 \leq i \leq r$. As an immediate consequence, we obtain for $1 \leq j, k \leq r$ and $1 \leq l \leq s$:

$$\frac{p_j}{p_k} = \lim_{\epsilon \to 0} \frac{\int_{A_j - \delta}^{A_j + \delta} e^{-\frac{2}{\epsilon} W_\epsilon(x)} dx}{\int_{A_k - \delta}^{A_k + \delta} e^{-\frac{2}{\epsilon} W_\epsilon(x)} dx} \quad \text{and} \quad \frac{q_l}{p_j} = \lim_{\epsilon \to 0} \frac{\int_{B_l - \delta}^{B_l + \delta} e^{-\frac{2}{\epsilon} W_\epsilon(x)} dx}{\int_{A_j - \delta}^{A_j + \delta} e^{-\frac{2}{\epsilon} W_\epsilon(x)} dx}.$$

By definition of the set $\mathcal{B}$, the weights $(q_l)_{l\geq 1}$ vanish. An adaptation of Lemma A.3 to the constant function $f\equiv 1$ yields

$$\lim_{\epsilon\to 0}\frac{\sqrt{\frac{\pi\epsilon}{W_\epsilon''(A_j^\epsilon)}}\,e^{-\frac{2U_\epsilon(A_j^\epsilon)}{\epsilon}}}{\sqrt{\frac{\pi\epsilon}{W_\epsilon''(A_k^\epsilon)}}\,e^{-\frac{2U_\epsilon(A_k^\epsilon)}{\epsilon}}}=\frac{p_j}{p_k}\quad\text{and}\quad\lim_{\epsilon\to 0}\frac{\sqrt{\frac{\pi\epsilon}{W_\epsilon''(B_l^\epsilon)}}\,e^{-\frac{2U_\epsilon(B_l^\epsilon)}{\epsilon}}}{\sqrt{\frac{\pi\epsilon}{W_\epsilon''(A_j^\epsilon)}}\,e^{-\frac{2U_\epsilon(A_j^\epsilon)}{\epsilon}}}=0.$$

Applying the function $x\to -\frac{1}{2}\log x$ to the previous equalities permits to prove the asymptotic estimates (2.6) and (2.7). $\square$

Remark 2.3. *Let us first note that the pseudo-potential W_ϵ doesn't generally reach its global minimum at each location A_j^ϵ respectively B_l^ϵ, defined in the statement of Lemma 2.1, even if each of these points converges to one location of the global minimum of W_0. The equation* (2.6) *emphasizes that the speed of convergence of $W_\epsilon(A_j^\epsilon)$ towards $W_0(A_j)$ is directly related to the weight p_j. Even if the elements of $\mathcal{B}$ don't have any impact on the limit measure u_0, they can influence the convergence's speed of the sequence of invariant measures u_ε for the self-stabilizing diffusion towards u_0.*

Let us introduce some assumptions in order to avoid the parasitism of $\mathcal{B}$ in the computation of the rate of convergence of any subsequence of invariant measures towards a limit measure u_0. In the following, this condition is assumed to be satisfied.
Let us recall the definition of $\mathcal{D}$: if $w_0:=\inf_{x\in\mathbb{R}}W_0(x)$ then $\mathcal{D}=W_0^{-1}(\{w_0\})$.

Assumption 2.1. *For each $D\in\mathcal{D}=\mathcal{A}\cup\mathcal{B}$, $W_0''(D)>0$. Moreover, for any element A_j^ε associated with $A_j\in\mathcal{A}$, $1\leq j\leq r$, (see Lemma 2.1 for the definition of $A_\cdot^\varepsilon$) and B_l^ε associated with B_l, $1\leq l\leq s$, we set*

$$\liminf_{\epsilon\to 0}\frac{W_\epsilon(B_l^\epsilon)-W_\epsilon(A_j^\epsilon)}{-\epsilon\log(\epsilon)}>1. \tag{2.8}$$

This condition is quite natural: it is related to the asymptotic estimate (2.4). In that development appear either terms induced from elements of $\mathcal{A}$ either from elements of $\mathcal{B}$. The condition expressed in (2.8) is interpreted as follows: the terms associated to $\mathcal{B}$ are negligible with respect to those of $\mathcal{A}$ of order $\varepsilon^{3/2}$. In other words, we assume that, for any $1\leq j\leq r$ and $1\leq l\leq s$,

$$\sqrt{\frac{\pi\epsilon}{W_\epsilon''(B_l^\epsilon)}}\,e^{-\frac{2W_\epsilon(B_l^\epsilon)}{\epsilon}}=\sqrt{\frac{\pi\epsilon}{W_\epsilon''(A_j^\epsilon)}}\,e^{-\frac{2W_\epsilon(A_j^\epsilon)}{\epsilon}}o(\epsilon),$$

which is equivalent to (2.8).
Example: Let us just introduce some example which satisfies Assumption 2.1. This example was already pointed out in [7]. The context is as follows: the environment function of the self-stabilizing process satisfies $V(x):=\frac{x^6}{6}-\frac{3}{2}x^4-\frac{17}{32}x^2$ while the interaction function equals $F(x):=\frac{x^4}{4}+\frac{x^2}{2}$. Two essential results

were already proven (see [7]): first, any family of symmetric invariant measures $\{u_\varepsilon,\, \varepsilon > 0\}$ satisfies the following weak convergence result:

$$\lim_{\epsilon\to 0} u_\epsilon = u_0 := \frac{26}{45}\delta_0 + \frac{19}{90}\left(\delta_{\frac{\sqrt{15}}{2}} + \delta_{-\frac{\sqrt{15}}{2}}\right). \tag{2.9}$$

We deduce, by the way, the expression of the limit pseudo-potential

$$W_0(x) = V(x) + \frac{26}{45}F(x) + \frac{19}{90}\Big\{F(x-x_0) + F(x+x_0) - 2F(x_0)\Big\},$$

where $x_0 = \frac{\sqrt{15}}{2}$.
Secondly, due to some convexity property of $W_0^{(4)}$, the global minimum of W_0 can only be reached at three locations, namely $A_1 = -x_0$, $A_2 = 0$ and $A_3 = x_0$.

- Therefore $\mathcal{B} = \emptyset$ which implies that it suffices to obtain $W_0''(A_i) > 0$, for $1 \le i \le 3$, in order to verify Assumption 2.1. After straightforward computations, we effectively obtain: $W_0''\left(\frac{\sqrt{15}}{2}\right) = W_0''\left(-\frac{\sqrt{15}}{2}\right) = 4W_0''(0) = \frac{75}{4} > 0$.
- In this example, Proposition 2.2 leads to some explicit computation of the first order development of $W_\varepsilon(A_1^\varepsilon)$ where $(A_1^\varepsilon)_{\varepsilon>0}$ is a sequence of local minimum locations for the potential W_ϵ which converges towards $A_1 = -x_0 = -\frac{\sqrt{15}}{2}$ (see Lemma 2.1). Let us note that the pseudo-potential W_ε associated to the symmetric invariant measure introduced in (2.9) admits exactly three local minima as ε is small. Indeed $\mathcal{D}$ admits three elements, which implies that W_ϵ admits at least three local minima in the small ε limit as it was proven in Lemma 2.1. Furthermore if W_ϵ admits more than 4 local minima, then W_ϵ' vanishes at least seven times. By Rolle's theorem this implies that $W_\epsilon^{(4)}$, which is a polynomial function of order 2, admits 3 zeros: this is of course a nonsense. Finally we obtain the existence of exactly three local minima of W_ϵ: $A_1^\varepsilon < A_2^\varepsilon < A_3^\varepsilon$. The symmetry of u^ε and consequently of W_ϵ, permits to know that $A_1^\varepsilon = -A_3^\varepsilon$ and $A_2^\varepsilon = 0$. Finally Proposition 2.2 and $W_\epsilon(0) = 0$ provide

$$W_\epsilon(A_1^\varepsilon) \approx \frac{\epsilon}{2}\log\left(1 + \frac{7}{19}\right) \quad \text{as } \varepsilon \to 0.$$

The next part of this paper concerns the rate of convergence of sequences of invariant measures for the self-stabilizing process towards the associated limit measure. The study shall of course depend on the limit measure considered. We focus our attention to various particular situations.

3 Convergence rate associated with $u_0 = \delta_0$

In this part of the study, $\{u_\varepsilon,\, \varepsilon > 0\}$ represents a sequence of *symmetric* invariant measures, with $2n$-th uniformly bounded moments, which converges to

$u_0 := \delta_0$. The aim is to establish the associated rate of convergence. Let us first note that the limit pseudo-potential defined in (2.1) is given by $W_0(x) = V(x) + F(x)$. Since the support of the limit measure is contained in the localization set of global minima for W_0 (see Theorem 3.6 in [7]), the origin is a global minimum. In this section, we shall assume that

$$\textbf{(H)} \qquad W_0 \text{ and } F'' \text{ are convex functions.}$$

By Lemma 2.1, 0 is the unique location of the global minimum of W_ε and of W_0. Therefore $\mathcal{B} = \emptyset$. Actually we do not impose that Assumption 2.1 is satisfied, we are going to study the rate of convergence in the two following cases:

- firstly $W_0''(0) > 0$, that is $\alpha > -V''(0)$, which corresponds to the situation where Assumption 2.1 is satisfied.
- secondly $W_0''(0) = 0$, i.e. $\alpha = -V''(0)$.

3.1 Convergence rate for the case: $\alpha > -V''(0)$

Let us recall that this situation corresponds to the lower-bound $W_0''(0) > 0$. Since we assume that Condition (H) is satisfied, $\mathcal{B}$ is empty and finally Assumption 2.1 is satisfied. Applying a Laplace type asymptotic result, we obtain easely the convergence rate of the sequence of *symmetric* invariant measures $\{u_\epsilon,\, \varepsilon > 0\}$ towards the limit measure $u_0 = \delta_0$ as $\varepsilon \to 0$.

Theorem 3.1. *Under the condition (H), for any function* $f \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ *with polynomial growth, we have:*

$$\lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}\Big\{ \langle f, u_\epsilon \rangle - \langle f, u_0 \rangle \Big\} = \frac{f''(0)}{4(\alpha + V''(0))} \tag{3.1}$$

Proof. We recall that u_ε is characterized by the exponential structure (2.2). Moreover due to Condition (H), $\mathcal{B} = \emptyset$ and 0 is the unique location of the global minimum of W_ϵ. Therefore, applying the third item of Lemma 2.1 with $A_1^\epsilon = 0$ and $W_\epsilon(A_1^\epsilon) = 0$, we get

$$\int_{\mathbb{R}} f(x) e^{-\frac{2}{\epsilon} W_\epsilon(x)} dx = \sqrt{\frac{\pi\epsilon}{W_\epsilon''(0)}} \Big\{ f(0) + \gamma_0(f)\epsilon + o(\epsilon) \Big\},$$

where $\gamma_0(f)$ is defined by (2.5). Since $W_0^{(3)}(0) = V^{(3)}(0) = 0$ and $W_0''(0) = \alpha + V''(0) > 0$, $\gamma_0(f)$ converges towards

$$\gamma(f) := -\frac{W_0^{(4)}(0)}{16 W_0''(0)^2} f(0) + \frac{f''(0)}{4 W_0''(0)}. \tag{3.2}$$

Hence

$$\int_{\mathbb{R}} f(x) \exp\left[-\frac{2}{\epsilon} W_\epsilon(x) \right] dx = \sqrt{\frac{\pi\epsilon}{W_\epsilon''(0)}} \Big\{ f(0) + \gamma(f)\epsilon + o(\epsilon) \Big\}. \tag{3.3}$$

First we apply (3.3) to the constant function $f \equiv 1$ and obtain

$$\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}W_\epsilon(x)\right] dx = \sqrt{\frac{\pi\epsilon}{W_\epsilon''(0)}}\Big\{1+\gamma(1)\epsilon+o(\epsilon)\Big\}. \tag{3.4}$$

Here $\gamma(1) = -\frac{W_0^{(4)}(0)}{16W_0''(0)^2}$. Let $f \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ with polynomial growth. We divide (3.3) by (3.4), the following estimate yields

$$\int_{\mathbb{R}} f(x)u_\epsilon(x)dx \quad = \quad \frac{f(0)+\gamma(f)\epsilon+o(\epsilon)}{1+\gamma(1)\epsilon+o(\epsilon)} = f(0) + \Big(\gamma(f) - f(0)\gamma(1)\Big)\epsilon + o(\epsilon),$$

where $\gamma(f)$ is defined by (3.2). Finally we get

$$\frac{\langle f, u_\epsilon\rangle - \langle f, u_0\rangle}{\epsilon} = \gamma(f) - f(0)\gamma(1) + o(1) = \frac{f''(0)}{4W_0''(0)} + o(1).$$

In order to complete the proof, it suffices to note that $W_0''(0) = \alpha + V''(0)$. □

3.2 Convergence rate for the case $\alpha = -V''(0)$

Let us recall that this situation is equivalent to $W_0''(0) = 0$. In other words, Assumption 2.1 is not satisfied. The aim of this subsection is to emphasize that the convergence rate is not always equal to ε. This rate was effectively presented in Section 3.1 and concerns most of the situations. The condition $W_0''(0) = 0$ changes drastically the asymptotic behavior of the self-stabilizing invariant measure. The asymptotic results shall be proved under some additional conditions, namely the convexity of both F'' and W_0, that is, Condition (H).
The computation of convergence rate will be based on successive derivations of the pseudo-potential: $W_\epsilon^{(2k)}(0)$. We therefore introduce:

$$k_0 \quad := \quad \min\left\{k \geq 1 \;\mid\; W_0^{(2k)}(0) > 0\right\}, \tag{3.5}$$

$$\Omega_\epsilon \quad := \quad \max_{1\leq j\leq k_0}\left\{\left|W_\epsilon^{(2j)}(0)\right|^{\frac{1}{2j}}\epsilon^{-\frac{1}{2j}}\right\} \tag{3.6}$$

$$\text{and} \quad M_{2r}(\epsilon) \quad := \quad \int_{\mathbb{R}} x^{2r}\exp\left[-\frac{2}{\epsilon}W_\epsilon(x)\right]dx. \tag{3.7}$$

The expression Ω_ϵ corresponds in fact to the suitable change of variable associated with the computation of $M_{2r}(\varepsilon)$. This result is detailed in the following proposition. Let us just note that $M_{2r}(\varepsilon)$ is well-defined. Indeed since u_ε is a symmetric invariant measure, Lemma 4.2 in [6] implies the following lower bound:

$$\int_0^x (F' * u_\varepsilon)(y)dy \geq 0.$$

It suffices then to use (2.1) and the growth property of V in order to prove the boundedness of the integrals $M_{2r}(\varepsilon)$, for $\varepsilon > 0$ and $r \in \mathbb{N}$.

Proposition 3.2. *Under the assumption (H), for all $r \in \mathbb{N}$, the following inequalities holds:*

$$0 < \liminf_{\epsilon \to 0} \Omega_\epsilon^{2r+1} M_{2r}(\epsilon) < \limsup_{\epsilon \to 0} \Omega_\epsilon^{2r+1} M_{2r}(\epsilon) < +\infty. \tag{3.8}$$

Proof. **Step 1. Preliminaries.** Since $\{u_\varepsilon,\ \varepsilon > 0\}$ is a sequence of symmetric invariant measures with uniformly bounded $2n$-th moments, $W_\epsilon^{(2r)}$ converges uniformly on each compact set to $W_0^{(2r)}$ (see the discussion before Lemma 2.1 and the statement of Proposition 3.3 in [7]). The definition of k_0 implies therefore: $|W_\epsilon^{(2k_0)}(0)|^{\frac{1}{2k_0}} \epsilon^{-\frac{1}{2k_0}} \to +\infty$ as $\varepsilon \to 0$. Consequently, $\Omega_\epsilon \to +\infty$.
Let $C_j(\epsilon) := \frac{W_\epsilon^{(2j)}(0)}{\epsilon \Omega_\epsilon^{2j}}$. By construction, the families $\{C_j(\epsilon)\}_{\epsilon>0}$ are bounded. Let us select a decreasing subsequence $(\epsilon_k)_{k\in\mathbb{N}}$ converging towards 0 such that, for any $1 \le j \le k_0$, we observe $C_j(\epsilon_k) \to C_j \in \mathbb{R}$.
In order to simplify the notations, we drop the index.
We define

$$\mathbb{J} := \Big\{ j \ \Big|\ 1 \le j \le k_0 ,\ C_j \neq 0 \Big\} = \Big\{ j_1, \dots, j_l \Big\} \tag{3.9}$$

with $1 \le j_1 < j_2 < \cdots < j_l \le k_0$.
Let us now focus our attention to the computation of the integral term $M_{2r}(\varepsilon)$ which can be split into two principal terms as follows:

$$\frac{M_{2r}(\varepsilon)}{2} = I_{2r}(\epsilon) + J_{2r}(\varepsilon) := \int_0^{\eta(\epsilon)} x^{2r} e^{-\frac{2W_\epsilon(x)}{\varepsilon}} dx + \int_{\eta(\epsilon)}^{+\infty} x^{2r} e^{-\frac{2W_\epsilon(x)}{\varepsilon}} dx, \tag{3.10}$$

$\eta(\epsilon)$ shall be specified in the following.
Step 2. Asymptotic analysis of $I_{2r}(\varepsilon)$. The mean value theorem applied to the function W_ϵ on $[0;\eta(\epsilon)]$ leads to:

$$\frac{W_\epsilon(x)}{\epsilon} = \frac{1}{\varepsilon} \sum_{j=1}^{k_0} \frac{1}{(2j)!} W_\epsilon^{(2j)}(0)\, x^{2j} + \frac{W_\epsilon^{(2k_0+2)}(y_x)}{(2k_0+2)!\ \epsilon} x^{2k_0+2},$$

with $y_x \in [0;\eta(\epsilon)]$. Using the definition and the convergence result related to $C_j(\epsilon)$, we get:

$$\begin{aligned}\frac{W_\epsilon(x)}{\epsilon} &= \sum_{k=1}^{l} \frac{C_{j_k}}{(2j_k)!} \Omega_\epsilon^{2j_k} x^{2j_k} + \frac{W_\epsilon^{(2k_0+2)}(y_x)}{(2k_0+2)!\ \epsilon} x^{2k_0+2} \\ &\quad + \sum_{j=1}^{k_0} \frac{1}{(2j)!} \operatorname{sgn}\left(C_j(\epsilon) - C_j\right) \left(|C_j(\epsilon) - C_j|^{\frac{1}{2j}} \right)^{2j} \Omega_\epsilon^{2j} x^{2j}. \end{aligned} \tag{3.11}$$

We shall find some suitable sequence $\{\eta(\epsilon),\ \varepsilon > 0\}$ (subsequence since the index was dropped), decreasing toward 0 and such that the first sum in the rhs of the previous expression is the principal term, all the others being negligible. For ϵ small enough and for all $x \in [0;\eta(\epsilon)]$, the second term is upper bounded by:

$$\frac{1}{\varepsilon} \left| W_\epsilon^{(2k_0+2)}(y_x) \right| x^{2k_0+2} \le \sup_{z\in[0;1]} \left| W_\epsilon^{(2k_0+2)}(z) \right| \eta(\epsilon)^{2k_0+2} \epsilon^{-1}.$$

Let us now introduce the parameter $\overline{\Omega}_\epsilon$ which tends to 0 in the small ε limit and which is defined by:

$$\overline{\Omega}_\epsilon := \max\left\{ \sup_{1\le j\le k_0} |C_j(\epsilon) - C_j|^{\frac{1}{2j}} \,;\ \frac{1}{\Omega_\epsilon};\ \epsilon^{\frac{1}{2k_0(2k_0+2)}} \right\}. \tag{3.12}$$

$\overline{\Omega}_\varepsilon$ is a good candidate for the construction of $\eta(\varepsilon)$; we set

$$\eta\,(\epsilon) := \Omega_\epsilon^{-1}\left(\overline{\Omega}_\epsilon\right)^{-\frac{1}{2}}. \tag{3.13}$$

Some straightforward considerations permit to observe that:

- Firstly, $\eta\,(\epsilon)$ tends to 0 as ε becomes small.
- Secondly, there exists $\rho(\epsilon) > 0$, satisfying $\lim_{\varepsilon\to 0}\rho(\varepsilon) = 0$, such that, for all $x \in [0;\eta(\epsilon)]$,

$$\frac{\left|W_\epsilon^{(2k_0+2)}(y_x)\right|}{\epsilon\,(2k_0+2)!}\, x^{2k_0+2} + \sum_{j=1}^{k_0} \frac{\left(|C_j(\epsilon) - C_j|^{\frac{1}{2j}}\right)^{2j}}{(2j)!}\, \Omega_\epsilon^{2j}\, x^{2j} < \rho(\epsilon). \tag{3.14}$$

Due to the suitable choice of the parameter $\eta(\varepsilon)$, see (3.13), the integral $I_{2r}(\varepsilon)$ defined by (3.10) is equivalent to the simpler integral

$$\int_0^{\eta(\epsilon)} x^{2r} \exp\left[-2\left(\sum_{k=1}^{l} \frac{C_{j_k}}{(2j_k)!}\Omega_\epsilon^{2j_k} x^{2j_k}\right)\right] dx,$$

in the small ε limit. The change of variable $x := \Omega_\epsilon^{-1} y$ provides

$$I_{2r}(\varepsilon) \approx \Omega_\epsilon^{-2r-1} \int_0^{\varphi(\epsilon)} y^{2r} \exp\left[-2\sum_{k=1}^{l} \frac{C_{j_k}}{(2j_k)!} y^{2j_k}\right] dy, \quad \text{as } \varepsilon \to 0, \tag{3.15}$$

where $\varphi(\epsilon) := \eta\,(\epsilon)\,\Omega_\epsilon = \left(\overline{\Omega}_\epsilon\right)^{-\frac{1}{2}} \to +\infty$ as $\varepsilon \to 0$. By definition $C_{j_l} \neq 0$, see (3.9). If $C_{j_l} > 0$, then $\Gamma_r := \int_{\mathbb{R}^+} x^{2r} \exp\left[-2\sum_{k=1}^{l} \frac{1}{(2j_k)!} C_{j_k} x^{2j_k}\right] dx < \infty$ and therefore, in the small ε limit, (3.15) leads to

$$I_{2r}(\varepsilon) \approx \Omega_\epsilon^{-2r-1}\Gamma_r. \tag{3.16}$$

To conclude the asymptotic analysis of $I_{2r}(\varepsilon)$, it remains to prove that $C_{j_l} > 0$. We shall prove it by *reductio ad absurdum*. Let us then assume that $C_{j_l} < 0$ which implies $\lim_{y\to+\infty} \sum_{k=1}^{l} \frac{C_{j_k}}{(2j_k)!}\, y^{2j_k} = -\infty$. Hence there exists $y_0 > 0$ such that $\sum_{k=1}^{l} \frac{C_{j_k}}{(2j_k)!}\, y_0^{2j_k} \le -1$. Due to the convergence of $\frac{W_\epsilon(\Omega_\epsilon^{-1} y)}{\epsilon}$ towards $\sum_{k=1}^{l} \frac{C_{j_k}}{(2j_k)!}\, y^{2j_k}$ for any $y \in \mathbb{R}$, we deduce that $W_\epsilon\left(\Omega_\epsilon^{-1} y_0\right) < 0$ for ϵ small enough. This contradicts the fact that 0 is the global minimum of W_ϵ in a neighborhood of 0, for ϵ small enough (see Lemma 2.1).

Step 3. Asymptotic analysis of $J_{2r}(\varepsilon)$. It is now sufficient to prove that $J_{2r}(\varepsilon)$ defined in (3.10) satisfies $J_{2r}(\varepsilon) = o(I_{2r}(\varepsilon)) = o\left\{\Omega_\epsilon^{-2r-1}\right\}$. We split this integral into three different parts depending on the support: $J_{2r}^\star(\varepsilon)$ for the support $[\eta(\varepsilon), \varepsilon^\lambda[$, $J_{2r}^\circ(\varepsilon)$ for $[\varepsilon^\lambda, \varepsilon^{-\mu}[$ and finally $J_{2r}^\Delta(\varepsilon)$ for $[\varepsilon^{-\mu}, +\infty[$ where $\lambda, \mu > 0$ shall be specified in the following.

3.1. Let us first estimate $J_{2r}^\Delta(\varepsilon)$. Due to the assumptions (F-2) and (V-4), we get the lower bound $W_\epsilon(x) \geq W_0(x) \geq C_4 x^4 - C_2 x^2 \geq \frac{x^2}{2}$ for large x. The first inequality in the previous formula is also related to the second item in the proof of Lemma 2.1. We apply the change of variable $x := \sqrt{\epsilon} y$, Lemma A.1 leads to:

$$J_{2r}^\Delta(\varepsilon) \leq \epsilon^{r+\frac{1}{2}} \int_{\epsilon^{-\mu-\frac{1}{2}}}^{\infty} y^{2r} e^{-y^2} dy \leq 2\epsilon^{1+(1-2r)\mu} \exp\left[-\epsilon^{-2\mu-1}\right],$$

for ϵ small enough. It remains to prove that the rhs is negligible with respect to Ω_ϵ^{-2r-1}. It suffices in fact to note that, by definition of Ω_ε, the following convergence result holds: $\varepsilon\Omega_\varepsilon \to 0$ as $\varepsilon \to 0$. Consequently, since $\mu > 0$,

$$\Omega_\epsilon^{2r+1} \epsilon^{1+(1-2r)\mu} \exp\left[-\epsilon^{-2\mu-1}\right] \to 0.$$

3.2. Secondly we estimate $J_{2r}^\circ(\varepsilon)$. We obtain:

$$J_{2r}^\circ(\varepsilon) = \int_{\epsilon^\lambda}^{\epsilon^{-\mu}} x^{2r} \exp\left[-\frac{2}{\epsilon} W_\epsilon(x)\right] dx \leq \epsilon^{-\mu(2r+1)} \exp\left[-\frac{2}{\epsilon} \inf_{z \in [\epsilon^\lambda; +\infty[} W_\epsilon(z)\right].$$

We note that 0 is the unique location of the global minimum for the pseudo-potential W_0 which implies that $\inf_{z\in[\epsilon^\lambda;+\infty[} W_\epsilon(z) = W_\epsilon\left(\epsilon^\lambda\right) \geq W_0\left(\epsilon^\lambda\right)$, for ϵ small enough. The mean value theorem provides $W_0\left(\epsilon^\lambda\right) \approx \frac{W_0^{(2k_0)}(0)}{(2k_0)!}\ \epsilon^{2k_0\lambda}$. Taking $\lambda = \frac{1}{2k_0+1}$ and $\mu > 0$, $J_{2r}^\circ(\varepsilon)$ is exponentially small in ϵ. By definition, $\sqrt{\epsilon} = o\left\{\Omega_\epsilon^{-1}\right\}$. Hence $J_{2r}^\circ(\varepsilon)$ is negligible.

3.3. We focus now our attention on the integral $J_{2r}^\star(\varepsilon)$ related to the support $[\eta(\varepsilon), \varepsilon^\lambda[$ where $\eta(\varepsilon)$ defined by (3.13) tends to 0 as ε becomes small. The change of variable $x := \Omega_\epsilon^{-1} y$ yields

$$J_{2r}^\star(\varepsilon) = \Omega_\epsilon^{-2r-1} \int_{a(\epsilon)}^{b(\epsilon)} y^{2r} \exp\Big[-\frac{2}{\varepsilon} W_\epsilon(\Omega_\epsilon^{-1} y)\Big] dy, \tag{3.17}$$

where $a(\epsilon) := \eta(\epsilon)\Omega_\varepsilon \to +\infty$ and $b(\epsilon) := \epsilon^\lambda \Omega_\epsilon$. Let us just prove that the integral introduced in (3.17) is negligible, that is tends to 0 in the small ε limit. An integration by parts permits to obtain:

$$\begin{aligned} J_{2r}^\star(\varepsilon)\Omega_\epsilon^{2r+1} \quad &= \quad \frac{a(\epsilon)^{2r} \exp\left[-\frac{2}{\varepsilon} W_\epsilon(\Omega_\epsilon^{-1} a(\epsilon))\right]}{\frac{2}{\varepsilon\Omega_\varepsilon} W_\epsilon'(\Omega_\varepsilon^{-1} a(\epsilon)) - \frac{2r}{a(\epsilon)}} - \frac{b(\epsilon)^{2r} \exp\left[-\frac{2}{\varepsilon} W_\epsilon(\Omega_\epsilon^{-1} b(\epsilon))\right]}{\frac{2}{\varepsilon\Omega_\varepsilon} W_\epsilon'(\Omega_\varepsilon^{-1} b(\epsilon)) - \frac{2r}{b(\epsilon)}} \\ &- \int_{a(\epsilon)}^{b(\epsilon)} y^{2r} \frac{\frac{2}{\varepsilon\Omega_\varepsilon^2} W_\epsilon''(\Omega_\varepsilon^{-1} y) + \frac{2r}{y^2}}{\left(\frac{2}{\varepsilon\Omega_\varepsilon} W_\epsilon'(\Omega_\varepsilon^{-1} y) - \frac{2r}{y}\right)^2} \exp\left[-\frac{2}{\varepsilon} W_\epsilon(\Omega_\varepsilon^{-1} y)\right] dy. \end{aligned}$$

Since F'' is a convex function, we obtain

$$W_\epsilon''(x) - W_0''(x) = \int_{\mathbb{R}_+} \Big(F''(x+z) + F''(x-z) - 2F''(x)\Big) u_\epsilon(z) dz \geq 0. \quad (3.18)$$

The main assumption in this section is $W_0''(0) = 0$. Moreover, since 0 is the unique global minimum location of the limit pseudo potential W_0, there exists some constant $\eta > 0$ such that $W_0''(x) \geq 0$ on the interval $]-\eta, \eta[$ and so, due to (3.18), $W_\epsilon''(\Omega_\varepsilon^{-1} y) \geq 0$ for $y \in [a(\varepsilon), b(\varepsilon)]$. Hence

$$J_{2r}^\star(\varepsilon)\Omega_\epsilon^{2r+1} \leq \frac{a(\epsilon)^{2r} e^{-\frac{2}{\varepsilon} W_\epsilon(\Omega_\epsilon^{-1} a(\epsilon))}}{\frac{2}{\varepsilon\Omega_\varepsilon} W_\epsilon'(\Omega_\varepsilon^{-1} a(\epsilon)) - \frac{2r}{a(\epsilon)}} - \frac{b(\epsilon)^{2r} e^{-\frac{2}{\varepsilon} W_\epsilon(\Omega_\varepsilon^{-1} b(\epsilon))}}{\frac{2}{\varepsilon\Omega_\varepsilon} W_\epsilon'(\Omega_\varepsilon^{-1} b(\epsilon)) - \frac{2r}{b(\epsilon)}}. \quad (3.19)$$

Moreover since the application $y \to W_\varepsilon'(\Omega_\varepsilon^{-1} y)$ is non decreasing on the interval $[a(\varepsilon), b(\varepsilon)]$, we get

$$\frac{2}{\varepsilon\Omega_\varepsilon} W_\epsilon'\Big(\Omega_\varepsilon^{-1} b(\epsilon)\Big) - \frac{2r}{b(\epsilon)} \geq \frac{2}{\varepsilon\Omega_\varepsilon} W_\epsilon'\Big(\Omega_\varepsilon^{-1} a(\epsilon)\Big) - \frac{2r}{a(\epsilon)}. \quad (3.20)$$

Let us prove now that the r.h.s. is positive for ε small enough. The mean value theorem leads to some similar development as (3.11) namely

$$\begin{aligned}\frac{W_\epsilon'(x)}{\varepsilon\Omega_\varepsilon} &= \sum_{k=1}^{l} \frac{C_{j_k}}{(2j_k-1)!} \Omega_\epsilon^{2j_k-1} x^{2j_k-1} + \frac{W_\epsilon^{(2k_0+2)}(y_x)}{(2k_0+1)!\, \epsilon\Omega_\varepsilon} x^{2k_0+1} \\ &+ \sum_{j=1}^{k_0} \frac{1}{(2j-1)!} \operatorname{sgn}(C_j(\epsilon) - C_j) \Big(|C_j(\epsilon) - C_j|^{\frac{1}{2j}}\Big)^{2j} \Omega_\epsilon^{2j-1} x^{2j-1},\end{aligned}$$

with $y_x \in [0, x]$. In particular, for $x = \Omega_\varepsilon^{-1} a(\varepsilon) = \eta_\varepsilon$, similar arguments as those used in (3.14) permit the existence of some function $\rho(\varepsilon) > 0$ satisfying $\lim_{\varepsilon\to 0} \rho(\varepsilon) = 0$, such that

$$\frac{\Big|W_\epsilon^{(2k_0+2)}(y_x)\Big|}{\epsilon\Omega_\varepsilon (2k_0+1)!} \eta(\varepsilon)^{2k_0+1} + \sum_{j=1}^{k_0} \frac{\Big(|C_j(\epsilon) - C_j|^{\frac{1}{2j}}\Big)^{2j}}{(2j-1)!} \Omega_\epsilon^{2j-1} \eta(\varepsilon)^{2j-1} < \rho(\epsilon).$$

We deduce that $\frac{W_\epsilon'(\Omega_\varepsilon^{-1} a(\varepsilon))}{\varepsilon\Omega_\varepsilon}$ is close to $P(\varepsilon) := \sum_{k=1}^{l} \frac{C_{j_k}}{(2j_k-1)!} a(\varepsilon)^{2j_k-1}$: for any $\delta > 0$ small enough, there exists $\varepsilon_0 > 0$, such that

$$\left| \frac{W_\epsilon'(\Omega_\varepsilon^{-1} a(\varepsilon))}{\varepsilon\Omega_\varepsilon} - P(\varepsilon) \right| \leq \delta, \quad \forall \varepsilon \leq \varepsilon_0.$$

Let us recall that $a(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Furthermore, in Step 2 we have proved that $C_{j_l} > 0$. Therefore, as $\varepsilon \to 0$, $P(\varepsilon) \to \infty$ and so do $\frac{W_\epsilon'(\Omega_\varepsilon^{-1} a(\varepsilon))}{\varepsilon\Omega_\varepsilon}$. Finally we deduce that the rhs of (3.20) is lower-bounded: for any $\delta > 0$ there exists ε_0 such that

$$\frac{2}{\varepsilon\Omega_\varepsilon} W_\epsilon'(\Omega_\varepsilon^{-1} a(\epsilon)) - \frac{2r}{a(\epsilon)} \geq \frac{1}{\delta}, \quad \forall \varepsilon \leq \varepsilon_0. \quad (3.21)$$

By (3.21), (3.20) and (3.19), there exists some $\delta > 0$ such that the following upper bound yields in the small ε limit:

$$J_{2r}^{\star}(\varepsilon)\Omega_\epsilon^{2r+1} \le \frac{a(\epsilon)^{2r} e^{-\frac{2}{\varepsilon} W_\epsilon(\Omega_\epsilon^{-1} a(\epsilon))}}{\frac{2}{\varepsilon\Omega_\varepsilon} W_\epsilon'(\Omega_\varepsilon^{-1} a(\epsilon)) - \frac{2r}{a(\epsilon)}} \le \delta a(\epsilon)^{2r} e^{-\frac{2}{\varepsilon} W_\epsilon(\Omega_\epsilon^{-1} a(\epsilon))}. \tag{3.22}$$

Let us prove now that the previous upper-bound becomes small as $\varepsilon \to 0$ which implies immediately the required asymptotic result: $J_{2r}^{\star}(\varepsilon) = o(\Omega_\epsilon^{-2r-1})$. It suffices in fact to get some estimate of $W_\epsilon(\Omega_\epsilon^{-1} a(\epsilon))/\varepsilon$. The procedure requires the arguments just used for the asymptotic estimation of the expression $W_\epsilon'(\Omega_\epsilon^{-1} a(\epsilon))\varepsilon^{-1}\Omega_\varepsilon^{-1}$. Indeed for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$\left| \frac{1}{\varepsilon} W_\epsilon(\Omega_\epsilon^{-1} a(\epsilon)) - Q(\varepsilon) \right| \le \delta, \quad \varepsilon \le \varepsilon_0, \tag{3.23}$$

with $Q(\varepsilon) := \sum_{k=1}^{l} \frac{C_{j_k}}{(2j_k)!} a(\varepsilon)^{2j_k}$. Since $C_{j_l} > 0$, the following limit holds $\lim_{\varepsilon\to 0} Q(\varepsilon) = +\infty$ and therefore (3.23) leads to $\frac{1}{\varepsilon} W_\epsilon(\Omega_\epsilon^{-1} a(\epsilon)) \ge \frac{C_{j_l}}{2(2j_l)!} a(\varepsilon)^{2j_l}$ for ε small enough. By (3.22), we finally get

$$0 \le J_{2r}^{\star}(\varepsilon)\Omega_\epsilon^{2r+1} \le \delta a(\epsilon)^{2r} \exp\Big[-\frac{C_{j_l}}{(2j_l)!} a(\epsilon)^{2j_l} \Big].$$

Since $a(\varepsilon) \to \infty$ as $\varepsilon \to 0$, the rhs in the preceding inequality tends to 0 and $J_{2r}^{\star}(\varepsilon) = o(\Omega_\epsilon^{-2r-1})$.

Step 4. Conclusion. In the first step, we have decomposed the moment $M_{2r}(\epsilon)$ (for some subsequence $(\epsilon_k)_{k\in\mathbb{N}}$) into two parts: $I_{2r}(\varepsilon)$ studied in the second step and $J_{2r}(\varepsilon)$ studied in the third step. We have proved that $J_{2r}(\varepsilon)$ is negligible with respect to $I_{2r}(\varepsilon)$. Hence the following asymptotic estimate holds

$$M_{2r}(\epsilon) = \Omega_\epsilon^{-2r-1} \int_{\mathbb{R}} x^{2r} \exp\left[-2 \sum_{k=1}^{l} \frac{1}{(2j_k)!} C_{j_k} x^{2j_k} \right] dx + o\left\{ \Omega_\epsilon^{-2r-1} \right\} \tag{3.24}$$

where the coefficient C_j depend on the sequence $\varepsilon = (\epsilon_k)_{k\in\mathbb{N}}$.

In order to archive the proof, we analyze, not only for some subsequence, the following expressions: $\liminf_{\epsilon\to 0} \Omega_\epsilon^{2r+1} M_{2r}(\epsilon)$ and $\limsup_{\epsilon\to 0} \Omega_\epsilon^{2r+1} M_{2r}(\epsilon)$ and prove (3.8) by *reductio ad absurdum*. If we assume that the lim sup is unbounded, then there exists some sequence $(\varepsilon_k)_{k\ge 0}$ which tends to 0 and such that

$$\lim_{k\to\infty} \Omega_{\epsilon_k}^{2r+1} M_{2r}(\epsilon_k) = +\infty.$$

Applying Step 1, we extract some subsequence $(\varepsilon_k')_{k\ge 0}$ such that $C_j(\varepsilon_k') \to C_j$ as $k \to \infty$ for all $1 \le j \le k_0$. For this subsequence we have already proved, see (3.24), that $\Omega_{\epsilon_k'}^{2r+1} M_{2r}(\epsilon_k')$ is bounded. We obtain the announced contradiction and therefore $\limsup_{\epsilon\to 0} \Omega_\epsilon^{2r+1} M_{2r}(\epsilon) < \infty$. The same argument is used to obtain the lower-bound. $\square$

According to Proposition 3.2, we observe that Ω_ε defined by (3.6) is essential in the description of the asymptotic estimation of $M_{2r}(\varepsilon)$, defined by

$$M_{2r}(\epsilon) = \int_{\mathbb{R}} x^{2r} \exp\left[-\frac{2}{\epsilon} W_\epsilon(x)\right] dx.$$

In particular $M_0(\varepsilon)$, corresponding to the normalization term for the invariant measure u_ε, see (2.2), converges towards 0 with the rate Ω_ε^{-1}.
We recall the expression of the limit pseudo-potential introduced by (2.1): $W_0(x) = V(x) + F(x)$ and the related parameter

$$k_0 = \min\left\{k \geq 1 \ | \ W_0^{(2k)}(0) > 0\right\}.$$

We introduce two other parameters: p_0 and m_0, defined by

$$p_0 := \inf\left\{k \geq 2 \ | \ F^{(2k)}(0) > 0\right\} \quad \text{and} \quad m_0 := \min\{k_0, p_0\}. \tag{3.25}$$

The aim is now to prove that the convergence rate of sequences $\{M_{2r}(\epsilon_k),\ k \geq 0\}$ is related to $\epsilon_k^{(2r+1)/(2m_0)}$. First of all, we present the following asymptotic result:

Proposition 3.3. *If F'' and W_0 are both convex functions, the following inequalities yield:*

$$0 < \liminf_{\epsilon \to 0} \Omega_\epsilon \epsilon^{\frac{1}{2m_0}} < \limsup_{\epsilon \to 0} \Omega_\epsilon \epsilon^{\frac{1}{2m_0}} < +\infty. \tag{3.26}$$

Proof. By definition of the parameter k_0, we have

$$W_\epsilon^{(2j)}(0) = \int_{\mathbb{R}} \left(F^{(2j)}(x) - F^{(2j)}(0)\right) u_\epsilon(x) dx, \quad 1 \leq j \leq k_0 - 1.$$

Since F is a polynomial function of degree $2n$,

$$W_\epsilon^{(2j)}(0) = \sum_{r=1}^{n-j} \frac{F^{(2r+2j)}(0)}{(2r)!} \ \frac{M_{2r}(\epsilon)}{M_0(\varepsilon)}. \tag{3.27}$$

For any $1 \leq j \leq n-1$, we define $s(j) := \inf\left\{r \geq 1 \ | \ F^{(2j+2r)}(0) \neq 0\right\}$. Applying Proposition 3.2 with $1 \leq r \leq k_0 - 1$, there exists a decreasing sequence $(\varepsilon_k)_{k \geq 0}$ such that $C_j(\varepsilon_k) = \frac{W_{\varepsilon_k}^{(2j)}(0)}{\varepsilon_k \Omega_{\varepsilon_k}^{2j}}$ converges towards some limit denoted by C_j, as $k \to \infty$, for all $1 \leq j \leq k_0$. Moreover, by (3.24), we obtain

$$W_{\epsilon_k}^{(2j)}(0) = \frac{F^{(2j+2s(j))}(0)}{(2s(j))!} \alpha_{s(j)} \, \Omega_{\epsilon_k}^{-2s(j)} (1 + o(1)) \quad \text{as } k \to \infty \tag{3.28}$$

$$\text{with} \quad \alpha_r = \frac{\int_{\mathbb{R}^+} x^{2r} \exp\left[-2\sum_{p=1}^{l} \frac{C_{j_p}}{(2j_p)!} x^{2j_p}\right] dx}{\int_{\mathbb{R}^+} \exp\left[-2\sum_{p=1}^{l} \frac{C_{j_p}}{(2j_p)!} x^{2j_p}\right] dx}. \tag{3.29}$$

The set of indexes $\mathbb{J} = \{j_k,\, 1 \le k \le l\}$ is defined by (3.9).
We distinguish two different cases:
First case: $p_0 > k_0$. By definition of the coefficient $C_j(\varepsilon_k)$ and using (3.28), we obtain the following asymptotic result:

$$C_j(\epsilon_k) = \frac{W_{\epsilon_k}^{(2j)}(0)}{\epsilon_k \Omega_{\epsilon_k}^{2j}} = \frac{F^{(2j+2s(j))}(0)}{(2s(j))!} \frac{\alpha_{s(j)}}{\epsilon_k \Omega_{\epsilon_k}^{2(j+s(j))}} + o\Big(\epsilon_k^{-1}\Omega_{\epsilon_k}^{-2(j+s(j))}\Big).$$

Since $j \ge 1$ and $s(j) \ge 1$, we get $j + s(j) \ge 2$. Using the definition of $s(j)$, we obtain $F^{(2j+2s(j))}(0) \neq 0$ which implies $j + s(j) \ge p_0$. Furthermore (3.6) yields $\Omega_{\epsilon_k} \ge W_{\epsilon_k}^{(2k_0)}(0)^{\frac{1}{2k_0}} \epsilon_k^{-\frac{1}{2k_0}}$. Therefore the following lower-bound holds

$$\epsilon_k \Omega_{\epsilon_k}^{2(j+s(j))} \ge \left(W_{\epsilon_k}^{(2k_0)}(0)^{\frac{1}{2k_0}}\right)^{2(j+s(j))} \epsilon_k^{1-\frac{p_0}{k_0}}.$$

The rhs of the preceding inequality becomes infinite as $k \to \infty$. This is due to the definition of k_0, see (3.5), and the inequality $p_0 > k_0$. Hence, for any $1 \le j \le k_0 - 1$, the sequence $C_j(\epsilon_k)$ tends to 0 as $k \to \infty$. In other words the set $\mathbb{J}$ is a singleton: $\mathbb{J} = \{k_0\}$. Finally for k large enough, we get

$$\Omega_{\epsilon_k} = W_{\epsilon_k}^{(2k_0)}(0)^{\frac{1}{2k_0}} \epsilon_k^{-\frac{1}{2k_0}} = W_{\epsilon_k}^{(2k_0)}(0)^{\frac{1}{2k_0}} \epsilon_k^{-\frac{1}{2m_0}}.$$

Second case: $p_0 \le k_0$. For all $j \le k_0 - 1$, (3.28) implies the following asymptotic estimation:

$$\Omega_{\epsilon_k} \ge \left|W_{\epsilon_k}^{(2j)}(0)\right|^{\frac{1}{2j}} \epsilon_k^{-\frac{1}{2j}} \approx \left|\frac{F^{(2j+2s(j))}(0)}{(2s(j))!}\alpha_{s(j)}\right|^{\frac{1}{2j}} \Omega_{\epsilon_k}^{-\frac{s(j)}{j}} \epsilon_k^{-\frac{1}{2j}} = K_j \Omega_{\epsilon_k}^{-\frac{s(j)}{j}} \epsilon_k^{-\frac{1}{2j}},$$

where K_j is some constant. Hence, for k large enough, $\Omega_{\epsilon_k} \epsilon_k^{\frac{1}{2(j+s(j))}} \ge K'_j$. In particular, for $j = p_0 - 1$, we have

$$\Omega_{\epsilon_k} \ge C \epsilon_k^{-\frac{1}{2p_0}}. \tag{3.30}$$

Hence there exists some constant $C' > 0$ such that, for any $1 \le j \le k_0 - 1$,

$$\Omega_{\epsilon_k}^{-\frac{s(j)}{j}} \epsilon_k^{-\frac{1}{2j}} \le C' \epsilon_k^{-\frac{p_0 - s(j)}{2jp_0}} \le C' \epsilon_k^{-\frac{1}{2p_0}}.$$

Therefore, for all $1 \le j \le k_0 - 1$, there exists some constant $C'' > 0$ such that

$$\left|W_{\epsilon_k}^{(2j)}(0)\right|^{\frac{1}{2j}} \epsilon_k^{-\frac{1}{2j}} \le C'' \epsilon_k^{-\frac{1}{2p_0}}, \quad \text{for } k \text{ large enough.}$$

In order to conclude it suffices to use the definition of Ω_{ε_k}, see (3.6). The term of highest degree in the construction of Ω_{ε_k} is $|W_{\epsilon_k}^{(2k_0)}(0)|^{\frac{1}{2k_0}} \epsilon_k^{-\frac{1}{2k_0}}$ which is of order $\epsilon_k^{-\frac{1}{2k_0}} = O(\epsilon_k^{-\frac{1}{2p_0}})$, since $p_0 \le k_0$. The others components satisfy

$$\sup_{1 \le j \le k_0 - 1} \left\{ \left|W_{\epsilon_k}^{(2j)}(0)\right|^{\frac{1}{2j}} \epsilon_k^{-\frac{1}{2j}} \right\} \le C' \epsilon_k^{-\frac{1}{2p_0}}.$$

These upper-bounds combined with (3.30) permits to prove the boundedness of the sequence $\{\Omega_{\epsilon_k}/\epsilon_k^{\frac{1}{2p_0}},\ k \geq 1\} = \{\Omega_{\epsilon_k}/\epsilon_k^{\frac{1}{2m_0}},\ k \geq 1\}$.
The result announced in (3.26) is a straightforward consequence of the convergence rates proved on subsequences. Indeed it suffices to adopt similar arguments as those developed in the proof of Proposition 3.2 (Step 4). $\square$

Remark 3.4. *In the proof of Proposition 3.2, the boundedness of the family* $\left(C_j(\epsilon) = \frac{W_\epsilon^{(2j)}(0)}{\epsilon(\Omega_\epsilon)^{2j}},\ \varepsilon > 0\right)$ *for all* $1 \leq j \leq k_0$ *and the asymptotic result* $\liminf_{\varepsilon\to 0} \sup_{1\leq j\leq k_0} |C_j(\varepsilon)| > 0$ *were the main starting arguments. Furthermore the inequalities presented in* (3.26) *imply that* $C_j'(\epsilon) := W_\epsilon^{(2j)}(0)\epsilon^{\frac{j}{m_0}-1}$ *satisfies these properties too. Therefore, in the following, we shall consider* $C_j'(\epsilon)$ *and its possible limit* C_j' *instead of* $C_j(\epsilon)$ *and* C_j*. In order to simplify the notations, we shall continue to write* $C_j(\epsilon)$ *and* C_j*.*

Using the preceding results concerning the asymptotic behavior of the moments $M_{2r}(\varepsilon)$ as $\varepsilon \to 0$, we shall now focus our attention to the convergence rate of the expression $\langle f, u_\varepsilon\rangle$ towards $\langle f, u_0\rangle$ for general functions f.

Theorem 3.5. *We assume that* F'' *and* W_0 *are convex functions. Let* $(\epsilon_k)_{k\geq 1}$ *be a decreasing sequence satisfying:*

- *1)* $\lim_{k\to\infty} \varepsilon_k = 0$
- *2) for any* $1 \leq j \leq k_0$*, the sequence* $\{W_{\epsilon_k}^{(2j)}(0)\epsilon_k^{\frac{j}{m_0}-1},\ k \geq 1\}$ *converges as* $k \to \infty$*. We denote by* C_j *the associated limit.*

Then, for any function $f \in C^4(\mathbb{R}, \mathbb{R})$ *with polynomial growth, we have the following asymptotic result:*

$$\epsilon_k^{-\frac{1}{m_0}}\Big(\langle f, u_{\epsilon_k}\rangle - \langle f, u_0\rangle\Big) \xrightarrow[k\to\infty]{} \frac{f''(0)}{2}\,\frac{\int_{\mathbb{R}} x^2 \exp\left[-2\sum_{j=1}^{k_0} \frac{C_j}{(2j)!}x^{2j}\right] dx}{\int_{\mathbb{R}} \exp\left[-2\sum_{j=1}^{k_0} \frac{C_j}{(2j)!}x^{2j}\right] dx}. \quad (3.31)$$

We recall that m_0 *is defined by* (3.25).

Proof. Let us introduce the function $f_+(x) := \frac{1}{2}(f(x) + f(-x))$. Therefore we obtain $\langle f, u_{\epsilon_k}\rangle - \langle f, u_0\rangle = 2\int_{\mathbb{R}^+}(f_+(x) - f_+(0))\,u_{\epsilon_k}(x)dx$. Applying the mean value theorem to f_+, there exists a function $x \to y_x \in [0, x]$, such that

$$\int_{\mathbb{R}_+}\Big(f_+(x) - f(0)\Big)u_{\epsilon_k}(x)dx = \frac{f''(0)\,M_2(\epsilon_k)}{4\,M_0(\epsilon_k)} + \frac{1}{24}\int_{\mathbb{R}_+} f_+^{(4)}(y_x)x^4 u_{\epsilon_k}(x)dx.$$

The integral term in the rhs can be upper-bounded by a finite combination of moments $\frac{M_{2r}(\epsilon_k)}{M_0(\epsilon_k)}$, with $r \geq 2$, since $y_x \in [0; x]$ and since f_+ is of polynomial growth. Taking into account Remark 3.4, we adapt Proposition 3.2 to our particular situation. Therefore $\int_{\mathbb{R}_+} f_+^{(4)}(y_x)x^4 u_{\epsilon_k}(x)dx = o(\epsilon_k^{\frac{1}{m_0}})$. Proposition 3.2 and especially the asymptotic equivalence (3.26) yields (3.31). $\square$

Let us now precise the limit just pointed out. The following study consists in describing the whole family of coefficients $(C_j,\, 1 \le j \le k_0)$.

Corollary 3.6. *We assume that both F'' and W_0 are convex functions. Let $f \in C^4(\mathbb{R}, \mathbb{R})$ be some function with polynomial growth. Let us recall that k_0, p_0 and m_0 are defined respectively by* (3.5) *and* (3.25).
1. *First case: $k_0 < p_0$. We have:*

$$\lim_{\varepsilon \to 0} \epsilon^{-\frac{1}{m_0}} \Big(\langle f, u_\epsilon \rangle - \langle f, u_0 \rangle \Big) = \frac{1}{2} \left(\frac{(2k_0)!}{2W_0^{(2k_0)}(0)} \right)^{\frac{1}{k_0}} \frac{\Gamma\left(\frac{3}{2k_0}\right)}{\Gamma\left(\frac{1}{2k_0}\right)} f''(0). \tag{3.32}$$

We note that this convergence concerns the whole family $\{u_\varepsilon,\, \varepsilon > 0\}$.
2. *Second case: $p_0 \le k_0$. Let us consider some decreasing sequence $(\epsilon_k)_{k \ge 1}$ which tends to 0 and satisfies: $C_j(\epsilon_k) = W_{\epsilon_k}^{(2j)}(0)\epsilon_k^{j/m_0 - 1}$ converges to C_j for all $1 \le j \le k_0$. Then*

- *$C_j > 0$ for $1 \le j \le p_0 - 1$,*
- *$C_j = 0$ for $p_0 \le j \le k_0 - 1$,*
- *$C_{k_0} = W_0^{(2k_0)}(0) 1_{\{p_0 = k_0\}}$.*

Proof. Set $s(j) = \min\{r \ge 1 \mid F^{(2j+2r)}(0) \ne 0\}$ for all $1 \le j < k_0$. Let us consider some decreasing sequence $(\varepsilon_k)_{k \ge 1}$ such that $C_j(\epsilon_k) = W_{\epsilon_k}^{(2j)}(0)\epsilon_k^{j/m_0 - 1}$ converges as $k \to \infty$. Using similar results as (3.28) and (3.29), we obtain

$$W_{\epsilon_k}^{(2j)}(0) = \frac{F^{(2j+2s(j))}(0)}{(2s(j))!} \alpha_{s(j)}\, \epsilon_k^{\frac{s(j)}{m_0}} (1 + o(1)) \quad \text{as } k \to \infty \tag{3.33}$$

$$\text{with} \quad \alpha_r := \frac{\int_{\mathbb{R}^+} x^{2r} \exp\left[-2 \sum_{j=1}^{k_0} \frac{C_j}{(2j)!}\, x^{2j} \right] dx}{\int_{\mathbb{R}^+} \exp\left[-2 \sum_{j=1}^{k_0} \frac{C_j}{(2j)!}\, x^{2j} \right] dx}. \tag{3.34}$$

1. If $k_0 < p_0$ then $C_j(\varepsilon_k) \to 0$ as $k \to \infty$ for all $1 \le j < k_0$. Indeed due to the inequality $j + s(j) \ge p_0 > k_0$, (3.33) leads to the asymptotic estimate:

$$C_j(\epsilon_k) = W_{\epsilon_k}^{(2j)}(0)\epsilon_k^{\frac{j}{k_0} - 1} \approx \frac{F^{(2j+2s(j))}(0)}{(2s(j))!} \alpha_{s(j)} \epsilon_k^{\frac{j+s(j)}{k_0} - 1} \to 0, \quad \text{as } k \to \infty.$$

Hence $C_j = 0$ for $1 \le j < k_0$. Moreover $C_{k_0} = W_{\epsilon_k}^{(2k_0)}(0)$. The rhs of (3.31) can be easily computed using some change of variable. We obtain (3.32) and the limit doesn't depend on the choice of the subsequence.
2. Let us consider now the case: $p_0 \le k_0$. By similar arguments as above we obtain that $C_j = 0$ for $p_0 \le j < k_0$ and $C_j = \frac{F^{(2p_0)}(0)}{(2s(j))!} \alpha_{s(j)} > 0$ for all $1 \le j < p_0$, since $j + s(j) = p_0$. □

Let us note that, in the case $p_0 \leq k_0$, the coefficients C_j corresponding to the limit values of special subsequences are linked together by the relation

$$C_j = \frac{F^{(2p_0)}(0)}{(2(p_0-j))!} \frac{\int_{\mathbb{R}^+} x^{2(p_0-j)} \exp\left[-2\sum_{l=1}^{p_0} \frac{C_l}{(2l)!} x^{2l}\right] dx}{\int_{\mathbb{R}^+} \exp\left[-2\sum_{l=1}^{p_0} \frac{C_l}{(2l)!} x^{2l}\right] dx}, \quad \text{for } 1 \leq j < p_0.$$

If we can prove that these relations admit some unique solution $(C_j,\ 1 \leq j < p_0)$ then the result of Corollary 3.6 is sharpened. Indeed the limit value does not depend on the choice of the subsequence. The prefactor in the convergence estimate is then uniquely determined. This is for instance the case for $p_0 = 2$ but, in general, this question is open. Let us finally observe that the rate of convergence in the particular case $p_0 = 2$ is $\varepsilon^{1/2}$ which is actually different from the rate (namely ε) described in Section 3.1. In other words the comparison between the interaction function and the potential landscape respectively represented by the growth coefficient α and $-V''(0)$ is essential for the study of the invariant measure convergence rate associated to the limit measure $u_0 = \delta_0$.

4 Convergence rate for $u_0 = \frac{1}{2}\delta_{-x_0} + \frac{1}{2}\delta_{x_0}$, $x_0 > 0$

In [7], the authors describe, in the self-stabilization framework and under some conditions (the convexity of both F'' and V''), the whole set of limit measures for sequences of symmetric invariant measures. In the previous section, we focus our attention to the convergence rate for sequences associated to the limit measure $u_0 = \delta_0$. This trivial discrete measure corresponds to particular *environment functions* V and *interaction functions* F. In Section 4, we are interested by other functions V and F which permit to deal with the following discrete limit measure $u_0 = \frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$.
According to Proposition 5.3 in [7], any limit measure associated to symmetric invariant measures and which support is reduced to the set $\{-x_0, x_0\}$ with $x_0 > 0$, satisfies the following properties:

$$\begin{cases} V'(x_0) + \frac{1}{2}F'(2x_0) = 0, \\ V''(x_0) + \frac{\alpha}{2} + \frac{1}{2}F''(2x_0) \geq 0. \end{cases} \tag{4.1}$$

Furthermore the support satisfies $x_0 \leq a$, where a was introduced in (V-3). In Section 3, we considered only the situation where the link between the function F and V is characterized by the inequality $\alpha := F''(0) \geq -V''(0)$. This condition is totally adapted to the existence of the limit measure δ_0. For the discrete measure $u_0 = \frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$, Remark 3.8 in [7] implies the following relation:

$$\alpha := F''(0) < \vartheta := \sup_{x\in\mathbb{R}} -V''(x).$$

In this section, we shall focus our attention to the rate of convergence of symmetric invariant measures u_ε towards the discrete measure u_0. Let us just note that the moments of symmetric invariant measures are uniformly bounded, see Lemma 5.2 in [7]. The material is organized in a similar way as Section 3: starting with the convergence of the pseudo-potential W_ε, defined by (2.1), towards W_0, given by

$$W_0(x) := V(x) + \frac{1}{2}F(x - x_0) + \frac{1}{2}F(x + x_0) - F(x_0),$$

we analyze the asymptotic behavior of the minimum locations and deduce the expected rate.

Lemma 4.1. *If V'' and F'' are convex functions then the global minimum of W_0 is reached exactly at two points x_0 and $-x_0$. Besides, $W_0''(x_0) > 0$ i.e. $k_0 := \min\left\{k \geq 1 \,|\; W_0^{(2k)}(x_0) > 0\right\} = 1$.*

Proof. Since V'' and F'' are convex functions, Theorem 5.4 of [7] ensures the uniqueness of x_0. If we assume that $W_0''(x_0) = 0$ then $W_0^{(3)}(x_0) = 0$, W_0 reaching some local minimum for $x = x_0$. However, the convexity property of W_0'' implies that $W_0^{(3)}$ is non-decreasing. Since $W_0^{(3)}(0) = 0$ due to the symmetry of W_0, we deduce that $W_0^{(3)}(x) = 0$, for all $x \in [0, x_0]$. Hence $W_0''(0) = W_0''(x_0)$ which is of course a nonsense since $W_0''(x_0) = 0$ and $W_0''(0) = \alpha - \vartheta < 0$. Indeed, $\vartheta := \sup_{x\in\mathbb{R}} -V''(x) = -V''(0)$ since V'' is a convex function. □

As a consequence, we obtain that the set $\mathcal{B}$ containing each location of the global minimum for the pseudo potential W_0 which does not belong to the support of u_0 is empty. From now on, we shall just assume that $W_0''(x_0) > 0$ and allow $\mathcal{B}$ not to be empty. The preliminary results obtained in Section 2 namely Lemma 2.1 permit to obtain directly the following asymptotic behavior: for ε small enough, there exists some unique x_0^ε in the neighborhood $\mathcal{V}$ of x_0 such that W_ε defined by (2.1) reaches its global minimum on $\mathcal{V}$ for $x = x_0^\varepsilon$. Moreover we get the following convergence: since $W_0''(x_0) > 0$, x_0^ϵ converges towards x_0 and

$$x_0^\epsilon = x_0 - \frac{W_\epsilon'(x_0)}{W_0''(x_0)} + o\left\{W_\epsilon'(x_0)\right\}. \tag{4.2}$$

This convergence can even be more precise.

Theorem 4.2. *If $W_0''(x_0) > 0$, under the condition* (2.8), *we get*

$$\lim_{\varepsilon\to 0} \frac{x_0^\epsilon - x_0}{\epsilon} = \frac{W_0^{(3)}(x_0)\left(F''(2x_0) - \alpha\right) - F^{(3)}(2x_0)W_0''(x_0)}{8W_0''(x_0)^2\left(V''(x_0) + F''(2x_0)\right)}. \tag{4.3}$$

The proof of Theorem 4.2 is essentially based on two lemmas: Lemma 4.3 and Lemma 4.4. The first one deals with some integral estimate in the spirit of (2.4) and permits to prove the second one which describes the asymptotic behavior of the following expression $W_\epsilon'(x_0)/\varepsilon$. It suffices then to consider (4.2) in order to finish the proof. The details are left to the reader.

Lemma 4.3. *Let us assume* (2.8). *For any function* $f \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ *with polynomial growth, the following estimate holds:*

$$\int_{\mathbb{R}} f(x)e^{-\frac{2}{\epsilon}W_\epsilon(x)}dx = 2\sqrt{\frac{\pi\epsilon}{W_\epsilon''(x_0^\epsilon)}}\, e^{-\frac{2W_\epsilon(x_0^\epsilon)}{\epsilon}}\Big\{f_+(x_0^\epsilon)+\gamma(f)\epsilon+o(\epsilon)\Big\} \tag{4.4}$$

where

$$\gamma(f) := \left(\frac{5\ \mathcal{W}_3^2}{48\ \mathcal{W}_2^3}-\frac{\mathcal{W}_4}{16\ \mathcal{W}_2^2}\right)f_+(x_0)-\frac{\mathcal{W}_3}{4\ \mathcal{W}_2^2}f_+'(x_0)+\frac{f_+''(x_0)}{4\ \mathcal{W}_2}. \tag{4.5}$$

Here $\mathcal{W}_k := W_0^{(k)}(x_0)$ *and* $f_+(x) := (f(x)+f(-x))/2$.

Proof. We recall that Lemma 2.1 provides directly the existence of $\pm x_0^\epsilon$. Moreover (2.4) combined with Assumption 2.1 permits to obtain (see the comments following Assumption 2.1):

$$\int_{\mathbb{R}} f(t)e^{\frac{-2W_\epsilon(t)}{\epsilon}}dt = 2\sqrt{\frac{\pi\epsilon}{W_\epsilon''(x_0^\epsilon)}}\, e^{-\frac{2W_\epsilon(x_0^\epsilon)}{\epsilon}}\left\{f_+(x_0^\epsilon)+\frac{\gamma_+(f)+\gamma_-(f)}{2}\epsilon+o(\epsilon)\right\}$$

with $\gamma_\pm(f) := f(\pm x_0)\left(\frac{5\ \mathcal{W}_{3,\epsilon}^2}{48\ \mathcal{W}_{2,\epsilon}^3}-\frac{\mathcal{W}_{4,\epsilon}}{16\ \mathcal{W}_{2,\epsilon}^2}\right)-f'(\pm x_0)\frac{\pm\mathcal{W}_{3,\epsilon}}{4\ \mathcal{W}_{2,\epsilon}^2}+\frac{f''(\pm x_0)}{4\ \mathcal{W}_{2,\epsilon}}$ and $\mathcal{W}_{k,\epsilon} := W_\epsilon^{(k)}(x_0^\epsilon)$. In order to prove (4.4), it suffices to note that x_0^ε converges towards x_0 and that $W_\epsilon^{(k)}$ converges uniformly towards $W_0^{(k)}$ (see Section 2) as $\varepsilon \to 0$. $\square$

Lemma 4.4. *Let* $W_0''(x_0) > 0$. *Under the condition* (2.8), *we have:*

$$\lim_{\varepsilon\to 0}\frac{W_\epsilon'(x_0)}{\epsilon} = \frac{F^{(3)}(2x_0)W_0''(x_0)-W_0^{(3)}(x_0)\left(F''(2x_0)-\alpha\right)}{8W_0''(x_0)\left(V''(x_0)+F''(2x_0)\right)}. \tag{4.6}$$

Proof. Since x_0 is the location of a local extremum for W_0, $W_0'(x_0)=0$. Hence, defining $\xi(z) := F'(x_0-z)$, we get

$$\begin{aligned} W_\epsilon'(x_0) &= W_\epsilon'(x_0)-W_0'(x_0) = \int_{\mathbb{R}}\xi(z)u_\epsilon(z)dz-\frac{1}{2}\Big(\xi(x_0)+\xi(-x_0)\Big) \\ &= \frac{\int_{\mathbb{R}}\xi(z)\exp\left[-\frac{2}{\epsilon}W_\epsilon(z)\right]dz}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}W_\epsilon(z)\right]dz}-\xi_+(x_0). \end{aligned}$$

By Lemma 4.3 the following estimates holds:

$$\int_{\mathbb{R}}\xi(x)e^{-\frac{2}{\epsilon}W_\epsilon(x)}dx = 2\sqrt{\frac{\pi\epsilon}{W_\epsilon''(x_0^\epsilon)}}\, e^{-\frac{2W_\epsilon(x_0^\epsilon)}{\epsilon}}\Big\{\xi_+(x_0^\epsilon)+\gamma(\xi)\epsilon+o(\epsilon)\Big\} \tag{4.7}$$

$$\int_{\mathbb{R}}e^{-\frac{2}{\epsilon}W_\epsilon(x)}dx = 2\sqrt{\frac{\pi\epsilon}{W_\epsilon''(x_0^\epsilon)}}\, e^{-\frac{2W_\epsilon(x_0^\epsilon)}{\epsilon}}\Big\{1+\gamma(1)\epsilon+o(\epsilon)\Big\}. \tag{4.8}$$

Let us divide (4.7) by (4.8). Therefore

$$\int_{\mathbb{R}}\xi(x)u_\epsilon(x)dx = \xi_+(x_0^\epsilon)+\Big(\gamma(\xi)-\xi_+(x_0^\epsilon)\gamma(1)\Big)\epsilon+o(\epsilon).$$

The definition of γ leads to:

$$\int_{\mathbb{R}} \xi(x) u_\epsilon(x) dx = \xi_+(x_0^\epsilon) + \left(-\frac{\mathcal{W}_3}{4\,\mathcal{W}_2^2}\ \xi_+'(x_0) + \frac{\xi_+''(x_0)}{4\,\mathcal{W}_2}\right)\epsilon + o(\epsilon). \tag{4.9}$$

Therefore, we have

$$\lim_{\varepsilon\to 0} \frac{W_\epsilon'(x_0)}{\varepsilon}\left\{1 - \frac{\xi_+(x_0^\varepsilon) - \xi_+(x_0)}{x_0^\varepsilon - x_0}\frac{x_0^\varepsilon - x_0}{W_\epsilon'(x_0)}\right\} = -\frac{\mathcal{W}_3}{4\,\mathcal{W}_2^2}\ \xi_+'(x_0) + \frac{\xi_+''(x_0)}{4\,\mathcal{W}_2}.$$

By (4.2), we get

$$\lim_{\varepsilon\to 0} \frac{W_\epsilon'(x_0)}{\varepsilon}\left\{1 + \frac{\xi_+'(x_0)}{W_0''(x_0)}\right\} = -\frac{\mathcal{W}_3}{4\,\mathcal{W}_2^2}\ \xi_+'(x_0) + \frac{\xi_+''(x_0)}{4\,\mathcal{W}_2}. \tag{4.10}$$

Since $\mathcal{W}_3 = W_0^{(3)}(x_0) = V^{(3)}(x_0) + F^{(3)}(x_0)/2$ and $\mathcal{W}_2 = W_0''(x_0) = V''(x_0) + \frac{\alpha}{2} + \frac{F''(2x_0)}{2}$, the announced limit (4.6) is proved. □

Finally we obtain the desired result concerning the convergence rate:

Theorem 4.5. *Let $W_0''(x_0) > 0$. Under the condition* (2.8)*, for any function $f \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ with polynomial growth, the following convergence rate holds*

$$\lim_{\varepsilon\to 0} \frac{\langle f, u_\epsilon\rangle - \langle f, u_0\rangle}{\epsilon} = \frac{f''(x_0) + f''(-x_0)}{8W_0''(x_0)} + \chi(x_0)\ \frac{f'(x_0) - f'(-x_0)}{8W_0''(x_0)}$$

where $\chi(x_0) := -\dfrac{V^{(3)}(x_0) + F^{(3)}(2x_0)}{V''(x_0) + F''(2x_0)}$.

Proof. Since $u_0 = \frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$, the difference $\langle f, u_\epsilon\rangle - \langle f, u_0\rangle$ equals

$$\int_{\mathbb{R}} f(x) u_\epsilon(x) dx - f_+(x_0) \quad \text{where } f_+(x) = \frac{1}{2}\Big(f(x) + f(-x)\Big).$$

Applying Lemma 4.3 to the functions f and 1, we obtain the estimate of the ratio. Hence

$$\int_{\mathbb{R}} f(x) u_\epsilon(x) dx = f_+(x_0^\epsilon) + \left(-\frac{\mathcal{W}_3}{4\,\mathcal{W}_2^2} f_+'(x_0) + \frac{f_+''(x_0)}{4\,\mathcal{W}_2}\right)\epsilon + o(\epsilon). \tag{4.11}$$

Therefore, defining

$$\mathcal{T} := \lim_{\varepsilon\to 0}\left\{\frac{\langle f, u_\varepsilon\rangle - \langle f, u_0\rangle}{\varepsilon} - \frac{f_+(x_0^\varepsilon) - f_+(x_0)}{x_0^\varepsilon - x_0}\frac{x_0^\varepsilon - x_0}{W_\varepsilon'(x_0)}\frac{W_\varepsilon'(x_0)}{\varepsilon}\right\},$$

we get $\mathcal{T} = \dfrac{f_+''(x_0)}{4W_0''(x_0)} - \dfrac{\mathcal{W}_3 f_+'(x_0)}{4\,\mathcal{W}_2^2}$. Obviously $\lim\limits_{\varepsilon\to 0} \dfrac{f_+(x_0^\varepsilon) - f_+(x_0)}{x_0^\varepsilon - x_0} = f_+'(x_0)$, (4.2) implies $\lim_{\varepsilon\to 0}(x_0^\varepsilon - x_0)/W_\varepsilon'(x_0) = -1/W_0''(x_0)$ and $\lim_{\varepsilon\to 0} W_\varepsilon'(x_0)/\varepsilon$ is determined by Lemma 4.4. Hence

$$\lim_{\varepsilon\to 0} \frac{\langle f, u_\varepsilon\rangle - \langle f, u_0\rangle}{\varepsilon} = \frac{f_+''(x_0)}{4W_0''(x_0)} + f_+'(x_0)\Delta(x_0),$$

with

$$\Delta(x_0) := \frac{W_0^{(3)}(x_0)\left(F''(2x_0)-\alpha\right)-F^{(3)}(2x_0)W_0''(x_0)}{8W_0''(x_0)^2\left(V''(x_0)+F''(2x_0)\right)}-\frac{\mathcal{W}_3}{4\,\mathcal{W}_2^2}$$
$$= -\frac{2W_0^{(3)}(x_0)+F^{(3)}(2x_0)}{8W_0''(x_0)\left(V''(x_0)+F''(2x_0)\right)}.$$

The proof is achieved since $W_0^{(3)}(x_0)=V^{(3)}(x_0)+\frac{1}{2}F^{(3)}(2x_0)$. □

5 Around the condition $\alpha = \vartheta = -V''(0)$

The study described in the preceding sections points out different rates of convergence which do not really depend on the corresponding discrete limit measure u_0. The difference comes essentially from the comparison between $\alpha := F''(0)$ and $-V''(0)$. Roughly speaking, if $\alpha \neq -V''(0)$ we obtain some rate like ε and if $\alpha = -V''(0)$ we get ε^{1/m_0} with $m_0 \geq 2$. The aim of this section is to prove that we can observed intermediate rates of convergence around the condition $\alpha = -V''(0)$ provided that the interaction function F depends on the small parameter ε. This section is based on some particular example, nevertheless extensions to general situations can easily be proved.
Let us first assume that V'' and F'' are convex functions. Consequently

$$\vartheta := \sup_{x\in\mathbb{R}} -V''(x) = -V''(0).$$

We consider symmetric stationary measures u_ϵ (whose $2n$-th moment is uniformly bounded) converging towards u_0, according to the results developed in the preceding sections (Theorem 3.1, Theorem 3.5 and Theorem 4.5), we get: for any $f \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ with polynomial growth,

$$\lim_{\epsilon\to 0}\frac{\log|\langle f,u_\epsilon\rangle-\langle f,u_0\rangle|}{\log\epsilon} = \Lambda(\alpha) := \begin{cases} 1 & \text{if } \alpha\neq\vartheta \\ 1/m_0 & \text{if } \alpha=\vartheta, \end{cases} \tag{5.1}$$

where m_0 is defined by (3.25). In order to produce intermediate rate, we construct some particular model. First we choose the reference environment (resp. interaction) function $V(x) := \frac{x^4}{4}-\frac{x^2}{2}$ (resp. $F(x) := \frac{x^4}{4}+\frac{x^2}{2}$). Obviously $\vartheta=\alpha=1$. The example is based on some small perturbation of this reference: we consider the association between V and one of the following interaction functions

$$F_\epsilon^+(x) := \frac{x^4}{4}+\frac{(1+\rho\epsilon^\eta)\,x^2}{2} \quad\text{or}\quad F_\epsilon^-(x) := \frac{x^4}{4}+\frac{(1-\rho\epsilon^\eta)\,x^2}{2},$$

with $\rho>0$. In other words, (1.2) leads to the study of symmetric invariant measures of the following self-stabilizing process:

$$\begin{cases} dX_t^\epsilon = \sqrt{\epsilon}dB_t - \left\{2\left(X_t^\epsilon\right)^3+\left(3m_2(\epsilon,t)\pm\rho\epsilon^\eta\right)X_t^\epsilon\right\}dt \\ m_2(\epsilon,t)=\mathbb{E}\left[\left(X_t^\epsilon\right)^2\right]. \end{cases} \tag{5.2}$$

We remove the odd moments appearing in the equation since we focus our attention to symmetric laws. There exists some unique strong solution for (5.2) (see [5]) and some symmetric invariant measure (see [6]) denoted by $u_\varepsilon^\pm$. We define $m_2^\pm(\epsilon) := \int_{\mathbb{R}} x^2 u_\epsilon^\pm(x)dx$. By (2.2), the second moment $m = m_2^\pm(\epsilon)$ satisfies

$$m = \frac{\int_{\mathbb{R}_+} x^2 \exp\left[-\frac{1}{\varepsilon}\left((3m \pm \rho\epsilon^\eta)x^2 + x^4\right)\right] dx}{\int_{\mathbb{R}_+} \exp\left[-\frac{1}{\varepsilon}\left((3m \pm \rho\epsilon^\eta)x^2 + x^4\right)\right] dx}. \tag{5.3}$$

Proposition 5.1. *For all $\eta > 0$ and $\rho > 0$, the sequence of symmetric invariant measures $(u_\epsilon^\pm, \varepsilon > 0)$ converges weakly to δ_0.*

Proof. Let us assume the existence of some positive constant C and some decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ converging towards 0 such that $m_2^\pm(\epsilon_k) > C$ for all $k \in \mathbb{N}$ (for notational simplicity, we shall drop the index k). We apply the following change of variable $x := \sqrt{\epsilon}y$ to (5.3) and get $m_2^\pm(\varepsilon) = \varepsilon\,\xi(m_2^\pm(\epsilon) \pm \rho\epsilon^\eta/3, \varepsilon)$ where

$$\xi(u,v) := \int_{\mathbb{R}_+} y^2 \nu_{u,v}(y)dy \quad \text{and} \quad \nu_{u,v}(y) := \frac{\exp\left[-3uy^2 - vy^4\right]}{\int_{\mathbb{R}} \exp\left[-3uz^2 - vz^4\right] dz}. \tag{5.4}$$

Let $u \in \mathbb{R}$, $v > 0$. We construct Y some random variable whose distribution is given by $\nu_{u,v}$. Then, by Jensen's inequality

$$\frac{\partial\xi}{\partial u}(u,v) = -3\mathbb{E}[Y^4] + 3\mathbb{E}[Y^2]^2 < 0; \quad \frac{\partial\xi}{\partial v}(u,v) = -\mathbb{E}[Y^6] + \mathbb{E}[Y^2]\mathbb{E}[Y^4] < 0.$$

Since $3m_2(\epsilon) \pm \rho\epsilon^\eta > C$, for ϵ small enough, we deduce that $m_2^\pm(\epsilon) \leq \epsilon\,\xi(C/3, 0)$. The r.h.s tends to 0 with ε which is a nonsense because $m_2^\pm(\varepsilon) > C$. Therefore we deduce that $m_2^\pm(\epsilon)$ converges to 0 as ε decreases. $\square$

In order to emphasize some intermediate rate of convergence, we shall just estimate the convergence of $m_2^\pm(\varepsilon)$ as $\varepsilon \to 0$ instead of the general expression $\langle f, u_\varepsilon^\pm\rangle$ since $\langle f, u_\varepsilon^\pm\rangle - \langle f, u_0^\pm\rangle$ is directly linked to $m_2^\pm(\varepsilon)$.

Proposition 5.2. *Let $\eta > 0$ and $\rho > 0$. Then*

$$\lim_{\epsilon\to 0} \frac{\log m_2^+(\epsilon)}{\log \epsilon} = \Lambda_+(\eta) := 1 - \min\left\{\eta; \frac{1}{2}\right\}. \tag{5.5}$$

Proof. Let us recall that $m_2^+(\varepsilon)$ satisfies $m_2^+(\varepsilon) = \varepsilon\,\xi(m_2^+(\epsilon) + \rho\epsilon^\eta/3, \varepsilon)$ where ξ is defined by (5.4). According to the proof of Proposition 5.1, the function ξ is decreasing with respect to both variables. Therefore we can compute some upper-bound of $m_2^+(\varepsilon)$ just by noting that $m_2^+(\epsilon) + \rho\epsilon^\eta/3 \geq m_2^+(\varepsilon)$ and $m_2^+(\epsilon) + \rho\epsilon^\eta/3 \geq \rho\varepsilon^\eta/3$.

- The first inequality leads to $m_2^+(\varepsilon) \leq \varepsilon\xi(m_2^+(\varepsilon), 0)$. The r.h.s can be computed by some change of variable: there exists some constant $c_0 > 0$ such that $m_2^+(\varepsilon) \leq c_0\sqrt{\varepsilon}$.

- The second inequality implies $m_2^+(\varepsilon) \le \varepsilon\xi(\rho\varepsilon^\eta/3, 0)$. The same arguments permit to obtain the existence of $c_1 > 0$ such that $m_2^+(\varepsilon) \le c_1\varepsilon^{1-\eta}$.

Hence, for ε small enough, we get the following bound

$$m_2^+(\varepsilon) \le \max\{c_1, c_2\}\varepsilon^{\Lambda_+(\eta)}. \tag{5.6}$$

Let us now prove the lower-bound. By (5.6), for ε small enough, there exists $c_3 > 0$ such that $m_2^+(\varepsilon)+\rho\varepsilon^\eta/3 \le c_3\varepsilon^{\min\{\eta,1/2\}}$. Since ξ is decreasing, we obtain: $m_2^+(\varepsilon) \ge \varepsilon\xi(c_3\varepsilon^{\min\{\eta,1/2\}}, \varepsilon^{\min\{2\eta,1\}})$. By some classical change of variable, we get immediately $m_2^+(\varepsilon) \ge c_4\varepsilon^{\Lambda_+(\eta)}$ with $c_4 > 0$. □

Let us now precise the asymptotic behavior of $m_2^+(\varepsilon)$.

Corollary 5.3. *Let $\eta > 0$ and $\rho > 0$. Then*

$$\lim_{\epsilon\to 0} \frac{m_2^+(\epsilon)}{\epsilon^{\Lambda_+(\eta)}} = \lambda_+(\eta) := \begin{cases} (2\rho)^{-1} & \textit{if} \quad 2\eta < 1 \\ x_\rho^+ & \textit{if} \quad 2\eta = 1 \\ x_0 & \textit{if} \quad 2\eta > 1. \end{cases} \tag{5.7}$$

Here x_0 (respectively x_ρ^+) is the unique solution of $x_0 = \xi(x_0, 1)$ (resp. $x_\rho^+ = \xi(x_\rho^+ + \rho/3, 1)$, see (5.4) *for the definition of ξ).*

Proof. **1.** Let us consider the first case: $\eta < 1/2$ i.e. $\Lambda_+(\eta) = 1 - \eta$. Applying the change of variable $x := \epsilon^{\frac{\eta}{2}}y$ to (5.3), we obtain

$$m_2^+(\varepsilon) = \varepsilon^\eta\,\xi\Big(\frac{m_2^+(\varepsilon)\varepsilon^{-\eta} + \rho/3}{\varepsilon^{1-2\eta}}, \frac{1}{\varepsilon^{1-2\eta}}\Big). \tag{5.8}$$

Since $m_2^+(\epsilon)\epsilon^{-\eta}$ tends to 0, we can use Lemma A.2 with $U(x) = \frac{\rho+3m_2(\epsilon)\epsilon^{-\eta}}{2}y^2 + \frac{y^4}{2}$ and $n = 2$. Thus the following estimate holds $m_2^+(\varepsilon) \approx \frac{\varepsilon^{1-\eta}}{2U''(0)} = \frac{\epsilon^{1-\eta}}{2\rho}$.
2. Let us consider now $\eta = 1/2$ which implies $\Lambda_+(\eta) = \frac{1}{2}$. By the same argument, we obtain (5.8). By (5.5) we know that $m_2^+(\varepsilon)/\sqrt{\varepsilon}$ is bounded w.r.t ε. Any limit value $\mathcal{M}$ satisfies $\mathcal{M} = \xi(\mathcal{M} + \rho/3, 1)$. Due to the strict monotonicity of ξ, this equation admits some unique solution denoted by x_ρ^+.
3. The arguments to prove the third case $\eta > 1/2$ are similar to those presented in the second one. The details are left to the reader. □

The same kind of convergence rate can be analyzed for $m_2^-(\varepsilon)$.

Proposition 5.4. *Let $\eta > 0$ and $\rho > 0$. Then*

$$\lim_{\epsilon\to 0} \frac{\log m_2^-(\epsilon)}{\log\epsilon} = \Lambda_-(\eta) = \min\left\{\eta; \frac{1}{2}\right\}. \tag{5.9}$$

Proof. **1.** Let us first assume that $\eta > 1/2$. Applying the change of variable $x := \epsilon^{\frac{1}{4}}y$ to (5.3) and by the definition of ξ (5.4) we obtain

$$\frac{m_2^-(\varepsilon)}{\sqrt{\varepsilon}} = \xi\Big(\frac{m_2^-(\varepsilon)}{\sqrt{\varepsilon}} - \rho\varepsilon^{\eta-1/2}/3, 1\Big). \tag{5.10}$$

Let us assume that $\liminf_{\epsilon\to 0} m_2^-(\epsilon)/\sqrt{\epsilon} = 0$, then we consider some sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ converging towards 0 such that $\lim_{\epsilon\to 0} m_2^-(\epsilon_k)/\sqrt{\epsilon_k} = 0$. Due to the continuity of the function ξ, we can let ε_k tend to 0 in (5.10). We immediately find some contradiction : $0 = \xi(0,1) = \Gamma(3/4)/\Gamma(1/4)$. Therefore we deduce that, for ε small enough, there exists some constant $c > 0$ such that $m_2^-(\varepsilon) \geq c\sqrt{\varepsilon}$. Since ξ is a decreasing function with respect to both variables, (5.10) leads, for ε small enough, to $m_2^-(\varepsilon) \leq \sqrt{\varepsilon}\xi(c/2,1)$ which achieves the proof in the first case. These arguments can also be used for the case $\eta = 1/2$, it suffices to replace $0 = \xi(0,1)$ by $0 = \xi(-\rho/3,1)$ in the contradiction statement.
2. Let us finally assume that $\eta < 1/2$. By some change of variable, (5.3) is equivalent to

$$\frac{m_2^-(\varepsilon)}{\varepsilon^\eta} = \xi\Big(\frac{m_2^-(\varepsilon)\varepsilon^{-\eta} - \rho/3}{\varepsilon^{1-2\eta}}, \varepsilon^{2\eta-1}\Big). \tag{5.11}$$

The monotonicity of ξ leads to $m_2^-(\varepsilon) \leq \varepsilon^\eta\xi(-\rho\varepsilon^{2\eta-1}/3, \varepsilon^{2\eta-1})$. Using Laplace's method, we estimate the r.h.s which is equivalent to $\frac{\rho}{2}\varepsilon^\eta$. We deduce that asymptotically $m_2^-(\varepsilon) \leq \rho\varepsilon^\eta$. Let us now assume that we can find some sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ converging towards 0 and such that $m_2^-(\varepsilon_k) \leq \rho\varepsilon_k^\eta/6$. By (5.11) we deduce the inequality $m_2^-(\varepsilon_k) \geq \varepsilon_k^\eta\xi(-\rho\varepsilon_k^{2\eta-1}/6, \varepsilon_k^{2\eta-1})$. By Laplace's method we find that the r.h.s. is equivalent to $\rho\varepsilon_k^\eta/4$, for large k, which contradicts the hypothesis. We deduce that $m_2^-(\varepsilon) \geq \rho\varepsilon^\eta/6$. □

Let us note that Proposition 5.2 and Proposition 5.4 point out that the behavior of the second moment is not symmetric with respect to the critical value $\alpha = \vartheta$. On one hand we find $\min\{\frac{1}{2};\eta\}$ and on the other hand $1 - \min\{\frac{1}{2};\eta\}$. Some heuristic argument which could explain this difference is based on the limit measure. We know that the measure considered when $\alpha \geq \vartheta$ is the trivial measure δ_0 whereas the support of the limit measure for $\alpha < \vartheta$ contains two points: u_ϵ^- is then farther from δ_0.
Let us precise the asymptotics for $m_2^-(\varepsilon)$.

Corollary 5.5. *Let $\eta > 0$ and $\rho > 0$. Then*

$$\lim_{\epsilon\to 0} \frac{m_2^-(\epsilon)}{\epsilon^{\Lambda_-(\eta)}} = \lambda_-(\eta) := \begin{cases} \rho/5 & \text{if } \ 2\eta < 1 \\ x_\rho^- & \text{if } \ 2\eta = 1 \\ x_0 & \text{if } \ 2\eta > 1. \end{cases} \tag{5.12}$$

Here x_0 (resp. x_ρ^-) is the unique solution of $x_0 = \xi(x_0,1)$ (resp. $x_\rho^- = \xi(x_\rho^- - \rho/3, 1)$, see (5.4) for the definition of ξ).

The proof of Corollary 5.5 is based on similar arguments as those used in Corollary 5.3. The proof is left to the reader.

Let us illustrate with some simulation: we choose $\varepsilon := 10^{-10}$. The following figures represent $\{\log(m_2^\pm(\varepsilon)) - \log(\lambda_\pm)\}/\log\varepsilon$ with respect to η. The continuous line represent $\min\{\eta, 1/2\}$ respectively $1 - \min\{\eta, 1/2\}$. The discontinuity appearing in the simulation is due to the prefactors in the asymptotic estimates.

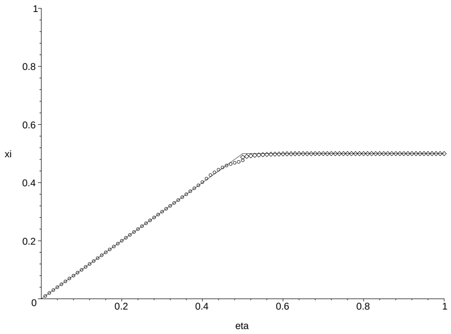

Figure 2: Description of m_2^-.

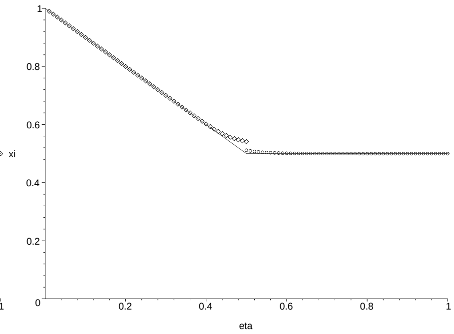

Figure 3: Description of m_2^+.

6 Convergence rate associated with $u_0^{\pm} = \delta_{\pm a}$

In the preceding sections, we deal with symmetric invariant measures associated to the self-stabilizing process (1.2). The aim of this section is to present convergence rates associated to non-symmetric stationary measures. For that purpose, we introduce the so-called *outlying* invariant measures, introduced in [6], which are concentrated around $\delta_{\pm a}$ in the small ε limit. Here a and $-a$ represent the locations of the global minimum of the environment potential V. In this section, we shall admit the existence of these extremal outlying stationary measures for ϵ small enough. In other words, we assume the existence of a sequence of stationary measures $\left(u_{\epsilon_k}^{\pm}\right)_{k\in\mathbb{N}^*}$ which converges to $\delta_{\pm a}$. We will drop the k for notational simplicity. Let us just note that this main assumption is satisfied in many situations. Let $2n$ be the degree of F. According to Theorem 4.6 in [6] and Proposition 4.1 in [7], we know that the following condition is sufficient in order to ensure this existence:

$$\sum_{p=0}^{2n-2} \frac{\left|F^{(p+2)}(a)\right|}{p!} a^p < \alpha + V''(a). \tag{6.1}$$

Let us denote by $W_\epsilon^{\pm}$ the pseudo-potential associated with these outlying measures (see (2.1) for the definition of the pseudo-potential). $(W_\varepsilon^{\pm})^{(j)}$ converges uniformly towards $(W_0^{\pm})^{(j)}$ as $\varepsilon \to 0$. The limit pseudo-potential is given by

$$W_0^{\pm} := V + F(. - (\pm a)) - F(a). \tag{6.2}$$

Let us also assume that V'' and F'' are convex functions. In particular, Condition (2.8) is satisfied since $\pm a$ is the unique location of the global minimum of $W_0^{\pm}$. In order to present the convergence rate of $u_\varepsilon^{\pm}$ towards $u_0^{\pm}$, we shall essentially apply the procedure presented in Section 4. By symmetry, it suffices to study u_ε^{+}, so in the following we delete the exponent symbol.
First of all, in order to apply Lemma 2.1, we just observe that $W_0''(a) =$

$\alpha + V''(a) > 0$ and deduce the following result: for ϵ small enough, W_ϵ reaches its global minimum only at the point a_ϵ which satisfies moreover:

$$a_\epsilon = a - \frac{W'_\epsilon(a)}{\alpha + V''(a)} + o\left\{W'_\epsilon(a)\right\}. \tag{6.3}$$

This convergence can even be more precise.

Theorem 6.1. *The distance between a and a_ε satisfies:*

$$\lim_{\varepsilon\to 0} \frac{a_\epsilon - a}{\epsilon} = -\frac{\alpha V^{(3)}(a)}{4V''(a)\left(\alpha + V''(a)\right)^2}. \tag{6.4}$$

The proof of this theorem is based on the decomposition: $\lim_{\varepsilon\to 0} \frac{a_\epsilon - a}{W'_\varepsilon(a)} \frac{W'_\varepsilon(a)}{\epsilon}$. The limit value of the first ratio is determined by (6.3). It suffices to study the second ratio.

Proposition 6.2. *The following convergence result holds:*

$$\lim_{\varepsilon\to 0} \frac{W'_\epsilon(a)}{\epsilon} = \frac{\alpha V^{(3)}(a)}{4V''(a)\left(\alpha + V''(a)\right)}. \tag{6.5}$$

Proof. Since a is the location of a local minimum of V, $W'_0(a) = 0$ and so

$$W'_\epsilon(a) = W'_\varepsilon(a) - W'_0(a) = \int_{\mathbb{R}} F'(a-z)u_\epsilon(z)dz.$$

We define $\xi(z) := F'(a-z)$ and proceed similarly to the proof of Lemma 4.4. Applying twice Lemma 2.1 to the functions $f(t) := \xi(t)$ and $f(t) := 1$ and computing the ratio permits to obtain

$$\int_{\mathbb{R}} \xi(x)u_\epsilon(x)dx \quad = \quad \xi(a_\epsilon) + \gamma_1(\xi)\epsilon - \xi(a_\epsilon)\gamma_1(1)\epsilon + o(\epsilon), \tag{6.6}$$

where γ_1 is defined by (2.5) with $A_1 = a$. In other words,

$$\int_{\mathbb{R}} \xi(x)u_\epsilon(x)dx \quad = \quad \xi(a_\epsilon) + \left(-\frac{\mathcal{W}_3}{4\,\mathcal{W}_2^2}\xi'(a) + \frac{\xi''(a)}{4\,\mathcal{W}_2}\right)\epsilon + o(\epsilon). \tag{6.7}$$

Therefore, we have

$$\lim_{\varepsilon\to 0} \frac{W'_\varepsilon(a)}{\varepsilon}\left\{1 - \frac{\xi(a_\varepsilon) - \xi(a)}{a_\varepsilon - a}\frac{a_\varepsilon - a}{W'_\varepsilon(a)}\right\} = -\frac{\mathcal{W}_3}{4\,\mathcal{W}_2^2}\xi'(a) + \frac{\xi''(a)}{4\,\mathcal{W}_2}.$$

It suffices in fact to replace in (4.9) ξ_+ by ξ, x_0 by a and x_0^ε by a^ε. The asymptotic result (4.10) is then satisfied. In order to finish the proof, let us note that $\mathcal{W}_3 = W^{(3)}(a) = V^{(3)}(a)$, $\mathcal{W}_2 = V''(a) + \alpha = W''_0(a)$, $\xi'(a) = -\alpha$ and $\xi''(a) = 0$. $\square$

Finally we obtain the wished convergence rate.

Theorem 6.3. *Let $f \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ with polynomial growth. Then,*

$$\lim_{\varepsilon\to 0}\frac{1}{\epsilon}\Big\{\langle f,u_\epsilon\rangle-\langle f,u_0\rangle\Big\}=\frac{V''(a)f''(a)-V^{(3)}(a)f'(a)}{4V''(a)\left(\alpha+V''(a)\right)}. \tag{6.8}$$

Proof. Let us recall that $u_0=\delta_a$. Hence $\langle f,u_\epsilon\rangle-\langle f,u_0\rangle=\int_{\mathbb{R}}f(x)u_\epsilon(x)dx-f(a)$. Obviously the proof is similar to that of Theorem 4.5. It suffices to replace f_+ by f, x_0 by a and x_0^ε by a_ε. So we obtain directly

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{\int_{\mathbb{R}}f(x)u_\epsilon(x)dx-f(a)\Big\}=\frac{f''(a)}{4W_0''(a)}+f'(a)\Delta(a), \tag{6.9}$$

where

$$\Delta(a):=\frac{-\alpha V^{(3)}(a)}{4V''(a)\left(\alpha+V''(a)\right)^2}-\frac{V^{(3)}(a)}{4\left(\alpha+V''(a)\right)^2}=-\frac{V^{(3)}(a)}{4V''(a)\left(\alpha+V''(a)\right)}.$$

The combination of both the definition of $\Delta(a)$ and (6.9) leads to (6.8). $\square$

Remark 6.4. *Theorem 4.8 in [6] can be presented as a consequence of Theorem 6.3 applied to the particular polynomial function $f(x):=x^k$. However the statement of the theorem is much more accurate. Indeed, on one hand, the authors proved in [6] that there exists an outlying stationary measure whose k-th first moments are closed to $a^k-ka^{k-1}\frac{aV^{(3)}(a)-(k-1)V''(a)}{4aV''(a)(\alpha+V''(a))}\varepsilon$. On the other hand, we prove in Theorem 6.3 that any stationary outlying measure around a has such moments.*

Remark 6.5. *In this section we consider general invariant measures converging towards a discrete limit measure with trivial support δ_a. In fact in the proof of Theorem 6.3 the value a does not play some crucial role: it suffices that it characterizes the limit measure. We deduce therefore that Theorem 6.3 can be applied to δ_0, it suffices to replace a by 0 in the statement. In other words, Theorem 3.1 which concerns only* symmetric *invariant measures can be extended to the whole set of invariant measures converging towards δ_0.*

7 Uniqueness problem for the stationary measures

The study about the convergence rate, developed in the preceding sections, permits to estimate the moments of the associated stationary measures. This feature is crucial for the uniqueness problem. Indeed, since F is a polynomial function of degree $2n$, the $2n-1$ first moments of an invariant measure characterize completely this measure (see the discussion introducing Section 4.3 in [6]). This essential property shall be used to discuss the uniqueness problem: in fact, we know that, under simple conditions, there exists several invariant measures for the self-stabilizing process (see [6]). However we want to precise the statement in order to describe the set of all invariant measures. This set was already explicitly presented in [6] for particular situations, namely when V'' is a convex function and F' is linear. Our aim is to extend this result to more general interaction functions.

We will now assume that the functions F and V satisfy (6.1) which implies the existence of the so-called *outlying* stationary measures: invariant measures converging towards $\delta_{\pm a}$. Moreover V'' and F'' shall be convex functions.

Let u_ϵ be an invariant measure for the self-stabilizing process and $\mu_1(\epsilon)$,..., $\mu_{2n-1}(\epsilon)$ its first $2n-1$ moments. Let us assume that u_ϵ converges to $u_0 \in \left\{\delta_a;\delta_{-a};\frac{1}{2}\delta_{x_0}+\frac{1}{2}\delta_{-x_0}\right\}$ where x_0 is the non-negative solution of

$$2V'(x_0)+F'(2x_0)=0 \quad \text{and } 2V''(x_0)+F''(2x_0)+\alpha\geq 0.$$

In [7] we proved that the condition $\alpha\geq -V''(0)$ is equivalent to $x_0=0$ and so $\frac{1}{2}\delta_{x_0}+\frac{1}{2}\delta_{-x_0}=\delta_0$. We denote by W_ϵ the pseudo-potential associated with u_ϵ and defined by (2.1), W_0 the limit pseudo-potential associated with u_0, and $m_k(0)$ the k-th moment of u_0.

For any measure u whose $2n$ first moments are bounded (we denoted these moments by $(m_p,\, 1\leq p\leq 2n-1)$), we have:

$$W_m(x):=V(x)+F*u(x)-F*u(0)=W_0(x)+Z_m(x)-Z_m(0), \tag{7.1}$$

with $Z_m(x):=\sum_{p=1}^{2n-1}\frac{(-1)^p}{p!}\left(m_p-m_p(0)\right)F^{(p)}(x)$. For all $k\geq 1$, we define the application φ_k^ϵ and the probability measure ν_m by

$$\varphi_k^\epsilon(m_1,\cdots,m_{2n-1}) \quad := \quad \frac{\int_{\mathbb{R}}x^k\exp\left[-\frac{2}{\epsilon}W_m(x)\right]dx}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}W_m(x)\right]dx}=:\int_{\mathbb{R}}x^k\nu_m(dx). \tag{7.2}$$

Moreover, if the two measures u and u_0 are symmetric, then Z_m and consequently W_m does not depend on the odd trivial moments. In this case, we consider the function ξ defined by

$$\xi_{2k}^\epsilon(m_2,\cdots,m_{2n-2})=\varphi_{2k}^\epsilon(0,m_2,0,\cdots,m_{2n-2},0). \tag{7.3}$$

Finally we introduce $\Phi^{(\epsilon)}:\mathbb{R}^{2n-1}\to\mathbb{R}^{2n-1}$ and $\Phi_0^{(\epsilon)}:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ given by

$$\Phi^{(\epsilon)}=(\varphi_1^\epsilon,\ldots,\varphi_k^\epsilon,\ldots,\varphi_{2n-1}^\epsilon)^T \text{ and } \Phi_0^{(\epsilon)}=(\xi_2^\epsilon,\ldots,\xi_{2k}^\epsilon,\ldots,\xi_{2n-2}^\epsilon)^T. \tag{7.4}$$

Key property: The measure u_ϵ is invariant if and only if the following vector $(\mu_1(\epsilon),\mu_2(\epsilon),\cdots,\mu_{2n-1}(\epsilon))$ is a fixed point of $\Phi^{(\epsilon)}$. It is invariant and symmetric if and only if $\mu_{2k+1}(\epsilon)=0$ for all $0\leq k\leq n-1$ and if the moments compose some vector $(\mu_2(\epsilon),\cdots,\mu_{2n-2}(\epsilon))$ which is a fixed point of $\Phi_0^{(\epsilon)}$.
Procedure: In order to obtain local uniqueness for asymmetric stationary measures, we shall use the uniform convergence on some compact set of $\Phi^{(\epsilon)}$ (and its derivatives) towards an application $\Phi^{(0)}$ (and its derivatives). Secondly, we shall prove that the differential of $\mathrm{Id}-\Phi^{(0)}$ is invertible on a small neighborhood of the limit point $(m_1(0),\cdots,m_{2n-1}(0))$ associated with u_0. Finally

we shall conclude by using the convergence rate which assures that the vector $(\mu_1(\epsilon), \cdots, \mu_{2n-1}(\epsilon))$ belongs to the observed compact set. We shall proceed in a similar way for the uniqueness of symmetric stationary measures with $\Phi_0^{(\epsilon)}$. We begin with a preliminary result:

Proposition 7.1. *Let $(\mu_1, \cdots, \mu_{2n-1}) \in \mathbb{R}^{2n-1}$ and $(\nu_2, \cdots, \nu_{2n-2}) \in \mathbb{R}^{n-1}$. We set $\mu_0 := 1 =: \nu_0$. For $C > 0$, we define two compact sets namely $\mathcal{P}_\epsilon := \prod_{p=1}^{2n-1} [\mu_p - C\epsilon; \mu_p + C\epsilon]$ and $\mathcal{Q}_\epsilon := \prod_{p=1}^{n-1} [\nu_{2p} - C\epsilon; \nu_{2p} + C\epsilon]$.*
1. *If the function $U_0(x) := V(x) + \sum_{p=0}^{2n-1} \frac{(-1)^p}{p!} \mu_p \left(F^{(p)}(x) - F^{(p)}(0)\right)$ reaches its global minimum at a unique location a_0 with $U_0''(a_0) > 0$ then for all $m \in \mathcal{P}_\epsilon$, $k \geq 1$ and $p \in [1; 2n-1]$, we have*

$$\frac{\partial \varphi_k^\epsilon}{\partial m_p}(m) = \frac{k\, a_0^{k-1}}{U_0''(a_0)} \frac{(-1)^{p-1}}{p!} F^{(p+1)}(a_0) + o_{\mathcal{P}_\epsilon}(1). \tag{7.5}$$

2. *If the function $T_0(x) := V(x) + \sum_{p=0}^{n-1} \frac{1}{(2p)!} \nu_{2p} \left(F^{(2p)}(x) - F^{(2p)}(0)\right)$ admits two global minima $\pm b_0$ with $T_0''(b_0) > 0$ then for all $\tilde{m} \in \mathcal{Q}_\epsilon$, $k \geq 1$ and $p \in [1; n-1]$, we have:*

$$\frac{\partial \xi_{2k}}{\partial m_{2p}}(\tilde{m}) = -\frac{2k\, b_0^{2k-1}}{T_0''(b_0)} \frac{1}{(2p)!} F^{(2p+1)}(b_0) + o_{\mathcal{Q}_\epsilon}(1). \tag{7.6}$$

Proof. **Step 1.** φ_k^ϵ is directly related to W_m. By (7.1) and since F is an even polynomial function of degree $2n$, we get

$$W_m(x) = W_0(x) + \sum_{p=1}^{2n-1} \frac{(-1)^p}{p!} (m_p - m_p(0)) \sum_{j \geq \frac{1+p}{2}}^{n} \frac{F^{(2j)}(0)}{(2j-p)!} x^{2j-p}.$$

Then, the derivative of (7.2) in the variable m_p satisfies

$$\frac{\partial \varphi_k^\epsilon}{\partial m_p}(m) = -\frac{2}{\epsilon} \frac{(-1)^p}{p!} \sum_{j \geq \frac{1+p}{2}}^{n} \frac{F^{(2j)}(0)}{(2j-p)!} \Big(\varphi_{2j+k-p}^\epsilon(m) - \varphi_{2j-p}^\epsilon(m) \varphi_k^\epsilon(m) \Big). \tag{7.7}$$

The derivative of ξ_{2k}^ε is computed in a similar way.
Step 2. Let $m \in \mathcal{P}_\epsilon$. For all $1 \leq i \leq 2n-1$, there exists $C_i \in [-C, C]$ such that $m_i = \mu_i + C_i \epsilon$. Then, for all $l \geq 1$:

$$\varphi_l^\epsilon(m) = \frac{\int_{\mathbb{R}} x^l \exp\left[-\frac{2}{\epsilon} U_0(x) - 2R_m(x)\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon} U_0(x) - 2R_m(x)\right] dx} \tag{7.8}$$

where $R_m(x) = \sum_{p=1}^{2n-1} \frac{(-1)^p}{p!} \frac{(m_p - \mu_p)}{\varepsilon} F^{(p)}(x) = \sum_{p=1}^{2n-1} \frac{(-1)^p}{p!} C_p F^{(p)}(x)$. According to Lemma A.5 in [6], we have the following asymptotic result which is uniform with respect to $m \in \mathcal{P}_\epsilon$:

$$\varphi_l^\epsilon(m) = a_0^l - l \frac{a_0^{l-2}}{4W_0''(a_0)} \left(a_0 \frac{U_0^{(3)}(a_0)}{U_0''(a_0)} - (l-1) + 4 a_0 R_m'(a_0) \right) \epsilon + o_{\mathcal{P}_\epsilon}(\epsilon).$$

We obtain an equivalence of the following expression directly linked to the derivative of φ_k^ϵ:

$$\varphi_{2j+k-p}^\epsilon(m) - \varphi_{2j-p}^\epsilon(m)\varphi_k^\epsilon(m) = \frac{k(2j-p)}{2U_0''(a_0)}\, a_0^{2j+k-p-2}\epsilon + o_{\mathcal{P}_\epsilon}(\epsilon).$$

Therefore, (7.7) becomes

$$\begin{aligned}
\frac{\partial\varphi_k}{\partial m_p}(m) &= \frac{2}{\epsilon}\frac{(-1)^{p-1}}{p!}\sum_{j\geq\frac{1+p}{2}}^{n}\frac{F^{(2j)}(0)}{(2j-p)!}\left\{\frac{k(2j-p)}{2U_0''(a_0)}\, a_0^{2j+k-p-2}\epsilon + o_{\mathcal{P}_\epsilon}(\epsilon)\right\} \\
&= \frac{k\, a_0^{k-1}(-1)^{p-1}}{U_0''(a_0)p!}\sum_{j\geq\frac{1+p}{2}}^{n}\frac{F^{(2j)}(0)}{(2j-p-1)!}a_0^{2j-p-1} + o_{\mathcal{P}_\epsilon}(1),
\end{aligned}$$

which provides (7.5) as announced.

Step 3. The proof of (7.6) is similar to the previous one. Let $\tilde{m} \in \mathcal{Q}_\epsilon$. For all $1 \leq i \leq n-1$, there exists $C_{2i} \in [-C;C]$ such that $\tilde{m}_{2i} = \nu_{2i} + C_{2i}\epsilon$. Then, for all $l \geq 1$, ξ_{2l}^ϵ satisfies the same expression that φ_{2l}^ϵ in (7.8) with the support of the integral reduced to $\mathbb{R}_+$, U_0 replaced by T_0 and R_m by $R_{\tilde{m}} = \sum_{p=1}^{n-1}\frac{1}{(2p)!}C_{2p}F^{(2p)}(x)$. We cannot apply directly Lemma A.5 in [6] since the support is reduced to $\mathbb{R}^+$ instead of $\mathbb{R}$. However the result can be adapted when b_0 – the unique minimum of T_0 on $\mathbb{R}^+$ – is positive. Therefore

$$\xi_{2j+2k-2p}^\epsilon(\tilde{m}) - \xi_{2j-2p}^\epsilon(\tilde{m})\xi_{2k}^\epsilon(\tilde{m}) = \frac{2k(2j-2p)}{2T_0''(b_0)}\, b_0^{2j+2k-2p-2}\epsilon + o_{\mathcal{Q}_\epsilon}(\epsilon)$$

Finally (7.6) is proved as follows:

$$\begin{aligned}
\frac{\partial\xi_{2k}}{\partial m_{2p}}(\tilde{m}) =& \frac{2}{\epsilon}\frac{1}{(2p)!}\sum_{j\geq p+1}^{n}\frac{F^{(2j)}(0)}{(2j-2p)!}\left\{\frac{2k(2j-2p)}{2T_0''(b_0)}\, b_0^{2j+2k-2p-2}\epsilon + o_{\mathcal{Q}_\epsilon}(\epsilon)\right\} \\
=& -\frac{2k\, b_0^{2k-1}}{T_0''(b_0)(2p)!}\sum_{j\geq p+1}^{n}\frac{F^{(2j)}(0)}{(2j-2p-1)!}\, b_0^{2j-2p-1} + o_{\mathcal{Q}_\epsilon}(1).
\end{aligned}$$

□

This preliminary result permits to estimate the differential and by the way answer some questions concerning the uniqueness problem.

7.1 Local uniqueness for outlying measures

In this section, we assume that the condition (6.1) is satisfied. Theorem 4.6 in [6] then provides the existence of outlying stationary measures u_ϵ^+ and u_ϵ^-, that is, invariant measures converging towards δ_a respectively δ_{-a} where a and $-a$ are the bottoms of the landscape V. We are going to prove that there exist exactly two extremal outlying measures for ϵ sufficiently small.

Theorem 7.2. *Let F'' and V'' be two convex functions. Let $(u_\epsilon)_{\epsilon>0}$ and $(v_\epsilon)_{\epsilon>0}$ two families of stationary measures converging to δ_a. Then there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$, $u_\epsilon = v_\epsilon$.*

By symmetry, the same result of *local uniqueness* holds for δ_{-a}.

Proof. **Step 1.** For all $1 \leq k \leq 2n-1$, we apply Theorem 6.3 to the function $f(x) := x^k$ and so get the existence of a constant $C > 0$ such that $(\mu_1(\epsilon), \cdots, \mu_{2n-1}(\epsilon))$ and $(\nu_1(\epsilon), \cdots, \nu_{2n-1}(\epsilon))$ belong to $\mathcal{P}_\epsilon$ for ϵ small enough. Here $\mu_k(\epsilon)$ (resp. $\nu_k(\epsilon)$) is the k-th moment of u_ϵ (resp. v_ϵ) and $\mathcal{P}_\epsilon := \prod_{i=1}^{2n-1} [a^i - C\epsilon; a^i + C\epsilon]$.
Step 2. Since u_ε and v_ε are invariant measures, each vector composed with the $2n-1$ first moments is solution of the equation: $\mu = \Phi^\varepsilon(\mu)$ where Φ^ε is defined by (7.4). Therefore let us prove that this equation admits some unique solution in $\mathcal{P}_\epsilon$, it suffices to point out that $\mathrm{Id} - \mathrm{Jac}\,\Phi^\varepsilon$ is invertible. Here $\mathrm{Jac}\,\Phi^\varepsilon$ represents the Jacobian matrix of the $2n-1$ dimensional function Φ^ε. According to Proposition 7.1 applied to $\mu_p = a^p$, $U_0 = W_0^+$ defined by (6.2) and satisfying $(W_0^+)''(a) = V''(a) + F''(0) = V''(a) + \alpha > 0$ (see condition (V-3) and (1.4)), we get

$$\frac{\partial \varphi_k^{(\epsilon)}}{\partial m_p}(m) = \frac{k a^{k-1}}{(W_0^+)''(a)} \frac{(-1)^{p-1}}{p!} F^{(p+1)}(a) + o_{\mathcal{P}_\epsilon}(1). \tag{7.9}$$

The Jacobian matrix then takes a simple expression. Indeed it suffices to prove that $(W_0^+)''(a)\mathrm{Id} + V_1 V_2^T$ is invertible, with $V_1(i) := i a^{i-1}$ and $V_2(j) := \frac{(-1)^j}{j!} F^{(j+1)}(a)$, $1 \leq i, j \leq 2n-1$. The proof of Lemma 4.7 in [6] solves this question: if $(W_0^+)''(a) + \langle V_1, V_2 \rangle \neq 0$ then the matrix considered is invertible. Let us note that $\langle V_1, V_2 \rangle = \sum_{i=1}^{2n-1} \frac{i a^{i-1} (-1)^i}{i!} F^{(i+1)}(a) = -F''(0) = -\alpha$. Hence $(W_0^+)''(a) + \langle V_1, V_2 \rangle = V''(a) > 0$ because of the hypothesis (V-3). By these arguments we have obtained that $\mu = \Phi^\varepsilon(\mu)$ admits a unique solution in $\mathcal{P}_\varepsilon$. In order to conclude it suffices to note that the $2n-1$ first moments characterize the stationary measure: F is a polynomial function of degree $2n$ and the invariant measures are defined by (2.2). □

7.2 Local uniqueness for symmetric measures

We shall divide the study into two parts. The first one concerns the limit measure $u_0 = \delta_0$ and the second one concerns $u_0 = \frac{1}{2}\delta_{-x_0} + \frac{1}{2}\delta_{x_0}$.

Let us now consider the limit measure δ_0. This discrete measure is effectively a limit measure when $\alpha > \vartheta$ (these parameters are defined by (1.4) and (1.3)). In this case, we get also the following property $\mathcal{B} = \emptyset$.

Theorem 7.3. *Let V'' and F'' be two convex functions. Let $\alpha > \vartheta$. There exists a unique symmetric invariant measure for ϵ small enough.*

Proof. **Step 1.** According to Theorem 4.5 of [6], we know that there exists at least one symmetric invariant measure u_ϵ. We know by Theorem 5.4 in [7] that

any such symmetric stationary converges weakly to δ_0 since $\alpha > \vartheta$.
Step 2. Let us consider now two symmetric invariant measures u_ϵ and v_ϵ converging towards δ_0.
By Theorem 3.1, there exists $C > 0$ such that the vectors $(\mu_2(\epsilon), \cdots, \mu_{2n-2}(\epsilon))$ and $(\nu_2(\epsilon), \cdots, \nu_{2n-2}(\epsilon))$ belong to $\mathcal{Q}_\epsilon$ for ϵ small enough. Here $\mu_{2k}(\epsilon)$ (resp. $\nu_{2k}(\epsilon)$) is the $2k$-th moment of u_ϵ (resp. v_ϵ) and $\mathcal{Q}_\epsilon := [-C\epsilon, C\epsilon]^{n-1}$.
As in the preceding proof, it suffices to prove that $\mathrm{Id} - \mathrm{Jac}\,\Phi_0^\varepsilon$ is locally invertible, where $\mathrm{Jac}\,\Phi_0^\varepsilon$ denotes the Jacobian matrix. Applying Proposition 7.1 with $\nu_{2p} = 0$ for all $1 \le p \le n-1$ and so $T_0 = W_0$ which admits one unique global minimum location: 0, we get for $\tilde{m} \in \mathcal{Q}_\varepsilon$, $\frac{\partial \xi_{2k}}{\partial m_{2p}}(\tilde{m}) = o_{\mathcal{Q}_\epsilon}(1)$. This implies directly that $\mathrm{Id} - \mathrm{Jac}\,\Phi_0^\varepsilon$ is invertible. Moreover since F is a polynomial function of degree $2n$, these moments characterize the measure, see (2.1). □

Let us finally consider the case $u_0 = \frac{1}{2}\,\delta_{-x_0} + \frac{1}{2}\,\delta_{x_0}$, $x_0 > 0$, associated with the study developed in Section 4. The discrete measure u_0 is a limit measure for families of symmetric invariant measures provided that $\alpha < \vartheta$ (these parameters are defined by (1.4) and (1.3)).

Theorem 7.4. *Let F'' and V'' be two convex functions and $\alpha < \vartheta$. For ϵ small enough, the self-stabilizing process* (1.2) *admits a unique symmetric invariant measure.*

Proof. We shall assume that $\deg(F) \ge 6$. Indeed, we have already proved (Theorem 3.2 in [6]) that, in the linear case (F' is linear), there exists a unique symmetric invariant measure for (1.2). Moreover, Example 4.2 in [6] points out that there exists a unique symmetric invariant measure for $\deg(F) = 4$.
According to Theorem 5.4 of [7], since V'' and F'' are convex functions, each sequence of symmetric stationary measures converges to the discrete measure $\frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$. Let $(u_\epsilon)_{\epsilon > 0}$ be such a sequence then it defines a fixed point of the application $\Phi_0^{(\varepsilon)}$ defined by (7.4). Moreover, by Theorem 4.5, we know that there exists $C > 0$ such that the $n-1$ first even moments of u_ε represented by $(\tilde{m}_2(\epsilon), \cdots, \tilde{m}_{2n-2}(\epsilon))$ belongs to the set $\mathcal{Q}_\epsilon := \prod_{p=1}^{n-1}[x_0^{2p} - C\epsilon, x_0^{2p} + C\epsilon]$. In order to prove the statement of the theorem, it suffices to prove that the equation $\mu = \Phi_0^{(\varepsilon)}(\mu)$ admits a unique symmetric solution in $\mathcal{Q}_\epsilon$. As explained in the two preceding proofs, the work just consists in verifying that $\mathrm{Id} - \mathrm{Jac}\,\Phi_0^{(\varepsilon)}$ is invertible where $\mathrm{Jac}\,\Phi_0^{(\varepsilon)}$ denotes the Jacobian matrix. Applying Proposition 7.1 with $\nu_{2p} = x_0^{2p}$ for all $1 \le p \le n-1$ and so $T_0 = W_0 = V + \frac{1}{2}F(.-x_0) + \frac{1}{2}F(.+x_0) - F(x_0)$ which reaches its global minimum for two locations $-x_0$ and x_0 (see (4.1)), we get for $\tilde{m} \in \mathcal{Q}_\varepsilon$

$$\frac{\partial \xi_{2k}^\epsilon}{\partial m_{2p}}(\tilde{m}) = -\frac{2k\, x_0^{2k-1}}{(2p)!W_0''(x_0)}\; F^{(2p+1)}(x_0) + o(1). \tag{7.10}$$

By similar arguments as those used in Theorem 7.3, we have just to verify that $W_0''(x_0) + \langle V_1, V_2\rangle \neq 0$ where $\langle V_1, V_2\rangle = \frac{1}{2}\left(F''(2x_0) - F''(0)\right)$. On one hand, the definition of x_0 (4.1) leads to $W_0''(x_0) \ge 0$, on the other hand the convexity of F'' which is a polynomial function of degree larger than 6 permits to obtain $\langle V_1, V_2\rangle > 0$. $\square$

Using the convergence rate from u_ϵ towards u_0, we prove that there exists a unique symmetric invariant measure for the self-stabilizing process (1.2) under the convexity property of V'' and F'' and when $\alpha > \vartheta$ or $\alpha < \vartheta$. The case $\alpha = \vartheta$ is more difficult since the convergence rate is not of order ϵ. It needs then some other kind of tools.
Let us note that the uniqueness of symmetric invariant measure was already studied in [1] where the authors considered the constant potential case $V(x) := 0$. They obtained uniqueness results for α large enough but ε fixed which is to relate to our situation where $\alpha > \vartheta$ but the noise intensity ε should be small. Their proof is essentially based on some contraction map which of course leads to local uniqueness. Our study handles directly with local uniqueness.

A Annex

Let us finally present some useful asymptotic results which are close to the classical Laplace's method. Let us first recall some preliminary asymptotic result (see [6]):

Lemma A.1. *Let $M > 0$. Let us assume that U is $\mathcal{C}^2([M,\infty[)$-continuous, $U(x) \neq 0$ and $U''(x) > 0$ for all $x \in [M,\infty[$ and $\lim_{x\to\infty} \frac{U''(x)}{(U'(x))^2} = 0$. If $x \to e^{-U(x)}$ is integrable on $\mathbb{R}$ then:*

$$\int_x^{+\infty} e^{-U(t)}dt \approx \frac{e^{-U(x)}}{U'(x)} \quad \text{and} \quad \int_M^x e^{U(t)}dt \approx \frac{e^{U(x)}}{U'(x)} \quad \text{as } x \to \infty. \tag{A.1}$$

Lemma A.2. *Set $\epsilon > 0$. Let U and G two $\mathcal{C}^\infty(\mathbb{R})$-continuous functions. We define $U_\mu := U + \mu G$ for μ belonging to some compact interval $\mathcal{I}$ of $\mathbb{R}$. Let us introduce some interval $[a,b]$ satisfying: $U_\mu'(a) \neq 0$, $U_\mu'(b) \neq 0$ and $U_\mu(x)$ reaches its global minimum on the interval $[a,b]$ in some unique point $x_\mu \in]a,b[$ for all $\mu \in \mathcal{I}$. We assume that there exists some exponent k_0 independent of $\mu \in \mathcal{I}$ such that $2k_0 = \min_{r\in\mathbb{N}^*}\left\{U_\mu^{(r)}(x_\mu) \neq 0\right\}$. Let f a $\mathcal{C}^4(\mathbb{R})$-continuous function. Then letting the parameter ϵ tend to 0, we get*

$$\int_a^b f(t)e^{-\frac{U_\mu(t)}{\epsilon}}dt = \frac{f(x_\mu)}{k_0}\left(\frac{\epsilon(2k_0)!}{U_\mu^{2k_0}(x_\mu)}\right)^{\frac{1}{2k_0}}\Gamma\left(\frac{1}{2k_0}\right)e^{-\frac{U_\mu(x_\mu)}{\epsilon}}(1+o_{\mathcal{I}}(1)),$$

where Γ represents the Euler function and $o_{\mathcal{I}}(1)$ converges towards 0 uniformly with respect to $\mu \in \mathcal{I}$.

Proof. The arguments are similar to those used in Lemma A.2 [6]. $\square$

Lemma A.3. *Let U_ϵ and $U \in \mathcal{C}^\infty([a;b],\mathbb{R})$ such that for all $i \in \mathbb{N}$, $U_\epsilon^{(i)}$ converges to $U^{(i)}$ uniformly on $[a;b]$ as $\epsilon \to 0$. If the global minimum of U*

is reached at some unique point x_0 on $[a;b]$ with $x_0 \in]a;b[$, then, for ϵ small enough,
1. *U_ϵ has a unique global minimum location x_ϵ on $[a;b]$ with $x_\epsilon \in]a;b[$.*
2. *$U''(x_0) > 0$ implies $U''_\epsilon(x_\epsilon) > 0$ and*

$$x_\epsilon = x_0 - \frac{U'_\epsilon(x_0)}{U''(x_0)} + o\left\{U'_\epsilon(x_0)\right\}. \tag{A.2}$$

3. *Furthermore, if $U''(x_0) > 0$, by taking the limit $\epsilon \to 0$, for all the function $f \in \mathcal{C}^4([a;b];\mathbb{R})$, we get*

$$\int_a^b f(t) e^{\frac{-2U_\epsilon(t)}{\epsilon}} dt = \sqrt{\frac{\pi\epsilon}{\mathcal{U}_2}}\, e^{-\frac{2U_\epsilon(x_\epsilon)}{\epsilon}} \Big\{ f(x_\epsilon) + \gamma_{x_0}(f)\epsilon + o(\epsilon) \Big\} \tag{A.3}$$

with

$$\gamma_{x_0}(f) := f(x_0)\left(\frac{5\,\mathcal{U}_3^2}{48\,\mathcal{U}_2^3} - \frac{\mathcal{U}_4}{16\,\mathcal{U}_2^2}\right) - f'(x_0)\frac{\mathcal{U}_3}{4\,\mathcal{U}_2^2} + \frac{f''(x_0)}{4\,\mathcal{U}_2}. \tag{A.4}$$

Here $\mathcal{U}_k := U_\epsilon^{(k)}(x_\epsilon)$.

Proof. **1.** We shall proceed using *reductio ad absurdum.* Let us assume that there exists a sequence $(\epsilon_k)_{k\ge 1}$ such that U_{ϵ_k} admits two different locations for the global minimum: $x_k^{(1)}$ and $x_k^{(2)}$ for all $k \ge 1$. Due to the uniform convergence of U_ϵ on $[a;b]$, both $x_k^{(1)}$ and $x_k^{(2)}$ tend to x_0 as $k \to \infty$. Hence, for any $\delta > 0$, there exists k_0 large enough, such that both $x_k^{(1)}$ and $x_k^{(2)}$ belong to $]x_0 - \delta; x_0 + \delta[$ for $k \ge k_0$. Moreover $U''(x_0) > 0$ by assumption and U''_ϵ converges uniformly on $[a;b]$; so there exist $\rho > 0$ and $\delta_0 > 0$ such that $U''_{\epsilon_k}(x) \ge \rho$ for all $x \in [x_0 - \delta_0; x_0 + \delta_0]$ and for k large enough. Consequently, the equation $U'(x) = 0$ does not admit several solutions on this interval. Taking $\delta = \delta_0$, we obtain the uniqueness of the global minimum location for U_ϵ and ϵ small enough.
2. The uniform convergence and the assumption $U''(x_0) > 0$ imply that $U''_\epsilon(x_\epsilon) > 0$ for ϵ small enough. Moreover we get the following convergence $U'_\epsilon(x_0) \to 0$ as $\epsilon \to 0$. Using the mean value theorem, we obtain as $\epsilon \to 0$:

$$\begin{aligned} U'_\epsilon(x_0) &= U'_\epsilon(x_\epsilon) + U''_\epsilon(x_\epsilon)\,(x_0 - x_\epsilon)\,(1 + o(1)) \\ &= U'_\epsilon(x_\epsilon) + U''(x_0)\,(x_0 - x_\epsilon)\,(1 + o(1)). \end{aligned}$$

Since $U'_\epsilon(x_\epsilon) = 0$, we obtain

$$x_\epsilon = x_0 - \frac{U'_\epsilon(x_0)}{U''(x_0)} + o\left\{U'_\epsilon(x_0)\right\}.$$

3. It suffices to adapt the proof of Lemma A.3 in [6]. The arguments are namely the same. □

We can extend the previous statement to integrals with unbounded supports.

Lemma A.4. *Let U_ϵ and $U \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ such that for all $i \in \mathbb{N}$, $U_\epsilon^{(i)}$ converges uniformly on all compact subset. If U has r global minimum locations $A_1 < \cdots < A_r$ and if there exist $R > 0$ and ϵ_c such that $U_\epsilon(x) > x^2$ for all $|x| > R$ and $\epsilon < \epsilon_c$, then, for ϵ small enough, we get:*
1. *U_ϵ has exactly one global minimum location A_i^ϵ on each interval I_i, where I_i represent the Voronoï cells corresponding to the central points A_i, with $1 \leq i \leq r$.*
2. *$U''(A_i) > 0$ implies $U''_\epsilon(A_i^\epsilon) > 0$ and*

$$A_i^\epsilon = A_i - \frac{U'_\epsilon(A_i)}{U''(A_i)} + o\left\{U'_\epsilon(A_i)\right\}. \tag{A.5}$$

3. *Furthermore, if $U''(A_i) > 0$ for all $1 \leq i \leq r$, then for any function $f \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ with polynomial growth, the following asymptotic development holds as $\epsilon \to 0$:*

$$\int_{\mathbb{R}} f(t) e^{-\frac{2U_\epsilon(t)}{\epsilon}} dt = \sum_{j=1}^{r} \sqrt{\frac{\pi\epsilon}{U''_\epsilon(A_j^\epsilon)}}\, e^{-\frac{2U_\epsilon(A_j^\epsilon)}{\epsilon}} \Big\{ f(A_j^\epsilon) + \gamma_j(f)\epsilon + o(\epsilon) \Big\} \tag{A.6}$$

with

$$\gamma_j(f) := f(A_j) \left(\frac{5\, \mathcal{U}_{3,j}^2}{48\, \mathcal{U}_{2,j}^3} - \frac{\mathcal{U}_{4,j}}{16\, \mathcal{U}_{2,j}^2} \right) - f'(A_j) \frac{\mathcal{U}_{3,j}}{4\, \mathcal{U}_{2,j}^2} + \frac{f''(A_j)}{4\, \mathcal{U}_{2,j}}, \tag{A.7}$$

and $\mathcal{U}_{k,j} := U_\epsilon^{(k)}(A_j^\epsilon)$.

Proof. For all $2 \leq j \leq r-1$, we apply Lemma A.3 on the interval I_j defined in the statement. We also apply Lemma A.3 on $[-R;R] \cap I_1$ and $[-R;R] \cap I_r$. Hence the result is proved on the integral $[-R;R]$. To conclude it suffices to note that the integral on $[-R;R]^c$ is negligible due to the polynomial growth of f and the gaussian behavior of $\exp\left[-\frac{2}{\epsilon}U_\epsilon\right]$. □

References

[1] S. Benachour, B. Roynette, D. Talay, and P. Vallois. Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos. *Stochastic Process. Appl.*, 75(2):173–201, 1998.

[2] S. Benachour, B. Roynette, and P. Vallois. Nonlinear self-stabilizing processes. II. Convergence to invariant probability. *Stochastic Process. Appl.*, 75(2):203–224, 1998.

[3] Patrick Cattiaux, Arnaud Guillin and Florient Malrieu. Probabilistic approach for granular media equations in the non-uniformly convex case. *Probability theory and related fields*, Vol 140, No 1-2, 19–40, 2008.

[4] Tadahisa Funaki. A certain class of diffusion processes associated with nonlinear parabolic equations. *Z. Wahrsch. Verw. Gebiete*, 67(3):331–348, 1984.

[5] Samuel Herrmann, Peter Imkeller, and Dierk Peithmann. Large deviations and a Kramers' type law for self-stabilizing diffusions. *Ann. Appl. Probab.*, 18(4):1379–1423, 2008.

[6] Samuel Herrmann and Julian Tugaut. Non-uniqueness of stationary measures for self-stabilizing diffusions. *Prépublications de l'Institut Elie Cartan*, 12, 2009. http://hal.archives-ouvertes.fr/hal-00368177/en/

[7] Samuel Herrmann and Julian Tugaut. Stationary measures for self-stabilizing processes: asymptotic analysis in the small noise limit. *Prépublications de l'Institut Elie Cartan*, 26, 2009. http://hal.archives-ouvertes.fr/hal-00397470/en/

[8] H. P. McKean, Jr. A class of Markov processes associated with nonlinear parabolic equations. *Proc. Nat. Acad. Sci. U.S.A.*, 56:1907–1911, 1966.

[9] Florient Malrieu. Logarithmic Sobolev inequalities for some nonlinear PDE's. *Stochastic Process. Appl.*, 95(1):109–132, 2001.

[10] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.

[11] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[12] Yozo Tamura. on asymptotic behaviors of the solution of a nonlinear diffusion equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 31(1):195–221, 1984.

[13] Yozo Tamura. Free energy and the convergence of distributions of diffusion processes of McKean type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(2):443–484, 1987.

[14] A. Yu. Veretennikov On ergodic measures for McKean-Vlasov stochastic equations. *Monte Carlo and Quasi-Monte Carlo Methods 2004*, 471–486, 2006.