Integration by parts formula and applications to equations with jumps

Emmanuelle Clement, Vlad Bally

Integration by parts formula and applications to equations with jumps

Vlad Bally* Emmanuelle Clément*

Preliminary version, November 2009

Abstract

We establish an integration by parts formula in an abstract framework in order to study the regularity of the law for processes solution of stochastic differential equations with jumps, including equations with discontinuous coefficients for which the Malliavin calculus developed by Bismut and Bichteler, Gravereaux and Jacod fails.

2000 MSC. Primary: 60H07, Secondary 60G51

Key words: Integration by parts formula, Malliavin Calculus, Stochastic Equations, Poisson Point Measures.

1 Introduction

This paper is made up of two parts. In a first part we give an abstract, finite dimensional version of Malliavin calculus. Of course Malliavin calculus is known to be an infinite dimensional differential calculus and so a finite dimensional version seems to be of a limited interest. We discuss later on the relation between the finite dimensional and the infinite dimensional frameworks and we highlight the interest of the finite dimensional approach.

In the second part of the paper we use the results from the first section in order to give sufficient conditions for the regularity of the law of X_t, where X is the Markov process with infinitesimal operator

$$Lf(x) = \langle \nabla f(x), g(x) \rangle + \int_{R^d} (f(x + c(z,x)) - f(x))\gamma(z,x)\mu(dz). \tag{1}$$

Suppose for the moment that γ does not depend on x. Then it is well known that the process X may be represented as the solution of a stochastic equation driven by a Poisson point measure with

*Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050, Université Paris-Est Marne-la-Vallée, 5 Bld Descartes, Champs-sur-marne, 77454 Marne-la-Vallée Cedex 2, France.

intensity measure $\gamma(z)\mu(dz)$. Sufficient conditions for the regularity of the law of X_t using a Malliavin calculus for Poisson point measures are given in $[B.G.J]$. But in our framework γ depends on x which roughly speaking means that the law of the jumps depends on the position of the particle when the jump occurs. Such processes are of interest in a lot of applications and unfortunately the standard Malliavin calculus developed in $[B.G.J]$ does not apply in this framework. After the classical paper of Bichteler Gravereaux and Jacod a huge work concerning the Malliavin calculus for Poisson point measures has been done and many different approaches have been developed. But as long as we know they do not lead to a solution for our problem. If X is an one dimensional process an analytical argument permits to solve the above problem , this is done in $[F.1], [F.2]$ and $[F.G]$ but the argument there seems difficult to extend in the multi-dimensional case.

We come now back to the relation between the finite dimensional and the infinite dimensional framework. This seems to be the more interesting point in our approach so we try to explain the main idea. In order to prove Malliavin's regularity criterion for the law of a functional F on the Wiener space the main tool is the integration by parts formula

$$E(\partial^\beta f(F)) = E(f(F)H_\beta) \tag{2}$$

where ∂^β denotes the derivative corresponding to a multi-index β and H_β is a random variable built using the Malliavin derivatives of F. Once such a formula is proved one may estimate the Fourier transform $\widehat{p}_F(\xi) = E(\exp(i\xi F))$ in the following way. First we remark that $\partial_x^\beta \exp(i\xi x) = (i\xi)^\beta \exp(i\xi x)$ (with an obvious abuse of notation) and then, using the integration by parts formula

$$\begin{aligned} |\widehat{p}_F(\xi)| &= \frac{1}{|\xi|^{|\beta|}} \left| E(\partial_x^\beta \exp(i\xi F)) \right| \\ &= \frac{1}{|\xi|^{|\beta|}} \left| E(\exp(i\xi F)H_\beta) \right| \leq \frac{1}{|\xi|^{|\beta|}} E\left|H_\beta\right|. \end{aligned}$$

If we know that $E|H_\beta| < \infty$ for every multi-index β then we have proved that $|\xi|^p |\widehat{p}_F(\xi)|$ is integrable for every $p \in \mathbb{N}$ and consequently the law of F is absolutely continuous with respect to the Lebesgue measure and has an infinitely differentiable density.

Let us come back to the infinite dimensional differential calculus which permits to built H_β. In order to define the Malliavin derivative of F one considers a sequence of simple functionals $F_n \to F$ in L^2 and, if $DF_n \to G$ in L^2, then one defines $DF = G$. The simple functionals F_n are functions of a finite number of random variables (increments of the Brownian motion) and the derivative DF_n is a gradient type operator defined in an elementary way. Then one may take the following alternative way in order to prove the regularity of the law of F. For each fixed n one proves the analogues of the integration by parts formula (2): $E(\partial^\beta f(F_n)) = E(f(F_n)H_\beta^n)$. As F_n is a function which depends on a finite number m of random variables, such a formula is obtained using standard integration by parts on $\mathbb{R}^m$ (this is done in the first section of this paper). Then the same calculus as above gives $|\widehat{p}_{F_n}(\xi)| \leq |\xi|^{-|\beta|} E\left|H_\beta^n\right|$. Passing to the limit one obtains

$$|\widehat{p}_F(\xi)| = \lim_n |\widehat{p}_{F_n}(\xi)| \leq |\xi|^{-|\beta|} \sup_n E\left|H_\beta^n\right|$$

and, if we can prove that $\sup_n E\left|H_\beta^n\right| < \infty$, we are done. Notice that here we do not need that $F_n \to F$ in L^2 but only in law. And also, we do not need to built H_β but only to prove that $\sup_n E\left|H_\beta^n\right| < \infty$. Anyway we are not very far from the standard Malliavin calculus. Things become different if $\sup_n E\left|H_\beta^n\right| = \infty$ and this is the case in our examples (because the Ornstein Uhlenbeck operators LF_n blow up as $n \to \infty$). But even in this case one may obtain estimates of the Fourier transform of F in the following way. One writes

$$|\widehat{p}_F(\xi)| \leq |\widehat{p}_F(\xi) - \widehat{p}_{F_n}(\xi)| + |\widehat{p}_{F_n}(\xi)| \leq |\xi| \times E\left|F - F_n\right| + |\xi|^{-|\beta|} E\left|H_\beta^n\right|.$$

And if one may obtain a good balance between the convergence to zero of the error $E\left|F - F_n\right|$ and the blow up to infinity of $E\left|H_\beta^n\right|$ then one obtains $|\widehat{p}_F(\xi)| \leq |\xi|^{-p}$ for some p. Examples in which such a balance works are given in Section 3. An other application of this methodology is given in [B.F] for the Boltzmann equation. In this case some specific and nontrivial difficulties appear due to the singularity and unboundedness of the coefficients of the equation.

The paper is organized as follows. In Section 2 we establish the abstract Malliavin calculus associated to a finite dimensional random variable and we obtain estimates of the weight H_β which appear in the integration by parts formula (we follow here some ideas which already appear in $[B], [B.B.M]$ and $[Ba.M]$). Section 3 is devoted to the study of the regularity of the law of the Markov process X of infinitesimal operator (1) and it contains our main results : Proposition 3 and Theorem 4. At last we provide in Section 4 the technical estimates which are needed to prove the results of section 3.

2 Integration by parts formula

2.1 Notations-derivative operators

Throughout this paper, we consider a sequence of random variables $(V_i)_{i\in\mathbb{N}^*}$ on a probability space $(\Omega, \mathcal{F}, P)$, a sub σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ and a random variable J, $\mathcal{G}$ measurable, with values in $\mathbb{N}$. We assume that the variables (V_i) and J satisfy the following integrability conditions : for all $p \geq 1$, $E(J^p) + E((\sum_{i=1}^J V_i^2)^p) < \infty$. Our aim is to establish a differential calculus based on the variables (V_i), conditionally on $\mathcal{G}$, and we first define the class of functions on which this differential calculus will apply. More precisely, we consider in this paper functions $f : \Omega \times \mathbb{R}^{\mathbb{N}^*} \to \mathbb{R}$ which can be written as

$$f(\omega, v) = \sum_{j=1}^{\infty} f^j(\omega, v_1, ..., v_j) 1_{\{J(\omega)=j\}} \tag{3}$$

where $f^j : \Omega \times \mathbb{R}^j \to \mathbb{R}$ are $\mathcal{G} \times \mathcal{B}(\mathbb{R}^j)-$measurable functions. We denote by $\mathcal{M}$ the class of functions f given by (3) such that there exists a random variable $C \in \cap_{q\geq 1} L^q(\Omega, \mathcal{G}, P)$ and a real number $p \geq 1$ satisfying $|f(\omega, v)| \leq C(\omega)(1 + (\sum_{i=1}^{J(\omega)} v_i^2)^p)$. So conditionally on $\mathcal{G}$, the functions of $\mathcal{M}$ have polynomial growth with respect to the variables (V_i). We need some more notations. Let $\mathcal{G}_i$ be the $\sigma-$algebra generated by $\mathcal{G} \cup \sigma(V_j, 1 \leq j \leq J, j \neq i)$ and let $(a_i(\omega))$ and $(b_i(\omega))$ be sequences of $\mathcal{G}_i$ measurable random variables satisfying $-\infty \leq a_i(\omega) < b_i(\omega) \leq +\infty$, for all $i \in \mathbb{N}^*$. Now let O_i be the open set of $\mathbb{R}^{\mathbb{N}^*}$ defined by $O_i = P_i^{-1}(]a_i, b_i[)$, where P_i is the coordinate map $P_i(v) = v_i$. We localize the differential calculus on the sets (O_i) by introducing some weights (π_i), satisfying the following hypothesis.

H0. For all $i \in \mathbb{N}^*$, $\pi_i \in \mathcal{M}$, $0 \leq \pi_i \leq 1$ and $\{\pi_i > 0\} \subset O_i$. Moreover for all $j \geq 1$, π_i^j is infinitely differentiable with bounded derivatives with respect to the variables $(v_1, \ldots, v_j)$.

We associate to these weights (π_i), the spaces $C_\pi^k \subset \mathcal{M}, k \in \mathbb{N}^*$ defined recursively as follows. For $k = 1$, C_π^1 denotes the space of functions $f \in \mathcal{M}$ such that for each $i \in \mathbb{N}^*$, f admits a partial derivative with respect to the variable v_i on the open set O_i. We then define

$$\partial_i^\pi f(\omega, v) := \pi_i(\omega, v) \frac{\partial}{\partial v_i} f(\omega, v)$$

and we assume that $\partial_i^\pi f \in \mathcal{M}$.

Note that the chain rule is verified : for each $\phi \in C^1(\mathbb{R}^d, \mathbb{R})$ and $f = (f^1, ..., f^d) \in (C_\pi^1)^d$ we have

$$\partial_i^\pi \phi(f) = \sum_{r=1}^{d} \partial_r \phi(f) \partial_i^\pi f^r.$$

Suppose now that C_π^k is already defined. For a multi-index $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^{*k}$ we define recursively $\partial_\alpha^\pi = \partial_{\alpha_k}^\pi ... \partial_{\alpha_1}^\pi$ and C_π^{k+1} is the space of functions $f \in C_\pi^k$ such that for every multi-index $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^{*k}$ we have $\partial_\alpha^\pi f \in C_\pi^1$. Note that if $f \in C_\pi^k$, $\partial_\alpha^\pi f \in \mathcal{M}$ for each α with $|\alpha| \leq k$.

Finally we define $C_\pi^\infty = \cap_{k\in\mathbb{N}^*} C_\pi^k$. Roughly speaking the space C_π^∞ is the analogue of C^∞ with partial derivatives ∂_i replaced by localized derivatives ∂_i^π.

Simple functionals. A random variable F is called a simple functional if there exists $f \in C_\pi^\infty$ such that $F = f(\omega, V)$, where $V = (V_i)$. We denote by $\mathcal{S}$ the space of the simple functionals. Notice that $\mathcal{S}$ is an algebra. It is worth to remark that conditionally on $\mathcal{G}$, $F = f^J(V_1, \ldots, V_J)$.

Simple processes. A simple process is a sequence of random variables $U = (U_i)_{i \in \mathbb{N}^*}$ such that for each $i \in \mathbb{N}^*$, $U_i \in \mathcal{S}$. Consequently, conditionally on $\mathcal{G}$, we have $U_i = u_i^J(V_1, \ldots, V_J)$. We denote by $\mathcal{P}$ the space of the simple processes and we define the scalar product

$$\langle U, V \rangle_J = \sum_{i=1}^{J} U_i V_i.$$

Note that $\langle U, V \rangle_J \in \mathcal{S}$.

We can now define the derivative operator and state the integration by parts formula.

□ **The derivative operator.** We define $D : \mathcal{S} \to \mathcal{P}$: by

$$DF := (D_i F) \in \mathcal{P} \quad \text{where } D_i F := \partial_i^\pi f(\omega, V).$$

Note that $D_i F = 0$ for $i > J$. For $F = (F^1, ..., F^d) \in \mathcal{S}^d$ the Malliavin covariance matrix is defined by

$$\sigma^{k,k'}(F) = \left\langle DF^k, DF^{k'} \right\rangle_J = \sum_{j=1}^{J} D_j F^k D_j F^{k'}.$$

We denote

$$\Lambda(F) = \{\det \sigma(F) \neq 0\} \quad \textit{and} \quad \gamma(F)(\omega) = \sigma^{-1}(F)(\omega), \omega \in \Lambda(F).$$

In order to derive an integration by parts formula, we need some additional assumptions on the random variables (V_i). The main hypothesis is that conditionally on $\mathcal{G}$, the law of the vector$(V_1, ..., V_J)$ admits a locally smooth density with respect to the Lebesgue measure on $\mathbb{R}^J$.

H1. i) Conditionally on $\mathcal{G}$, the vector $(V_1, ..., V_J)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^J$ and we note p_J the conditional density.

ii) The set $\{p_J > 0\}$ is open in $\mathbb{R}^J$ and on $\{p_J > 0\}$ $\ln p_J \in C_\pi^\infty$.

iii) $\forall q \geq 1$, there exists a constant C_q such that

$$(1 + |v|^q) p_J \leq C_q$$

where $|v|$ stands for the euclidian norm of the vector $(v_1, \ldots, v_J)$.

Assumption iii) implies in particular that conditionally on $\mathcal{G}$, the functions of $\mathcal{M}$ are integrable with respect to p_J and that for $f \in \mathcal{M}$:

$$E_{\mathcal{G}}(f(\omega, V)) = \int_{\mathbb{R}^J} f^J \times p_J(\omega, v_1, ..., v_J) dv_1 ... dv_J.$$

□ **The divergence operator** Let $U = (U_i)_{i\in\mathbb{N}^*} \in \mathcal{P}$ with $U_i \in \mathcal{S}$. We define $\delta : \mathcal{P} \to \mathcal{S}$ by

$$\begin{aligned} \delta_i(U) \quad &: \quad = -(\partial_{v_i}(\pi_i U_i) + U_i 1_{\{p_J>0\}} \partial_i^\pi \ln p_J), \\ \delta(U) \quad &= \quad \sum_{i=1}^J \delta_i(U) \end{aligned}$$

For $F \in \mathcal{S}$, let $L(F) = \delta(DF)$.

2.2 Duality and integration by parts formulae

In our framework the duality between δ and D is given by the following proposition.

Proposition 1 *Assume **H0** and **H1**, then* $\forall F \in \mathcal{S}$ *and* $\forall U \in \mathcal{P}$ *we have*

$$E_{\mathcal{G}}(\langle DF, U\rangle_J) = E_{\mathcal{G}}(F\delta(U)). \tag{4}$$

Proof: By definition, we have $E_{\mathcal{G}}(\langle DF, U\rangle_J) = \sum_{i=1}^J E_{\mathcal{G}}(D_iF \times U_i)$ and from **H1**

$$E_{\mathcal{G}}(D_iF \times U_i) = \int_{\mathbb{R}^J} \partial_{v_i}(f^J)\pi_i \; u_i^J \; p_J(\omega, v_1, ..., v_J) dv_1...dv_J$$

recalling that $\{\pi_i > 0\} \subset O_i$, we obtain from Fubini's theorem

$$E_{\mathcal{G}}(D_iF \times U_i) = \int_{\mathbb{R}^{J-1}} \left(\int_{a_i}^{b_i} \partial_{v_i}(f^J)\pi_i \; u_i^J \; p_J(\omega, v_1, ..., v_J) dv_i \right) dv_1..dv_{i-1}dv_{i+1}...dv_J.$$

By using the classical integration by parts formula, we have

$$\int_{a_i}^{b_i} \partial_{v_i}(f^J)\pi_i \; u_i^J \; p_J(\omega, v_1, ..., v_J) dv_i = [f^J \pi_i u_i^J p_J]_{a_i}^{b_i} - \int_{a_i}^{b_i} f^J \partial_{v_i}(u_i^J \pi_i p_J) dv_i.$$

Now if $-\infty < a_i < b_i < +\infty$, we have $\pi_i(a_i) = 0 = \pi_i(b_i)$ and $[f^J \pi_i u_i^J p_J]_{a_i}^{b_i} = 0$. Moreover since f^J, u_i^J and π_i belong to $\mathcal{M}$, we deduce from H_1 iii) that $\lim_{|v_i|\to+\infty}(f^J \pi_i u_i^J p_J) = 0$ and we obtain that for all a_i, b_i such that $-\infty \leq a_i < b_i \leq +\infty$:

$$\int_{a_i}^{b_i} \partial_{v_i}(f^J)\pi_i \; u_i^J \; p_J(\omega, v_1, ..., v_J) dv_i = - \int_{a_i}^{b_i} f^J \partial_{v_i}(u_i^J \pi_i p_J) dv_i,$$

Observing that $\partial_{v_i}(u_i^J \pi_i p_J) = (\partial_{v_i}(u_i^J \pi_i) + u_i^J 1_{\{p_J>0\}} \partial_i^\pi(\ln p_J))p_J$, the proposition is proved.

$\diamond$

We have the following straightforward computation rules.

Lemma 1 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function and $F = (F^1, ..., F^d) \in \mathcal{S}^d$. Then $\phi(F) \in \mathcal{S}$ and*

$$D\phi(F) = \sum_{r=1}^{d} \partial_r \phi(F) DF^r. \tag{5}$$

If $F \in \mathcal{S}$ and $U \in \mathcal{P}$ then

$$\delta(FU) = F\delta(U) - \langle DF, U \rangle_J \,. \tag{6}$$

Moreover, for $F = (F^1, ..., F^d) \in \mathcal{S}^d$, we have

$$L\phi(F) = \sum_{r=1}^{d} \partial_r \phi(F) LF^r - \sum_{r,r'=1}^{d} \partial_{r,r'} \phi(F) \left\langle DF^r, DF^{r'} \right\rangle_J . \tag{7}$$

The first equality is a consequence of the chain rule, the second one follows from the definition of the divergence operator δ. Combining these equalities (7) follows.

We can now state the main results of this section.

Theorem 1 *We assume **H0** and **H1**. Let $F = (F^1, ..., F^d) \in \mathcal{S}^d$, $G \in \mathcal{S}$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth bounded function with bounded derivatives. Let $\Lambda \in \mathcal{G}, \Lambda \subset \Lambda(F)$ such that*

$$E(|\det \gamma(F)|^p \, 1_\Lambda) < \infty \quad \forall p \geq 1. \tag{8}$$

Then, for every $r = 1, ..., d$,

$$E_{\mathcal{G}} \left(\partial_r \phi(F) G \right) 1_\Lambda = E_{\mathcal{G}} \left(\phi(F) H_r(F, G) \right) 1_\Lambda \tag{9}$$

with

$$H_r(F, G) = \sum_{r'=1}^{d} \delta(G\gamma^{r',r}(F) DF^{r'}) = \sum_{r'=1}^{d} \left(G\delta(\gamma^{r',r}(F) DF^{r'}) - \gamma^{r',r} \left\langle DF^{r'}, DG \right\rangle_J \right). \tag{10}$$

Proof: Using the chain rule

$$\begin{aligned} \left\langle D\phi(F), DF^{r'} \right\rangle_J &= \sum_{j=1}^{J} D_j \phi(F) D_j F^{r'} \\ &= \sum_{j=1}^{J} (\sum_{r=1}^{d} \partial_r \phi(F) D_j F^r) D_j F^{r'} = \sum_{r=1}^{d} \partial_r \phi(F) \sigma^{r,r'}(F) \end{aligned}$$

so that $\partial_r \phi(F) 1_\Lambda = 1_\Lambda \sum_{r'=1}^{d} \left\langle D\phi(F), DF^{r'} \right\rangle_J \gamma^{r',r}(F)$. Since $F \in \mathcal{S}^d$ it follows that $\phi(F) \in \mathcal{S}$ and $\sigma^{r,r'}(F) \in \mathcal{S}$. Moreover, since $\det \gamma(F) 1_\Lambda \in \cap_{p \geq 1} L^p$ it follows that $\gamma^{r,r'}(F) 1_\Lambda \in \mathcal{S}$. So $G\gamma^{r',r}(F) DF^{r'} 1_\Lambda \in$

$\mathcal{P}$ and the duality formula gives:

$$\begin{aligned} E_{\mathcal{G}}\left(\partial_r\phi(F)G\right)1_\Lambda &= \sum_{r'=1}^{d} E_{\mathcal{G}}\left(\left\langle D\phi(F), G\gamma^{r',r}(F)DF^{r'}\right\rangle_J\right)1_\Lambda \\ &= \sum_{r'=1}^{d} E_{\mathcal{G}}\left(\phi(F)\delta(G\gamma^{r',r}(F)DF^{r'})\right)1_\Lambda . \end{aligned}$$

$\diamond$

We can extend this integration by parts formula.

Theorem 2 *Under the assumptions of Theorem 1, we have for every multi-index* $\beta = (\beta_1, \dots, \beta_q) \in \{1, \dots, d\}^q$

$$E_{\mathcal{G}}\left(\partial_\beta\phi(F)G\right)1_\Lambda = E_{\mathcal{G}}\left(\phi(F)H^q_\beta(F,G)\right)1_\Lambda \tag{11}$$

where the weights H^q *are defined recursively by (10) and*

$$H^q_\beta(F,G) = H_{\beta_1}\left(F, H^{q-1}_{(\beta_2,\dots,\beta_q)}(F,G)\right). \tag{12}$$

Proof: The proof is straightforward by induction. For $q = 1$, this is just Theorem 1. Now assume that Theorem 2 is true for $q \geq 1$ and let us prove it for $q+1$. Let $\beta = (\beta_1, \dots, \beta_{q+1}) \in \{1, \dots, d\}^{q+1}$, we have

$$E_{\mathcal{G}}\left(\partial_\beta\phi(F)G\right)1_\Lambda = E_{\mathcal{G}}\left(\partial_{(\beta_2,\dots,\beta_{q+1})}(\partial_{\beta_1}\phi(F))G\right)1_\Lambda = E_{\mathcal{G}}\left(\partial_{\beta_1}\phi(F)H^q_{(\beta_2,\dots,\beta_{q+1})}(F,G)\right)1_\Lambda$$

and the result follows. $\diamond$

2.3 Estimations of H^q

2.3.1 Iterated derivative operators, Sobolev norms

In order to estimate the weights H^q appearing in the integration by parts formulae of the previous section, we need first to define iterations of the derivative operator. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a multi-index, with $\alpha_i \in \{1, \dots, J\}$, for $i = 1, \dots, k$ and $|\alpha| = k$. For $F \in \mathcal{S}$, we define recursively $D^k_{(\alpha_1,\dots,\alpha_k)}F = D_{\alpha_k}(D^{k-1}_{(\alpha_1,\dots,\alpha_{k-1})}F)$ and

$$D^kF = \left(D^k_{(\alpha_1,\dots,\alpha_k)}F\right)_{\alpha_i\in\{1,\dots,J\}}.$$

Remark that $D^kF \in \mathbb{R}^{J\otimes k}$ and consequently we define the norm of D^kF as

$$|D^kF| = \sqrt{\sum_{\alpha_1,\ldots,\alpha_k=1}^{J} |D^k_{(\alpha_1,\ldots,\alpha_k)}F|^2}.$$

Moreover, we introduce the following norms, for $F \in \mathcal{S}$:

$$|F|_{1,l} = \sum_{k=1}^{l} |D^kF|, \quad |F|_l = |F| + |F|_{1,l} = \sum_{k=0}^{l} |D^kF|. \tag{13}$$

For $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$:

$$|F|_{1,l} = \sum_{r=1}^{d} |F^r|_{1,l}, \quad |F|_l = \sum_{r=1}^{d} |F^r|_l,$$

and similarly for $F = (F^{r,r'})_{r,r'=1,\ldots,d}$

$$|F|_{1,l} = \sum_{r,r'=1}^{d} |F^{r,r'}|_{1,l}, \quad |F|_l = \sum_{r,r'=1}^{d} |F^{r,r'}|_l.$$

Finally for $U = (U_i)_{i\leq J} \in \mathcal{P}$, we have $D^kU = (D^kU_i)_{i\leq J}$ and we define the norm of D^kU as

$$|D^kU| = \sqrt{\sum_{i=1}^{J} |D^kU_i|^2}.$$

We can remark that for $k = 0$, this gives $|U| = \sqrt{\langle U, U\rangle_J}$. Similarly to (13), we set

$$|U|_{1,l} = \sum_{k=1}^{l} |D^kU|, \quad |U|_l = |U| + |U|_{1,l} = \sum_{k=0}^{l} |D^kU|.$$

Observe that for $F, G \in \mathcal{S}$, we have $D(F \times G) = DF \times G + F \times DG$. This leads to the following useful inequalities

Lemma 2 *Let $F, G \in \mathcal{S}$ and $U, V \in \mathcal{P}$, we have*

$$|F \times G|_l \leq 2^l \sum_{l_1+l_2\leq l} |F|_{l_1}|G|_{l_2}, \tag{14}$$

$$|\langle U, V\rangle_J|_l \leq 2^l \sum_{l_1+l_2\leq l} |U|_{l_1}|V|_{l_2}. \tag{15}$$

We can remark that the first inequality is sharper than the following one $|F \times G|_l \leq C_l|F|_l|G|_l$. Moreover from (15) with $U = DF$ and $V = DG$ ($F, G, \in \mathcal{S}$) we deduce

$$|\langle DF, DG\rangle_J|_l \leq 2^l \sum_{l_1+l_2\leq l} |F|_{1,l_1+1}|G|_{1,l_2+1} \tag{16}$$

and as an immediate consequence of (14) and (16), we have for $F, G, H \in \mathcal{S}$:

$$|H \langle DF, DG \rangle_J |_l \leq 2^{2l} \sum_{l_1+l_2+l_3 \leq l} |F|_{1,l_1+1} |G|_{1,l_2+1} |H|_{l_3}. \tag{17}$$

Proof: We just prove (15), since (14) can be proved on the same way. We first give a bound for $D^k \langle U, V \rangle_J = (D^k_\alpha \langle U, V \rangle_J)_{\alpha \in \{1,\dots,J\}^k}$. For a multi-index $\alpha = (\alpha_1, ..., \alpha_k)$, with $\alpha_i \in \{1, \dots, J\}$, we note $\alpha(\Gamma) = (\alpha_i)_{i \in \Gamma}$, where $\Gamma \subset \{1, \dots, k\}$ and $\alpha(\Gamma^c) = (\alpha_i)_{i \notin \Gamma}$. We have

$$D^k_\alpha \langle U, V \rangle_J = \sum_{i=1}^J D^k_\alpha (U_i V_i) = \sum_{k'=0}^k \sum_{|\Gamma|=k'} \sum_{i=1}^J D^{k'}_{\alpha(\Gamma)} U_i \times D^{k-k'}_{\alpha(\Gamma^c)} V_i.$$

Let $W^{i,\Gamma} = (W^{i,\Gamma}_\alpha)_{\alpha \in \{1,\dots,J\}^k} = (D^{k'}_{\alpha(\Gamma)} U_i \times D^{k-k'}_{\alpha(\Gamma^c)} V_i)_{\alpha \in \{1,\dots,J\}^k}$, we have the equality in $\mathbb{R}^{J \otimes k}$:

$$D^k \langle U, V \rangle_J = \sum_{k'=0}^k \sum_{|\Gamma|=k'} \sum_{i=1}^J W^{i,\Gamma}.$$

This gives

$$|D^k \langle U, V \rangle_J | \leq \sum_{k'=0}^k \sum_{|\Gamma|=k'} |\sum_{i=1}^J W^{i,\Gamma}|,$$

where

$$|\sum_{i=1}^J W^{i,\Gamma}| = \sqrt{\sum_{\alpha_1,\dots,\alpha_k=1}^J |\sum_{i=1}^J W^{i,\Gamma}_\alpha|^2}.$$

But from Cauchy Schwarz inequality, we have

$$|\sum_{i=1}^J W^{i,\Gamma}_\alpha|^2 = |\sum_{i=1}^J D^{k'}_{\alpha(\Gamma)} U_i \times D^{k-k'}_{\alpha(\Gamma^c)} V_i|^2 \leq \sum_{i=1}^J |D^{k'}_{\alpha(\Gamma)} U_i|^2 \times \sum_{i=1}^J |D^{k-k'}_{\alpha(\Gamma^c)} V_i|^2.$$

Consequently we obtain

$$\begin{aligned} |\sum_{i=1}^J W^{i,\Gamma}| &\leq \sqrt{\sum_{\alpha_1,\dots,\alpha_k=1}^J \sum_{i=1}^J |D^{k'}_{\alpha(\Gamma)} U_i|^2 \times \sum_{i=1}^J |D^{k-k'}_{\alpha(\Gamma^c)} V_i|^2} \\ &= |D^{k'} U| \times |D^{k-k'} V|. \end{aligned}$$

This last equality results from the fact that we sum on different index sets (Γ and Γ^c). This gives

$$\begin{aligned} \left| D^k \langle U, V \rangle_J \right| &\leq \sum_{k'=0}^k \sum_{|\Gamma|=k'} \left| D^{k'} U \right| \left| D^{k-k'} V \right| = \sum_{k'=0}^k C^{k'}_k \left| D^{k'} U \right| \left| D^{k-k'} V \right| \\ &\leq \sum_{k'=0}^k C^{k'}_k |U|_{k'} |V|_{k-k'} \leq 2^k (\sum_{l_1+l_2=k} |U|_{l_1} |V|_{l_2}). \end{aligned}$$

Summing on $k = 0, ..., l$ we deduce (16).

$\diamond$

2.3.2 Estimation of $|\gamma(F)|_l$

We give in this section an estimation of the derivatives of $\gamma(F)$ in terms of $\det\sigma(F)$ and the derivatives of F. We assume that $\omega\in\Lambda(F)$.

In what follows $C_{l,d}$ is a constant depending eventually on the order of derivation l and the dimension d.

Proposition 2 *Let $F\in\mathcal{S}^d$, we have $\forall l\in\mathbb{N}$*

$$\begin{aligned}|\gamma(F)|_l &\le C_{l,d}\sum_{l_1+l_2\le l}|F|_{1,l_2+1}^{2(d-1)}\left(\frac{1}{|\det\sigma(F)|}+\sum_{k=1}^{l_1}\frac{|F|_{1,l_1+1}^{2kd}}{|\det\sigma(F)|^{k+1}}\right) &(18)\\ &\le C_{l,d}\frac{1}{|\det\sigma(F)|^{l+1}}(1+|F|_{1,l+1}^{2d(l+1)}). &(19)\end{aligned}$$

Before proving Proposition 2, we establish a preliminary lemma.

Lemma 3 *for every $G\in S$, $G>0$ we have*

$$\left|\frac{1}{G}\right|_l\le C_l\left(\frac{1}{G}+\sum_{k=1}^{l}\frac{1}{G^{k+1}}\sum_{\substack{k\le r_1+\ldots+r_k\le l\\ r_1,\ldots,r_k\ge1}}\prod_{i=1}^{k}|D^{r_i}G|\right)\le C_l\left(\frac{1}{G}+\sum_{k=1}^{l}\frac{1}{G^{k+1}}|G|_{1,l}^k\right). \tag{20}$$

Proof: For $F\in\mathcal{S}^d$ and $\phi:\mathbb{R}^d\to\mathbb{R}$ a $\mathcal{C}^\infty$ function, we have from the chain rule

$$D^k_{(\alpha_1,\ldots,\alpha_k)}\phi(F)=\sum_{|\beta|=1}^{k}\partial_\beta\phi(F)\sum_{\Gamma_1\cup\ldots\cup\Gamma_{|\beta|}=\{1,\ldots,k\}}\left(\prod_{i=1}^{|\beta|}D^{|\Gamma_i|}_{\alpha(\Gamma_i)}F^{\beta_i}\right), \tag{21}$$

where $\beta\in\{1,\ldots,d\}^{|\beta|}$ and $\sum_{\Gamma_1\cup\ldots\cup\Gamma_{|\beta|}}$ denotes the sum over all partitions of $\{1,\ldots,k\}$ with length $|\beta|$. In particular, for $G\in\mathcal{S}$, $G>0$ and for $\phi(x)=1/x$, we obtain

$$|D^k_\alpha(\frac{1}{G})|\le C_k\sum_{k'=1}^{k}\frac{1}{G^{k'+1}}\sum_{\Gamma_1\cup\ldots\cup\Gamma_{k'}=\{1,\ldots,k\}}\left(\prod_{i=1}^{k'}|D^{|\Gamma_i|}_{\alpha(\Gamma_i)}G|\right). \tag{22}$$

We deduce then that

$$\begin{aligned}|D^k(\frac{1}{G})| &\le C_k\sum_{k'=1}^{k}\frac{1}{G^{k'+1}}\sum_{\Gamma_1\cup\ldots\cup\Gamma_{k'}=\{1,\ldots,k\}}\left|\prod_{i=1}^{k'}D^{|\Gamma_i|}_{\alpha(\Gamma_i)}G\right|_{\mathbb{R}^{J\otimes k}},\\ &= C_k\sum_{k'=1}^{k}\frac{1}{G^{k'+1}}\sum_{\Gamma_1\cup\ldots\cup\Gamma_{k'}=\{1,\ldots,k\}}\left(\prod_{i=1}^{k'}|D^{|\Gamma_i|}G|\right),\\ &= C_k\sum_{k'=1}^{k}\frac{1}{G^{k'+1}}\sum_{\substack{r_1+\ldots+r_{k'}=k\\ r_1,\ldots,r_{k'}\ge1}}\left(\prod_{i=1}^{k'}|D^{r_i}G|\right),\end{aligned}$$

and the first part of (20) is proved. The proof of the second part is straightforward. $\diamond$

With this lemma, we can prove Proposition 2.

Proof: Proposition 2. We have on $\Lambda(F)$

$$\gamma^{r,r'}(F) = \frac{1}{\det \sigma(F)} \hat{\sigma}^{r,r'}(F),$$

where $\hat{\sigma}(F)$ is the algebraic complement of $\sigma(F)$. But recalling that $\sigma^{r,r'}(F) = \left\langle D^r F, D^{r'} F \right\rangle_J$ we have

$$|\det \sigma(F)|_l \leq C_{l,d} |F|_{1,l+1}^{2d} \quad \text{and} \quad |\hat{\sigma}(F)|_l \leq C_{l,d} |F|_{1,l+1}^{2(d-1)}. \tag{23}$$

Applying inequality (14), this gives

$$|\gamma(F)|_l \leq C_{l,d} \sum_{l_1+l_2 \leq l} |(\det \sigma(F))^{-1}|_{l_1} |\hat{\sigma}(F)|_{l_2}.$$

From Lemma 3 and (23), we have

$$|(\det \sigma(F))^{-1}|_{l_1} \leq C_{l_1} \left(\frac{1}{|\det \sigma(F)|} + \sum_{k=1}^{l_1} \frac{|F|_{1,l_1+1}^{2kd}}{|\det \sigma(F)|^{k+1}} \right).$$

Putting together these inequalities, we obtain the inequality (18) and consequently (19). $\diamond$

2.3.3 Some bounds on H^q

Now our goal is to establish some estimates for the weights H^q in terms of the derivatives of G, F, LF and $\gamma(F)$.

Theorem 3 *For $F \in \mathcal{S}^d$, $G \in \mathcal{S}$ and for all $q \in \mathbb{N}^*$ there exists an universal constant $C_{q,d}$ such that for every multi-index $\beta = (\beta_1, .., \beta_q)$*

$$\begin{aligned} \left| H_\beta^q(F,G) \right| &\leq \frac{C_{q,d} \left|G\right|_q (1 + |F|_{q+1})^{(6d+1)q}}{|\det \sigma(F)|^{3q-1}} (1 + \sum_{j=1}^q \sum_{k_1+\ldots+k_j \leq q-j} \prod_{i=1}^j |L(F)|_{k_i}), \\ &\leq \frac{C_{q,d} \left|G\right|_q (1 + |F|_{q+1})^{(6d+1)q}}{|\det \sigma(F)|^{3q-1}} (1 + |LF|_{q-1}^q). \end{aligned}$$

Proof: For $F \in \mathcal{S}^d$, we define the linear operator $T_r : \mathcal{S} \to \mathcal{S}, r = 1, ..., d$ by

$$T_r(G) = \langle DG, (\gamma(F) DF)^r \rangle ,$$

where $(\gamma(F)DF)^r=\sum_{r'=1}^d \gamma^{r',r}(F)DF^{r'}$. Notice that

$$T_r(G\times G')=GT_r(G')+G'T_r(G). \tag{24}$$

Moreover, for a multi-index $\beta=(\beta_1,..,\beta_q)$ we define by induction $T_\beta(G)=T_{\beta_q}(T_{(\beta_1,...,\beta_{q-1})}(G))$. We also make the convention that if β is the void multi-index, then $T_\beta(G)=G$. Finally we denote by $L_r^\gamma(F)=\sum_{r'=1}^d \delta(\gamma^{r',r}(F)DF^{r'})$. With this notation we have

$$\begin{aligned} H_r(F,G) &= GL_r^\gamma(F)-T_r(G),\\ H_\beta^q(F,G) &= H_{\beta_1}(F,H^{q-1}_{(\beta_2,...,\beta_q)}(F,G)). \end{aligned}$$

We will now give an explicite expression of $H_\beta^q(F,G)$. In order to do this we have to introduce some more notation. Let $\Lambda_j=\{\lambda_1,\ldots,\lambda_j\}\subset\{1,\ldots,q\}$ such that $|\Lambda_j|=j$. We denote by $\mathcal{P}(\Lambda_j)$ the set of the partitions $\Gamma=(\Gamma_0,\Gamma_1,...,\Gamma_j)$ of $\{1,...,q\}\setminus\Lambda_j$. Notice that we accept that $\Gamma_i, i=0,1,...,j$ may be void sets. Moreover, for a multi-index $\beta=(\beta_1,..,\beta_q)$ we denote by $\Gamma_i(\beta)=(\beta_{k_i^1},...,\beta_{k_i^p})$ where $\Gamma_i=\{k_i^1,...,k_i^p\}$. With this notation we can prove by induction and using (24) that

$$H_\beta^q(F,G)=T_\beta(G)+\sum_{j=1}^q\sum_{\Lambda_j\subset\{1,...q\}}\sum_{\Gamma\in\mathcal{P}(\Lambda_j)} c_{\beta,\Gamma}T_{\Gamma_0(\beta)}(G)\prod_{i=1}^j T_{\Gamma_i(\beta)}(L^\gamma_{\beta_{\lambda_i}}(F)) \tag{25}$$

where $c_{\beta,\Gamma}\in\{-1,0,1\}$.

We first give an estimation of $|T_\beta(G)|_l$, for $l\geq 0$ and $\beta=(\beta_1,\ldots,\beta_q)$. We proceed by induction. For $q=1$ and $1\leq r\leq d$, we have

$$|T_r(G)|_l=|\langle DG,(\gamma(F)DF)^r\rangle|_l$$

and using (17) we obtain

$$|T_r(G)|_l\leq C_l\sum_{l_1+l_2+l_3\leq l}|\gamma(F)|_{l_1}|G|_{l_2+1}|F|_{l_3+1}\leq |G|_{l+1}|F|_{l+1}\sum_{l_1=0}^l|\gamma(F)|_{l_1},$$

where C_l is a constant which depends on l only. We obtain then by induction for every multi-index $\beta=(\beta_1,\ldots,\beta_q)$

$$|T_\beta(G)|_l\leq C_{l,q}|G|_{l+q}|F|^q_{l+q}\sum_{l_1+...+l_q\leq l+q-1}\prod_{i=1}^q|\gamma(F)|_{l_i}. \tag{26}$$

In particular this gives for $l=0$

$$|T_\beta(G)|\leq C_q|G|_q|F|_q^qP_q(\gamma(F)),$$

with

$$P_q(\gamma(F)) = \sum_{l_1+\ldots+l_q \leq q-1} \prod_{i=1}^{q} |\gamma(F)|_{l_i}, \quad q \geq 1.$$

To complete the notation, we note $P_0(\gamma(F)) = 1$. We obtain

$$\left| T_{\Gamma_i(\beta)}(L^{\gamma}_{\beta_{\lambda_i}}(F)) \right| \leq C_q \left| L^{\gamma}_{\beta_{\lambda_i}}(F) \right|_{|\Gamma_i(\beta)|} |F|^{|\Gamma_i(\beta)|}_{|\Gamma_i(\beta)|} P_{|\Gamma_i(\beta)|}(\gamma(F)).$$

We turn now to the estimation of $|L^{\gamma}_r(F)|_l$. From the properties of the divergence operator δ (see Lemma 1)

$$\delta(\gamma(F)DF) = \gamma(F)\delta(DF) - \langle D\gamma(F), DF \rangle_J .$$

It follows from (14) and (16) that

$$|L^{\gamma}_r(F)|_l \leq C_l \, |\gamma(F)|_{l+1} \, (|\delta(DF)|_l + |F|_{l+1}) \leq C_l \, |\gamma(F)|_{l+1} \, (1 + |LF|_l)(1 + |F|_{l+1}),$$

and we get

$$\left| T_{\Gamma_i(\beta)}(L^{\gamma}_{\beta_{\lambda_i}}(F)) \right| \leq C_q \, |\gamma(F)|_{|\Gamma_i(\beta)|+1} \, (1 + |LF|_{|\Gamma_i(\beta)|})(1 + |F|_{|\Gamma_i(\beta)|+1}) |F|^{|\Gamma_i(\beta)|}_{|\Gamma_i(\beta)|} P_{|\Gamma_i(\beta)|}(\gamma(F)). \quad (27)$$

Reporting these inequalities in (25) and recalling that $|\Gamma_0(\beta)| + \ldots + |\Gamma_j(\beta)| = q - j$ we deduce :

$$\begin{aligned} |H^q_\beta(F,G)| \leq \; & |T_\beta(G)| + C_{q,d} \sum_{j=1}^{q} \sum_{k_0+\cdots+k_j=q-j} |G|_{k_0} |F|^{k_0}_{k_0} P_{k_0}(\gamma(F)) \left(\prod_{i=1}^{j} |\gamma(F)|_{k_i+1} P_{k_i}(\gamma(F)) \right. \\ & \left. |F|^{k_i}_{k_i} (1 + |F|_{k_i+1})(1 + |LF|_{k_i}) \right) \end{aligned} \quad (28)$$

Now, for $q \geq 1$, we have from (19) :

$$P_q(\gamma(F)) \leq C_q \frac{1}{|\det \sigma(F)|^{2q-1}} (1 + |F|_q)^{4dq},$$

so the following inequality holds for $q \geq 0$:

$$P_q(\gamma(F)) \leq C_q \frac{1}{|\det \sigma(F)|^{2q}} (1 + |F|_q)^{4dq}.$$

We obtain then for $k_0, k_1, \ldots, k_j \in \mathbb{N}$ such that $k_0 + \ldots + k_j = q - j$

$$\prod_{i=0}^{j} P_{k_i}(\gamma(F)) \leq C_q \frac{1}{|\det \sigma(F)|^{2(q-j)}} (1 + |F|_{q-j})^{4d(q-j)} \quad (29)$$

and once again from (19)

$$\prod_{i=1}^{j} |\gamma(F)|_{k_i+1} \leq C_q \frac{1}{|\det \sigma(F)|^{q+j}} (1 + |F|_{q-j+2})^{2d(q+j)} \quad (30)$$

it yields finally

$$\prod_{i=0}^{j} P_{k_i}(\gamma(F)) \prod_{i=1}^{j} |\gamma(F)|_{k_i+1} \leq C_q \frac{1}{|\det \sigma(F)|^{3q-j}} (1 + |F|_{q-j+2})^{6dq-2dj}.$$

Turning back to (28), it follows that

$$\left| H_\beta^q(F,G) \right| \leq \frac{C_{q,d} \, |G|_q \, (1 + |F|_{q+1})^{(6d+1)q}}{|\det \sigma(F)|^{3q-1}} (1 + \sum_{j=1}^{q} \sum_{k_1+\ldots+k_j \leq q-j} \prod_{i=1}^{j} |L(F)|_{k_i}),$$

and Theorem 3 is proved. $\diamond$

3 Stochastic equations with jumps

3.1 Notations and hypotheses

We consider a Poisson point process p with state space $(E, B(E))$, where $E = \mathbb{R}^d \times \mathbb{R}_+$. We refer to [I.W] for the notation. We denote by N the counting measure associated to p, we have $N([0,t) \times A) = \#\{0 \leq s < t; p_s \in A\}$ for $t \geq 0$ and $A \in B(E)$. We assume that the associated intensity measure is given by $\widehat{N}(dt, dz, du) = dt \times d\mu(z) \times 1_{[0,\infty)}(u)du$ where $(z,u) \in E = \mathbb{R}^d \times \mathbb{R}_+$ and $\mu(dz) = h(z)dz$.

We are interested in the solution of the d dimensional stochastic equation

$$X_t = x + \int_0^t \int_E c(z, X_{s-}) 1_{\{u < \gamma(z, X_{s-})\}} N(ds, dz, du) + \int_0^t g(X_s) ds. \tag{31}$$

We remark that the infinitesimal generator of the Markov process X_t is given by

$$L\psi(x) = g(x) \nabla \psi(x) + \int_{\mathbb{R}^d} (\psi(x + c(z,x)) - \psi(x)) K(x, dz)$$

where $K(x, dz) = \gamma(z,x) h(z) dz$ depends on the variable $x \in \mathbb{R}^d$. See [F.1] for the proof of existence and uniqueness of the solution of the above equation.

Our aim is to give sufficient conditions in order to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure and has a smooth density. In this section we make the following hypotheses on the functions γ, g, h and c.

Hypothesis 3.0 We assume that γ, g, h and c are infinitely differentiable functions in both variables z and x. Moreover we assume that g and its derivatives are bounded and that $\ln h$ has bounded derivatives

Hypothesis 3.1. We assume that there exist two functions $\overline{\gamma}, \underline{\gamma} : \mathbb{R}^d \to \mathbb{R}_+$ such that

$$\overline{C} \geq \overline{\gamma}(z) \geq \gamma(z,x) \geq \underline{\gamma}(z) \geq 0, \quad \forall x \in \mathbb{R}^d$$

where $\overline{C}$ is a constant.

Hypothesis 3.2. **i)** We assume that there exists a non negative and bounded function $\overline{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^d} \overline{c}(z) d\mu(z) < \infty$ and

$$|c(z,x)| + \left|\partial_z^\beta \partial_x^\alpha c(z,x)\right| \le \overline{c}(z) \quad \forall z, x \in \mathbb{R}^d.$$

We need this hypothesis in order to estimate the Sobolev norms.

ii) There exists a measurable function $\widehat{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^d} \widehat{c}(z) d\mu(z) < \infty$ and

$$\left\| \nabla_x c \times (I + \nabla_x c)^{-1}(z,x) \right\| \le \widehat{c}(z), \quad \forall (z,x) \in \mathbb{R}^d \times \mathbb{R}^d.$$

In order to simplify the notations we assume that $\widehat{c}(z) = \overline{c}(z)$.

iii) There exists a non negative function $\underline{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that for every $z \in \mathbb{R}^d$

$$\sum_{r=1}^d \langle \partial_{z_r} c(z,x), \xi \rangle^2 \ge \underline{c}^2(z) \left|\xi\right|^2, \quad \forall \xi \in \mathbb{R}^d$$

and we assume that there exists $\theta > 0$ such that

$$\underline{\lim}_{a \to +\infty} \frac{1}{\ln a} \int_{\{\underline{c}^2 \ge 1/a\}} \underline{\gamma}(z) d\mu(z) = \theta.$$

Remark : assumptions ii) and iii) give sufficient conditions to prove the non degeneracy of the Malliavin covariance matrix as defined in the previous section. In particular the second part of iii) implies that $\underline{c}^2$ is a (p,t) broad function (see $[B.G.J.]$) for $p/t < \theta$. Notice that we may have $\underline{c}(z) = 0$ for some $z \in \mathbb{R}^d$.

We add to these hypotheses some assumptions on the derivatives of γ and $\ln \gamma$ with respect to x and z. For $l \ge 1$ we use the notation :

$$\overline{\gamma}^{x,l}(z) = \sup_x \sup_{1 \le |\beta| \le l} |\partial_{\beta,x} \gamma(z,x)|,$$

$$\overline{\gamma}_{\ln}^{x,l}(z) = \sup_x \sup_{1 \le |\beta| \le l} |\partial_{\beta,x} \ln \gamma(z,x)|,$$

$$\overline{\gamma}_{\ln}^{z,l}(z) = \sup_x \sup_{1 \le |\beta| \le l} |\partial_{\beta,z} \ln \gamma(z,x)|.$$

Hypothesis 3.3. We assume that $\ln \gamma$ has bounded derivatives with respect to z (that is $\overline{\gamma}_{\ln}^{z,l}(z)$ is bounded) and that γ has bounded derivatives with respect to x such that $\forall z \in \mathbb{R}^d$, $\overline{\gamma}^{x,l}(z) \le \overline{\gamma}^{x,l}$; moreover we assume that

$$\sup_{z^* \in \mathbb{R}^d} \int_{B(z^*,1)} \overline{\gamma}(z) d\mu(z) < +\infty.$$

We complete this hypothesis with two alternative hypotheses.

a) (weak dependence on x) We assume that $\forall l \geq 1$

$$\int_{\mathbb{R}^d} \overline{\gamma}_{\ln}^{x,l}(z)\overline{\gamma}(z)d\mu(z) < \infty.$$

b) (strong dependence on x) We assume that $\ln \gamma$ has bounded derivatives with respect to x such that $\forall l \geq 1$

$$\forall z \in \mathbb{R}^d, \quad \overline{\gamma}_{\ln}^{x,l}(z) \leq \overline{\gamma}_{\ln}^{x,l}.$$

Remark : if μ is the Lebesgue measure (case $h = 1$) and if γ does not depend on z then $\overline{\gamma}_{\ln}^{x,l}$ is constant and consequently hypothesis 3.3.a fails. Conversely, if $\gamma(z, x) = \gamma(z)$ then hypothesis 3.3.a is satisfied as soon as $\ln \gamma$ has bounded derivatives. This last case corresponds to the standard case where the law of the amplitude of the jumps does not depend on the position of X_t. Under Hypothesis 3.3.a we are in a classical situation where the divergence does not blow up and this leads to an integration by part formula with bounded weights (see Proposition 4 and Lemma 11). On the contrary under assumption 3.3.b, the divergence can blow up as well as the weights appearing in the integration by part formula.

3.2 Main results and examples

Our methodology to study the regularity of the law of the random variable X_t is based on the following result. Let $\hat{p}_X(\xi) = E(e^{i\langle \xi, X\rangle})$ be the Fourier transform of a d-dimensional random variable X then using the Fourier inversion formula, one can prove that if $\int_{\mathbb{R}^d} |\xi|^p |\hat{p}_X(\xi)| d\xi < \infty$ for $p > 0$ then the law of X is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and its density is $\mathcal{C}^{[p]}$, where $[p]$ denotes the entire part of p.

To apply this result, we just have to bound the Fourier transform of X_t in terms of $1/|\xi|$. This is done in the next proposition. The proof of this proposition needs a lot of steps that we detail in the next sections and it will be given later.

Proposition 3 *Let* $B_M = \{z \in \mathbb{R}^d; |z| < M\}$*, then under hypotheses* 3.0., 3.1. 3.2. *and* 3.3 *we have for all* $M \geq 1$*, for* $q \geq 1$ *and* $t > 0$ *such that* $4d(3q-1)/t < \theta$

a) if 3.3.a *holds*

$$|\hat{p}_{X_t}(\xi)| \quad \leq \quad t\int_{B_{M-1}^c} \underline{c}^2(z)\underline{\gamma}(z)d\mu(z)\frac{1}{2}|\xi|^2 + |\xi|\, te^{Ct}\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)d\mu(z) + \frac{C_q}{|\xi|^q}.$$

b) if 3.3.b *holds*

$$|\hat{p}_{X_t}(\xi)| \leq t\int_{B^c_{M-1}} \underline{c}^2(z)\underline{\gamma}(z)d\mu(z)\frac{1}{2}|\xi|^2 + |\xi|\, te^{Ct}\int_{B^c_M} \overline{c}(z)\overline{\gamma}(z)d\mu(z) + \frac{C_q(1+\mu(B_{M+1})^q)}{|\xi|^q}.$$

We can remark that if $\theta = +\infty$ then the result holds $\forall q \geq 1$ and $\forall t > 0$.

By choosing M judiciously as a function of ξ in the inequalities given in Proposition 3, we obtain $|\hat{p}_{X_t}(\xi)| \leq C/|\xi|^p$ for some $p > 0$ and this permits us to deduce some regularity for the density of X_t. The next theorem precise the optimal choice of M with respect to ξ and permits us to derive the regularity of the law of the process X_t.

Theorem 4 *We assume that hypotheses* 3.0., 3.1., 3.2 *and* 3.3. *hold.*

a) *Assuming* 3.3.a*, the law of X_t admits a density $\mathcal{C}^k$ if $t > (3k+3d-1)\frac{4d}{\theta}$. In the case $\theta = \infty$, the law of X_t admits a density $\mathcal{C}^\infty$.*

b) *Assuming* 3.3.b *and the two following hypotheses*

A1 *: $\exists p_1, p_2 > 0$ such that :*

$$\limsup_M M^{p_1}\int_{B^c_M} \overline{c}(z)\overline{\gamma}(z)d\mu(z) < +\infty;$$

$$\limsup_M M^{p_2}\int_{B^c_M} \underline{c}^2(z)\underline{\gamma}(z)d\mu(z) < +\infty;$$

A2 *: $\exists \rho > 0$ such that $\mu(B_M) \leq CM^\rho$ where $B_M = \{z \in \mathbb{R}^d; |z| < M\}$;*

case 1*: if $\theta = +\infty$ then the law of X_t admits a density $\mathcal{C}^k$ with $k < \min(p_1/\rho - 1 - d, p_2/\rho - 2 - d)$ if $\min(p_1/\rho - 1 - d, p_2/\rho - 2 - d) \geq 1$.*

case 2 *: if $0 < \theta < \infty$ let $q^*(t,\theta) = [\frac{1}{3}(\frac{t\theta}{4d} + 1)]$; then the law of X_t admits a density $\mathcal{C}^k$ for $k < \sup_{0<r<1/\rho} \min(rp_1 - 1 - d, rp_2 - 2 - d, q^*(t,\theta)(1 - r\rho) - d)$, if for some $0 < r < 1/\rho$, $\min(rp_1 - 1 - d, rp_2 - 2 - d, q^*(t,\theta)(1 - r\rho) - d) \geq 1$.*

Proof:

a) Assuming 3.3.a and letting M go to infinity in the right-hand side of the inequality given in Proposition 3 , we deduce

$$|\hat{p}_{X_t}(\xi)| \leq C/|\xi|^q,$$

and the result follows.

b) From $A1$, for M large enough, we have

$$\int_{B^c_M} \overline{c}(z)\overline{\gamma}(z)d\mu(z) \leq C/M^{p_1}$$

and

$$\int_{B^c_{M-1}} \underline{c}^2(z)\underline{\gamma}(z)d\mu(z) \le C/M^{p_2}.$$

Now assuming 3.3.b and $A2$ and choosing $M = |\xi|^r$, for $0 < r < 1/\rho$, we obtain from Proposition 3

$$|\hat{p}_{X_t}(\xi)| \le C\left(\frac{1}{|\xi|^{rp_1-1}} + \frac{1}{|\xi|^{rp_2-2}} + \frac{1}{|\xi|^{q(1-r\rho)}}\right),$$

for q and t such that $4d(3q-1)/t < \theta$. Now if $\theta = \infty$, we obtain for q large enough

$$|\hat{p}_{X_t}(\xi)| \le C\left(\frac{1}{|\xi|^{rp_1-1}} + \frac{1}{|\xi|^{rp_2-2}}\right).$$

In the case $\theta < \infty$, the best choice of q is $q^*(t,\theta)$. This achieves the proof of theorem 4. $\diamond$

We end this section with some examples in order to illustrate the results of Theorem 4.

Example 1. In this example we assume that $h = 1$ so $\mu(dz) = dz$ and that $\underline{\gamma}(z)$ is equal to a constant $\underline{\gamma} > 0$. We also assume that Hypothesis 3.3.b holds. We have $\mu(B_M) = r_d M^d$ where r_d is the volume of the unit ball in $\mathbb{R}^d$ so $\rho = d$. We will consider two types of behaviour for c.

i) Exponential decay: we assume that $\overline{c}(z) = e^{-b|z|^c}$ and $\underline{c}(z) = e^{-a|z|^c}$ for some constants $0 < b \le a$ and $c > 0$. We have

$$\int_{\{\underline{c}^2 > 1/u\}} \underline{\gamma}(z)d\mu(z) = \frac{\underline{\gamma} r_d}{(2a)^{d/c}} \times (\ln u)^{d/c}.$$

We deduce then

$$\theta = 0 \quad if \quad c > d, \quad \theta = \infty \quad if \quad 0 < c < d \quad and \quad \theta = \frac{\underline{\gamma} r_d}{2a} \quad if \quad c = d. \tag{32}$$

If $c > d$, hypothesis 3.2.iii fails, this is coherent with the result of $[B.G.J]$. Now observe that

$$\int_{B^c_M} \underline{c}^2(z)\underline{\gamma}(z)d\mu(z) + \int_{B^c_M} \overline{c}(z)\overline{\gamma}(z)d\mu(z) \le e^{-\eta|z|^c}$$

for some $\eta > 0$ so $p_1 = p_2 = \infty$. In the case $0 < c < d$ we obtain a density C^∞ for every $t > 0$. In the case $c = d$ we have $q^*(t,\theta) = [\frac{1}{3}(1 + \frac{\underline{\gamma} r_d}{8da} \times t)]$. If $t < 8da(3d+2)/(\underline{\gamma} r_d)$ we obtain nothing and if $t \ge 8da(3d+2)/(\underline{\gamma} r_d)$ we obtain a density $\mathcal{C}^k$ where k is the largest integer less than $[\frac{1}{3}(1+\frac{\underline{\gamma} r_d}{8da} \times t)] - d$.

ii) Polynomial decay. We assume that $\overline{c}(z) = b/(1+|z|^p)$ and $\underline{c}(z) = a/(1+|z|^p)$ for some constants $0 < a \le b$ and $p > d$. We have

$$\int_{\{\underline{c}^2 > 1/u\}} \underline{\gamma}(z)d\mu(z) = \underline{\gamma} r_d \times (a\sqrt{u} - 1)^{d/p}$$

so $\theta=\infty$ and our result works for every $t>0$. Hence a simple computation gives

$$\int_{B_M^c} \underline{c}^2(z)\underline{\gamma}(z)d\mu(z) \leq \frac{C}{M^{2p-d}}, \quad \int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)d\mu(z) \leq \frac{C}{M^{p-d}}$$

and then $p_1=p-d$ and $p_2=2p-d$. If $p\geq d(d+3)$ then $\min(p/d-2-d, 2p/d-3-d)\geq 1$ and we obtain a density $\mathcal{C}^k$ with $k<\frac{p}{d}-d-2$. Conversely if $p<d(d+3)$, we can say nothing about the regularity of the density of X_t. We give now an example where the function γ satisfies Hypothesis 3.3.a.

Example 2. As in the preceding example, we assume $h=1$. We consider the function $\gamma(z,x)=\exp(-\alpha(x)/(1+|z|^q))$ for some $q>d$. We assume that α is a smooth function which is bounded and has bounded derivatives and moreover there exists two constants such that $\overline{\alpha}\geq\alpha(x)\geq\underline{\alpha}>0$. Notice that the derivatives with respect to x of $\ln\gamma(z,x)$ are bounded by $C/(1+|z|^q)$ which is integrable with respect to the Lebesgue measure if $q>d$. So Hypothesis 3.3.a is true. Moreover we check that $\underline{\gamma}(z)=\exp(-\overline{\alpha}/(1+|z|^q))$.

i) Exponential decay. We take c as in Example 1.i). It follows that

$$\int_{\{\underline{c}^2>1/u\}} \underline{\gamma}(z)d\mu(z) \geq \exp(-\overline{\alpha})\frac{r_d}{(2a)^{d/c}} \times (\ln u)^{d/c}.$$

So we obtain once again θ as in (32). In the case $c>d$ we can say nothing, in the case $c<d$ we obtain a density C^∞ and in the case $c=d$ we have $\theta=\frac{r_d}{2a}$ and we obtain a density C^k if $t>\frac{8ad(3k+3d-1)}{r_d}$. In particular we have no results if $t\leq\frac{8ad(3d-1)}{r_d}$. Notice that the only difference with respect to the previous example concerns the case $c=d$ when we have a slight gain.

ii) Polynomial decay. At last we take c as in the example 1.ii). We check that $\theta=\infty$ so we obtain a density C^∞ , which is a better result than the one of the previous example.

Example 3. We consider the process (Y_t) solution of the stochastic equation

$$dY_t = f(Y_t)dL_t,$$

where L_t is a Lévy process with intensity measure $|y|^{-(1+\rho)}1_{\{|y|\leq 1\}}dy$, with $0<\rho<1$. The infinitesimal generator of Y is given by

$$L\psi(x) = \int_{\{|y|\leq 1\}} (\psi(x+f(x)y)-\psi(x))\frac{dy}{|y|^{1+\rho}}.$$

If we introduce some function $g(x)$ in this operator we obtain

$$L\psi(x) = \int_{\{|y|\leq 1\}} (\psi(x+f(x)y)-\psi(x))g(x)\frac{dy}{|y|^{1+\rho}}.$$

We are interested to represent this operator through a stochastic equation. In order to come back in our framework, we translate the integrability problem from 0 to ∞ by the change of variables $z = y^{-1}$ and we obtain

$$L\psi(x) = \int_{\{|z|\geq 1\}} (\psi(x + f(x)z^{-1}) - \psi(x))g(x)\frac{dz}{|z|^{1-\rho}}.$$

This operator can be viewed as the infinitesimal generator of the process (X_t) solution of

$$X_t = x + \int_0^t \int_{\mathbb{R}\times\mathbb{R}_+} f(X_{s-})z^{-1}1_{\{u<g(X_{s-})\}}N(ds, dz, du).$$

We have $E = \mathbb{R}\times\mathbb{R}_+$, $d\mu(z) = \frac{1}{|z|^{1-\rho}}1_{\{|z|\geq 1\}}dz$, $c(z,x) = f(x)z^{-1}$ and $\gamma(z,x) = g(x)$. We make the following assumptions. There exist two constants $\underline{f}$ and $\overline{f}$ such that $\forall x$ $\underline{f} \leq f(x) \leq \overline{f}$ and we suppose that all derivatives of f are bounded by $\overline{f}$. Moreover we assume that there exist two constants $\underline{g}$ and $\overline{g}$ such that g and its derivative are bounded by $\overline{g}$ and $0 < \underline{g} \leq g(x)$, $\forall x$. Consequently it is easy to check that hypotheses 3.0., 3.1., 3.2. and 3.3.b are satisfied, with $\theta = +\infty$. Moreover we have $\mu(B_M) \leq CM^\rho$ and $A2$ holds with $p_1 = 1 - \rho$ and $p_2 = 2 - \rho$. Consequently we deduce that the law of X_t admits a density $\mathcal{C}^k$ with $k < 1/\rho - 3$ if $1/\rho - 3 \geq 1$.

The next sections are the successive steps to prove proposition 3 .

3.3 Approximation of X_t

In order to prove that the process X_t, solution of (31), has a smooth density, we will apply the differential calculus and the integration by parts formula of section 2. But since the random variable X_t can not be viewed as a simple functional, the first step consists in approximate it. We describe in this section our approximation procedure. We consider a non-negative and smooth function $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ such that $\varphi(z) = 0$ for $|z| > 1$ and $\int_{\mathbb{R}^d}\varphi(z)dz = 1$. And for $M \in \mathbb{N}$ we denote $\Phi_M(z) = \varphi * 1_{B_M}$ with $B_M = \{z \in \mathbb{R}^d : |z| < M\}$. Then $\Phi_M \in C_b^\infty$ and we have $1_{B_{M-1}} \leq \Phi_M \leq 1_{B_{M+1}}$. We denote by X_t^M the solution of the equation

$$X_t^M = x + \int_0^t \int_E c_M(z, X_{s-}^M)1_{\{u<\gamma(z,X_{s-}^M)\}}N(ds, dz, du) + \int_0^t g(X_s^M)ds. \tag{33}$$

where $c_M(z,x) := c(z,x)\Phi_M(z)$. Observe that equation (33) is obtained from (31) replacing the coefficient c by the truncating one c_M. Let $N_M(ds, dz, du) := 1_{B_{M+1}}(z) \times 1_{[0,2\overline{C}]}(u)N(ds, dz, du)$. Since $\{u < \gamma(z, X_{s-}^M)\} \subset \{u < 2\overline{C}\}$ and $\Phi_M(z) = 0$ for $|z| > M + 1$, we may replace N by N_M in the above equation and consequently X_t^M is solution of the equation

$$X_t^M = x + \int_0^t \int_E c_M(z, X_{s-}^M)1_{\{u<\gamma(z,X_{s-}^M)\}}N_M(ds, dz, du) + \int_0^t g(X_s^M)ds.$$

Since the intensity measure $\widehat{N}_M$ is finite we may represent the random measure N_M by a compound Poisson process. Let $\lambda_M = 2\overline{C} \times \mu(B_{M+1}) = t^{-1}E(N_M(t,E))$ and let J_t^M a Poisson process of parameter λ_M. We denote by $T_k^M, k \in \mathbb{N}$ the jump times of J_t^M. We also consider two sequences of independent random variables $(Z_k^M)_{k\in\mathbb{N}}$ and $(U_k)_{k\in\mathbb{N}}$ respectively in $\mathbb{R}^d$ and $\mathbb{R}_+$ which are independent of J^M and such that

$$Z_k \sim \frac{1}{\mu(B_{M+1})} 1_{B_{M+1}}(z)d\mu(z), \quad \textit{and} \quad U_k \sim \frac{1}{2\overline{C}} 1_{[0,2\overline{C}]}(u)du.$$

To simplify the notation, we omit the dependence on M for the variables (T_k^M) and (Z_k^M). Then equation (33) may be written as

$$X_t^M = x + \sum_{k=1}^{J_t^M} c_M(Z_k, X_{T_k-}^M) 1_{(U_k,\infty)}(\gamma(Z_k, X_{T_k-}^M)) + \int_0^t g(X_s^M)ds. \tag{34}$$

Lemma 4 *Assume that hypotheses* 3.0., 3.1., 3.2 *and* 3.3. *hold true then we have*

$$E\left|X_t^M - X_t\right| \le \varepsilon_M := te^{Ct} \int_{\{|z|>M\}} \overline{c}(z)\overline{\gamma}(z)d\mu(z), \tag{35}$$

for some constant C.

Proof: We have $E\left|X_t^M - X_t\right| \le I_M^1 + I_M^2$ with

$$\begin{aligned} I_M^1 &= E\int_0^t \int_{\mathbb{R}^d} \int_0^{\overline{C}} \left| c(z,X_s)1_{\{u<\gamma(z,X_s)\}} - c_M(z,X_s^M)1_{\{u<\gamma(z,X_s^M)\}} \right| dud\mu(z)ds \\ I_M^2 &= E\int_0^t \left|g(X_s) - g(X_s^M)\right| ds. \end{aligned}$$

Since $|\nabla_x c(z,x)| \le \overline{c}(z)$ we have $I_M^1 \le I_M^{1,1} + I_M^{1,2}$ with

$$\begin{aligned} I_M^{1,1} &= E\int_0^t \int_{\mathbb{R}^d} \int_0^{\overline{C}} \left|c(z,X_s) - c_M(z,X_s^M)\right| 1_{\{u<\overline{\gamma}(z)\}} dud\mu(z)ds \\ &\le t\int_{\mathbb{R}^d} \overline{c}(z)\overline{\gamma}(z)(1-\Phi_M(z))d\mu(z) + \int_{\mathbb{R}^d} \overline{c}(z)\overline{\gamma}(z)dz \times E\int_0^t \left|X_s - X_s^M\right| ds \end{aligned}$$

and, since $|\nabla_x \gamma(z,x)| \le \overline{\gamma}^{x,1}$

$$\begin{aligned} I_M^{1,2} &= E\int_0^t \int_{\mathbb{R}^d} \int_0^{\overline{C}} \overline{c}(z) \left|1_{\{u<\gamma(z,X_s)\}} - 1_{\{u<\gamma(z,X_s^M)\}}\right| dud\mu(z)ds \\ &= E\int_0^t \int_{\mathbb{R}^d} \overline{c}(z) \left|\gamma(z,X_s) - \gamma(z,X_s^M)\right| d\mu(z)ds \\ &\le \int_{\mathbb{R}^d} \overline{c}(z)\overline{\gamma}^{x,1} d\mu(z) \times E\int_0^t \left|X_s - X_s^M\right| ds. \end{aligned}$$

A similar inequality holds for I_M^2 so we obtain

$$E\left|X_t^M - X_t\right| \leq t \times \int_{R^d} \overline{\gamma}(z)\overline{c}(z)(1-\Phi_M(z))d\mu(z) + C\int_0^t E\left|X_s - X_s^M\right| ds.$$

We conclude by using Gronwall's lemma. $\diamond$

The random variable X_t^M solution of (34) is a function of $(Z_1 \ldots, Z_{J_t^M})$ but it is not a simple functional, as defined in section 2 because the coefficient $c_M(z,x)1_{(u,\infty)}(\gamma(z,x))$ is not differentiable with respect to z. In order to avoid this difficulty we use the following alternative representation. Let $z_M^* \in \mathbb{R}^d$ such that $|z_M^*| = M+3$. We define

$$\begin{aligned} q_M(z,x) \quad &: = \varphi(z-z_M^*)\theta_{M,\gamma}(x) + \frac{1}{2\overline{C}\mu(B_{M+1})}1_{B_{M+1}}(z)\gamma(z,x)h(z) \\ \theta_{M,\gamma}(x) \quad &: = \frac{1}{\mu(B_{M+1})}\int_{\{|z|\leq M+1\}}(1-\frac{1}{2\overline{C}}\gamma(z,x))\mu(dz). \end{aligned} \tag{36}$$

We recall that φ is the function defined at the beginning of this subsection : a non-negative and smooth function with $\int \varphi = 1$ and which is null outside the unit ball. Moreover from hypothesis 3.1, $0 \leq \gamma(z,x) \leq \overline{C}$ and then $1 \geq \theta_{M,\gamma}(x) \geq 1/2$. By construction the function q_M satisfies $\int q_M(x,z)dz = 1$. Hence we can check that

$$E(f(X_{T_k}^M) \mid X_{T_k-}^M = x) = \int_{R^d} f(x+c_M(z,x))q_M(z,x)dz. \tag{37}$$

In fact the left hand side term of (37) is equal to $I+J$ with

$$\begin{aligned} I &= E(f(X_{T_k}^M)1_{\{U_k \geq \gamma(Z_k, X_{T_k-}^M)\}} \mid X_{T_k-}^M = x) \quad \mathit{and} \\ J &= E(f(X_{T_k}^M)1_{\{U_k < \gamma(Z_k, X_{T_k-}^M)\}} \mid X_{T_k-}^M = x). \end{aligned}$$

A simple calculation leads to

$$I = f(x)P(U_k \geq \gamma(Z_k,x)) = f(x)\theta_{M,\gamma}(x) = \int_{|z|>M+1} f(x+c_M(z,x))q_M(z,x)dz$$

where the last equality results from the fact that $c_M(z,x) = 0$ for $|z| > M+1$. Moreover one can easily see that $J = \int_{|z|\leq M+1} f(x+c_M(z,x))q_M(z,x)dz$ and (37) is proved.

From the relation (37) we construct a process $(\overline{X}_t^M)$ equal in law to (X_t^M) on the following way.

We denote by $\Psi_t(x)$ the solution of $\Psi_t(x) = x + \int_0^t g(\Psi_s(x))ds$. We assume that the times $T_k, k \in \mathbb{N}$ are fixed and we consider a sequence $(z_k)_{k\in\mathbb{N}}$ with $z_k \in \mathbb{R}^d$. Then we define $x_t, t \geq 0$ by $x_0 = x$ and, if x_{T_k} is given, then

$$\begin{aligned} x_t &= \Psi_{t-T_k}(x_{T_k}) \quad T_k \leq t < T_{k+1}, \\ x_{T_{k+1}} &= x_{T_{k+1}^-} + c_M(z_{k+1}, x_{T_{k+1}^-}). \end{aligned}$$

We remark that for $T_k \leq t < T_{k+1}, x_t$ is a function of $z_1, ..., z_k$. Notice also that x_t solves the equation

$$x_t = x + \sum_{k=1}^{J_t^M} c_M(z_k, x_{T_k^-}) + \int_0^t g(x_s)ds.$$

We consider now a sequence of random variables $(\overline{Z}_k), k \in \mathbb{N}^*$ and we denote $\mathcal{G}_k = \sigma(T_p, p \in \mathbb{N}) \vee \sigma(\overline{Z}_p, p \leq k)$ and $\overline{X}_t^M = x_t(\overline{Z}_1, ..., \overline{Z}_{J_t^M})$. We assume that the law of $\overline{Z}_{k+1}$ conditionally on $\mathcal{G}_k$ is given by

$$P(\overline{Z}_{k+1} \in dz \mid \mathcal{G}_k) = q_M(x_{T_{k+1}^-}(\overline{Z}_1, ..., \overline{Z}_k), z)dz = q_M(\overline{X}_{T_{k+1}^-}^M, z)dz.$$

Clearly $\overline{X}_t^M$ satisfies the equation

$$\overline{X}_t^M = x + \sum_{k=1}^{J_t^M} c_M(\overline{Z}_k, \overline{X}_{T_k-}^M) + \int_0^t g(\overline{X}_s^M)ds \tag{38}$$

and $\overline{X}_t^M$ has the same law as X_t^M. Moreover we can prove a little bit more.

Lemma 5 *For a locally bounded and measurable function $\psi : \mathbb{R}^d \to \mathbb{R}$ let*

$$\overline{S}_t(\psi) = \sum_{k=1}^{J_t^M} (\Phi_M\psi)(\overline{Z}_k), \quad S_t(\psi) = \sum_{k=1}^{J_t^M} (\Phi_M\psi)(Z_k)1_{\{\gamma(Z_k, X^M(T_k-)) > U_k\}},$$

then $(\overline{X}_t^M, \overline{S}_t(\psi))_{t\geq 0}$ has the same law as $(X_t^M, S_t(\psi))_{t\geq 0}$.

Proof: Observing that $(\overline{X}_t^M, \overline{S}_t(\psi))_{t\geq 0}$ solves a system of equations similar to (38) but in dimension $d+1$, it suffices to prove that $(\overline{X}_t^M)_{t\geq 0}$ has the same law as $(X_t^M)_{t\geq 0}$. This readily follows from

$$E(f(X_{T_{k+1}}^M) \mid X_{T_{k+1}-}^M = x) = E(f(\overline{X}_{T_{k+1}}^M) \mid \overline{X}_{T_{k+1}-}^M = x)$$

which is a consequence of (37).

$\diamond$

Remark 1 *Looking at the infinitesimal generator L of X it is clear that the natural approximation of X_t is $\overline{X}_t^M$ instead of X_t^M. But we use the representation given by X_t^M for two reasons. First it is easier to obtain estimates for this process because we have a stochastic equation and so we may use the stochastic calculus associated to a Poisson point measure. Moreover, having this equation in mind, gives a clear idea about the link with other approaches by Malliavin calculus to the solution* *of a stochastic equation with jumps: we mainly think to [B.G.J]. Remark that X_t is solution of an equation with discontinuous coefficients so the approach developped by [B.G.J] does not work. And if we consider the equation of $\overline{X}_t^M$ then the underlying point measure depends on the solution of the equation so it is no more a Poisson point measure.*

3.4 The integration by parts formula

The random variable $\overline{X}_t^M$ constructed previously is a simple functional but unfortunately its Malliavin covariance matrix is degenerated. To avoid this problem we use a classical regularization procedure. Instead of the variable $\overline{X}_t^M$, we consider the regularized one F_M defined by

$$F_M = \overline{X}_t^M + \sqrt{U_M(t)} \times \Delta, \tag{39}$$

where Δ is a $d-$dimensional standard gaussian variable independent of the variables $(\overline{Z}_k)_{k\geq 1}$ and $(T_k)_{k\geq 1}$ and $U_M(t)$ is defined by

$$U_M(t) = t\int_{B_{M-1}^c} \underline{c}^2(z)\underline{\gamma}(z)d\mu(z). \tag{40}$$

We observe that $F_M \in \mathcal{S}^d$ where $\mathcal{S}$ is the space of simple functionals for the differential calculus based on the variables $(\overline{Z}_k)_{k\in\mathbb{N}}$ with $\overline{Z}_0 = (\Delta^r)_{1\leq r\leq d}$ and $\overline{Z}_k = (\overline{Z}_k^r)_{1\leq r\leq d}$ and we are now in the framework of section 2 by taking $\mathcal{G} = \sigma(T_k, k \in \mathbb{N})$ and defining the weights (π_k) by $\pi_0^r = 1$ and $\pi_k^r = \Phi_M(\overline{Z}_k)$ for $1 \leq r \leq d$. Conditionally on $\mathcal{G}$, the density of the law of $(\overline{Z}_1, ..., \overline{Z}_{J_t^M})$ is given by

$$p_M(\omega, z_1, ..., z_{J_t^M}) = \prod_{j=1}^{J_t^M} q_M(z_j, \Psi_{T_j - T_{j-1}}(\overline{X}_{T_{j-1}}^M))$$

where $\overline{X}_{T_{j-1}}^M$ is a function of $z_i, 1 \leq i \leq j-1$. We can check that p_M satisfies the hypothesis H1 of section 2.

To clarify the notation, the derivative operator can be written in this framework for $F \in \mathcal{S}$ by $DF = (D_{k,r}F)$ where $D_{k,r} = \pi_k^r \partial_{\overline{Z}_k^r}$ for $k \geq 0$ and $1 \leq r \leq d$. Consequently we deduce that $D_{k,r}F_M^{r'} = D_{k,r}\overline{X}_t^{M,r'}$, for $k \geq 1$ and $D_{0,r}F_M^{r'} = \sqrt{U_M(t)}\delta_{r,r'}$ with $\delta_{r,r'} = 0$ if $r \neq r'$, $\delta_{r,r'} = 1$ otherwise.

The Malliavin covariance matrix of $\overline{X}_t^M$ is equal to

$$\sigma(\overline{X}_t^M)^{i,j} = \sum_{k=1}^{J_t^M} \sum_{r=1}^{d} D_{k,r}\overline{X}_t^{M,i} D_{k,r}\overline{X}_t^{M,j}$$

for $1 \leq i,j \leq d$ and finally the Malliavin covariance matrix of F_M is given by

$$\sigma(F_M) = \sigma(\overline{X}_t^M) + U_M(t) \times Id.$$

Using the results of section 2, we can state an integration by part formula and give a bound for the weight $H^q(F_M, 1)$ in terms of the Sobolev norms of F_M, the divergence LF_M and the determinant of the inverse of the Malliavin covariance matrix $\det \sigma(F_M)$. The control of these last three quantities is rather technical and is studied in detail in section 4.

Proposition 4 *Assume hypotheses* 3.0. 3.1. 3.2. *and let* $\phi : \mathbb{R}^d \to \mathbb{R}$ *be a bounded smooth function with bounded derivatives. For every multi-index* $\beta = (\beta_1, \dots \beta_q) \in \{1, \dots, d\}^q$ *such that* $4d(3q-1)/t < \theta$

a) if 3.3.a *holds then*

$$|E(\partial_\beta \phi(F_M))| \leq C_q \|\phi\|_\infty . \tag{41}$$

b) if 3.3.b *holds then*

$$|E(\partial_\beta \phi(F_M))| \leq C_q \|\phi\|_\infty (1 + \mu(B_{M+1})^q), \tag{42}$$

Remark : if $\theta = \infty$ then $\forall t > 0$, we have an integration by parts formula for any order of derivation q. Conversely if θ is finite, we need to have t large enough to integrate q times by part.

Proof: The integration by parts formula (11) gives, for every smooth $\phi : \mathbb{R}^d \to \mathbb{R}$ and every multi-index $\beta = (\beta_1, ..., \beta_q)$

$$E(\partial_\beta \phi(F_M)) = E(\phi(F_M) H^q_\beta(F_M, 1)),$$

and consequently

$$|E(\partial_\beta \phi(F_M))| \quad \leq \quad \|\phi\|_\infty E(|H^q_\beta(F_M, 1)|).$$

So we just have to bound $|H^q_\beta(F_M, 1)|$. From the second part of Theorem 3 we have

$$|H^q(F_M, 1)| \leq C_q \frac{1}{|\det \sigma(F_M)|^{3q-1}} (1 + |F_M|_{q+1}^{(6d+1)q})(1 + |LF_M|_{q-1}^q).$$

Now from Lemma 13 (see section 4), we have :

a) assuming 3.3.a, for $l, p \geq 1$,

$$E|LF_M|_l^p \leq C_{l,p};$$

b) assuming 3.3.b, for $l, p \geq 1$,

$$E|LF_M|_l^p \leq C_{l,p}(1 + \mu(B_{M+1})^p).$$

Hence from Lemma 9, for $l, p \geq 1$

$$E|F_M|_l^p \leq C_{l,p};$$

and from Lemma 16 , we have for $p \geq 1$, $t > 0$ such that $2dp/t < \theta$

$$E\frac{1}{\det \sigma(F_M))^p} \leq C_p.$$

The final result is then a straightforward consequence of Cauchy-Schwarz inequality. $\diamond$

3.5 Estimates for the Fourier transform of X_t

In this section, we prove Proposition 3.

Proof: The proof consists first to approximate X_t by $\overline{X}_t^M$ and then to apply the integration by parts formula.

Approximation. We have

$$\left|E(e^{i\langle \xi, X_t\rangle})\right| \leq |\xi|\, E\left|X_t - \overline{X}_t^M\right| + \left|E(e^{i\left\langle \xi, \overline{X}_t^M\right\rangle} - e^{i\langle \xi, F_M\rangle})\right| + \left|E(e^{i\langle \xi, F_M\rangle})\right|.$$

From (35) we deduce

$$E(\left|X_t - \overline{X}_t^M\right|) \leq \varepsilon_M = te^{Ct}\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)d\mu(z).$$

Moreover

$$E(e^{i\left\langle \xi, \overline{X}_t^M\right\rangle} - e^{i\langle \xi, F_M\rangle}) \quad = \quad E(e^{i\left\langle \xi, \overline{X}_t^M\right\rangle}(1 - e^{i\left\langle \xi, \sqrt{U_M(t)}\Delta\right\rangle})) = E(e^{i\left\langle \xi, \overline{X}_t^M\right\rangle})(1 - e^{-\frac{1}{2}|\xi|^2 U_M(t)}),$$

so that

$$\left|E(e^{i\left\langle \xi, \overline{X}_t^M\right\rangle} - e^{i\langle \xi, F_M\rangle})\right| \leq U_M(t)\frac{1}{2}\left|\xi\right|^2.$$

We conclude that

$$\left|E(e^{i\langle \xi, X_t\rangle})\right| \leq U_M(t)\frac{1}{2}\left|\xi\right|^2 + |\xi|\, te^{Ct}\int_{B_M^c} \overline{c}(z)d\mu(z)) + \left|E(e^{i\langle \xi, F_M\rangle})\right|.$$

Integration by parts. We denote $e_\xi(x) = \exp(i\left\langle \xi, x\right\rangle)$ and we have $\partial_\beta e_\xi(x) = i^{|\beta|}\xi_{\beta_1}\ldots\xi_{\beta_q}e_\xi(x)$. Consequently

a) assuming 3.3.a and applying (41) for β such that $|\beta| = q$ we obtain

$$\left|E(e^{i\ \langle \xi, F_M\rangle})\right| \leq \frac{C_q}{|\xi|^q},$$

b) assuming 3.3.b, we obtain similarly from (42)

$$|\xi_{\beta_1}\dots\xi_{\beta_q}|\left|E(e^{i\ \langle\xi,F_M\rangle})\right| = |E(\partial_\beta e_\xi(F_M))| \leq C_q(1+\mu(B_{M+1})^q),$$

and then

$$\left|E(e^{i\ \langle\xi,F_M\rangle})\right| \leq \frac{C_q}{|\xi|^q}(1+\mu(B_{M+1})^q),$$

and the proposition is proved.

$\diamond$

4 Sobolev norms-Divergence-Covariance matrix

4.1 Sobolev norms

We prove in this section that $\forall l \geq 1$ and $\forall p \geq 1$ $E|F_M|_l^p \leq C_{l,p}$. We begin this section with a preliminary lemma which will be also useful to control the covariance matrix.

4.1.1 Preliminary

We consider a Poisson point measure $N(ds,dz,du)$ on $\mathbb{R}^d\times\mathbb{R}_+$ with compensator $\mu(dz)\times 1_{(0,\infty)}(u)du$ and two non negative measurable functions $f,g:\mathbb{R}^d\to\mathbb{R}_+$. For a measurable set $B\subset\mathbb{R}^d$ we denote $B_g=\{(z,u): z\in B, u<g(z)\}\subset\mathbb{R}^d\times\mathbb{R}_+$ and we consider the process

$$N_t(1_{B_g}f) := \int_0^t\int_{B_g} f(z)N(ds,dz,du).$$

Moreover we note $\nu_g(dz)=g(z)d\mu(z)$ and

$$\alpha_{g,f}(s)=\int_{\mathbb{R}^d}(1-e^{-sf(z)})d\nu_g(dz), \quad \beta_{B,g,f}(s)=\int_{B^c}(1-e^{-sf(z)})d\nu_g(dz).$$

We have the following result.

Lemma 6 *Let $\phi(s)=Ee^{-sN_t(f1_{B_g})}$ the Laplace transform of the random variable $N_t(f1_{B_g})$ then we have*

$$\phi(s)=e^{-t(\alpha_{g,f}(s)-\beta_{B,g,f}(s))}.$$

Proof: From Itô's formula we have

$$\exp(-sN_t(f1_{B_g})) = 1-\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}_+}\exp(-s(N_{r-}(f1_{B_g})))(1-\exp(-sf(z)1_{B_g}(z,u)))dN(r,z,u)$$

and consequently

$$E(\exp(-sN_t(f1_{B_g}))) = 1 - \int_0^t E(\exp(-s(N_{r-}(f1_{B_g})) \int_{\mathbb{R}^d\times\mathbb{R}_+} (1-\exp(-sf(z)1_{B_g}(z,u)))d\mu(z)dudr.$$

But

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}_+} (1-\exp(-sf(z)1_{B_g}(z,u)))d\mu(z)du &= \int_{\mathbb{R}^d\times\mathbb{R}_+} 1_{B_g}(z,u)(1-\exp(-sf(z)))d\mu(z)du \\ &= \int_{\mathbb{R}^d} 1_B(z)(1-\exp(-sf(z))) \int_{\mathbb{R}_+} 1_{\{u<g(z)\}} dud\mu(z) \\ &= \int_B (1-\exp(-sf(z)))g(z)d\mu(z) = \alpha_{g,f}(s) - \beta_{B,g,f}(s),\end{aligned}$$

It follows that

$$E(\exp(-sN_t(f1_{B_g}))) = \exp(-t(\alpha_{g,f}(s) - \beta_{B,g,f}(s))).$$

$\diamond$

4.1.2 Bound for $|\overline{X}_t^M|_l$

In this section, we use the notation $\overline{c}_1(z) = \sup_x |\nabla_x c(z,x)|$. Under hypothesis 3.3.$i$ we have $\overline{c}_1(z) \leq \overline{c}(z)$, but we introduce this notation to highlight the dependence on the first derivative of the function c.

Lemma 7 *Let* $(\overline{X}_t^M)$ *the process solution of equation* (38) *then under hypotheses* 3.0., 3.1. *and* 3.2. *we have* $\forall l \geq 1$,

$$\sup_{s\leq t} |\overline{X}_s^M|_{1,l} \leq C_l(1 + \sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k))^{l\times l!} \sup_{s\leq t} (\mathcal{E}_s^M)^{l\times l!}$$

where C_l *is an universal constant and where* $\mathcal{E}_t^M$ *is solution of the linear equation*

$$\mathcal{E}_t^M = 1 + C_l \sum_{k=1}^{J_t^M} \overline{c}_1(\overline{Z}_k)\mathcal{E}_{T_k-}^M + C_l \int_0^t \mathcal{E}_s^M ds. \tag{43}$$

Consequently $\forall l, p \geq 1$

$$\sup_M E \sup_{s\leq t} |\overline{X}_s^M|_{1,l}^p < \infty$$

Before proving this lemma we first give a result which is a straightforward consequence of lemma 1 and formula (21).

Lemma 8 *Let* $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ *a* $\mathcal{C}^\infty$ *function and* $F \in \mathcal{S}^d$ *then* $\forall l \geq 1$ *we have*

$$|\phi(F)|_{1,l} \leq |\nabla\phi(F)||F|_{1,l} + C_l \sup_{2\leq|\beta|\leq l} |\partial_\beta \phi(F)||F|_{1,l-1}^l.$$

We proceed now to the proof of Lemma 7.

Proof: We first recall that from hypothesis 3.0., g and its derivatives are bounded and from hypothesis 3.2.i) the coefficient c as well as its derivatives are bounded by the function $\overline{c}$. Now the truncated coefficient c_M of equation (38) is equal to $c_M = c \times \phi_M$ where ϕ_M is a $\mathcal{C}^\infty$ bounded function with derivatives uniformly bounded with respect to M. Consequently using Lemma 8 we obtain for $l \geq 1$

$$|\overline{X}_t^M|_{1,l} \leq C_l \left(A_{t,l-1} + \sum_{k=1}^{J_t^M} \overline{c}_1(\overline{Z}_k)|\overline{X}_{T_k-}^M|_{1,l} + \int_0^t |\overline{X}_s^M|_{1,l} ds \right),$$

with

$$A_{t,l-1} = \sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k)(|\overline{Z}_k|_{1,l} + |\overline{Z}_k|_{1,l-1}^l + |\overline{X}_{T_k-}^M|_{1,l-1}^l) + \int_0^t |\overline{X}_s^M|_{1,l-1}^l ds.$$

This gives

$$\forall s \leq t \quad |\overline{X}_s^M|_{1,l} \leq A_{t,l-1}\mathcal{E}_s^M, \tag{44}$$

Under hypotheses 3.0. 3.1. and 3.2. we have

$$\forall p \geq 1 \quad E(\sup_{s \leq t} |\mathcal{E}_t^M|^p) \leq C_p.$$

Now one can easily check that for $l \geq 1$

$$|\overline{Z}_k|_{1,l} \leq |\pi_k|_{l-1},$$

but since $\pi_k = \phi_M(\overline{Z}_k)$ we deduce from Lemma 8 that

$$|\overline{Z}_k|_{1,l} \leq 1 + C_l(|\overline{Z}_k|_{1,l-1} + |\overline{Z}_k|_{1,l-2}^{l-1}).$$

Observing that $|\overline{Z}_k|_{1,1} = |D\overline{Z}_k| = |\pi_k| \leq 1$ we conclude that $\forall l \geq 1$

$$|\overline{Z}_k|_{1,l} \leq C_l.$$

This gives

$$A_{t,l-1} \leq tC_l(1 + \sup_{s \leq t} |\overline{X}_s^M|_{1,l-1})^l (1 + \sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k)). \tag{45}$$

From this inequality we can prove easily Lemma 7 by induction. For $l = 1$ we remark that

$$\forall s \leq t \quad |\overline{X}_s^M|_{1,1} \leq A_{t,0}\mathcal{E}_s^M, \quad \text{with} \quad A_{t,0} = \sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k),$$

and the result is true. To complete the proof of lemma 7, we prove that $\forall p \geq 1$

$$E\left(\sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k)\right)^p \leq C_p.$$

We have the equality in law

$$\sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k) \backsimeq \int_0^t \int_E \overline{c}(z) 1_{u<\gamma(z,X^M_{T_k-})} 1_{B_{M+1}}(z) 1_{[0,2\overline{C}]}(u) N(ds,dz,du),$$

moreover using the notations of section 4.1.1. we have

$$\int_0^t \int_E \overline{c}(z) 1_{u<\gamma(z,X^M_{T_k-})} 1_{B_{M+1}}(z) 1_{[0,2\overline{C}]}(u) N(ds,dz,du) \leq N_t(1_{B_{\overline{\gamma}}}\overline{c})$$

with $B_{\overline{\gamma}} = \{(z,u); z \in B_{M+1}; 0 < u < \overline{\gamma}(z)\}$. From Lemma 6 it follows that

$$Ee^{-sN_t(1_{B_{\overline{\gamma}}}\overline{c})} = \exp(-t \int_{B_{M+1}} (1 - e^{-s\overline{c}(z)}) \overline{\gamma}(z) d\mu(z))$$

and since from hypotheses 3.1. and 3.2., $\int_{\mathbb{R}^d} |\overline{c}(z)\overline{\gamma}(z)| d\mu(z) < \infty$ we deduce that $\forall p \geq 1$, $EN_t(1_{B_{\overline{\gamma}}}\overline{c})^p = t^p(\int_{B_{M+1}} |\overline{c}(z)\overline{\gamma}(z)| d\mu(z))^p \leq C_p$ where the constant C_p does not depend on M. This achieves the proof of Lemma 7. $\diamond$

4.1.3 Bound for $|F_M|_l$

Lemma 9 *Under hypotheses* 3.0., 3.1. *and* 3.2. *we have*

$$\forall l, p \geq 1 \quad E|F_M|_l^p \leq C_{l,p}.$$

We have $F_M = \overline{X}_t^M + \sqrt{U_M(t)}\Delta$ and then $|F_M|_l \leq |\overline{X}_t^M|_l + \sqrt{U_M(t)}|\Delta|_l$. But $|\Delta|_l \leq |\Delta| + d$ and $U_M(t) \leq t \int_{\mathbb{R}^d} \overline{c}^2(z)\overline{\gamma}(z) d\mu(z) < \infty$. So the conclusion of Lemma 9 follows from Lemma 7.

4.2 Divergence

In this section our goal is to bound $|LF_M|_l$ for $l \geq 0$. From the definition of the divergence operator L we have $LF_M^r = L\overline{X}_t^{M,r} - \Delta^r$ and then

$$|LF_M|_l \leq |L\overline{X}_t^M|_l + |\Delta| + d,$$

so we just have to bound $|L\overline{X}_t^M|_l$. We proceed as in the previous section and we first state a lemma similar to Lemma 8.

Lemma 10 *Let $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ a $\mathcal{C}^\infty$ function and $F \in \mathcal{S}^d$ then $\forall l \geq 1$ we have*

$$\begin{aligned} |L\phi(F)|_l &\leq |\nabla\phi(F)||LF|_l + C_l \sup\nolimits_{2\leq|\beta|\leq l+2} |\partial_\beta\phi(F)|(1+|F|_l^l)(|LF|_{l-1} + |F|_{1,l+1}^2), \\ &\leq |\nabla\phi(F)||LF|_l + C_l \sup\nolimits_{2\leq|\beta|\leq l+2} |\partial_\beta\phi(F)|(1+|F|_{l+1}^{l+2})(1+|LF|_{l-1}). \end{aligned}$$

For $l = 0$, we have

$$|L\phi(F)| \leq \nabla\phi(F)||LF| + \sup_{\beta=2} |\partial_\beta\phi F||F|_{1,1}^2.$$

The proof follows from (7) and Lemma 8 and we omit it.

Next we give a bound for $|L\overline{Z}_k|_l$. We recall the notation

$$\overline{\gamma}_{\ln}^{z,l}(z) = \sup_x \sup_{1\leq|\beta|\leq l} |\partial_{\beta,z} \ln\gamma(z,x)|, \quad \overline{h}_{\ln}^l(z) = \sup_{1\leq|\beta|\leq l} |\partial_\beta \ln h(z)|, \quad \overline{\theta}_{\ln}^l = \sup_x \sup_{1\leq|\beta|\leq l} |\partial_\beta \ln\theta_{M,\gamma}(x)|,$$

$$\overline{\gamma}_{\ln}^{x,l}(z) = \sup_x \sup_{1\leq|\beta|\leq l} |\partial_{\beta,x} \ln\gamma(z,x)|, \quad \overline{\gamma}^{x,l} = \sup_z \sup_x \sup_{1\leq|\beta|\leq l} |\partial_{\beta,x}\gamma(z,x)|.$$

Lemma 11 *Assuming hypotheses 3.0., 3.1., 3.2 and 3.3., we have $\forall l \geq 0$ and $\forall k \leq J_t^M$*

$$|L\overline{Z}_k|_l \leq C_l(\overline{\gamma}_{\ln}^{z,l+1}(\overline{Z}_k) + \overline{h}_{\ln}^{z,l+1}(\overline{Z}_k) + \sup_{s\leq t} |\overline{X}_s^M|_{l+1}^{l+1} \sum_{j=k+1}^{J_t^M} \overline{\theta}_{\ln}^{l+1} 1_{B(z_M^*,1)}(\overline{Z}_j) + \overline{\gamma}_{\ln}^{x,l+1}(\overline{Z}_j))),$$

with $\overline{\theta}_{\ln}^l \leq C_l(\overline{\gamma}^{x,l})^l$.

In addition, if we assume 3.3.a., we obtain $\forall p \geq 1$

$$E \sup_{k\leq J_t^M} |L\overline{Z}_k|_l^p \leq C_{p,l}.$$

On the other hand, assuming 3.3.b, we have $\forall p \geq 1$

$$E \sup_{k\leq J_t^M} |L\overline{Z}_k|_l^p \leq C_{p,l}(1+\mu(B_{M+1})^p)$$

Proof: We first recall that we have proved in the preceding section that $\forall l \geq 1$, $|\overline{Z}_k|_l \leq C_l$. Now $L\overline{Z}_k^r = \delta(D\overline{Z}_k^r)$ and since $D_{k,r}\overline{Z}_k^r = \pi_k$ we obtain

$$L\overline{Z}_k^r = -\partial_{k,r}(\pi_k^2) - \pi_k D_{k,r} \ln p_M,$$

this leads to

$$|L\overline{Z}_k^r|_l \leq C_l(1 + |D_{k,r} \ln p_M|_l).$$

Recalling that $\ln p_M = \sum_{j=1}^{J_t^M} \ln q_M(\overline{Z}_j, \overline{X}^M_{T_j-})$ and that $\overline{X}^M_{T_j-}$ depends on $\overline{Z}_k$ for $k \leq j-1$ we obtain

$$D_{k,r} \ln p_M = D_{k,r} \ln q_M(\overline{Z}_k, \overline{X}^M_{T_k-}) + \sum_{j=k+1}^{J_t^M} D_{k,r} \ln q_M(\overline{Z}_j, \overline{X}^M_{T_j-})$$

But on $\{\pi_k > 0\}$, we have $q_M(\overline{Z}_k, \overline{X}^M_{T_k-}) = C\gamma(\overline{Z}_k, \overline{X}^M_{T_k-})h(\overline{Z}_k)$, and then

$$D_{k,r} \ln q_M(\overline{Z}_k, \overline{X}^M_{T_k-}) = D_{k,r} \ln \gamma(\overline{Z}_k, \overline{X}^M_{T_k-}) + D_{k,r} \ln h(\overline{Z}_k).$$

Now for $j \geq k+1$, if $|\overline{Z}_j - z^*_M| < 1$ then

$$\ln q_M(\overline{Z}_j, \overline{X}^M_{T_j-}) = \ln \varphi(\overline{Z}_j - z^*_M) + \ln \theta_{M,\gamma}(\overline{X}^M_{T_j-})$$

consequently

$$D_{k,r} \ln q_M(\overline{Z}_j, \overline{X}^M_{T_j-}) = D_{k,r} \ln \theta_{M,\gamma}(\overline{X}^M_{T_j-}),$$

and if $\overline{Z}_j \in B_{M+1}$ then

$$D_{k,r} \ln q_M(\overline{Z}_j, \overline{X}^M_{T_j-}) = D_{k,r} \ln \gamma(\overline{Z}_j, \overline{X}^M_{T_j-})$$

and finally

$$D_{k,r} \ln q_M(\overline{Z}_j, \overline{X}^M_{T_j-}) = D_{k,r} \ln \theta_{M,\gamma}(\overline{X}^M_{T_j-}) 1_{B(z^*_M,1)}(\overline{Z}_j) + D_{k,r} \ln \gamma(\overline{Z}_j, \overline{X}^M_{T_j-}) 1_{B_{M+1}}(\overline{Z}_j).$$

It is worth to note that this random variable is a simple variable as defined in section 2.

Putting this together, it yields

$$\begin{aligned} |D_{k,r} \ln p_M|_l &\leq |D_{k,r} \ln \gamma(\overline{Z}_k, \overline{X}^M_{T_k-})|_l + |D_{k,r} \ln h(\overline{Z}_k)|_l \\ &+ \sum_{j=k+1}^{J_t^M} (|D_{k,r} \ln \theta_{M,\gamma}(\overline{X}^M_{T_j-}) 1_{B(z^*_M,1)}(\overline{Z}_j)|_l + |D_{k,r} \ln \gamma(\overline{Z}_j, \overline{X}^M_{T_j-})|_l). \end{aligned}$$

Applying Lemma 8, this gives

$$|D_{k,r} \ln p_M|_l \leq (\overline{\gamma}^{z,l+1}_{\ln}(\overline{Z}_k) + \overline{h}^{z,l+1}_{\ln}(\overline{Z}_k))|\overline{Z}_k|^{l+1}_{1,l+1} + \sum_{j=k+1}^{J_t^M} (\overline{\theta}^{l+1}_{\ln} 1_{B(z^*_M,1)}(\overline{Z}_j) + \overline{\gamma}^{x,l+1}_{\ln}(\overline{Z}_j))|\overline{X}^M_{T_j-})|^{l+1}_{1,l+1}.$$

We obtain then, for $k \leq J_t^M$

$$|L\overline{Z}_k|_l \leq C_l(\overline{\gamma}^{z,l+1}_{\ln}(\overline{Z}_k) + \overline{h}^{z,l+1}_{\ln}(\overline{Z}_k) + \sup_{s \leq t} |\overline{X}^M_s|^{l+1}_{l+1} \sum_{j=k+1}^{J_t^M} (\overline{\theta}^{l+1}_{\ln} 1_{B(z^*_M,1)}(\overline{Z}_j) + \overline{\gamma}^{x,l+1}_{\ln}(\overline{Z}_j))).$$

Now from the definition of $\theta_{M,\gamma}$, we have

$$\partial_\beta \theta_{M,\gamma}(x) = -\frac{1}{2\overline{C}\mu(B_{M+1})} \int_{B_{M+1}} \partial_{\beta,x} \gamma(z,x) d\mu(z).$$

Then assuming 3.3. and recalling that $1/2 \leq \theta_{M,\gamma}(x) \leq 1$, we obtain

$$\overline{\theta}^{l}_{\ln} \leq C_l(\overline{\gamma}^{x,l})^l$$

this finally gives

$$|L\overline{Z}_k|_l \leq C_l(\overline{\gamma}^{z,l+1}_{\ln}(\overline{Z}_k) + \overline{h}^{z,l+1}_{\ln}(\overline{Z}_k) + \sup_{s\leq t}|\overline{X}^M_s|^{l+1}_{l+1}\sum_{j=k+1}^{J^M_t}((\overline{\gamma}^{x,l+1})^{l+1}1_{B(z^*_M,1)}(\overline{Z}_j) + \overline{\gamma}^{x,l+1}_{\ln}(\overline{Z}_j))).$$

The first part of Lemma 11 is proved. Moreover, we can check that from 3.3, we have $\forall p \geq 1$

$$E(\sum_{j=1}^{J^M_t} 1_{B(z^*_M,1)}(\overline{Z}_j))^p \leq t^p \sup_{z^*}(\int_{B(z^*,1)} \overline{\gamma}(z)d\mu(z))^p < \infty.$$

Now assuming 3.3.a, we have $\forall p \geq 1$

$$E(\sum_{j=1}^{J^M_t} \overline{\gamma}^{x,l+1}_{\ln}(\overline{Z}_j))^p \leq t^p(\int \overline{\gamma}^{x,l+1}_{\ln}(z)\overline{\gamma}(z)d\mu(z))^p < \infty,$$

then the second part of Lemma 11 follows from Lemma 7 and Cauchy-Schwarz inequality. At last, assuming 3.3.b, we check that $\sum_{j=1}^{J^M_t} \overline{\gamma}^{x,l+1}_{\ln}(\overline{Z}_j) \leq \overline{\gamma}^{x,l+1}_{\ln} J^M_t$, and the third part follows easily. $\diamond$

We can now state the main lemma of this section.

Lemma 12 *Assuming hypotheses* 3.0., 3.1. *and* 3.2., *we have* $\forall l \geq 0$

$$\sup_{s\leq t}|L\overline{X}^M_s|_l \leq B^M_{t,l}(1 + \sup_{k\leq J^M_t}|L\overline{Z}_k|_l),$$

where $B^M_{t,l}$ *is a random variable such that* $\forall p \geq 1$, $E(B^M_{t,l})^p \leq C_p$ *for a constant* C_p *independent on* M. *More precisely we have*

$$B^M_{t,l} \leq C_l(1 + \sum_{k=1}^{J^M_t}\overline{c}(\overline{Z}_k))^{l+1}(1 + \sup_{s\leq t}|\overline{X}^M_s|^{l+2}_{l+1})^{l+1}\sup_{s\leq t}(\mathcal{E}^M_s)^{l+1},$$

where $\mathcal{E}_s$ *is solution of (43).*

Proof: We proceed by induction. From equation (38) we have

$$L\overline{X}^M_t = \sum_{k=1}^{J^M_t} Lc_M(\overline{Z}_k, \overline{X}^M_{T_k-}) + \int_0^t Lg(\overline{X}^M_s)ds.$$

For $l=0$, the second part of Lemma 10 gives

$$|L\overline{X}_t^M| \leq B_{t,0} + C\left(\sum_{k=1}^{J_t^M} \overline{c}_1(\overline{Z}_k)|L\overline{X}_{T_k-}^M| + \int_0^t |L\overline{X}_s^M|ds\right)$$

with

$$B_{t,0} = C\left(\sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k)(|L\overline{Z}_k| + |\overline{Z}_k|_{1,1}^2 + |\overline{X}_{T_k-}^M|_{1,1}^2) + \int_0^t |\overline{X}_s^M|_{1,1}^2 ds\right).$$

This gives

$$\forall s \leq t, \quad |L\overline{X}_s^M| \leq B_{t,0}\mathcal{E}_s^M,$$

where $\mathcal{E}_s^M$ is solution of (43) and

$$B_{t,0} \leq C(1 + \sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k))(1 + \sup_{s\leq t} |\overline{X}_s^M|_1^2)(1 + \sup_{k\leq J_t^M} |L\overline{Z}_k|).$$

Consequently Lemma 12 is proved for $l=0$.

For $l>0$, we obtain similarly from Lemma 10

$$|L\overline{X}_t^M|_l \leq B_{t,l-1} + C_l\left(\sum_{k=1}^{J_t^M} \overline{c}_1(\overline{Z}_k)|L\overline{X}_{T_k-}^M|_l + \int_0^t |L\overline{X}_s^M|_l ds\right)$$

with

$$\begin{aligned} B_{t,l-1} \quad &= \quad C_l \textstyle\sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k)(|L\overline{Z}_k|_l + 1 + |L\overline{X}_{T_k-}^M|_{l-1})(1 + |\overline{Z}_k|_{l+1}^{l+2} + |\overline{X}_{T_k-}^M|_{l+1}^{l+2}) \\ &\quad\ + C_l \textstyle\int_0^t (1 + |L\overline{X}_{T_k-}^M|_{l-1})(1 + |\overline{X}_s^M|_{l+1}^{l+2})ds. \end{aligned}$$

We deduce then that

$$\begin{aligned} B_{t,l-1} \quad &\leq \quad C_l(1 + \sup_{s\leq t} |L\overline{X}_s^M|_{l-1})(1 + \sup_{s\leq t} |\overline{X}_s^M|_{l+1}^{l+2})(1 + \textstyle\sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k)) \\ &\quad\ + C_l \sup_{k\leq J_t^M} |L\overline{Z}_k|_l (1 + \sup_{s\leq t} |\overline{X}_s^M|_{l+1}^{l+2}) \textstyle\sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k), \end{aligned}$$

now from the induction hypothesis, we have

$$\begin{aligned} B_{t,l-1} \quad &\leq \quad C_l(1 + \sup_{s\leq t} |\overline{X}_s^M|_{l+1}^{l+2})^{l+1}(1 + \textstyle\sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k))^{l+1} \sup_{s\leq t}(\mathcal{E}_s^M)^l (1 + \sup_{k\leq J_t^M} |L\overline{Z}_k|_{l-1}) \\ &\quad\ + C_l \sup_{k\leq J_t^M} |L\overline{Z}_k|_l (1 + \sup_{s\leq t} |\overline{X}_s^M|_{l+1}^{l+2}) \textstyle\sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k), \end{aligned}$$

this leads to

$$\forall s \leq t \quad |L\overline{X}_s^M|_l \leq B_{t,l}^M (1 + \sup_{k\leq J_t^M} |L\overline{Z}_k|_l),$$

with

$$B_{t,l}^M \leq C_l(1 + \sup_{s\leq t} |\overline{X}_s^M|_{l+1}^{l+2})^{l+1}(1 + \sum_{k=1}^{J_t^M} \overline{c}(\overline{Z}_k))^{l+1} \sup_{s\leq t}(\mathcal{E}_s^M)^{l+1}.$$

From Lemma 7, we observe that $E(B_{t,l}^M)^p < C_p$.

$\diamond$

Finally recalling that

$$|LF_M|_l \leq |L\overline{X}_t^M|_l + |\Delta| + d$$

and combining Lemma 7, Lemma 11 and Lemma 12 we deduce easily the following lemma.

Lemma 13 *Assuming hypotheses* 3.0., 3.1. *and* 3.2., *we have* $\forall l, p \geq 1$

a) if 3.3.a *holds,* $E|LF_M|_l^p \leq C_{l,p}$*;*

b) if 3.3.b *holds,* $E|LF_M|_l^p \leq C_{l,p}(1 + \mu(B_{M+1})^p)$.

4.3 The covariance matrix

4.3.1 Preliminaries

We consider an abstract measurable space E, a measure ν on this space and a non negative measurable function $f : E \to \mathbb{R}_+$ such that $\int f d\nu < \infty$. For $t > 0$ and $p \geq 1$ we note

$$\alpha_f(t) = \int_E (1 - e^{-tf(a)}) d\nu(a) \quad and \quad I_t^p(f) = \int_0^\infty s^{p-1} e^{-t\alpha_f(s)} ds.$$

Lemma 14 *i) Suppose that for* $p \geq 1$ *and* $t > 0$

$$\underline{\lim}_{u\to\infty} \frac{1}{\ln u} \alpha_f(u) > p/t \tag{46}$$

then $I_t^p(f) < \infty$.

ii) A sufficient condition for (46) *is*

$$\underline{\lim}_{u\to\infty} \frac{1}{\ln u} \nu(f \geq \frac{1}{u}) > p/t. \tag{47}$$

In particular, if $\underline{\lim}_{u\to\infty} \frac{1}{\ln u} \nu(f \geq \frac{1}{u}) = \infty$ *then* $\forall p \geq 1$ *and* $\forall t > 0$, $I_t^p(f) < +\infty$.

We remark that if ν is finite then (47) can not be satisfied.

Proof: i) From (46) one can find $\varepsilon > 0$ such that as s goes to infinity $s^{p-1} e^{-t\alpha_f(s)} \leq 1/s^{1+\varepsilon}$ and consequently $I_t^p(f) < \infty$.

ii) With the notation $n(dz) = \nu \circ f^{-1}(dz)$ we have

$$\alpha_f(u) = \int_0^\infty (1 - e^{-uz}) dn(z) = \int_0^\infty e^{-y} n(\frac{y}{u}, \infty) dy.$$

Using Fatou's lemma and (47) we obtain

$$\underline{\lim}_{u\to\infty}\frac{1}{\ln u}\int_0^\infty e^{-y}n(\frac{y}{u},\infty)dy \quad \geq \quad \int_0^\infty e^{-y}\underline{\lim}_{u\to\infty}\frac{1}{\ln u}n(\frac{y}{u},\infty)dy > p/t.$$

$\diamond$

We come now back to the framework of section 4.1.1 and we consider the Poisson point measure $N(ds,dz,du)$ on $\mathbb{R}^d\times\mathbb{R}_+$ with compensator $\mu(dz)\times 1_{(0,\infty)}(u)du$. We recall that

$$N_t(1_{B_g}f) := \int_0^t\int_{B_g} f(z)N(ds,dz,du),$$

for $f,g:\mathbb{R}^d\to\mathbb{R}_+$ and $B_g=\{(z,u): z\in B, u<g(z)\}\subset\mathbb{R}^d\times\mathbb{R}_+$ and that

$$\alpha_{g,f}(s)=\int_{\mathbb{R}^d}(1-e^{-sf(z)})d\nu_g(dz),\quad \beta_{B,g,f}(s)=\int_{B^c}(1-e^{-sf(z)})d\nu_g(dz).$$

We have the following result.

Lemma 15 *Let $U_t=t\int_{B^c}f(z)d\nu_g(z)$, then $\forall p\geq 1$*

$$E(\frac{1}{(N_t(1_{B_g}f)+U_t)^p})\leq\frac{1}{\Gamma(p)}\int_0^\infty s^{p-1}\exp(-t\alpha_{g,f}(s))ds=\frac{1}{\Gamma(p)}I_t^p(f). \tag{48}$$

Suppose moreover that for some $0<\theta\leq\infty$

$$\underline{\lim}_{a\to\infty}\frac{1}{\ln a}\nu_g(f\geq\frac{1}{a})=\theta, \tag{49}$$

then for every $t>0$ and $p\geq 1$ such that $p/t<\theta$

$$E(\frac{1}{(N_t(1_{B_g}f)+U_t)^p})<\infty.$$

Observe that if $\nu(B)<\infty$ then $E\frac{1}{(N_t(1_{B_g}f)^p}=\infty$

Proof: By a change of variables we obtain for every $\lambda>0$

$$\lambda^{-p}\Gamma(p)=\int_0^\infty s^{p-1}e^{-\lambda s}ds.$$

Taking the expectation in the previous equality with $\lambda=N_t(f1_{B_g})+U_t$ we obtain

$$E(\frac{1}{(N_t(f1_{B_g})+U_t)^p})=\frac{1}{\Gamma(p)}\int_0^\infty s^{p-1}E(\exp(-s(N_t(f1_{B_g})+U_t))ds.$$

Now from Lemma 6 we have

$$E(\exp(-sN_t(f1_{B_g})))=\exp(-t(\alpha_{g,f}(s)-\beta_{B,g,f}(s))).$$

Moreover, from the definition of U_t one can easily check that $\exp(-sU_t)\leq\exp(-t\beta_{B,g,f}(s))$ and then

$$E(\exp(-s(N_t(f1_{B_g})+U_t))\leq\exp(-t\alpha_{g,f}(s))$$

this achieves the proof of (48). The second part of the lemma follows directily from lemma 14. $\diamond$

4.3.2 The Malliavin covariance matrix

In this section, we prove that under some additional assumptions on p and t, $E(\det\sigma(F_M))^{-p} \leq C_p$, for the Malliavin covariance matrix $\sigma(F_M)$ defined in section 3.4.

We first remark that from Hypothesis 3.2 ii) the tangent flow of equation (38) is invertible and that the moments of all order of this inverse are finite. More precisely we define $Y_t^M, t \geq 0$ and $\widehat{Y}_t^M, t \geq 0$ as the matrix solutions of the equations

$$Y_t^M = I + \sum_{k=1}^{J_t^M} \nabla_x c_M(\overline{Z}_k, \overline{X}_{T_k-}^M) Y_{T_k-}^M + \int_0^t \nabla_x g(\overline{X}_s^M) Y_s^M ds, \tag{50}$$

$$\widehat{Y}_t^M = I - \sum_{k=1}^{J_t^M} \nabla_x c_M (I + \nabla_x c_M)^{-1}(\overline{Z}_k, \overline{X}_{T_k-}^M) \widehat{Y}_{T_k-}^M - \int_0^t \nabla_x g(\overline{X}_s^M) \widehat{Y}_s^M ds. \tag{51}$$

Then $\widehat{Y}_t^M \times Y_t^M = I, \forall t \geq 0$. Moreover we can prove under hypotheses 3.0, 3.1 and 3.2. that $\forall p \geq 1$

$$E(\sup_{s\leq t}(\left\|\widehat{Y}_s^M\right\|^p + \left\|Y_s^M\right\|^p)) \leq K_p < \infty \tag{52}$$

where K_p is a constant.

Lemma 16 *Assuming hypothesis* 3.0, 3.1, 3.2 *we have for* $p \geq 1$, $t > 0$ *such that* $2dp/t < \theta$

$$E(\frac{1}{(\det\sigma(F_M))^p}) \leq C_p, \tag{53}$$

where the constant C_p *does not depend on* M.

Proof: We first give a lower bound for the lowest eigenvalue of the matrix $\sigma(\overline{X}_t^M)$.

$$\rho_t := \inf_{|\xi|=1} \left\langle \sigma(\overline{X}_t^M)\xi, \xi \right\rangle = \inf_{|\xi|=1} \sum_{k=1}^{J_t^M} \sum_{r=1}^{d} \left\langle D_{k,r}\overline{X}_t^M, \xi \right\rangle^2.$$

But from equation (38) we have

$$D_{k,r}\overline{X}_t^M = \sum_{k'=1}^{J_t^M} \nabla_z c_M(\overline{Z}_{k'}, \overline{X}_{T_{k'}^-}^M) D_{k,r}\overline{Z}_{k'} + \sum_{k'=1}^{J_t^M} \nabla_x c_M(\overline{Z}_{k'}, \overline{X}_{T_{k'}^-}^M) D_{k,r}\overline{X}_{T_{k'}^-}^M + \int_0^t \nabla_x g(\overline{X}_s^M) D_{k,r}\overline{X}_s^M ds$$

where $\nabla_z c_M = (\partial_{z_r} c_M^{r'})_{r',r}$ and $\nabla_x c_M = (\partial_{x_r} c_M^{r'})_{r',r}$. Since $D_{k,r}\overline{Z}_{k'} = 0$ for $k \neq k'$ we obtain

$$D_{k,r}\overline{X}_t^{M,r'} = (Y_t^M \nabla_z c_M(\overline{Z}_k, \overline{X}_{T_k^-}^M) D_{k,r}\overline{Z}_k)_{r',r} = \pi_k (Y_t^M \nabla_z c_M(\overline{Z}_k, \overline{X}_{T_k^-}^M))_{r',r}.$$

We deduce that

$$\sum_{r=1}^{d}\left\langle D_{k,r}\overline{X}_t^M,\xi\right\rangle^2=\sum_{r=1}^{d}\pi_k^2\left\langle \partial_{z^r}c_M(\overline{Z}_k,\overline{X}^M_{T_k^-}),(Y_t^M)^*\xi\right\rangle^2,$$

but recalling that $\pi_k\geq 1_{B_{M-1}}(\overline{Z}_k)$ and $c_M=c$ on B_{M-1} we obtain

$$\sum_{r=1}^{d}\left\langle D_{k,r}\overline{X}_t^M,\xi\right\rangle^2\geq\sum_{r=1}^{d}1_{B_{M-1}}(\overline{Z}_k)\left\langle \partial_{z^r}c(\overline{Z}_k,\overline{X}^M_{T_k^-}),(Y_t^M)^*\xi\right\rangle^2,$$

and consequently using hypothesis 3.2.iii)

$$\rho_t\geq\inf_{|\xi|=1}\sum_{k=1}^{J_t^M}1_{B_{M-1}}(\overline{Z}_k)\underline{c}^2(\overline{Z}_k)|(Y_t^M)^*\xi|^2\geq\left\|\widehat{Y}_t^M\right\|^{-2}\sum_{k=1}^{J_t^M}1_{B_{M-1}}(\overline{Z}_k)\underline{c}^2(\overline{Z}_k).$$

Now since $\sigma(F_M)=\sigma(\overline{X}_t^M)+U_M(t)$ we have

$$E\left|\frac{1}{\det\sigma(F_M)}\right|^p\leq E\left|\frac{1}{\rho_t+U_M(t)}\right|^{dp}\leq E\left(\frac{1+\left\|\widehat{Y}_t^M\right\|^2}{\sum_{k=1}^{J_t^M}1_{B_{M-1}}(\overline{Z}_k)\underline{c}^2(\overline{Z}_k)+U_M(t)}\right)^{dp}.$$

Now observe that the denominator of the last fraction is equal in law to

$$\sum_{k=1}^{J_t^M}1_{B_{M-1}}(Z_k)\underline{c}^2(Z_k)1_{U_k<\gamma(Z_k,X^M_{T_k-})}+U_M(t)\geq N_t(1_{B^M_{\underline{\gamma}}}\underline{c}^2)+U_M(t),$$

with $B^M_{\underline{\gamma}}=\{(z,u);z\in B_{M-1};0<u<\underline{\gamma}(z)\}$. Assuming hypothesis 3.2.iii, we can apply lemma 15 with $f=\underline{c}^2$ and $d\nu(z)=\underline{\gamma}(z)d\mu(z)$. This gives for $p'\geq 1$ such that $p'/t<\theta$

$$E\left(\frac{1}{N_t(1_{B^M_{\underline{\gamma}}}\underline{c}^2)+U_M(t)}\right)^{p'}\leq C_{p'}.$$

Finally since the moments of $\left\|\widehat{Y}_t^M\right\|$ are bounded uniformly on M the result of lemma 16 follows from Cauchy-Schwarz inequality.

$\diamond$

5 References

[B] V. Bally: An elementary introduction to Malliavin calculus. Preprint No 4718, INRIA February 2003.

[B.B.M] V. Bally, M-P. Bavouzet and M. Messaud: Integration by parts formula for locally smooth laws and applications to sensitivity computations. Annals of Applied Probability 2007, Vol. 17, No. 1, 33-66.

[B.F] V.Bally and N.Fournier: regularization properties of the 2D homogeneous Boltzmann equation without cutoff. Preprint.

[Ba.M] M-P. Bavouzet and M. Messaud: Computation of Greeks using Malliavin calculus in jump type market models. Electronic Journal of Probability, 11/ 276-300, 2006.

[Bi] J.M. Bismut: Calcul des variations stochastiques et processus de sauts. Z. Wahrsch. Verw. Gebite, No 2, 147-235, 1983.

[B.G.J] K. Bichteler, J.B. Gravereaux and J. Jacod: Malliavin calculus for processes with jumps. Gordon and Breach, 1987.

[Bou] N. Bouleau: Error calculus for finance and Physics, the language of Dirichlet forms. De Gruyer, 2003.

[F.1] N. Fournier: Jumping SDE's: Absolute continuity using monotonicity. SPA, 98 (2), pp 317-330, 2002.

[F.2] N. Fournier: Smoothness of the law of some one-dimensional jumping SDE's with non constant rate of jump. In preparation.

[F.G] N. Fournier and J.S. Giet: On small particles in coagulation-fragmentation equations. J. Statist. Phys. 111 (5/6) pp 1299-1329, 2003.

[I.W] N. Ikeda and S. Watanabe: Stochastic Differential Equations and Diffusion processes. North-Holland, 1989.

[L] R. Léandre: Régularité de processus de sauts dégénérés. Ann. Inst. H. Poincaré, Proba. Stat., 21, No 2, 125-146, 1985.

[N] D. Nualart: Malliavin calculus and related topics. Springer Verlag, 1995.