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To cite this version:

HAL Id: hal-00430141
https://hal.archives-ouvertes.fr/hal-00430141
Submitted on 5 Nov 2009

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ON REPRESENTATIONS OF THE FEASIBLE SET IN CONVEX OPTIMIZATION

JEAN B. LASERRE

Abstract. We consider the convex optimization problem \( \min_x \{ f(x) : g_j(x) \leq 0, \, j = 1, \ldots, m \} \) where \( f \) is convex, the feasible set \( K \) is convex and Slater’s condition holds, but the functions \( g_j \)'s are not necessarily convex. We show that for any representation of \( K \) that satisfies a mild nondegeneracy assumption, every minimizer is a Karush-Kuhn-Tucker (KKT) point and conversely every KKT point is a minimizer. That is, the KKT optimality conditions are necessary and sufficient as in convex programming where one assumes that the \( g_j \)'s are convex. So in convex optimization, and as far as one is concerned with KKT points, what really matters is the geometry of \( K \) and not so much its representation.

1. Introduction

Given differentiable functions \( f, g_j : \mathbb{R}^n \to \mathbb{R}, \, j = 1, \ldots, m \), consider the following convex optimization problem:

\[
(1.1) \quad f^* := \inf_x \{ f(x) : x \in K \}
\]

where \( f \) is convex and the feasible set \( K \subset \mathbb{R}^n \) is convex and represented in the form:

\[
(1.2) \quad K = \{ x \in \mathbb{R}^n : g_j(x) \leq 0, \, j = 1, \ldots, m \}.
\]

Convex optimization usually refers to minimizing a convex function over a convex set without precisng its representation (see e.g. Ben-Tal and Nemirovsky [1, Definition 5.1.1] or Bertsekas et al. [3, Chapter 2]), and it is well-known that convexity of the function \( f \) and of the set \( K \) imply that every local minimum is a global minimum. An elementary proof only uses the geometry of \( K \), not its representation by the defining functions \( g_j \); see e.g. Bertsekas et al. [3, Prop. 2.1.2].

The convex set \( K \) may be represented by different choices of the (not necessarily convex) defining functions \( g_j, \, j = 1, \ldots, m \). For instance, the set

\[
K := \{ x \in \mathbb{R}^2 : 1 - x_1 x_2 \leq 0; \, x \geq 0 \}
\]

is convex but the function \( x \mapsto 1 - x_1 x_2 \) is not convex on \( \mathbb{R}^2_+ \). Of course, depending on the choice of the defining functions \( g_j \), several properties may or may not hold. In particular, the celebrated Karush-Kuhn-Tucker (KKT) optimality conditions depend on the representation of \( K \). Recall that \( x \in K \) is a KKT point if

\[
(1.3) \quad \nabla f(x) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x) = 0 \quad \text{and} \quad \lambda_j g_j(x) = 0, \, j = 1, \ldots, m,
\]

for some nonnegative vector \( \lambda \in \mathbb{R}^m \). (More precisely \( (x, \lambda) \) is a KKT point.)
Convex programming refers to the situation where \( f \) is convex and the defining functions \( g_j \) of \( K \) are also convex. See for instance Ben-Tal and Nemirovsky [1, p. 335], Berkovitz [2, p. 179], Boyd and Vandenberghe [3, p. 7], Bertsekas et al. [2, §3.5.5], Nesterov and Nemirovskii [6, p. 217-218], and Hiriart-Urruty [5].

A crucial feature of convex programming is that when Slater’s condition holds \(^1\), the KKT optimality conditions (1.3) are necessary and sufficient, which shows that a representation of the convex set \( K \) with convex functions \( (g_j) \) has some very attractive features.

The purpose of this note is to show that in fact, when \( K \) is convex and as far as one is concerned with KKT points, what really matters is the geometry of \( K \) and not so much its representation. Indeed, we show that if \( K \) is convex and Slater’s condition holds then the KKT optimality conditions (1.3) are also necessary and sufficient for all representations of \( K \) that satisfy a mild nondegeneracy condition, no matter if the \( g_j \)’s are convex. So this attractive feature is not specific to representations of \( K \) with convex functions.

That a KKT point is a local (hence global) minimizer follows easily from the convexity of \( K \). More delicate is the fact that any local (hence global) minimizer is a KKT point. Various constraint qualifications are usually required to hold at a minimizer, and when the \( g_j \)’s are convex the simple Slater’s condition is enough. Here we show that Slater’s condition is also sufficient for all representations of \( K \) that satisfy a mild additional nondegeneracy assumption on the boundary of \( K \). Moreover under Slater’s condition this mild nondegeneracy assumption is automatically satisfied if the \( g_j \)’s are convex.\(^2\)

2. Main result

Let \( K \subset \mathbb{R}^n \) be as in (1.2). We first start with the following non degeneracy assumption:

**Assumption 2.1** (nondegeneracy). For every \( j = 1, \ldots, m \),

\[
(2.1) \quad \nabla g_j(x) \neq 0, \quad \text{whenever } x \in K \text{ and } g_j(x) = 0.
\]

Observe that under Slater’s condition, (2.1) is automatically satisfied if \( g_j \) is convex. Indeed if \( g_j(x) = 0 \) and \( \nabla g_j(x) = 0 \) then by convexity 0 is the global minimum of \( g_j \) on \( \mathbb{R}^n \). Hence there is no \( x_0 \in K \) with \( g_j(x_0) < 0 \). We next state the following characterization of convexity.

**Lemma 2.2.** With \( K \subset \mathbb{R}^n \) as in (1.2), let Assumption 2.1 and Slater’s condition both hold for \( K \). Then \( K \) is convex if and only if for every \( j = 1, \ldots, m \):

\[
(2.2) \quad \langle \nabla g_j(x), y - x \rangle \leq 0, \quad \forall x, y \in K \text{ with } g_j(x) = 0.
\]

*Proof. Only if part.* Assume that \( K \) is convex and \( \langle \nabla g_j(x), y - x \rangle > 0 \) for some \( j \in \{1, \ldots, m\} \) and some \( x, y \in K \) with \( g_j(x) = 0 \). Then \( g_j(x + t(y - x)) > 0 \) for all sufficiently small \( t \), in contradiction with \( x + t(y - x) \in K \) for all \( 0 \leq t \leq 1 \) (by convexity of \( K \)).

*If part.* By (2.2), at every point \( x \) on the boundary of \( K \), there exists a supporting hyperplane for \( K \). As \( K \) is closed with nonempty interior, by [8, Th. 1.3.3] the set \( K \) is convex. \( \square \)

\(^1\)Slater’s condition holds for \( K \) if for some \( x_0 \in K, g_j(x_0) < 0 \) for every \( j = 1, \ldots, m \).

\(^2\)The author wishes to thank Prof. L. Tunçel for providing him with the reference [8].
Theorem 2.3. Consider the nonlinear programming problem (1.1) and let Assumption 2.1 and Slater’s condition both hold. If \( f \) is convex then every minimizer is a KKT point and conversely, every KKT point is a minimizer.

**Proof.** Let \( x^* \in K \) be a minimizer (hence a global minimizer) with \( f^* = f(x^*) \). We first prove that \( x^* \) is a KKT point. The Fritz-John optimality conditions state that

\[
\lambda_0 \nabla f(x^*) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x^*) = 0; \quad \lambda_j g_j(x^*) = 0, \quad j = 1, \ldots, m,
\]

for some non trivial nonnegative vector \( 0 \neq \lambda \in \mathbb{R}^{m+1} \). See e.g. Hiriart-Urruty [6, Th. page 77] or Polyak [8, Th. 1, p. 271]. We next prove that \( \lambda_0 \neq 0 \). Suppose that \( \lambda_0 = 0 \) and let \( J := \{ j \in \{1, \ldots, m \} : \lambda_j > 0 \} \). As \( \lambda \neq 0 \) and \( \lambda_0 = 0 \), the set \( J \) is nonempty. Next, as \( g_j(x_0) < 0 \) for every \( j = 1, \ldots, m \), there is some \( \rho > 0 \) such that \( B(x_0, \rho) := \{ z \in \mathbb{R}^n : \|z - x_0\| < \rho \} \subseteq K \) and \( g_j(z) < 0 \) for all \( z \in B(x_0, \rho) \) and all \( j \in J \). Therefore we obtain

\[
\sum_{j \in J} \lambda_j \langle \nabla g_j(x^*), z - x^* \rangle = 0 \quad \forall z \in B(x_0, \rho),
\]

which, by Lemma 2.2 implies that \( \langle \nabla g_j(x^*), z - x^* \rangle = 0 \) for every \( j \in J \) and every \( z \in B(x_0, \rho) \). But this clearly implies that \( \nabla g_j(x^*) = 0 \) for every \( j \in J \), in contradiction with Assumption 2.1. Hence \( \lambda_0 > 0 \) and we may and will set \( \lambda_0 = 1 \), so that the KKT conditions hold at \( x^* \).

Conversely, let \( x \in K \) be an arbitrary KKT point, i.e., \( x \in K \) satisfies

\[
\nabla f(x) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x) = 0; \quad \lambda_j g_j(x) = 0, \quad j = 1, \ldots, m,
\]

for some nonnegative vector \( \lambda \in \mathbb{R}^m \). Suppose that there exists \( y \in K \) with \( f(y) < f(x) \). Then we obtain the contradiction:

\[
0 > f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle \quad \text{[by convexity of } f] \]

\[
= -\sum_{j=1}^{m} \lambda_j \langle \nabla g_j(x), y - x \rangle \geq 0
\]

where the last inequality follows from \( \lambda \geq 0 \) and Lemma 2.2. Hence \( x \) is a minimizer. \( \square \)

Hence if \( K \) is convex and both Assumption 2.1 and Slater’s condition hold, there is a one-to-one correspondence between KKT points and minimizers. That is, the KKT optimality conditions are necessary and sufficient for all representations of \( K \) that satisfy Slater’s condition and Assumption 2.1.

However there is an important additional property when all the defining functions \( g_j \) are convex. Dual methods of the type

\[
\sup_{x \in \mathbb{R}^n_+} \left\{ \inf_{x \in \mathbb{R}^n} \left( f(x) + \sum_{j=1}^{m} \lambda_j g_j(x) \right) \right\},
\]

are well defined because \( x \mapsto f(x) + \sum_{j=1}^{m} \lambda_j g_j(x) \) is a convex function. In particular, the Lagrangian \( x \mapsto L_f(x) := f(x) - f^* + \sum_{j=1}^{m} \lambda_j g_j(x) \), defined from an
arbitrary KKT point \((x^*, \lambda) \in K \times \mathbb{R}^m_+\), is convex and nonnegative on \(\mathbb{R}^n\), with \(x^*\) being a global minimizer. If the \(g_j\)’s are not convex this is not true in general.

**Example 1.** Let \(n = 2\) and consider the problem

\[
P: \quad f^* = \min \{ f(x) : a - x_1 x_2 \leq 0; \ Ax \leq b; x \geq 0 \},
\]

where \(a > 0\), \(A \in \mathbb{R}^{m \times n}\), \(b \in \mathbb{R}^m\), and \(f\) is convex and differentiable. The set

\[
K := \{ x \in \mathbb{R}^2 : a - x_1 x_2 \leq 0; \ Ax \leq b; x \geq 0 \}
\]

is convex and it is straightforward to check that Assumption 2.1 holds. Therefore, by Theorem 2.3 if Slater’s condition holds, every KKT point is a global minimizer. However, the Lagrangian

\[
x \mapsto f(x) - f^* + \psi(a - x_1 x_2) + \langle \lambda, Ax - b \rangle - \langle \mu, x \rangle,
\]

with nonnegative \((\psi, \lambda, \mu) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n\), may not be convex whenever \(\psi \neq 0\) (for instance if \(f\) is linear). On the other hand, notice that \(K\) has the equivalent convex representation

\[
K := \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} x_1 / \sqrt{a} \\ \sqrt{a} x_2 \end{bmatrix} \succeq 0; \ Ax \leq b \right\},
\]

where for a real symmetric matrix \(B\), the notation \(B \succeq 0\) stands for \(B\) is positive semidefinite.

A topic of further investigation is concerned with computational efficiency. Can efficient algorithms be devised for some class of convex problems (1.1) where the defining functions \(g_j\) of \(K\) are not necessarily convex?

**References**


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