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ON REPRESENTATIONS OF THE FEASIBLE SET IN CONVEX OPTIMIZATION

JEAN B. LASERRE

Abstract. We consider the convex optimization problem \( \min_x \{ f(x) : g_j(x) \leq 0, \; j = 1, \ldots, m \} \) where \( f \) is convex, the feasible set \( K \) is convex and Slater’s condition holds, but the functions \( g_j \)'s are not necessarily convex. We show that for any representation of \( K \) that satisfies a mild nondegeneracy assumption, every minimizer is a Karush-Kuhn-Tucker (KKT) point and conversely every KKT point is a minimizer. That is, the KKT optimality conditions are necessary and sufficient as in convex programming where one assumes that the \( g_j \)'s are convex. So in convex optimization, and as far as one is concerned with KKT points, what really matters is the geometry of \( K \) and not so much its representation.

1. Introduction

Given differentiable functions \( f, g_j : \mathbb{R}^n \to \mathbb{R}, \; j = 1, \ldots, m \), consider the following convex optimization problem:

\[
(1.1) \quad f^* := \inf_x \{ f(x) : x \in K \}
\]

where \( f \) is convex and the feasible set \( K \subset \mathbb{R}^n \) is convex and represented in the form:

\[
(1.2) \quad K = \{ x \in \mathbb{R}^n : g_j(x) \leq 0, \; j = 1, \ldots, m \}.
\]

Convex optimization usually refers to minimizing a convex function over a convex set without precising its representation (see e.g. Ben-Tal and Nemirovsky [1, Definition 5.1.1] or Bertsekas et al. [3, Chapter 2]), and it is well-known that convexity of the function \( f \) and of the set \( K \) imply that every local minimum is a global minimum. An elementary proof only uses the geometry of \( K \), not its representation by the defining functions \( g_j \); see e.g. Bertsekas et al. [3, Prop. 2.1.2].

The convex set \( K \) may be represented by different choices of the (not necessarily convex) defining functions \( g_j, \; j = 1, \ldots, m \). For instance, the set

\[
K := \{ x \in \mathbb{R}^2 : 1 - x_1 x_2 \leq 0; \; x \geq 0 \}
\]

is convex but the function \( x \mapsto 1 - x_1 x_2 \) is not convex on \( \mathbb{R}^2_+ \). Of course, depending on the choice of the defining functions \( (g_j) \), several properties may or may not hold.

In particular, the celebrated Karush-Kuhn-Tucker (KKT) optimality conditions depend on the representation of \( K \). Recall that \( x \in K \) is a KKT point if

\[
(1.3) \quad \nabla f(x) + \sum_{j=1}^m \lambda_j \nabla g_j(x) = 0 \quad \text{and} \quad \lambda_j g_j(x) = 0, \; j = 1, \ldots, m,
\]

for some nonnegative vector \( \lambda \in \mathbb{R}^m \). (More precisely \( (x, \lambda) \) is a KKT point.)

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Convex programming refers to the situation where $f$ is convex and the defining functions $g_j$ of $K$ are also convex. See for instance Ben-Tal and Nemirovsky [1, p. 335], Berkovitz [4, p. 179], Boyd and Vandenberghe [4, p. 7], Bertsekas et al. [2, §3.5.5], Nesterov and Nemirovskii [6, p. 217-218], and Hiria rt-Urruty [5].

A crucial feature of convex programming is that when Slater’s condition holds\(^1\), the KKT optimality conditions (1.3) are necessary and sufficient, which shows that a representation of the convex set $K$ with convex functions ($g_j$) has some very attractive features.

The purpose of this note is to show that in fact, when $K$ is convex and as far as one is concerned with KKT points, what really matters is the geometry of $K$ and not so much its representation. Indeed, we show that if $K$ is convex and Slater’s condition holds then the KKT optimality conditions (1.3) are also necessary and sufficient for all representations of $K$ that satisfy a mild nondegeneracy condition, no matter if the $g_j$’s are convex. So this attractive feature is not specific to representations of $K$ with convex functions.

That a KKT point is a local (hence global) minimizer follows easily from the convexity of $K$. More delicate is the fact that any local (hence global) minimizer is a KKT point. Various constraint qualifications are usually required to hold at a minimizer, and when the $g_j$’s are convex the simple Slater’s condition is enough. Here we show that Slater’s condition is also sufficient for all representations of $K$ that satisfy a mild additional nondegeneracy assumption on the boundary of $K$. Moreover under Slater’s condition this mild nondegeneracy assumption is automatically satisfied if the $g_j$’s are convex.

2. Main result

Let $K \subset \mathbb{R}^n$ be as in (1.2). We first start with the following non degeneracy assumption:

**Assumption 2.1** (nondegeneracy). For every $j = 1, \ldots, m$,

$$\nabla g_j(x) \neq 0, \quad \text{whenever } x \in K \text{ and } g_j(x) = 0. \quad (2.1)$$

Observe that under Slater’s condition, (2.1) is automatically satisfied if $g_j$ is convex. Indeed if $g_j(x) = 0$ and $\nabla g_j(x) = 0$ then by convexity 0 is the global minimum of $g_j$ on $\mathbb{R}^n$. Hence there is no $x_0 \in K$ with $g_j(x_0) < 0$. We next state the following characterization of convexity.

**Lemma 2.2.** With $K \subset \mathbb{R}^n$ as in (1.2), let Assumption 2.1 and Slater’s condition both hold for $K$. Then $K$ is convex if and only if for every $j = 1, \ldots, m$:

$$\langle \nabla g_j(x), y - x \rangle \leq 0, \quad \forall x, y \in K \quad \text{with } g_j(x) = 0. \quad (2.2)$$

**Proof.** Only if part. Assume that $K$ is convex and $\langle \nabla g_j(x), y - x \rangle > 0$ for some $j \in \{1, \ldots, m\}$ and some $x, y \in K$ with $g_j(x) = 0$. Then $g_j(x + t(y - x)) > 0$ for all sufficiently small $t$, in contradiction with $x + t(y - x) \in K$ for all $0 \leq t \leq 1$ (by convexity of $K$).

If part. By (2.2), at every point $x$ on the boundary of $K$, there exists a supporting hyperplane for $K$. As $K$ is closed with nonempty interior, by [8][Th. 1.3.3] the set $K$ is convex. \(\square\)

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\(^1\)Slater’s condition holds for $K$ if for some $x_0 \in K$, $g_j(x_0) < 0$ for every $j = 1, \ldots, m$.

\(^2\)The author wishes to thank Prof. L. Tuncel for providing him with the reference [8].
Theorem 2.3. Consider the nonlinear programming problem \((P)\) and let Assumption 2.1 and Slater’s condition both hold. If \(f\) is convex then every minimizer is a KKT point and conversely, every KKT point is a minimizer.

Proof. Let \(x^* \in K\) be a minimizer (hence a global minimizer) with \(f^* = f(x^*)\). We first prove that \(x^*\) is a KKT point. The Fritz-John optimality conditions state that

\[
\lambda_0 \nabla f(x^*) + \sum_{j=1}^m \lambda_j \nabla g_j(x^*) = 0; \quad \lambda_j g_j(x^*) = 0, \quad j = 1, \ldots, m,
\]

for some non trivial nonnegative vector \(0 \neq \lambda \in \mathbb{R}^{m+1}\). See e.g. Hiriart-Urruty \[1\] Th. page 77] or Polyak \[6\] Theor. 1, p. 271]. We next prove that \(\lambda_0 \neq 0\). Suppose that \(\lambda_0 = 0\) and let \(J := \{j \in \{1, \ldots, m\} : \lambda_j > 0\}\). As \(\lambda \neq 0\) and \(\lambda_0 = 0\), the set \(J\) is nonempty. Next, as \(g_j(x_0) < 0\) for every \(j = 1, \ldots, m\), there is some \(\rho > 0\) such that \(B(x_0, \rho) := \{z \in \mathbb{R}^n : \|z - x_0\| < \rho\} \subset K\) and \(g_j(z) < 0\) for all \(z \in B(x_0, \rho)\) and all \(j \in J\). Therefore we obtain

\[
\sum_{j \in J} \lambda_j \langle \nabla g_j(x^*), z - x^* \rangle = 0 \quad \forall z \in B(x_0, \rho),
\]

which, by Lemma 2.2, implies that \(\langle \nabla g_j(x^*), z - x^* \rangle = 0\) for every \(j \in J\) and every \(z \in B(x_0, \rho)\). But this clearly implies that \(\nabla g_j(x^*) = 0\) for every \(j \in J\), in contradiction with Assumption 2.1. Hence \(\lambda_0 > 0\) and we may and will set \(\lambda_0 = 1\), so that the KKT conditions hold at \(x^*\).

Conversely, let \(x \in K\) be an arbitrary KKT point, i.e., \(x \in K\) satisfies

\[
\nabla f(x) + \sum_{j=1}^m \lambda_j \nabla g_j(x) = 0; \quad \lambda_j g_j(x) = 0, \quad j = 1, \ldots, m,
\]

for some nonnegative vector \(\lambda \in \mathbb{R}^m\). Suppose that there exists \(y \in K\) with \(f(y) < f(x)\). Then we obtain the contradiction:

\[
0 > f(y) - f(x) \\
\geq \langle \nabla f(x), y - x \rangle \quad \text{[by convexity of } f]\n\]

\[
= - \sum_{j=1}^m \lambda_j \langle \nabla g_j(x), y - x \rangle \geq 0
\]

where the last inequality follows from \(\lambda \geq 0\) and Lemma 2.2. Hence \(x\) is a minimizer.

Hence if \(K\) is convex and both Assumption 2.1 and Slater’s condition hold, there is a one-to-one correspondence between KKT points and minimizers. That is, the KKT optimality conditions are necessary and sufficient for all representations of \(K\) that satisfy Slater’s condition and Assumption 2.1.

However there is an important additional property when all the defining functions \(g_j\) are convex. Dual methods of the type

\[
\sup_{\lambda \in \mathbb{R}^m_+} \left\{ \inf_X f(x) + \sum_{j=1}^m \lambda_j g_j(x) \right\},
\]

are well defined because \(x \mapsto f(x) + \sum_{j=1}^m \lambda_j g_j(x)\) is a convex function. In particular, the Lagrangian \(x \mapsto L_f(x) := f(x) - f^* + \sum_{j=1}^m \lambda_j g_j(x)\), defined from an
arbitrary KKT point \((x^*, \lambda) \in K \times \mathbb{R}^m_+\), is convex and nonnegative on \(\mathbb{R}^n\), with \(x^*\) being a global minimizer. If the \(g_j\)'s are not convex this is not true in general.

**Example 1.** Let \(n = 2\) and consider the problem

\[
P : \quad f^* = \min \{ f(x) : a - x_1 x_2 \leq 0; \ Ax \leq b; \ x \geq 0 \},
\]

where \(a > 0\), \(A \in \mathbb{R}^{m \times n}\), \(b \in \mathbb{R}^m\), and \(f\) is convex and differentiable. The set

\[
K := \{x \in \mathbb{R}^2 : a - x_1 x_2 \leq 0; \ Ax \leq b; \ x \geq 0 \}
\]

is convex and it is straightforward to check that Assumption 2.1 holds. Therefore, by Theorem 2.3 if Slater’s condition holds, every KKT point is a global minimizer. However, the Lagrangian

\[
x \mapsto f(x) - f^* + \psi(a - x_1 x_2) + \langle \lambda, Ax - b \rangle - \langle \mu, x \rangle,
\]

with nonnegative \((\psi, \lambda, \mu) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n\), may not be convex whenever \(\psi \neq 0\) (for instance if \(f\) is linear). On the other hand, notice that \(K\) has the equivalent convex representation

\[
K := \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} x_1 \sqrt{a} \\ x_2 \sqrt{a} \end{bmatrix} \succeq 0; \ Ax \leq b \right\},
\]

where for a real symmetric matrix \(B\), the notation \(B \succeq 0\) stands for \(B\) is positive semidefinite.

A topic of further investigation is concerned with computational efficiency. Can efficient algorithms be devised for some class of convex problems (1.1) where the defining functions \(g_j\) of \(K\) are not necessarily convex?

**References**


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