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ON THE HYDROSTATIC STOKES APPROXIMATION WITH NON HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS.

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ABSTRACT. We deal with the hydrostatic Stokes approximation with non homogeneous Dirichlet boundary conditions. After having investigated the homogeneous case, we build a shifting operator of boundary values related to the divergence operator, and solve the non homogeneous problem in a cylindrical type domain

KEY WORDS. Hydrostatic approximation, De Rham's lemma, shifting operator, Primitive Equations, non homogeneous Dirichlet conditions.

AMS SUBJECT CLASSIFICATIONS. 35Q30, 35B40, 76D05, 34C35

1. INTRODUCTION

Let us consider $\Omega \subset \mathbb{R}^3$ a bounded domain defined by

$$(1.1) \quad \Omega = \{x = (x', x_3) \in \mathbb{R}^3 / x' \in \omega \text{ and } -h(x') < x_3 < 0\},$$

where $\omega \subset \mathbb{R}^2$ is a bounded Lipschitz-continuous domain and where h is a Lipschitz-continuous map over ω , chosen such that Ω has a Lipschitz-continuous boundary Γ . The boundary Γ split into three parts, each one with a non negative measure: the surface Γ_S , the bottom Γ_B , and sidewalls Γ_L , defined by:

$$\begin{aligned} \Gamma_S &= \omega \times \{0\}, & \Gamma_B &= \{(x', -h(x')) / x' \in \omega\}, \\ \Gamma_L &= \{x \in \mathbb{R}^3 / x' \in \partial\omega \text{ and } -h(x') < x_3 < 0\}. \end{aligned}$$

Finally, we denote by \mathbf{n} the unit outward vector normal to Γ .

FIGURE 1.1. The domain Ω

Let $\mathbf{f}' = (f_1, f_2) : \Omega \rightarrow \mathbb{R}^2$, $\Phi : \Omega \rightarrow \mathbb{R}$, and $\mathbf{g} = (\mathbf{g}', g_3) : \Gamma \rightarrow \mathbb{R}^3$ be given functions, Φ and \mathbf{g} satisfying adequate compatibility conditions. In this paper, we study the non homogeneous version of the hydrostatic Stokes approximation, also called in some more general cases Primitive Equations for the Ocean, consisting in finding a pair of functions (\mathbf{u}, p) defined on Ω , with $\mathbf{u} = (\mathbf{u}', u_3)$ such that

$$(\mathcal{SH}) \left\{ \begin{array}{l} -\Delta \mathbf{u}' + \nabla' p = \mathbf{f}', \quad \frac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \mathbf{u} = \Phi \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma. \end{array} \right.$$

Here $\nabla' = (\partial_{x_1}, \partial_{x_2})^1$ stands for the gradient operator with respect to the variables x_1 and x_2 .

When Φ and g_3 vanish, Problem (\mathcal{SH}) and its generalizations to the non linear and time dependent cases, have been studied by many authors, from different points of view, and it would be too long to list them all here. But, to our knowledge, M. Laydi, O. Besson, R. Touzani have worked on the 2D linear case in [?] and on the 3D nonlinear version in [?]. Then, P. Azérad, F. Guillén have investigated the linear and nonlinear time dependent cases in [?, ?]. Such results have been obtained by performing an asymptotic analysis of the Stokes and Navier-Stokes equations, set in a thin domain, with an anisotropic viscosity. This first method can be seen as the physical approach of the problem (\mathcal{SH}) .

Thanks to the particular geometry of the computational domain Ω , other authors have considered (\mathcal{SH}) and its generalizations as a well-posed reduced Stokes or Navier-Stokes system. Let us illustrate these second method on Problem (\mathcal{SH}) . The simplifications of (\mathcal{SH}) arise from the hydrostatic pressure hypothesis:

$$(1.2) \quad \frac{\partial p}{\partial x_3} = 0 \text{ in } \Omega,$$

ensuring that p_S , the pressure at $x_3 = 0$, is in fact the true unknown. Moreover, by integrating with respect to x_3 the incompressibility equation:

$$(1.3) \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega,$$

and taking into account the boundary conditions over u_3 , it appears that the unknown u_3 is given by the vector field \mathbf{u}' . Thus, the equations retained are the following one

$$(1.4) \quad \left\{ \begin{array}{l} -\Delta \mathbf{u}' + \nabla' p_S = \mathbf{f}' \quad \text{in } \Omega, \\ \nabla' \cdot \int_{-h(x')}^0 \mathbf{u}'(x', x_3) dx_3 = 0 \quad \text{in } \omega, \end{array} \right.$$

¹In order to give a nice reading of the paper, we denote by the symbol ∂_{x_i} the operator $\frac{\partial}{\partial x_i}$, in the text sentences.

coming with various types of boundary conditions over \mathbf{u}' . Then, we get back to u_3 and the global pressure p by setting

$$x \in \Omega, \quad u_3(x) = \int_{x_3}^0 \nabla' \cdot \mathbf{u}'(x', \xi) d\xi, \quad p(x) = p_S(x').$$

However, studying (1.4) yields real difficulties when the mapping h vanishes on $\partial\omega$. This is why most of the works dedicated to its study comes with the following assumption on the mapping h : *there is a constant $\alpha > 0$, such that*

$$(1.5) \quad \inf_{x' \in \omega} h(x') \geq \alpha.$$

Under this assumption, J.L. Lions, R. Temam and S. Wang have introduced System (1.4) and studied its weak solutions, in space dimension 3, in [?, ?]. These results are synthesized by R. Lewandowski in [?], where the author studies some close models to (1.4). With this second approach, it is possible to go further in the analysis of the hydrostatic Stokes and Navier-Stokes approximation. For example, M. Ziane brings in [?], the first regularity results on the stationary linear case where he consider various boundary conditions. Together, R. Temam and M. Ziane, extend this regularity result to a more general problem. To our knowledge, their work [?] is the reference in terms of regularity results for the stationary case. Concerning the study of the time dependent case, we refer to the work of M. A. Rodriguez-Bellido, F. Guillén González, and N. Masmoudi [?, ?].

To finish, some numerical studies have been done, and even if it is not our purpose here, let us briefly mention R. Lewandowski, T. Chacón Rebollo, E. Chacón Vera [?], F. Guillén-Gonzalez, M.A. Rodriguez-Bellido, N. Masmoudi [?], R. Medar [?].

Let us introduce the results obtained in this paper. In the mayor part of the paper, assumption (1.5) is not needed. Indeed, we propose an optimal functional framework, based on weighted spaces in order to control the behavior of the mapping h over $\partial\omega$. Then, thanks to assumption (1.5), we prove the equivalence between an adapted reduced problem (7.1) and (\mathcal{SH}) . In particular, we improve the boundary condition over u_3 usually considered in the literature, which is:

$$(1.6) \quad u_3 n_3 = 0 \quad \text{in } \Gamma,$$

by studying the Dirichlet condition

$$(1.7) \quad u_3 = g_3 \quad \text{on } \Gamma,$$

in a sense to be defined in Proposition 6.2, even in the incompressible case. This leads to the main result of this paper. Before stating it, we need to introduce the space

$$\mathbf{X} = H^1(\Omega)^2 \times H(\partial_{x_3}, \Omega),$$

endowed with the norm $\|\mathbf{u}\|_{\mathbf{X}} = \|\mathbf{u}'\|_{H^1(\Omega)} + \|u_3\|_{H(\partial_{x_3}, \Omega)}$, where $H(\partial_{x_3}, \Omega)$ and its norm are defined in the paragraph 2.4.

Theorem 1.1. *Assume (1.5). Let $\mathbf{f}' \in L^2(\Omega)^2$, $\Phi \in L^2(\Omega)$, $\mathbf{g}' \in H^{1/2}(\Gamma)^2$ and $g_3 \in L^2(\Gamma, |n_3| d\sigma)$. Assume the following compatibility condition:*

$$(1.8) \quad \int_{\Gamma} \mathbf{g} \cdot \mathbf{n} d\sigma = \int_{\Omega} \Phi dx.$$

Then, there is a unique pair $(\mathbf{u}, p) \in \mathbf{X} \times (L^2(\Omega)/\mathbb{R})$ solution to Problem (\mathcal{SH}) and satisfying the estimate,

$$\|\mathbf{u}\|_{\mathbf{X}} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leq C \left(\|\mathbf{f}'\|_{L^2(\Omega)^2} + \|\Phi\|_{L^2(\Omega)} + \|\mathbf{g}'\|_{H^{1/2}(\Gamma)^2} + \|g_3\|_{L^2(\Gamma, |n_3| d\sigma)} \right),$$

where $C > 0$ is a constant depending only on Ω .

We propose, for this paper, the following organization. In Section 2, we set the functional framework. In particular, we recall some properties of the spaces related to the divergence operator and the well-known lemma of De Rham. We will also recall the definition and structure of the anisotropic space $H(\partial_{x_3}, \Omega)$, which gives the classical regularity of the unknown u_3 . To finish, we introduce weighted Lebesgue and Sobolev spaces in order to offset, if necessary, the degeneracy of h on $\partial\omega$.

In Section 3, we focus on the usual integration operators M and F needed, in Section 4, to study some properties of spaces related to the operator $\nabla' \cdot M$. In particular, we give an adapted lemma of De Rham, proved by Ortégón Gallego in [?], and we build a lifting operator in Lemma 4.8 related to the non local constraint $\nabla' \cdot M\mathbf{u}' = \phi$, necessary to handle non homogeneous conditions for some Stokes type system.

In Section 5, we start the study of Problem (\mathcal{SH}) by investigating the homogeneous case (5.1). In particular we reduce (\mathcal{SH}) to the problem (5.2) thanks to Lemma (5.2), and then we solve Problem (5.2) in Lemma 5.3.

We keep on in Section 6 with the non homogeneous case. Here, Assumption (1.5) is necessary to give a meaning of the boundary condition $u_3 = g_3$ on Γ in (1.7), see Proposition 6.2, and to build a lifting operator of boundary values in Theorem 6.5 to get back to the homogeneous case, studied in the previous section. As a consequence, Section 5 and 6 constitute the proof of the main result of this paper, Theorem 1.1.

The results we have established in the previous sections allow to prove, with no adding computations, complementary results. We propose to collect them in Section 7.

To finish, we give in Section 8 another proof of Theorem 5.1 based, this time, on the asymptotic study of the penalized problem.

2. FUNCTIONAL FRAMEWORK

2.1. Usual settings. We define $\mathcal{D}(\Omega)$ to be the linear space of infinitely differentiable functions, with compact support in Ω , and $\mathcal{D}'(\Omega)$ the space of distributions. Then we set

$$\mathcal{D}(\overline{\Omega}) = \{\varphi|_{\Omega} / \varphi \in \mathcal{D}(\mathbb{R}^3)\}.$$

Let us recall that $L^2(\Omega)$ denotes the space of the (almost everywhere classes of) measurable functions u such that $\int_{\Omega} u^2 dx < \infty$, which is a Hilbert space for the usual norm $\|u\|_{L^2(\Omega)} = (\int_{\Omega} u^2 dx)^{1/2}$. We also mention the space

$$L_0^2(\Omega) = \left\{ u \in L^2(\Omega) / \int_{\Omega} u dx = 0 \right\},$$

which is a closed subspace of $L^2(\Omega)$, and isomorphic to the quotient space $L^2(\Omega)/\mathbb{R}$.

We denote by $H^1(\Omega)$ the Sobolev space $\{u \in L^2(\Omega) / \nabla u \in L^2(\Omega)\}$ which is also a Hilbert space. In particular, since Ω has a Lipschitz-continuous boundary Γ , any function $u \in H^1(\Omega)$ bears a trace, still denoted by u , in $H^{1/2}(\Gamma)$ thus defined by

$$H^{1/2}(\Gamma) = \{ \mu \in L^2(\Gamma) / \exists u \in H^1(\Omega) \text{ such that } u = \mu \text{ on } \Gamma \},$$

where $L^2(\Gamma)$ denotes the space of measurable functions $\mu : \Gamma \rightarrow \mathbb{R}$ square integrable for the surface measure $d\sigma$, equipped with

$$\|\mu\|_{L^2(\Gamma)} = \left(\int_{\Gamma} \mu^2 d\sigma \right)^{1/2}.$$

Recall that $H^{1/2}(\Gamma)$ is a Hilbert space endowed with the norm

$$\|\mu\|_{H^{1/2}(\Gamma)} = \inf \left\{ \|u\|_{H^1(\Omega)} / u \in H^1(\Omega), u = \mu \text{ on } \Gamma \right\},$$

and that there is a constant $C > 0$ depending only on Ω such that

$$\forall \mu \in H^{1/2}(\Gamma), \quad \|\mu\|_{L^2(\Gamma)} \leq C \|\mu\|_{H^{1/2}(\Gamma)}.$$

Notation 2.1. Throughout the paper, we denote by C any positive constant depending at most on Ω or ω .

As usual $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$ where

$$H_0^{-1}(\Omega) = \{ u \in H^1(\Omega) / u = 0 \text{ on } \Gamma \} = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}.$$

To finish, we define, for $\Gamma_0 \subset \Gamma$ with a positive measure, the space

$$H_{\Gamma_0}^1(\Omega) = \{ u \in H^1(\Omega) / u = 0 \text{ on } \Gamma \setminus \Gamma_0 \},$$

which is a Hilbert space endowed with the standard $H^1(\Omega)$ norm. Then, we set

$$H_{00}^{1/2}(\Gamma_0) = \{ g \in L^2(\Gamma_0) / \exists u \in H_{\Gamma_0}^1(\Omega) \text{ such that } u = g \text{ on } \Gamma_0 \},$$

which is a Hilbert space for the quotient norm given by

$$\|g\|_{H_{00}^{1/2}(\Gamma_0)} = \inf \left\{ \|u\|_{H^1(\Omega)} / u \in H_{\Gamma_0}^1(\Omega) \text{ and } u = g \text{ on } \Gamma_0 \right\}.$$

As usual, we denote by $H^{-1/2}(\Gamma_0)$ its dual space, and $\langle \cdot, \cdot \rangle_{\Gamma_0}$ stands for the duality pairing between $H^{-1/2}(\Gamma_0)$ and $H_{00}^{1/2}(\Gamma_0)$.

2.2. Computations of surface integrals. For given geometry of Ω see (1.1) we would like to replace any integral over Γ_S or Γ_B by one defined over ω .

Notation 2.2. For any function $\mu : \Gamma \rightarrow \mathbb{R}$, we define the functions μ_S (or $(\mu)_S$) and μ_B (or $(\mu)_B$) by setting

$$x' \in \omega, \quad \mu_S(x') = \mu(x', 0), \quad \mu_B(x') = \mu(x', -h(x')).$$

Proposition 2.3. *The mapping $\mu \mapsto (\mu_S, \mu_B)$ is linear and continuous from $L^2(\Gamma)$ into $L^2(\omega)^2$. Moreover, one has by definition of the measure $d\sigma$:*

$$(2.1) \quad \int_{\Gamma_S} \mu d\sigma = \int_{\omega} \mu_S dx' \quad \text{and} \quad \int_{\Gamma_B} \mu d\sigma = \int_{\omega} \mu_B (1 + |\nabla h|^2)^{1/2} dx'.$$

Proof. This result holds by a change of variables. □

Remark 2.4. Notice that the integrals in (2.1) are well defined since ω is bounded. Next, the third component of the normal n_3 checks $n_3 = 1$ on Γ_S , $n_3 = 0$ on Γ_L and $(n_3)_B (1 + |\nabla h|^2)^{1/2} = -1$ on ω . Moreover, $(n_i)_B (1 + |\nabla h|^2)^{1/2} = -\partial_{x_i} h$ on ω , for $i = 1, 2$. Therefore, for all $\mu \in L^2(\Gamma)$,

$$(2.2) \quad \int_{\Gamma} \mu n_3 d\sigma = \int_{\omega} \mu_S dx' - \int_{\omega} \mu_B dx'. \quad \text{and} \quad \int_{\Gamma_B} \mu n_i d\sigma = - \int_{\omega} \mu \frac{\partial h}{\partial x_i} dx'.$$

2.3. Properties of spaces related to the divergence operator. Any results concerning this paragraph can be found in [?] from page 22 to 25. For any vector field $\mathbf{v} = (v_1, v_2, v_3)$, we define the divergence operator by

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \sum_{i=1}^3 \partial_{x_i} v_i.$$

Let us introduce the space $\mathbf{V} = \{\mathbf{v} \in H_0^1(\Omega)^3 / \nabla \cdot \mathbf{v} = 0\}$, and state the Lemma of De Rham.

Lemma 2.5. *If $\mathbf{f} \in H^{-1}(\Omega)^3$ satisfies*

$$\forall \mathbf{v} \in \mathbf{V}, \quad \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}(\Omega)^3, H_0^1(\Omega)^3} = 0,$$

then, there is $p \in L^2(\Omega)$ such that $\nabla p = \mathbf{f}$ in Ω . Since Ω is connected, p is unique up to an additive constant.

Observing that for any $\mathbf{v} \in H_0^1(\Omega)^3$ one has $\int_{\Omega} \nabla \cdot \mathbf{v} dx = 0$, the range space of the divergence operator div is contained in $L_0^2(\Omega)$. This observation and Lemma 2.5 yield two isomorphisms

$$(2.3) \quad \nabla : L^2(\Omega)/\mathbb{R} \longrightarrow \mathbf{V}^\circ \quad \text{and} \quad \operatorname{div} : \mathbf{V}^\perp \longrightarrow L_0^2(\Omega).$$

Here, \mathbf{V}^\perp denotes the orthogonal of \mathbf{V} for the scalar product

$$\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx = \sum_{i=1}^3 \int_{\Omega} \nabla u_i \cdot \nabla v_i \, dx.$$

Besides, \mathbf{V}° denotes the polar space of \mathbf{V} :

$$\mathbf{V}^\circ = \left\{ \mathbf{f} \in H^{-1}(\Omega)^3 / \forall \mathbf{v} \in \mathbf{V}, \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}(\Omega)^3, H_0^1(\Omega)^3} = 0 \right\}.$$

From the second isomorphism of (2.3), one deduces that for any $\Phi \in L^2(\Omega)$ and any $\mathbf{g} \in H^{1/2}(\Gamma)^3$ such that

$$\int_{\Omega} \Phi \, dx = \int_{\Gamma} \mathbf{g} \cdot \mathbf{n} \, d\sigma,$$

there is $\mathbf{u} \in H^1(\Omega)^3$ such that

$$(2.4) \quad \nabla \cdot \mathbf{u} = \Phi \text{ in } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \Gamma.$$

2.4. The anisotropic space $H(\partial_{x_3}, \Omega)$. Let us recall some useful results in the sequel, that can be found in [?], for example. Let us consider the space

$$H(\partial_{x_3}, \Omega) = \left\{ u \in L^2(\Omega) / \frac{\partial u}{\partial x_3} \in L^2(\Omega) \right\},$$

which is a Hilbert space for the norm $\|u\|_{H(\partial_{x_3}, \Omega)} = (\|u\|_{L^2(\Omega)}^2 + \|\partial_{x_3} u\|_{L^2(\Omega)}^2)^{1/2}$. Let us recall that the space $\mathcal{D}(\overline{\Omega})$ is dense in $H(\partial_{x_3}, \Omega)$. As a consequence, one has the following result.

Proposition 2.6. *The mapping $\gamma_3 : u \mapsto un_3|_{\Gamma}$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended in a unique way to a linear and continuous mapping, still denoted γ_3 , from $H(\partial_{x_3}, \Omega)$ into $H^{-1/2}(\Gamma)$.*

By extension, $\gamma_3 u$ is denoted un_3 . Moreover, we derive the following Green's formula:

$$(2.5) \quad \begin{cases} \forall u \in H(\partial_{x_3}, \Omega), \forall v \in H^1(\Omega), \\ \int_{\Omega} u \frac{\partial v}{\partial x_3} \, dx = - \int_{\Omega} v \frac{\partial u}{\partial x_3} \, dx + \langle un_3, v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}. \end{cases}$$

Now, let us consider the space

$$H_0(\partial_{x_3}, \Omega) = \{ u \in H(\partial_{x_3}, \Omega) / un_3 = 0 \text{ in } H^{-1/2}(\Gamma) \}.$$

Since $\mathcal{D}(\Omega)$ is dense in $H_0(\partial_{x_3}, \Omega)$ (see [?] pages 376 and 377 Theorem 2.2), the following Green's formula holds

$$(2.6) \quad \begin{cases} \forall u \in H(\partial_{x_3}, \Omega), \forall v \in H_0(\partial_{x_3}, \Omega), \\ \int_{\Omega} u \frac{\partial v}{\partial x_3} \, dx = - \int_{\Omega} v \frac{\partial u}{\partial x_3} \, dx, \end{cases}$$

as well as the inequality of Poincaré

$$(2.7) \quad \forall u \in H_0(\partial_{x_3}, \Omega), \quad \|u\|_{L^2(\Omega)} \leq \|h\|_{L^\infty(\omega)} \left\| \frac{\partial u}{\partial x_3} \right\|_{L^2(\Omega)}.$$

2.5. Weighted Lebesgue and Sobolev spaces. In this Section and until Section 5, we do not need to work under assumption (1.5) and therefore, we allow the mapping h to vanish on ω , which leads to more general situations. In this case, we offset the degeneracy of the domain Ω by working with weighted Lebesgue spaces.

Let $\alpha \in \{-1/2, 1/2\}$, and let consider the following weighted Lebesgue space

$$L_{h^\alpha}^2(\omega) = \{p : \omega \rightarrow \mathbb{R} / h^\alpha p \in L^2(\omega)\}.$$

The space $L_{h^\alpha}^2(\omega)$ is a Hilbert space for the norm

$$\|p\|_{L_{h^\alpha}^2(\omega)} = \|h^\alpha p\|_{L^2(\omega)}.$$

Note that one has the following embeddings

$$L_{1/\sqrt{h}}^2(\omega) \hookrightarrow L^2(\omega) \hookrightarrow L_{\sqrt{h}}^2(\omega).$$

One can prove that the space $\mathcal{D}(\omega)$ is dense in $L_{h^\alpha}^2(\omega)$. Moreover, the dual space of $L_{h^\alpha}^2(\omega)$ is the space $L_{h^{-\alpha}}^2(\omega)$.

Remark 2.7. We shall identify a function p on ω with one defined on Ω . This is why, for convenience, we set

$$x \in \Omega, \quad \tilde{p}(x) = p(x').$$

Therefore, note that p belongs to $L_{\sqrt{h}}^2(\omega)$ if and only if \tilde{p} belongs to $L^2(\Omega)$, with

$$\|p\|_{L_{\sqrt{h}}^2(\omega)} = \|\tilde{p}\|_{L^2(\Omega)}.$$

3. DEFINITION AND PROPERTIES OF THE OPERATORS M AND F .

Notation 3.1. Let u be a function defined in Ω . We introduce the following operators

$$\begin{aligned} x' \in \omega, \quad Mu(x') &= \int_{-h(x')}^0 u(x', x_3) dx_3, \\ x \in \Omega, \quad Fu(x) &= \int_{x_3}^0 u(x', \xi) d\xi, \\ x \in \Omega, \quad Gu(x) &= \int_{-h(x')}^{x_3} u(x', \xi) d\xi. \end{aligned}$$

Proposition 3.2. *The operator M is linear and continuous from $L^2(\Omega)$ into $L^2_{1/\sqrt{h}}(\omega)$, and from $H^1(\Omega)$ into $H^1(\omega)$. Then, one has for $i = 1, 2$:*

$$(3.1) \quad \forall u \in H^1(\Omega), \quad \frac{\partial}{\partial x_i}(Mu) = M\left(\frac{\partial u}{\partial x_i}\right) + \frac{\partial h}{\partial x_i}(u|_{\Gamma})_B \text{ in } \omega;$$

$$(3.2) \quad \forall u \in H^1_0(\Omega), \quad \frac{\partial}{\partial x_i}(Mu) = M\left(\frac{\partial u}{\partial x_i}\right) \text{ in } \omega.$$

Moreover,

$$(3.3) \quad \forall u \in H_0(\partial_{x_3}, \Omega), \quad M\left(\frac{\partial u}{\partial x_3}\right) = 0 \text{ in } \omega,$$

holds.

Proof. Let $u \in \mathcal{D}(\Omega)$. The inequality of Cauchy-Schwarz gives for all x' in ω

$$(3.4) \quad \frac{|Mu(x')|^2}{h(x')} \leq \int_{-h(x')}^0 |u(x', x_3)|^2 dx_3 < \infty.$$

As a consequence, the operator M is linear and continuous on $\mathcal{D}(\Omega)$ for the $L^2(\Omega)$ norm. By density, we can extend in a unique way the operator M to a linear and continuous one, still denoted by M , from $L^2(\Omega)$ into $L^2_{1/\sqrt{h}}(\omega)$.

Next, let u in $H^1(\Omega)$ and $i = 1, 2$. One has for any $\psi \in \mathcal{D}(\omega)$:

$$\int_{\omega} Mu \frac{\partial \psi}{\partial x_i} dx' = \int_{\Omega} u \frac{\partial \tilde{\psi}}{\partial x_i} dx = - \int_{\Omega} \frac{\partial u}{\partial x_i} \tilde{\psi} dx + \int_{\Gamma} un_i \tilde{\psi} d\sigma,$$

as $\widetilde{\partial_{x_i} \psi} = \partial_{x_i} \tilde{\psi}$ in Ω . Then, relation (2.2) gives

$$\int_{\Gamma} un_i \tilde{\psi} d\sigma = - \int_{\omega} (u|_{\Gamma})_B \frac{\partial h}{\partial x_i} \psi dx',$$

as $\tilde{\psi}$ does not depend on x_3 and since $\tilde{\psi} = 0$ on Γ_L . Consequently

$$\int_{\omega} Mu \frac{\partial \psi}{\partial x_i} dx' = - \int_{\omega} \left[M\left(\frac{\partial u}{\partial x_i}\right) + (u|_{\Gamma})_B \frac{\partial h}{\partial x_i} \right] \psi dx'.$$

Thus (3.1) holds in $\mathcal{D}(\omega)$ and in $L^2(\omega)$, since h is Lipschitz-continuous. The same arguments give the continuity of M from $H^1(\Omega)$ into $H^1(\omega)$. When u belongs to $H^1_0(\Omega)$, the function $(u|_{\Gamma})_B$ vanishes on ω . Therefore, we deduce (3.2) from relation (3.1).

To finish, relation (3.3) is straightforward with a computation in the sense of distributions on ω and using relation (2.6). \square

Remark 3.3. The assumption $u \in H^1(\Omega)$ is not sufficient to get $(u|_{\Gamma})_B(\partial_{x_i} h) \in L^2_{1/\sqrt{h}}(\omega)$ and conclude that $\partial_{x_i}(Mu) \in L^2_{1/\sqrt{h}}(\omega)$.

Proposition 3.4. *The operator F is linear and continuous from $L^2(\Omega)$ into $L^2(\Omega)$ and the operator G is the adjoint operator to F . Next, the operator F is continuous from $L^2(\Omega)$ into $H(\partial_{x_3}, \Omega)$, with in particular:*

$$(3.5) \quad \forall u \in L^2(\Omega), \quad \frac{\partial}{\partial x_3}(Fu) = -u \text{ in } \Omega.$$

Moreover,

$$(3.6) \quad \forall u \in H_0(\partial_{x_3}, \Omega), \quad F\left(\frac{\partial u}{\partial x_3}\right) = -u \text{ in } \Omega,$$

holds.

Proof. Let $u \in L^2(\Omega)$. Thanks to the theorem of Fubini and since for almost all $x \in \Omega$

$$\begin{aligned} |Fu(x)|^2 &\leq |x_3| \int_{x_3}^0 |u(x', x_3)|^2 dx_3 \\ &\leq h(x') \int_{-h(x')}^0 |u(x', x_3)|^2 dx_3, \end{aligned}$$

we deduce that $Fu \in L^2(\Omega)$ and $\|Fu\|_{L^2(\Omega)} \leq \|h\|_\infty \|u\|_{L^2(\Omega)}$. Hence F is linear and continuous from $L^2(\Omega)$ into $L^2(\Omega)$.

Then, G is the adjoint to F for the scalar product of $L^2(\Omega)$ since, thanks to Fubini's theorem for any u and v in $L^2(\Omega)$

$$\begin{aligned} \int_{\Omega} v Fu dx &= \int_{\omega} \left[\int_{-h(x')}^0 \int_{x_3}^0 u(x', \xi) v(x', x_3) d\xi dx_3 \right] dx' \\ &= \int_{\omega} \int_{-h(x')}^0 u(x', \xi) \left[\int_{-h(x')}^{\xi} v(x', x_3) dx_3 \right] d\xi dx' \\ &= \int_{\Omega} u Gv dx. \end{aligned}$$

Next, one has for any $\varphi \in \mathcal{D}(\Omega)$

$$\int_{\Omega} \frac{\partial \varphi}{\partial x_3} Fu dx = \int_{\Omega} u G\left(\frac{\partial \varphi}{\partial x_3}\right) dx = \int_{\Omega} u \varphi dx.$$

Hence (3.5) holds in $\mathcal{D}'(\Omega)$ and $\partial_{x_3}(Fu) \in L^2(\Omega)$. Moreover, we deduce from the previous claim that the operator F is continuous from $L^2(\Omega)$ into $H(\partial_{x_3}, \Omega)$. To finish, we use the same arguments as above and relation (2.6) to prove (3.6). \square

Lemma 3.5. *Let u in $H^1(\Omega)$, and set $\mathcal{G} = u|_{\Gamma}$. Then:*

$$M\left(\frac{\partial u}{\partial x_3}\right) = \mathcal{G}_S - \mathcal{G}_B \text{ in } \omega, \quad \text{and} \quad F\left(\frac{\partial u}{\partial x_3}\right) = \widetilde{\mathcal{G}}_S - u, \quad G\left(\frac{\partial u}{\partial x_3}\right) = u - \widetilde{\mathcal{G}}_B \text{ in } \Omega.$$

Proof. Proposition 2.3 implies that \mathcal{G}_S and \mathcal{G}_B belong to $L^2(\omega)$. Then, one gets from relation (2.2), that for any $\psi \in \mathcal{D}(\omega)$:

$$\begin{aligned} \int_{\omega} M\left(\frac{\partial u}{\partial x_3}\right) \psi \, dx' &= \int_{\Omega} \tilde{\psi} \frac{\partial u}{\partial x_3} \, dx = \int_{\Gamma} \tilde{\psi} \mathcal{G} n_3 \, d\sigma \\ &= \int_{\omega} \psi \mathcal{G}_S \, dx' - \int_{\omega} \psi \mathcal{G}_B \, dx'. \\ &= \int_{\omega} (\mathcal{G}_S - \mathcal{G}_B) \psi \, dx', \end{aligned}$$

and the first relation is proved. Let us deal with the second one. By Remark 2.7, the function $\widetilde{\mathcal{G}}_S - u$ belongs to $L^2(\Omega)$. Next, Proposition 3.4 and (2.2), prove that for any $\varphi \in \mathcal{D}(\Omega)$:

$$\begin{aligned} \int_{\Omega} F\left(\frac{\partial u}{\partial x_3}\right) \varphi \, dx &= \int_{\Omega} \frac{\partial u}{\partial x_3} G\varphi \, dx \\ &= - \int_{\Omega} u\varphi \, dx + \int_{\omega} \mathcal{G}_S(G\varphi)_S \, dx' - \int_{\omega} \mathcal{G}_B(G\varphi)_B \, dx. \end{aligned}$$

Since $(G\varphi)_S = M\varphi$ and $(G\varphi)_B = 0$ in ω , we deduce that

$$\int_{\Omega} F\left(\frac{\partial u}{\partial x_3}\right) \varphi \, dx = - \int_{\Omega} u\varphi \, dx + \int_{\Omega} \widetilde{\mathcal{G}}_S \varphi \, dx,$$

from which one deduces the second relation. To prove the last one, note that

$$Gu = \widetilde{Mu} - Fu \text{ in } \Omega.$$

□

Proposition 3.6. *Let $u \in L^2(\Omega)$. Then, one has*

$$(Fu) n_3 = 0 \text{ in } H^{-1/2}(\Gamma_S \cup \Gamma_L), \quad (Fu) n_3 = \widetilde{Mu} n_3 \text{ in } H^{-1/2}(\Gamma_B).$$

Proof. Let $u \in L^2(\Omega)$. By Proposition 3.4, one has $Fu \in H(\partial_{x_3}, \Omega)$, and Proposition 2.6 ensures $(Fu) n_3 \in H^{-1/2}(\Gamma)$. Then, Lemma 3.5 gives for any $v \in H^1(\Omega)$:

$$\begin{aligned} \langle (Fu) n_3, v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} &= \int_{\Omega} \frac{\partial v}{\partial x_3} Fu \, dx - \int_{\Omega} uv \, dx \\ &= \int_{\Omega} u G\left(\frac{\partial v}{\partial x_3}\right) \, dx - \int_{\Omega} uv \, dx \\ &= \int_{\Omega} u (v - \widetilde{(v|_{\Gamma})_B}) \, dx - \int_{\Omega} uv \, dx. \\ &= - \int_{\omega} (v|_{\Gamma})_B Mu \, dx'. \end{aligned}$$

Therefore we get

$$(3.7) \quad \forall v \in H^1(\Omega), \quad \langle (Fu) n_3, v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = - \int_{\omega} (v|_{\Gamma})_B Mu \, dx'.$$

As a consequence, one deduces that

$$\forall v \in H_{\Gamma_S \cup \Gamma_L}^1(\Omega), \quad \langle (Fu) n_3, v \rangle_{\Gamma_S \cup \Gamma_L} = 0,$$

and the first equality is proved. Let us focus on the second one. Since $\partial_{x_3} \widetilde{Mu} = 0$ in Ω , the function \widetilde{Mu} belongs to $H(\partial_{x_3}, \Omega)$. Hence, one has $\widetilde{Mu} n_3 \in H^{-1/2}(\Gamma)$. Moreover, from Green's formula (2.5), one gets for any $v \in H^1(\Omega)$

$$\left\langle \widetilde{Mu} n_3, v \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \int_{\Omega} \widetilde{Mu} \frac{\partial v}{\partial x_3} \, dx = \int_{\omega} Mu M \left(\frac{\partial v}{\partial x_3} \right) \, dx.$$

Then, one uses Lemma 3.5 to deduce

$$\begin{aligned} \left\langle \widetilde{Mu} n_3, v \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} &= - \int_{\omega} Mu (v|_{\Gamma})_B \, dx' + \int_{\omega} Mu (v|_{\Gamma})_S \, dx' \\ &= \langle (Fu) n_3, v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} + \int_{\omega} Mu (v|_{\Gamma})_S \, dx'. \end{aligned}$$

Therefore, one has for any $v \in H_{\Gamma_B}^1(\Omega)$,

$$\langle (Fu) n_3, v \rangle_{\Gamma_B} = \left\langle \widetilde{Mu} n_3, v \right\rangle_{\Gamma_B},$$

which proves the last equality. \square

As a consequence of Proposition 3.6, Proposition 3.2 and relation (3.7), the following result holds.

Corollary 3.7. *Let $u \in L^2(\Omega)$. Then, the following assertions are equivalent:*

- i) $Mu = 0$ in $L_{1/\sqrt{h}}^2(\omega)$.
- ii) $(Fu)n_3 = 0$ in $H^{-1/2}(\Gamma)$.

4. SOME PROPERTIES RELATED TO THE MEAN DIVERGENCE OPERATOR

For $\mathbf{v}' = (v_1, v_2)$, we define the mean divergence operator by

$$\nabla' \cdot M\mathbf{v}' = \sum_{i=1,2} \partial_{x_i} (Mv_i).$$

Let us introduce the space

$$\mathbf{V}_M = \{ \mathbf{v}' \in H_0^1(\Omega)^2 / \nabla' \cdot M\mathbf{v}' = 0 \text{ in } \omega \}.$$

Since \mathbf{V}_M is a closed subspace of $H_0^1(\Omega)^2$, we have the decomposition

$$(4.1) \quad H_0^1(\Omega)^2 = \mathbf{V}_M \oplus \mathbf{V}_M^\perp,$$

where \mathbf{V}_M^\perp denotes the orthogonal of \mathbf{V}_M for the scalar product $\int_\Omega \nabla \mathbf{u}' : \nabla \mathbf{v}' dx$.

Next, let us mention here the work of Ortegón Gallego [?], to establish a kind of Lemma of De Rham, useful in the study of problem related to the non local constraint $\nabla' \cdot M \mathbf{u}' = 0$ in ω (see Problem (5.2)). We start by giving the definition of distributions independent on x_3 (see [?] Theorem 1 page 337).

Proposition 4.1. *Let $T \in \mathcal{D}'(\Omega)$. Then, the following assertions are equivalent.*

1. *The distribution T does not depend on x_3 in the sense $\partial_{x_3} T = 0$ in Ω .*
2. *There is a unique distribution $S \in \mathcal{D}'(\omega)$ such that*

$$(4.2) \quad \forall \varphi \in \mathcal{D}(\Omega), \quad \langle T, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \langle S, M\varphi \rangle_{\mathcal{D}'(\omega), \mathcal{D}(\omega)}.$$

Besides, the regularity of the distribution S depends on the one of T .

Proposition 4.2. *We keep the notations of Proposition 4.1. Let $T \in \mathcal{D}'(\Omega)$ and $S \in \mathcal{D}'(\omega)$ satisfying (4.2). If $T \in H^{-1}(\Omega)$, then*

$$S \in H^{-1}(\overset{\circ}{K}), \quad \text{for any compact set } K \subset \omega.$$

If moreover, $\text{ess inf}_\omega h > 0$, then $S \in H^{-1}(\omega)$.

Remark 4.3. By adapting Proposition 4.2, one can prove that if $T \in L^2(\Omega)$ then S is locally in $L^2(\omega)$. Then, from relation (4.2) one deduces that $T = \tilde{S}$ in $L^2(\Omega)$, and consequently $S \in L^2_{\sqrt{h}}(\omega)$ by Remark 2.7.

By now, let us state and prove of the expected De Rham-like lemma, useful in the sequel.

Lemma 4.4. *If $\mathbf{f}' \in H^{-1}(\Omega)^2$ satisfies*

$$\forall \mathbf{v}' \in \mathbf{V}_M, \quad \langle \mathbf{f}', \mathbf{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = 0,$$

then, there is $q \in L^2_{\sqrt{h}}(\omega)$, unique up to an additive constant, such that $\nabla' \tilde{q} = \mathbf{f}'$ in Ω .

Proof. Let us set $\mathbf{f} = (\mathbf{f}', 0)$. Thanks to (3.2) and (3.3), any $\mathbf{v} \in \mathbf{V}$ is such that $\mathbf{v}' \in \mathbf{V}_M$. As a consequence, the distribution \mathbf{f} is exactly in the statement of Lemma 2.5. Therefore, there is a unique function p in $L^2(\Omega)/\mathbb{R}$ such that

$$\nabla' p = \mathbf{f}' \text{ and } \frac{\partial p}{\partial x_3} = 0 \text{ in } \Omega.$$

Then, since $\partial_{x_3} p = 0$ in Ω , we deduce from Proposition 4.1 and Remark 4.3, that there is $q \in L^2_{\sqrt{h}}(\omega)$, unique up to an additive constant, such that $p = \tilde{q}$. Hence q satisfies $\nabla' \tilde{q} = \mathbf{f}'$ in Ω . \square

As a consequence of Lemma 4.4, we give a characterization of the space \mathbf{V}_M^\perp . Previously, we require a definition.

Definition 4.5. Let $(-\Delta)^{-1} : H^{-1}(\Omega)^2 \rightarrow H_0^1(\Omega)^2$ denotes the linear and continuous Green's operator related to Dirichlet's homogeneous problem for $-\Delta$ in Ω , *i.e.* $\mathbf{u}' = (-\Delta)^{-1} \mathbf{f}'$ if and only if \mathbf{u}' is the solution to:

$$-\Delta \mathbf{u}' = \mathbf{f}' \text{ in } \Omega, \quad \mathbf{u}' = 0 \text{ on } \Gamma.$$

Corollary 4.6. *We have*

$$(4.3) \quad \mathbf{V}_M^\perp = \left\{ (-\Delta)^{-1} \nabla' \tilde{q} / q \in L_{\sqrt{h}}^2(\omega) \right\}.$$

Proof. First, let us check that for $q \in L_{\sqrt{h}}^2(\omega)$, $\mathbf{u}' = (-\Delta)^{-1} \nabla' \tilde{q}$ belongs to \mathbf{V}_M^\perp . One has for any $\mathbf{v}' \in \mathbf{V}_M$:

$$\int_{\Omega} \nabla \mathbf{u}' : \nabla \mathbf{v}' dx = \langle -\Delta \mathbf{u}', \mathbf{v}' \rangle = - \int_{\Omega} \tilde{q} \nabla' \cdot \mathbf{v}' dx = - \int_{\omega} q M(\nabla' \cdot \mathbf{v}') dx' = 0,$$

thanks to (3.2) and the fact that $\nabla' \cdot M \mathbf{v}' = 0$. Conversely, let $\mathbf{u}' \in \mathbf{V}_M^\perp$ and consider mapping \mathbf{l}' in $H^{-1}(\Omega)^2$ by

$$\langle \mathbf{l}', \mathbf{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = \int_{\Omega} \nabla \mathbf{u}' : \nabla \mathbf{v}' dx.$$

Then \mathbf{l}' vanishes on \mathbf{V}_M , and thanks to Lemma 4.4 there is $q \in L_{\sqrt{h}}^2(\omega)$ such that

$$\forall \mathbf{v}' \in H_0^1(\Omega)^2, \quad \int_{\Omega} \nabla \mathbf{u}' : \nabla \mathbf{v}' dx = \langle \nabla' \tilde{q}, \mathbf{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2}.$$

Hence $\mathbf{u}' = (-\Delta)^{-1} \nabla' \tilde{q}$. □

Next, we want to characterize the range space of the operator $\nabla' \cdot M$. Observe that Green's formula and (3.2) yield:

$$\forall \mathbf{v}' \in H_0^1(\Omega)^2, \quad \int_{\omega} \nabla' \cdot M \mathbf{v}' dx' = 0.$$

Thus the range space of $\nabla' \cdot M$ is contained in a proper, closed subspace of $L_{1/\sqrt{h}}^2(\omega)$, that is

$$L_{1/\sqrt{h},0}^2(\omega) := L_{1/\sqrt{h}}^2(\omega) \cap L_0^2(\omega).$$

Corollary 4.7. *The following operators are isomorphisms*

$$(4.4) \quad \nabla' : L_{\sqrt{h}}^2(\omega) / \mathbb{R} \longrightarrow \mathbf{V}_M^\circ \quad \text{and} \quad \nabla' \cdot M : \mathbf{V}_M^\perp \longrightarrow L_{1/\sqrt{h},0}^2(\omega).$$

Proof. We already know that ∇' is linear and continuous. Moreover, it is a bijection thanks to Lemma 4.4. As $L^2_{\sqrt{h}}(\omega)/\mathbb{R}$ and \mathbf{V}_M° are both Banach spaces, it follows that ∇' is an isomorphism.

Therefore, $\nabla' \cdot M$ is an isomorphism from $(\mathbf{V}_M^\circ)'$ onto $(L^2_{\sqrt{h}}(\omega)/\mathbb{R})'$. On the one hand, one can prove that the space $(\mathbf{V}_M^\circ)'$ can be identified to \mathbf{V}_M^\perp , by using similar arguments as in the proof of Corollary 2.4 in [?]. On the other hand, note that $(L^2_{\sqrt{h}}(\omega)/\mathbb{R})'$ can be identified with $L^2_{1/\sqrt{h}}(\omega) \perp \mathbb{R}$, where the orthogonality is taken in the following sense

$$\forall p \in L^2_{1/\sqrt{h}}(\omega), \quad \langle p, 1 \rangle_{L^2_{1/\sqrt{h}}(\omega), L^2_{\sqrt{h}}(\omega)} = \int_{\omega} p \, dx' = 0.$$

It follows that $L^2_{1/\sqrt{h}}(\omega) \perp \mathbb{R}$ can be identified to $L^2_{1/\sqrt{h},0}(\omega)$. \square

We finish this section by giving an adaptation of the lift operator defined in relation (2.4), by building a lift operator of boundary values related to the operator $\nabla' \cdot M$. Previously, we need to give the Stokes formula related to the operator $\nabla' \cdot M$, that is a consequence of relations (3.1) and (2.2):

$$(4.5) \quad \forall \mathbf{u}' \in H^1(\Omega)^2, \quad \int_{\omega} \nabla' \cdot M \mathbf{u}' \, dx' = \int_{\Gamma_L} \mathbf{u}' \cdot \mathbf{n}' \, d\sigma.$$

In order to state the last result of this section, let us consider the following subspace of $H^{1/2}(\Gamma)$,

$$(4.6) \quad H_h^{1/2}(\Gamma) = \left\{ g \in H^{1/2}(\Gamma) / \frac{g}{\sqrt{h}} \in L^2(\Gamma) \right\},$$

that we equip with the graph norm

$$\|g\|_{H_h^{1/2}(\Gamma)} = \left(\|g\|_{H^{1/2}(\Gamma)}^2 + \left\| \frac{g}{\sqrt{h}} \right\|_{L^2(\Gamma)}^2 \right)^{1/2}.$$

Lemma 4.8. *Let $\mathbf{g}' \in H_h^{1/2}(\Gamma)^2$ and $\phi \in L^2_{1/\sqrt{h}}(\omega)$, satisfying the following compatibility condition:*

$$\int_{\omega} \phi \, dx' = \int_{\Gamma_L} \mathbf{g}' \cdot \mathbf{n}' \, d\sigma.$$

Then, there is $\mathbf{u}' \in H^1(\Omega)^2$ such that

$$(4.7) \quad \nabla' \cdot M \mathbf{u}' = \phi \text{ in } \omega, \quad \mathbf{u}' = \mathbf{g}' \text{ on } \Gamma.$$

Moreover, there is a constant $C > 0$ depending only on Ω such that

$$(4.8) \quad \|\mathbf{u}'\|_{H^1(\Omega)^2} \leq C \left(\|\mathbf{g}'\|_{H_h^{1/2}(\Gamma)^2} + \|\phi\|_{L^2_{1/\sqrt{h}}(\omega)} \right).$$

Proof. Let $\mathbf{g}' \in H_h^{1/2}(\Gamma)^2$. Then, there is \mathbf{v}' in $H^1(\Omega)^2$ such that $\mathbf{v}' = \mathbf{g}'$ on Γ , and

$$(4.9) \quad \frac{\mathbf{v}'}{\sqrt{h}} \in L^2(\Gamma).$$

Firstly, note that (4.9) and (2.2) ensures that $(\mathbf{v}'|_\Gamma)_B \cdot \nabla h$ belongs to $L_{1/\sqrt{h}}^2(\omega)$, with the estimates

$$(4.10) \quad \|(\mathbf{v}'|_\Gamma)_B \cdot \nabla h\|_{L_{1/\sqrt{h}}^2(\omega)} \leq C \|\mathbf{v}'\|_{H^{1/2}(\Gamma_h)^2}.$$

Secondly, according to Stokes formula (4.5), one observes that

$$\int_\omega \nabla' \cdot M \mathbf{v}' dx' = \int_{\Gamma_L} \mathbf{g}' \cdot \mathbf{n}' d\sigma = \int_\omega \phi dx'.$$

From what precedes, the function $\nabla' \cdot M \mathbf{v}' - \phi$ is in $L_{1/\sqrt{h},0}^2(\omega)$, and the isomorphism given in (4.4) provides a unique \mathbf{z}' in \mathbf{V}_M^\perp satisfying

$$\nabla' \cdot M \mathbf{z}' = \nabla' \cdot M \mathbf{v}' - \phi \quad \text{in } \omega.$$

Moreover, there is a constant $C > 0$ depending only on Ω such that

$$\|\mathbf{z}'\|_{H^1(\Omega)^2} \leq C \|\nabla' \cdot M \mathbf{v}' - \phi\|_{L_{1/\sqrt{h}}^2(\omega)}.$$

One has by (3.1), (4.10) and since $\mathbf{v}' = \mathbf{g}'$ in $H_h^{1/2}(\Gamma)$:

$$\begin{aligned} \|\mathbf{z}'\|_{H^1(\Omega)^2} &\leq C \left(\|M(\nabla' \cdot \mathbf{v}')\|_{L_{1/\sqrt{h}}^2(\omega)} + \|(\mathbf{v}'|_\Gamma)_B \cdot \nabla h\|_{L_{1/\sqrt{h}}^2(\omega)} + \|\phi\|_{L_{1/\sqrt{h}}^2(\omega)} \right) \\ &\leq C \left(\|\mathbf{v}'\|_{H^1(\Omega)^2} + \|\mathbf{g}'\|_{H_h^{1/2}(\Gamma)^2} + \|\phi\|_{L_{1/\sqrt{h}}^2(\omega)} \right). \end{aligned}$$

By taking the infimum on the functions $\mathbf{v}' \in H^1(\Omega)^2$ such that $\mathbf{v}' = \mathbf{g}'$ on Γ , we deduce that

$$(4.11) \quad \|\mathbf{z}'\|_{H^1(\Omega)^2} \leq C \left(\|\mathbf{g}'\|_{H_h^{1/2}(\Gamma)^2} + \|\phi\|_{L_{1/\sqrt{h}}^2(\omega)} \right).$$

Consequently, $\mathbf{u}' = \mathbf{v}' - \mathbf{z}'$ satisfies (4.7) and (4.8). \square

5. RESOLUTION OF PROBLEM (\mathcal{SH}) WITH HOMOGENEOUS DIRICHLET CONDITIONS

Given $\mathbf{f}' : \Omega \rightarrow \mathbb{R}^2$, one wishes to solve the homogeneous hydrostatic Stokes problem, that consists in seeking $\mathbf{u} : \Omega \rightarrow \mathbb{R}^3$ and $p : \Omega \rightarrow \mathbb{R}$ formally solution to

$$(5.1) \quad \begin{cases} -\Delta \mathbf{u}' + \nabla' p = \mathbf{f}', & \frac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}' = 0, \quad u_3 n_3 = 0 & & \text{on } \Gamma. \end{cases}$$

Before giving the main result of this section, let us denote by \mathbf{X}_0 the space

$$\mathbf{X}_0 = H_0^1(\Omega)^2 \times H_0(\partial_{x_3}, \Omega).$$

Then, let $\mathbf{f}' \in H^{-1}(\Omega)^2$ and consider the following auxiliary problem

$$(5.2) \quad \begin{aligned} & \text{Find } (\mathbf{u}', p_S) \in H_0^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R}) \text{ such that:} \\ & \begin{cases} -\Delta \mathbf{u}' + \nabla' \widetilde{p}_S = \mathbf{f}' & \text{in } \Omega, \\ \nabla' \cdot M \mathbf{u}' = 0 & \text{in } \omega, \\ \mathbf{u}' = 0 & \text{on } \Gamma. \end{cases} \end{aligned}$$

Theorem 5.1. *Let $\mathbf{f}' \in H^{-1}(\Omega)^2$, and let (\mathbf{u}', p_S) be the unique solution to Problem (5.2). Moreover, let us set*

$$(5.3) \quad x \in \Omega, \quad p(x) = \widetilde{p}_S(x'), \quad u_3(x) = F(\nabla' \cdot \mathbf{u}')(x).$$

Then, the pair $(\mathbf{u}, p) \in \mathbf{X}_0 \times (L^2(\Omega)/\mathbb{R})$ is the unique solution to Problem (5.1). Besides, there is a constant $C > 0$ depending at most on Ω such that

$$(5.4) \quad \|\mathbf{u}'\|_{H^1(\Omega)^2} + \|u_3\|_{H(\partial_{x_3}, \Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leq C \|\mathbf{f}'\|_{H^{-1}(\Omega)^2}.$$

In order to prove Theorem 5.1, we establish thanks to Lemma 5.2 that (5.1) and (5.2), combined with (5.3), are equivalent. Finally, we prove in Lemma 5.3, that Problem (5.2) has a unique solution (\mathbf{u}', p_S) in the space $H_0^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$. Note that we give in the appendix another approach to solve Problem (5.1), that can be seen as the physical approach of this problem.

Lemma 5.2. *Let $\mathbf{u} \in H^1(\Omega)_0^2 \times H(\partial_{x_3}, \Omega)$. Then the following assertions are equivalent:*

- i) $\nabla \cdot \mathbf{u} = 0$ in Ω , $u_3 n_3 = 0$ in $H^{-1/2}(\Gamma)$.*
- ii) $\nabla' \cdot (M \mathbf{u}') = 0$ in ω , $u_3 = F(\nabla' \cdot \mathbf{u}')$ in Ω .*

Proof. Assume *i*. Then, (3.3) and (3.6) give

$$M(\nabla' \cdot \mathbf{u}') = 0 \text{ in } \omega, \quad u_3 = F(\nabla' \cdot \mathbf{u}') \text{ in } \Omega.$$

Moreover, thanks to (3.2) one has $M(\nabla' \cdot \mathbf{u}') = \nabla' \cdot M \mathbf{u}'$, which proves *ii*. Conversely, one has by relation (3.5)

$$\frac{\partial u_3}{\partial x_3} = -\nabla' \cdot \mathbf{u}' \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega.$$

Besides, and since $M(\nabla' \cdot \mathbf{u}') = 0$, Proposition 3.7 ensures that

$$n_3 F(\nabla' \cdot \mathbf{u}') = 0 \text{ in } H^{-1/2}(\Gamma),$$

hence $u_3 n_3 = 0$ in $H^{-1/2}(\Gamma)$. □

According to Lemma 5.2 and the fact that p does not depend on x_3 , see Remark 4.3, solving Problem (5.1) reduces to solve Problem (5.2). Then, we get back to p and u_3 thanks to relation (5.3).

Lemma 5.3. *Let \mathbf{f}' in $H^{-1}(\Omega)^2$. There is a unique solution (\mathbf{u}', p_S) in the space $H_0^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ to Problem (5.2). Moreover, there is a constant $C > 0$ depending only on Ω such that*

$$(5.5) \quad \|\mathbf{u}'\|_{H^1(\Omega)^2} + \|p_S\|_{L^2_{\sqrt{h}}(\omega)/\mathbb{R}} \leq C \|\mathbf{f}'\|_{H^{-1}(\Omega)^2}.$$

Proof. Let $(\mathbf{u}', p_S) \in H_0^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ be a solution to (5.2), and let $\mathbf{v}' \in \mathbf{V}_M$. Then, one has thanks to (3.2)

$$\begin{aligned} \langle \nabla' \widetilde{p}_S, \mathbf{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} &= - \int_{\Omega} \widetilde{p}_S \nabla' \cdot \mathbf{v}' dx = - \int_{\omega} p_S M(\nabla' \cdot \mathbf{v}') dx' \\ &= - \int_{\omega} p_S \nabla' \cdot M \mathbf{v}' dx' = 0, \end{aligned}$$

since $\nabla' \cdot M \mathbf{v}' = 0$ in ω . As a consequence, \mathbf{u}' satisfies the following variational formulation:

$$(5.6) \quad \begin{cases} \text{Find } \mathbf{u}' \in \mathbf{V}_M \text{ such that:} \\ \forall \mathbf{v}' \in \mathbf{V}_M, \int_{\Omega} \nabla \mathbf{u}' : \nabla \mathbf{v}' dx = \langle \mathbf{f}', \mathbf{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2}. \end{cases}$$

Conversely, any solution $\mathbf{u}' \in \mathbf{V}_M$ to (5.6) is such that

$$\forall \mathbf{v}' \in \mathbf{V}_M, \quad \langle \Delta \mathbf{u}' + \mathbf{f}', \mathbf{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = 0,$$

hence $\Delta \mathbf{u}' + \mathbf{f}'$ is exactly in the statement of Lemma 4.4. Therefore, there is a unique p_S in $(L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ such that

$$(5.7) \quad \nabla \widetilde{p}_S = \Delta \mathbf{u}' + \mathbf{f}' \text{ in } \Omega,$$

wher \mathbf{u}' is a solution to Problem (5.2). As a consequence, we have proved that a pair $(\mathbf{u}', p) \in H_0^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ is a solution to (5.1) if and only if \mathbf{u}' is a solution to (5.6). To prove the existence and uniqueness of a solution to (5.1), note that Lax-Milgram's lemma provides a unique \mathbf{u}' in \mathbf{V}_M satisfying (5.6). Then, by taking $\mathbf{v} = \mathbf{u}$ in (5.6), we deduce that

$$(5.8) \quad \|\nabla \mathbf{u}'\|_{L^2(\Omega)} \leq C \|\mathbf{f}'\|_{H^{-1}(\Omega)}.$$

To finish, the first isomorphism of (4.4) relation (5.8) and (5.7), give the following estimates

$$\|p_S\|_{L^2_{\sqrt{h}}(\omega)/\mathbb{R}} \leq C \|\nabla \widetilde{p}_S\|_{L^2(\Omega)} \leq C \|\mathbf{f}'\|_{H^{-1}(\Omega)^2}$$

Here, we recall that $C > 0$ denotes any constant depending only on Ω . \square

Proof of Theorem 5.1. Thanks to Proposition 5.3 and Lemma 5.2, one has solved (5.1). Moreover we have one part of the estimate (5.4), the one concerning \mathbf{u}' and p . To get the missing one on $u_3 = F(\nabla' \cdot \mathbf{u}')$, we use Proposition 3.4, and to obtain the one $p = \widetilde{p}_S$, we use Remark 2.7. \square

6. RESOLUTION OF PROBLEM (\mathcal{SH}) WITH NON HOMOGENEOUS DIRICHLET CONDITIONS

Throughout this section, we assume that Ω has sidewalls. More precisely, we make the assumption² (1.5) on the mapping h . Note, in this case, that the spaces $L^2_{h^\alpha}(\omega)$ of Paragraph 2.5 are algebraically and topologically equal to $L^2(\omega)$.

FIGURE 6.1. The domain Ω with sidewalls

6.1. An optimal trace operator for the space $H(\partial_{x_3}, \Omega)$. Let us recall that $n_3 = 0$ on Γ_L , $n_3 = 1$ on Γ_S and $n_3 < 0$ on Γ_B . As a consequence, since n_3 has a constant sign on Γ_B , n_3 can be considered as a weight for the surface measure $d\sigma$. This is why we introduce the following Lebesgue space

$$L^2(\Gamma, |n_3| d\sigma) = \left\{ \mu : \Gamma \rightarrow \mathbb{R}, |n_3| d\sigma - \text{measurable} / \mu |n_3|^{1/2} \in L^2(\Gamma) \right\},$$

endowed with the norm

$$\|\mu\|_{L^2(\Gamma, |n_3| d\sigma)} = \left\| \mu |n_3|^{1/2} \right\|_{L^2(\Gamma)}.$$

Remark 6.1. Let $\mu \in L^2(\Gamma, |n_3| d\sigma)$. Then, since $n_3^2 \leq |n_3|$ on Γ , one has $\mu n_3 \in L^2(\Gamma)$, with

$$\|\mu n_3\|_{L^2(\Gamma)} \leq \|\mu\|_{L^2(\Gamma, |n_3| d\sigma)}.$$

Moreover, one deduces, from the above inequality and (2.2), that μ_S and μ_B both belong to $L^2(\omega)$, with

$$\begin{aligned} \|\mu_S\|_{L^2(\omega)} &\leq \|\mu n_3\|_{L^2(\Gamma_S)} \leq C \|\mu\|_{L^2(\Gamma, |n_3| d\sigma)}, \\ \|\mu_B\|_{L^2(\omega)} &\leq \|\mu n_3\|_{L^2(\Gamma_B)} \leq C \|\mu\|_{L^2(\Gamma, |n_3| d\sigma)}. \end{aligned}$$

Next, let $\lambda, \mu \in L^2(\Gamma, |n_3| d\sigma)$. One proves, by the inequality of Hölder, that $\lambda \mu n_3$ belongs to $L^1(\Gamma)$. To finish this remark, note that for $\mu \in L^2(\Gamma, |n_3| d\sigma)$ one has:

$$(6.1) \quad \int_{\Gamma} \mu |n_3| d\sigma = \int_{\omega} \mu_S dx' + \int_{\omega} \mu_B dx',$$

thanks to relation (2.2).

Proposition 6.2. *The linear mapping $\gamma : u \mapsto u|_{\Gamma}$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended in a unique way to a linear and continuous mapping, denoted in the same way, from $H(\partial_{x_3}, \Omega)$ into $L^2(\Gamma, |n_3| d\sigma)$.*

²Far from now, this assumption was not needed.

Proof. Let $\theta = \theta(x_3)$ in $\mathcal{D}(\mathbb{R})$ such that

$$\mathbb{I}_{[-\|h\|_{L^\infty(\omega)}, -2\alpha/3]} \leq \theta \leq \mathbb{I}_{[-\|h\|_{L^\infty(\omega)}, -\alpha/3]},$$

where α is defined in (1.5). Then, let us set

$$x \in \Omega, \quad \theta^B(x) = \theta(x_3), \quad \theta^S(x) = 1 - \theta(x_3).$$

Therefore $|n_3| \theta^S = n_3 \theta^S$ and $|n_3| \theta^B = -n_3 \theta^B$ over Γ . Next, let $u \in \mathcal{D}(\overline{\Omega})$. Then

$$\begin{aligned} \int_{\Gamma} u^2 |n_3| d\sigma &= \int_{\Gamma} u^2 (\theta^S + \theta^B) |n_3| d\sigma \\ &= \int_{\Gamma} u^2 \theta^S n_3 d\sigma - \int_{\Gamma} u^2 \theta^B n_3 d\sigma \\ &= \int_{\Gamma} u^2 (\theta^S - \theta^B) n_3 d\sigma. \end{aligned}$$

From Green's formula, we deduce

$$\begin{aligned} \int_{\Gamma} u^2 |n_3| d\sigma &= 2 \int_{\Omega} (\theta^S - \theta^B) u \frac{\partial u}{\partial x_3} dx + \int_{\Omega} u^2 \frac{\partial}{\partial x_3} (\theta^S - \theta^B) dx \\ &\leq C_{\theta} \|u\|_{H(\partial_{x_3}, \Omega)}^2, \end{aligned}$$

where $C_{\theta} > 0$ is a constant depending on θ . Consequently, the mapping γ is linear and continuous for the norm of $H(\partial_{x_3}, \Omega)$. The result holds by an extension argument. \square

From the density of $\mathcal{D}(\overline{\Omega})$ in $H(\partial_{x_3}, \Omega)$ and from Remark 6.1, the following Green's formula holds:

$$(6.2) \quad \forall u, v \in H(\partial_{x_3}, \Omega), \quad \int_{\Omega} u \frac{\partial v}{\partial x_3} dx = - \int_{\Omega} v \frac{\partial u}{\partial x_3} dx + \int_{\Gamma} u v n_3 d\sigma.$$

Remark 6.3. Let $u, v \in H(\partial_{x_3}, \Omega)$. Thanks to Remark 6.1 the functions $(u|_{\Gamma})_S$, $(u|_{\Gamma})_B$ and $(v|_{\Gamma})_S$, $(v|_{\Gamma})_B$ belong to $L^2(\omega)$. Consequently, one has by relation (2.2):

$$(6.3) \quad \int_{\Gamma} u v n_3 d\sigma = \int_{\omega} (u|_{\Gamma})_S (v|_{\Gamma})_S dx' - \int_{\omega} (u|_{\Gamma})_B (v|_{\Gamma})_B dx'.$$

To finish, one can adapt Lemma 3.5, with Remark 6.1, Green's formula (6.2) and relation (6.3), to prove the following result.

Lemma 6.4. *Let u in $H(\partial_{x_3}, \Omega)$. and set $\mathcal{G} = u|_{\Gamma}$. Then:*

- i) $M\left(\frac{\partial u}{\partial x_3}\right) = \mathcal{G}_S - \mathcal{G}_B$ in $L^2(\omega)$.
- ii) $F\left(\frac{\partial u}{\partial x_3}\right) = \widetilde{\mathcal{G}}_S - u$ in $L^2(\Omega)$.
- iii) $G\left(\frac{\partial u}{\partial x_3}\right) = u - \widetilde{\mathcal{G}}_B$ in $L^2(\Omega)$.

6.2. Lift operator of boundary values. To solve Problem (\mathcal{SH}) , one wishes to bring back to the homogeneous case of Section 5, where $\Phi = 0$ and $\mathbf{g} = 0$.

Theorem 6.5. *Let $\Phi \in L^2(\Omega)$, $\mathbf{g}' \in H^{1/2}(\Gamma)^2$ and $g_3 \in L^2(\Gamma, |n_3| d\sigma)$. Assume the following compatibility condition:*

$$(6.4) \quad \int_{\Gamma} \mathbf{g} \cdot \mathbf{n} d\sigma = \int_{\Omega} \Phi dx.$$

Then, there is \mathbf{u} in \mathbf{X} satisfying

$$(6.5) \quad \nabla \cdot \mathbf{u} = \Phi \text{ in } \Omega, \quad \mathbf{u}' = \mathbf{g}' \text{ in } \Gamma, \quad u_3 = g_3 \text{ in } \Gamma \text{ } |n_3| d\sigma - a.e.$$

Moreover, there is a constant $C > 0$ such that

$$(6.6) \quad \|\mathbf{u}\|_{\mathbf{X}} \leq C \left(\|\Phi\|_{L^2(\Omega)} + \|\mathbf{g}'\|_{H^{1/2}(\Gamma)^2} + \|g_3\|_{L^2(\Gamma, |n_3| d\sigma)} \right).$$

Remark 6.6. The compatibility condition (6.4) comes from the following Stokes formula:

$$\forall \mathbf{u} \in \mathbf{X}, \quad \int_{\Omega} \nabla \cdot \mathbf{u} dx = \int_{\Gamma} \mathbf{g} \cdot \mathbf{n} d\sigma,$$

which is satisfied thanks to Remark 6.3.

The proof of this theorem is straightforward, after having established the following lemma.

Lemma 6.7. *We keep the notations of Theorem 6.5, and we set*

$$(6.7) \quad \phi = M\Phi + (g_3)_B - (g_3)_S + \mathbf{g}'_B \cdot \nabla h \text{ in } \omega \quad \text{and} \quad U = \widetilde{(g_3)_S} - F\Phi \text{ in } \Omega.$$

1. Then $\phi \in L^2(\omega)$ and satisfies with \mathbf{g}' the compatibility condition

$$(6.8) \quad \int_{\omega} \phi dx' = \int_{\Gamma_L} \mathbf{g}' \cdot \mathbf{n}' d\sigma.$$

2. The function $\mathbf{u} \in \mathbf{X}$ satisfies (6.5) if and only if

$$(6.9) \quad \nabla' \cdot M\mathbf{u}' = \phi \text{ in } \omega,$$

with $u_3 = F(\nabla' \cdot \mathbf{u}') + U$ in Ω .

Proof. 1. Thanks to Proposition 2.3, Proposition 3.2 and Remark 6.1, one deduces that ϕ belongs to $L^2(\omega)$. Next, relation (6.8) is a direct application of relations (6.3) and (2.2).

2. Assume that $\nabla \cdot \mathbf{u} = \Phi$ and $u_3 = g_3$ in $L^2(\Gamma, |n_3| d\sigma)$. Then, Lemma 6.4 gives

$$(6.10) \quad M(\nabla' \cdot \mathbf{u}') = \phi - \mathbf{g}'_B \cdot \nabla h \text{ in } \omega \quad \text{and} \quad u_3 = F(\nabla' \cdot \mathbf{u}') + U \text{ in } \Omega.$$

Moreover, thanks to relation (3.1), one has $\nabla' \cdot M\mathbf{u}' = M(\nabla' \cdot \mathbf{u}') + \mathbf{g}'_B \cdot \nabla h$, which proves (6.9). Conversely, Proposition 3.4 ensures that

$$\frac{\partial u_3}{\partial x_3} = -\nabla' \cdot \mathbf{u}' + \Phi \quad \text{and} \quad \nabla \cdot \mathbf{u} = \Phi \text{ in } \Omega.$$

Finally, let us establish that $u_3 = g_3$ in $L^2(\Gamma, |n_3| d\sigma)$. Since $n_3 < 0$ over Γ_B , we deduce from Proposition 3.6 and Proposition 6.2 that

$$[F(\nabla' \cdot \mathbf{u}' - \Phi)]_{\Gamma_S} = 0, \quad [F(\nabla' \cdot \mathbf{u}' - \Phi)]_{\Gamma_B} = M(\nabla' \cdot \mathbf{u}' - \Phi) \quad \text{on } \omega.$$

As a consequence, one obtains, thanks to relation (6.1) and by definition of u_3 (6.10), for any $\mu \in L^2(\Gamma, |n_3| d\sigma)$:

$$\begin{aligned} \int_{\Gamma} u_3 \mu |n_3| d\sigma &= \int_{\omega} (u_3|_{\Gamma})_S \mu_S dx' + \int_{\omega} (u_3|_{\Gamma})_B \mu_B dx' \\ &= \int_{\omega} (g_3)_S \mu_S dx' + \int_{\omega} (g_3)_B \mu_B dx' + \int_{\omega} M(\nabla' \cdot \mathbf{u}' - \Phi) \mu_B dx'. \end{aligned}$$

Since $M(\nabla' \cdot \mathbf{u}' - \Phi) = (g_3)_B - (g_3)_S$, one gets

$$\begin{aligned} \int_{\Gamma} u_3 \mu |n_3| d\sigma &= \int_{\omega} (g_3)_S \mu_S dx' + \int_{\omega} (g_3)_B \mu_B dx' \\ &= \int_{\Gamma} g_3 \mu |n_3| d\sigma. \end{aligned}$$

As a consequence, $u_3 = g_3$ in $L^2(\Gamma, |n_3| d\sigma)$, which ends the proof. \square

Proof of Theorem 6.5. Let $\Phi \in L^2(\Omega)$, $\mathbf{g}' \in H^{1/2}(\Gamma)^2$, $g_3 \in L^2(\Gamma, |n_3| d\sigma)$ and (6.7). From what precedes and thanks to Remark 6.1, the function ϕ is exactly in the statement of Lemma 4.8, with

$$(6.11) \quad \|\phi\|_{L^2(\omega)} \leq C \left(\|\Phi\|_{L^2(\Omega)} + \|\mathbf{g}'\|_{H^{1/2}(\Gamma)^2} + \|g_3\|_{L^2(\Gamma, |n_3| d\sigma)} \right).$$

Therefore, there is $\mathbf{u}' \in H^1(\Omega)^2$ such that

$$\nabla' \cdot M\mathbf{u}' = \phi \text{ in } \omega, \quad \mathbf{u}' = \mathbf{g}' \text{ on } \Gamma,$$

and satisfying the following estimate

$$(6.12) \quad \|\mathbf{u}'\|_{H^1(\Omega)^2} \leq C \left(\|\mathbf{g}'\|_{H^{1/2}(\Gamma)^2} + \|\phi\|_{L^2(\omega)} \right).$$

Then, we set $u_3 = F(\nabla' \cdot \mathbf{u}') + U$. The vector field $\mathbf{u} = (\mathbf{u}', u_3)$ belongs to \mathbf{X} and satisfies (6.5) thanks to Lemma 6.7. Moreover, we deduce the estimates (6.6) thanks to Proposition 3.4, (6.11) and (6.12). \square

Therefore, Theorem 6.5 combined with Theorem 5.1 proves the main result stated in Theorem 1.1, that we recall below.

Theorem. *Assume (1.5). Let $\mathbf{f}' \in L^2(\Omega)^2$, $\Phi \in L^2(\Omega)$, $\mathbf{g}' \in H^{1/2}(\Gamma)^2$ and $g_3 \in L^2(\Gamma, |n_3| d\sigma)$. Assume the following compatibility condition:*

$$\int_{\Gamma} \mathbf{g} \cdot \mathbf{n} d\sigma = \int_{\Omega} \Phi dx.$$

Then, there is a unique pair $(\mathbf{u}, p) \in \mathbf{X} \times (L^2(\Omega)/\mathbb{R})$ solution to problem (\mathcal{SH}) . Moreover, there is a constant $C > 0$ depending only on Ω such that

$$\|\mathbf{u}\|_{\mathbf{X}} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leq C \left(\|\mathbf{f}'\|_{L^2(\Omega)^2} + \|\Phi\|_{L^2(\Omega)} + \|\mathbf{g}'\|_{H^{1/2}(\Gamma)^2} + \|g_3\|_{L^2(\Gamma, |n_3| d\sigma)} \right).$$

7. AUXILIARY RESULTS

The results we have established in the previous sections allow to prove, with no additional computation, the following corollaries. Firstly, thanks to Lemma 4.8 and Theorem 5.3, we solved the linear version of the Primitive Equations with non homogeneous Dirichlet conditions in an arbitrary domain Ω defined as in (1.1).

Corollary 7.1. *Let Ω as in (1.1). Let $\mathbf{f}' \in L^2(\Omega)^2$, $\phi \in L^2_{1/\sqrt{h}}(\omega)$ and $\mathbf{g}' \in H_h^{1/2}(\Gamma)^2$ (see (4.6)) such that*

$$\int_{\Gamma_L} \mathbf{g}' \cdot \mathbf{n}' d\sigma = \int_{\omega} \phi dx.$$

Then, there is a unique pair $(\mathbf{u}', p) \in H^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ such that

$$(7.1) \quad \begin{cases} -\Delta \mathbf{u}' + \nabla' p_S = \mathbf{f}' & \text{in } \Omega, \\ \nabla' \cdot M \mathbf{u}' = \phi & \text{in } \omega, \\ \mathbf{u}' = \mathbf{g}' & \text{on } \Gamma. \end{cases}$$

Moreover, there is a constant $C > 0$ depending only on Ω such that

$$\|\mathbf{u}'\|_{H^1(\Omega)^2} + \|p_S\|_{L^2_{\sqrt{h}}(\omega)/\mathbb{R}} \leq C \left(\|\mathbf{f}'\|_{L^2(\Omega)^2} + \|\phi\|_{L^2_{1/\sqrt{h}}(\omega)} + \|\mathbf{g}'\|_{H_h^{1/2}(\Gamma)^2} \right).$$

Secondly, we have proved that we can reduce (\mathcal{SH}) to an equivalent problem, when Ω is a cylindrical-type domain.

Corollary 7.2. *Assume (1.5). Let $\mathbf{f}' \in L^2(\Omega)^2$, $\Phi \in L^2(\Omega)$, $\mathbf{g}' \in H^{1/2}(\Gamma)^2$, and $g_3 \in L^2(\Gamma, |n_3| d\sigma)$. Assume the following compatibility condition:*

$$\int_{\Gamma} \mathbf{g}' \cdot \mathbf{n} d\sigma = \int_{\Omega} \Phi dx.$$

Next, let us set

$$\phi = M\Phi + (g_3)_B - (g_3)_S + \mathbf{g}'_B \cdot \nabla h + \mathbf{g}'_B \cdot \nabla h \quad \text{and} \quad U = \widetilde{(g_3)_S} - F\Phi.$$

1. Then $\phi \in L^2(\omega)$ satisfies with \mathbf{g}' the compatibility condition

$$\int_{\omega} \phi dx' = \int_{\Gamma_L} \mathbf{g}' \cdot \mathbf{n}' d\sigma.$$

2. Any pair $(\mathbf{u}, p) \in \mathbf{X} \times (L^2(\Omega)/\mathbb{R})$ is solution to (\mathcal{SH}) if and only if $(\mathbf{u}', p_S) \in H^1(\Omega)^2 \times (L^2(\omega)/\mathbb{R})$ is a solution to (7.1) with

$$x \in \Omega, \quad p(x) = \widetilde{p}_S(x'), \quad u_3(x) = F(\nabla' \cdot \mathbf{u}')(x) + U(x).$$

Finally, we can complete Proposition 6.2 about the trace of functions of $H(\partial_{x_3}, \Omega)$.

Corollary 7.3. *Assume (1.5). The mapping $\gamma : u \mapsto u|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$, can be extended in a unique and continuous way to a linear mapping, still denoted γ , from $H(\partial_{x_3}, \Omega)$ into $L^2(\Gamma, |n_3| d\sigma)$.*

Moreover for any $g \in L^2(\Gamma, |n_3| d\sigma)$, there is $u \in H(\partial_{x_3}, \Omega)$ such that $u = g$ in $L^2(\Gamma, |n_3| d\sigma)$ and

$$\|u\|_{H(\partial_{x_3}, \Omega)} \leq C \|g\|_{L^2(\Gamma, |n_3| d\sigma)},$$

where $C > 0$ is a depending at most on Ω .

8. ANNEXE

We keep the notations of Theorem 5.1. As we have explained it in the beginning of section 5, we give another approach to prove Theorem 5.1, consisting in considering the pair (\mathbf{u}, p) as the solution of an asymptotic problem. Let $\varepsilon \in]0, 1]$ and consider the problem:

Find $\mathbf{u}_\varepsilon \in H_0^1(\Omega)^3$ and $p_\varepsilon \in L^2(\Omega)/\mathbb{R}$ such that:

$$(8.1) \quad \begin{cases} -\Delta \mathbf{u}'_\varepsilon + \nabla' p_\varepsilon = \mathbf{f}', & -\varepsilon^2 \Delta u_3^\varepsilon + \frac{\partial p_\varepsilon}{\partial x_3} = 0, & \nabla \cdot \mathbf{u}_\varepsilon = 0 & \text{in } \Omega, \\ & & \mathbf{u}_\varepsilon = 0 & \text{on } \Gamma. \end{cases}$$

Theorem 8.1. *Let \mathbf{f}' in $H^{-1}(\Omega)^2$, let $\varepsilon \in]0, 1]$ and $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H_0^1(\Omega)^3 \times (L^2(\Omega)/\mathbb{R})$ the solution to Problem (8.1). Then, the sequence $(\mathbf{u}_\varepsilon, p_\varepsilon)_\varepsilon$ converges to the unique solution (\mathbf{u}, p) , in the space $\mathbf{X}_0 \times (L^2(\Omega)/\mathbb{R})$, of Problem (5.1), and satisfying (5.4).*

The sequel is dedicated to the proof of Theorem 8.1. Firstly, let us prove that Problem (5.1) has at most one solution.

Lemma 8.2. *For any $\mathbf{f}' \in H^{-1}(\Omega)^2$, Problem (5.1) has at most one solution (\mathbf{u}, p) in the space $\mathbf{X}_0 \times (L^2(\Omega)/\mathbb{R})$.*

Proof. Let us consider a possible solution (\mathbf{u}, p) related to the data $\mathbf{f}' = 0$. By multiplying the first equation of (5.1) by \mathbf{u}' , and by using Green's formula, one has

$$\int_\Omega \nabla \mathbf{u}' : \nabla \mathbf{u}' dx = \int_\Omega p \nabla' \cdot \mathbf{u}' dx = - \int_\Omega p \frac{\partial u_3}{\partial x_3} dx,$$

since $\nabla \cdot \mathbf{u} = 0$ in Ω . As $u_3 \in H_0(\partial_{x_3}, \Omega)$ and $\partial_{x_3} p = 0$ in Ω , relation (2.6) gives:

$$\int_{\Omega} \nabla \mathbf{u}' : \nabla \mathbf{u}' dx = \int_{\Omega} u_3 \frac{\partial p}{\partial x_3} dx = 0,$$

and, therefore, $\nabla \mathbf{u}' = 0$ in Ω . Since $\mathbf{u}' = 0$ on Γ and since Ω is connected, one has $\mathbf{u}' = 0$ in Ω . As $\nabla \cdot \mathbf{u} = 0$ in Ω , we deduce that $\partial_{x_3} u_3 = 0$ in Ω , and from the inequality (2.7) we get $u_3 = 0$ in Ω . Then, as $\nabla' p = \Delta \mathbf{u}' = 0$ in Ω , one obtains that $\nabla p = 0$ in Ω , hence $p = 0$ in $L^2(\Omega)/\mathbb{R}$, since we recall that Ω is connected. As a conclusion, Problem (5.1) has at most one solution when $\mathbf{f}' = 0$ which is $\mathbf{u} = 0$ and $p = 0$. Consequently, Problem (5.1) has at most one solution in $\mathbf{X}_0 \times (L^2(\Omega)/\mathbb{R})$. \square

Secondly, we give an asymptotic analysis of Problem (8.1). We prove in Lemma 8.3 that for any $\varepsilon \in]0, 1]$ there is a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon)$ to Problem (8.1). Then, we get interested in *a priori* estimates on the sequence $(\mathbf{u}_\varepsilon, p_\varepsilon)_\varepsilon$ in Lemma 8.4, enough, in a third time, to go through the limit in the equations (8.1) as ε goes to 0.

Lemma 8.3. *Let $\varepsilon \in]0, 1]$. For every \mathbf{f}' in $H^{-1}(\Omega)^2$, Problem (8.1) has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon)$ in the space $H_0^1(\Omega)^3 \times (L^2(\Omega)/\mathbb{R})$.*

Proof. For any solution $(\mathbf{u}_\varepsilon, p_\varepsilon)$ of (8.1), \mathbf{u}_ε satisfies the following variational problem:

$$(8.2) \quad \begin{cases} \text{Find } \mathbf{u}_\varepsilon \in \mathbf{V} \text{ such that:} \\ \int_{\Omega} \nabla \mathbf{u}'_\varepsilon : \nabla \mathbf{v}' dx + \varepsilon^2 \int_{\Omega} \nabla u_3^\varepsilon \cdot \nabla v_3 dx = \langle \mathbf{f}', \mathbf{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2}. \end{cases}$$

We refer to paragraph 2.3 for the definition of \mathbf{V} . Conversely, let us consider any solution $\mathbf{u}_\varepsilon \in \mathbf{V}$ to (8.2). Then, one has for any $\varphi \in \mathbf{V}$:

$$\langle \Delta \mathbf{u}'_\varepsilon + \mathbf{f}', \varphi' \rangle + \langle \varepsilon^2 \Delta u_3^\varepsilon, \varphi_3 \rangle = 0,$$

where the duality is taken in the sense of $H^{-1}(\Omega)^2$, $H_0^1(\Omega)^2$. Therefore, the distribution $\mathbf{T} = (\Delta \mathbf{u}'_\varepsilon + \mathbf{f}', \varepsilon^2 \Delta u_3^\varepsilon)$ is exactly in the statement of Lemma 2.5, hence there is a unique function p_ε in $L^2(\Omega)/\mathbb{R}$ such that

$$\nabla p_\varepsilon = \mathbf{T} \text{ in } \Omega.$$

As a consequence, any pair $(\mathbf{u}_\varepsilon, p_\varepsilon)$ in the space $H_0^1(\Omega)^3 \times (L^2(\Omega)/\mathbb{R})$ is a solution to (8.1) if and only if \mathbf{u}_ε satisfies (8.2). Therefore, we easily deduce, thanks to Lax-Milgram's lemma, that there is a unique \mathbf{u}_ε in \mathbf{V} satisfying (8.2). Hence, Problem (8.1) has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon)$ in the space $H_0^1(\Omega)^3 \times (L^2(\Omega)/\mathbb{R})$. \square

Lemma 8.4. *Let $\varepsilon \in]0, 1]$, let $\mathbf{f}' \in H^{-1}(\Omega)^2$, and $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H_0^1(\Omega)^3 \times (L^2(\Omega)/\mathbb{R})$ the solution to Problem (8.1). Then, there is a constant $C > 0$, independent on ε , such that:*

$$(8.3) \quad \|\mathbf{u}'_\varepsilon\|_{H^1(\Omega)^2} + \varepsilon \|u_3^\varepsilon\|_{H^1(\Omega)} + \|u_3^\varepsilon\|_{H(\partial_{x_3}, \Omega)} + \|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C \|\mathbf{f}'\|_{H^{-1}(\Omega)^2}.$$

Proof. By taking $\mathbf{v} = \mathbf{u}_\varepsilon$ in (8.2), we deduce that

$$\|\nabla \mathbf{u}'_\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla u_3^\varepsilon\|_{L^2(\Omega)}^2 \leq C \|\mathbf{f}'\|_{H^{-1}(\Omega)} \|\nabla \mathbf{u}'_\varepsilon\|_{L^2(\Omega)},$$

which leads to

$$(8.4) \quad \|\nabla \mathbf{u}'_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla u_3^\varepsilon\|_{L^2(\Omega)} \leq C \|\mathbf{f}'\|_{H^{-1}(\Omega)^2}.$$

Since $\nabla \cdot \mathbf{u}_\varepsilon = 0$ in Ω , we deduce thanks to (8.4) that

$$\left\| \frac{\partial u_3^\varepsilon}{\partial x_3} \right\|_{L^2(\Omega)} = \|\nabla' \cdot \mathbf{u}'_\varepsilon\|_{L^2(\Omega)} \leq C \|\mathbf{f}'\|_{H^{-1}(\Omega)^2}.$$

As u_3^ε belongs to $H_0(\partial_{x_3}, \Omega)$, Poincaré's inequality (2.7) and the above inequality imply that

$$\|u_3^\varepsilon\|_{L^2(\Omega)} \leq C \|\mathbf{f}'\|_{H^{-1}(\Omega)^2}.$$

Then, we are interested in the estimate of $\|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}}$. The gradient operator is an isomorphism from $L^2(\Omega)/\mathbb{R}$ onto \mathbf{V}° (see (4.4)). As a consequence, since $p_\varepsilon \in L^2(\Omega)/\mathbb{R}$, there is a constant $C > 0$ such that

$$\begin{aligned} \|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} &\leq C \|\nabla p_\varepsilon\|_{H^{-1}(\Omega)} \\ &\leq C \left(\|\mathbf{f}'\|_{H^{-1}(\Omega)^2} + \|\Delta \mathbf{u}'_\varepsilon\|_{H^{-1}(\Omega)^2} + \varepsilon^2 \|\Delta u_3^\varepsilon\|_{H^{-1}(\Omega)} \right) \\ &\leq C \left(\|\mathbf{f}'\|_{H^{-1}(\Omega)^2} + \|\nabla \mathbf{u}'_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla u_3^\varepsilon\|_{L^2(\Omega)} \right), \end{aligned}$$

by reading equations of (8.1). As a consequence, thanks to (8.4), $(p_\varepsilon)_\varepsilon$ is bounded in $L^2(\Omega)/\mathbb{R}$, and we get (8.3). \square

Remark 8.5. We keep the notations of Lemma 8.4. By writing the third equation of (8.1) in the following way

$$\frac{\partial p_\varepsilon}{\partial x_3} = \varepsilon^2 \Delta u_3^\varepsilon,$$

we deduce thanks to (8.4) that there is a constant $C > 0$ independent on ε such that

$$(8.5) \quad \left\| \frac{\partial p_\varepsilon}{\partial x_3} \right\|_{H^{-1}(\Omega)} \leq \varepsilon C.$$

Proof of Theorem 5.1. From Lemma 8.4, we proved that $(\mathbf{u}_\varepsilon, p_\varepsilon)$ is bounded in the space $\mathbf{X}_0 \times (L^2(\Omega)/\mathbb{R})$. Since it is a reflexive space, there is a pair (\mathbf{u}, p) in the space $\mathbf{X}_0 \times (L^2(\Omega)/\mathbb{R})$ and a subsequence of $(\mathbf{u}_\varepsilon, p_\varepsilon)$, denoted in the same way, that converges weakly towards the couple (\mathbf{u}, p) in the space $\mathbf{X}_0 \times (L^2(\Omega)/\mathbb{R})$. In particular, one has

$$-\Delta \mathbf{u}'_\varepsilon + \nabla' p_\varepsilon \rightharpoonup -\Delta \mathbf{u}' + \nabla' p = \mathbf{f}' \text{ in } H^{-1}(\Omega)^2.$$

Then we deduce from relation (8.5) that

$$\frac{\partial p_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial p}{\partial x_3} = 0 \text{ in } H^{-1}(\Omega).$$

To finish, one has

$$\nabla \cdot \mathbf{u}_\varepsilon \rightharpoonup \nabla \cdot \mathbf{u} = 0 \text{ in } L^2(\Omega).$$

As a conclusion, the weak limit (\mathbf{u}, p) is a solution to (5.1) and satisfies (5.4) by taking the infimum limit in (8.3). We achieve the proof by using Lemma 8.2, ensuring that (\mathbf{u}, p) is the unique solution to (5.1). Consequently all the sequence $(\mathbf{u}_\varepsilon, p_\varepsilon)$ converges to (\mathbf{u}, p) in $\mathbf{X}_0 \times (L^2(\Omega)/\mathbb{R})$. \square