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Weighted L^p's theory for a Stokes problem in a perturbed half-space and in an aperture domain

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Abstract The purpose of this work is to solve the Stokes problem with a Dirichlet boundary condition in a perturbed half-space and in an aperture domain, two unbounded geometries with noncompact boundaries. We study the existence and the uniqueness of generalized solutions in weighted L^p's theory with $1 < p < \infty$. We study too the case of strong solutions and very weak solutions.
Key words Weighted Sobolev spaces ; Stokes operator ; Dirichlet boundary condition ; Aperture domain ; Perturbed half-space.

1 Introduction and preliminaries

Many problems in fluid dynamics, such as flows past obstacles, around corners or through pipes or apertures, are first conceptualized by Stokes or Navier-Stokes equations in unbounded domains. In a previous paper ([4]), we have solved the Stokes system in a particular unbounded domain, namely an exterior domain in the half-space. We have given results in weighted Sobolev spaces which are well-adapted to these problems because they satisfy an optimal Poincaré-type inequality. Here, we want to study the Stokes system in two other types of unbounded geometry. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, an unbounded domain with a noncompact and sufficienly smooth boundary $\Gamma = \partial\Omega$. Either

- Ω is a perturbed half-space, *i.e.* it is obtained by an arbitrary modification of the half-space

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_n > 0\},$$

or

- Ω is an aperture domain, *i.e.* Ω consists of two half-spaces separated by some wall of thickness $d > 0$ and connected by some hole (aperture).

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We can remind that the aperture domain became interesting since Heywood ([17], [18], in the Hilbert case) found the important role of the flux condition

$$\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma$$

as an additional boundary condition in order to get uniqueness for the Stokes problem with a Dirichlet boundary condition (see also [9], [20], [23] or [24]). Here, we denote by M a $(n-1)$-dimensionnal smooth manifold in the hole dividing Ω in an upper domain Ω_+ and a lower domain Ω_- such that $\Omega = \Omega_+ \cup \Omega_- \cup M$ is a disjoint decomposition and $M = \partial\Omega_+ \cap \partial\Omega_-$. Further, $\boldsymbol{n}$ is the unit normal vector on M directed to Ω_- and $\boldsymbol{u} \cdot \boldsymbol{n}$ the normal component of the velocity field $\boldsymbol{u}$.

To study a Stokes problem, several families of spaces are used, like the completion of $\mathcal{D}(\Omega)$ for the norm of the gradient in $L^p(\Omega)$ (in [17] and [13] for instance), which has the inconvenient that, when $p \geq n$ and $\Omega = \mathbb{R}^n$, some very treacherous Cauchy sequences exist in $\mathcal{D}(\mathbb{R}^n)$ that do not converge to distributions, a behaviour carefully described in 1954 by Deny and Lions [11]. An other family of spaces is the subspace in $L^p_{loc}(\Omega)$ of functions whose gradients belong to $L^p(\Omega)$ (see [12]) which have an imprecision at infinity inherent to the L^p_{loc} norm. We have chosen to go on working with the weighted Sobolev spaces because these spaces allow us to describe the behaviour of functions and not just of their gradient, which is vital from the mathematical and the numerical point of view. Moreover, on such geometries, we notice that even the boundary is not bounded. So, we have to introduce weights even in the spaces of traces. We can cite Hanouzet [19] who has given the first results for such spaces in 1971 and Amrouche, Nečasová [7] who have extended these results in 2001 to weighted Sobolev spaces which possess logarithmic weights (we just remind that logarithmic weights allow us to have a Poincaré-type inequality even in the "critical" cases ; see below for more details). Let us quote too Maz'ya-Plamenevskii-Stupyalis [22] and Amrouche, Nečasová and Raudin [8] who have solved Stokes systems in weighted Sobolev spaces with noncompact boundary.

In this paper, we state that we will concentrate only on the basic weights for the sake of simplicity and because they are the most usual.

Now, we give a precise definition of a perturbed half-space. Let Ω an open and connected domain such that

$$\mathbb{R}^n_+ \subset \Omega \subset \mathbb{R}^n.$$

There exists an open ball $B \subset \mathbb{R}^n$ such that

$$\Omega \cup B = \mathbb{R}^n_+ \cup B.$$

We set $\Gamma = \partial\Omega$ the boundary of Ω that we suppose of class $C^{1,1}$; then, we can choose some bounded subdomain $G \subset \Omega$ with boundary ∂G of class $C^{1,1}$ such that $\Omega \cap B \subset G$. The ball B can be chosen centered on the origin and sufficiently large so that there exists another ball B_0 centered on the origin with closure $\overline{B}_0 \subset B$ such that

$$\Omega \cup B_0 = \mathbb{R}^n_+ \cup B_0.$$

Then, we define the following domains : $\Sigma = \Gamma \cap \mathbb{R}^{n-1}$, $D = \mathbb{R}^{n-1} \setminus \Sigma$ and $S = \Gamma \setminus \Sigma$. Moreover, we define also $\omega = \Omega \setminus \overline{\mathbb{R}^n_+}$ (we notice that $\partial\omega = D \cup S$) and ω' the symmetric region of ω with respect to $\mathbb{R}^{n-1}$. Finally, we choose B_0 and B sufficiently large so that they satisfy $\omega' \subset B_0 \subset B$ and we introduce the following partition of the unity

$$\psi_1,\ \psi_2 \in C^\infty(\mathbb{R}^n),\ 0 \leq \psi_1, \psi_2 \leq 1,\ \psi_1 + \psi_2 = 1 \text{ in } \mathbb{R}^n,$$
$$\psi_1 = 1 \text{ in } B_0, \quad \text{supp } \psi_1 \subset B.$$

Next, we give the definition of an aperture domain. For this, we set $d \in \mathbb{R}^+_*$ and $\mathbb{R}^n_{-d} = \{\boldsymbol{x} \in \mathbb{R}^n, x_n < -d\}$. Let Ω an open domain such that

$$\mathbb{R}^n_+ \cup \mathbb{R}^n_{-d} \subset \Omega \subset \mathbb{R}^n.$$

There exists an open ball $B \subset \mathbb{R}^n$ such that

$$\Omega \cup B = \mathbb{R}^n_+ \cup \mathbb{R}^n_{-d} \cup B.$$

We set $\Gamma = \partial\Omega$ the boundary of Ω that we suppose of class $C^{1,1}$; then, we can choose some bounded subdomain $G \subset \Omega$ with boundary ∂G of class $C^{1,1}$ such that $\Omega \cap B \subset G$. The ball B can be chosen centered on $(0, \ldots, 0, -\frac{d}{2})$ and sufficiently large so that there exists another ball B_0 centered on $(0, \ldots, 0, -\frac{d}{2})$ with closure $\overline{B}_0 \subset B$ such that

$$\Omega \cup B_0 = \mathbb{R}^n_+ \cup \mathbb{R}^n_{-d} \cup B_0.$$

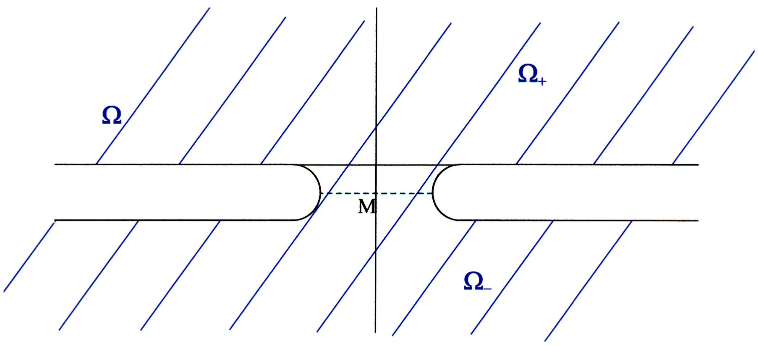
We define two disjoints subdomains Ω_+ and Ω_- of Ω and an $(n-1)$-dimensional smooth manifold M with the following properties :

$$\Omega = \Omega_+ \cup \Omega_- \cup M, \qquad M = \partial\Omega_+ \cap \partial\Omega_-,$$

and

$$\Omega_+ \cup B = \mathbb{R}^n_+ \cup B, \qquad \Omega_- \cup B = \mathbb{R}^n_{-d} \cup B.$$

We can notice that Ω_+ and Ω_- are perturbed half-spaces.



Finally, we define here again the following partition of unity :

$$\psi_1,\ \psi_2 \in C^\infty(\mathbb{R}^n),\ 0 \leq \psi_1, \psi_2 \leq 1,\ \psi_1 + \psi_2 = 1 \text{ in } \mathbb{R}^n,$$
$$\psi_1 = 1 \text{ in } B_0, \quad \text{supp } \psi_1 \subset B.$$

We complete this introduction with a short review of the weighted Sobolev spaces and their trace spaces. In all this article, we suppose that the dimension n is greater than or equal to 3. For any integer q we denote by $\mathcal{P}_q$ the space of polynomials in n variables, of degree less than or equal to q, with the convention that $\mathcal{P}_q$ is reduced to $\{0\}$ when q is negative.
For any real number $p \in]1, +\infty[$, we denote by p' the dual exponent of p :

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be a typical point of $\mathbb{R}^n$, $\boldsymbol{x'} = (x_1, \dots, x_{n-1})$ and let $r = |\boldsymbol{x}| = (x_1^2 + \cdots + x_n^2)^{1/2}$ denote its distance to the origin. We shall use two basic weights :

$$\rho(r) = (1 + r^2)^{1/2} \quad \text{and} \quad lg\ r = ln(2 + r^2).$$

As usual, $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions with compact support in Ω, $\mathcal{D}'(\Omega)$ its dual space, called the space of distributions and $\mathcal{D}(\overline{\Omega})$ the space of restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^n)$. We define also

$$L^p_{loc}(\overline{\Omega}) = \{u, \text{ for any compact } K \subset \overline{\Omega},\ u \in L^p(K)\}.$$

Then, for any integers $n \geq 3$ and $m \geq 0$ and real numbers $p > 1$ and α, setting

$$k = k(m, n, p, \alpha) = \begin{cases} -1 & \text{if } \dfrac{n}{p} + \alpha \notin \{1, \dots, m\}, \\ \\ m - \dfrac{n}{p} - \alpha & \text{if } \dfrac{n}{p} + \alpha \in \{1, \dots, m\}, \end{cases}$$

we define the following space :

$$\begin{aligned} W^{m,p}_\alpha(\Omega) = \{u \in \mathcal{D}'(\Omega);& \\ &\forall \lambda \in \mathbb{N}^n : 0 \leqslant |\lambda| \leqslant k,\ \rho^{\alpha - m + |\lambda|}(lg\ r)^{-1} D^\lambda u \in L^p(\Omega); \\ &\forall \lambda \in \mathbb{N}^n : k + 1 \leqslant |\lambda| \leqslant m,\ \rho^{\alpha - m + |\lambda|} D^\lambda u \in L^p(\Omega)\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm :

$$\begin{aligned} \|u\|_{W^{m,p}_\alpha(\Omega)} = (&\sum_{0 \leqslant |\lambda| \leqslant k} \|\rho^{\alpha - m + |\lambda|}(lg\ r)^{-1} D^\lambda u\|^p_{L^p(\Omega)} \\ &+ \sum_{k+1 \leqslant |\lambda| \leqslant m} \|\rho^{\alpha - m + |\lambda|} D^\lambda u\|^p_{L^p(\Omega)})^{1/p}. \end{aligned}$$

We also define the semi-norm :

$$|u|_{W^{m,p}_\alpha(\Omega)} = (\sum_{|\lambda| = m} \|\rho^\alpha D^\lambda u\|^p_{L^p(\Omega)})^{1/p}.$$

<u>Remark</u> : In this paper, we will work sometimes in classical Sobolev spaces. Let us remind the notations of these spaces that we will use in the sequel (see [10]

for more informations). We define classical Sobolev spaces for any nonnegative integers n and m and real numbers $p > 1$ setting :

$$W^{m,p}(\Omega) = \{u \in \mathcal{D}'(\Omega);\ \forall\lambda \in \mathbb{N}^n,\ |\lambda| \leq m,\ D^\lambda u \in L^p(\Omega)\}.$$

We equipped this space with its natural norm. When $m = 1$ and $p = 2$, we set

$$H^1(\Omega) = W^{1,2}(\Omega),$$

then $H^{\frac{1}{2}}(\Gamma)$ is the space of traces of functions in $H^1(\Omega)$ and $H_0^1(\Omega)$ is the subspace of functions in $H^1(\Omega)$ whose the trace is equal to zero on Γ. $\square$

The weights defined previously are chosen so that that the space $\mathcal{D}(\overline{\Omega})$ is dense in $W_\alpha^{m,p}(\Omega)$ and so that the following Poincaré-type inequalities hold in the following spaces :

Theorem 1.1. *Let α be a real number, $m \geq 1$ an integer and $q' = \min(q, m-1)$, where q is the highest degree of the polynomials contained in $W_\alpha^{m,p}(\Omega)$. Then :*
i) the semi-norm $|\ .\ |_{W_\alpha^{m,p}(\Omega)}$ defined on $W_\alpha^{m,p}(\Omega)/\mathcal{P}_{q'}$ is a norm equivalent to the quotient norm.
ii) the semi-norm $|\ .\ |_{W_\alpha^{m,p}(\Omega)}$ is a norm on $\overset{\circ}{W}{}_\alpha^{m,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W_\alpha^{m,p}(\Omega)}}$, which is equivalent to the full norm $\|\ .\ \|_{W_\alpha^{m,p}(\Omega)}$.

We prove this theorem using the previous partitions of unity and results in bounded domains [21] and in the half-space [7]. We denote $W_{-\alpha}^{-m,p'}(\Omega)$ the dual space of $\overset{\circ}{W}{}_\alpha^{m,p}(\Omega)$ and we notice that it is a space of distributions.

Now, we define the traces of functions of $W_\alpha^{m,p}(\Omega)$. For any $\sigma \in\ ]0,1[$, we set

$$\omega_1 = \begin{cases} \rho & \text{if } \dfrac{n}{p} \neq \sigma, \\ \rho(lg\rho)^{1/\sigma} & \text{if } \dfrac{n}{p} = \sigma. \end{cases}$$

If Ω is a perturbed half-space, we define the space

$$W_0^{\sigma,p}(\Gamma) = \{u, \omega_1^{-\sigma} u \in L^p(\Sigma), u \in L^p(S), \int_{\Gamma\times\Gamma} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}}\ d\boldsymbol{x}d\boldsymbol{y} < \infty\}.$$

It is a reflexive Banach space equipped with its natural norm

$$(\|\frac{u}{\omega_1^\sigma}\|_{L^p(\Sigma)}^p + \|u\|_{L^p(S)}^p + \int_{\Gamma\times\Gamma} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}}\ d\boldsymbol{x}d\boldsymbol{y})^{1/p}.$$

If Ω is an aperture domain, we define the space

$$\begin{aligned} W_0^{\sigma,p}(\Gamma) = \{u, \omega_1^{-\sigma} u \in & L^p(\Gamma \cap\ {}^cB_0), u \in L^p(\Gamma \cap B_0), \\ & \int_{\Gamma\times\Gamma} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}}\ d\boldsymbol{x}d\boldsymbol{y} < \infty\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm

$$(\|\frac{u}{\omega_1^\sigma}\|_{L^p(\Gamma\cap\ {}^cB_0)}^p + \|u\|_{L^p(\Gamma\cap B_0)}^p + \int_{\Gamma\times\Gamma} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}}\ d\boldsymbol{x}d\boldsymbol{y})^{1/p}.$$

Now, when Ω is either a perturbed half-space or an aperture domain, for any $s \in \mathbb{R}^+$, we set

$$W_0^{s,p}(\Gamma) = \{u \in W_{[s]-s}^{[s],p}(\Gamma),\ \forall |\lambda| = [s],\ D^\lambda u \in W_0^{s-[s],p}(\Gamma)\}.$$

It is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W_0^{s,p}(\Gamma)} = \|u\|_{W_{[s]-s}^{[s],p}(\Gamma)} + \sum_{|\lambda|=s} \|D^\lambda u\|_{W_0^{s-[s],p}(\Gamma)}.$$

Then, for any $s \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, we set

$$W_\alpha^{s,p}(\Gamma) = \{u \in W_{[s]+\alpha-s}^{[s],p}(\Gamma),\ \forall |\lambda| = [s],\ \rho^\alpha D^\lambda u \in W_0^{s-[s],p}(\Gamma)\}.$$

Next, in both cases, we have the following traces lemma :

Lemma 1.2. *For any integer $m \geq 1$ and real number α, we define the mapping*

$$\begin{aligned} \gamma : \mathcal{D}(\overline{\Omega}) \to\ & (\mathcal{D}(\Gamma))^m \\ u \mapsto\ & (\gamma_0 u, \dots, \gamma_{m-1} u) \end{aligned}$$

where for any $k = 0, \dots, m-1$, $\gamma_k u = \dfrac{\partial^k u}{\partial \boldsymbol{n}^k}$. Then, γ can be extended by continuity to a linear and continuous mapping still denoted by γ from $W_\alpha^{m,p}(\Omega)$ to $\prod_{j=0}^{m-1} W_\alpha^{m-j-\frac{1}{p},p}(\Gamma)$. Moreover, γ is onto and

$$Ker\ \gamma = \overset{\circ}{W}{}_\alpha^{m,p}(\Omega).$$

Proof- We prove first this lemma for the perturbed half-space and in the basic case of a function in $W_0^{1,p}(\Omega)$, the generalization being obvious.

i) First, let u be in $\mathcal{D}(\overline{\Omega})$. We set $u_i = \psi_i u$ for $i = 1, 2$ and we have

$$\|\gamma u_2\|_{W_0^{1-\frac{1}{p},p}(\Gamma)} = \|\gamma u_2\|_{W_0^{1-\frac{1}{p},p}(\Sigma)} \leq\ C\ \|u_2\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq\ C\ \|u\|_{W_0^{1,p}(\Omega)},$$

because $\gamma u_2 = 0$ on S, $\Sigma \subset \mathbb{R}^{n-1}$ and that γ is continuous from $W_0^{1,p}(\mathbb{R}^n_+)$ to $W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ (see [7]). Next, noticing that $\gamma u_1 = 0$ on $\Gamma \cap {}^cB$, that $\Gamma \cap B \subset \partial G$ and that, thanks to results in bounded domains, γ is continuous from $W^{1,p}(G)$ to $W^{1-\frac{1}{p},p}(\partial G)$, we have

$$\|\gamma u_1\|_{W_0^{1-\frac{1}{p},p}(\Gamma)} \leq\ C\ \|\gamma u_1\|_{W^{1-\frac{1}{p},p}(\Gamma \cap B)} \leq\ C\ \|u_1\|_{W^{1,p}(G)} \leq\ C\ \|u\|_{W_0^{1,p}(\Omega)}.$$

Finally, we deduce from this, by density, that γ can be extended by continuity to a linear and continuous mapping from $W_0^{1,p}(\Omega)$ to $W_0^{1-\frac{1}{p},p}(\Gamma)$.

ii) Now, we want to show that γ is onto. Let g be in $W_0^{1-\frac{1}{p},p}(\Gamma)$. We set $g_i = \psi_i g$, $i = 1, 2$ and

$$\widetilde{g_2} = g_2 \text{ on } \Sigma, \quad \widetilde{g_2} = 0 \text{ on } D.$$

We have $\widetilde{g_2} \in W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ and there exists (see [7]) $v \in W_0^{1,p}(\mathbb{R}^n_+)$ such that $v = \widetilde{g_2}$ on $\mathbb{R}^{n-1}$. We define the function $\widetilde{v}$ by $\widetilde{v} = v$ in $\mathbb{R}^n_+$ and $\widetilde{v} = 0$ in ω. Then, $\widetilde{v} \in W_0^{1,p}(\Omega)$ and $\widetilde{v} = g_2$ on Γ. We set also

$$\widetilde{g_1} = g_1 \text{ on } \partial G \cap \Gamma, \quad \widetilde{g_1} = 0 \text{ on } \partial G \cap \mathbb{R}^n_+.$$

We have $\widetilde{g_1} \in W^{1-\frac{1}{p},p}(\partial G)$ and there exists, thanks to results in bounded domains, $w \in W^{1,p}(G)$ such that $w = \widetilde{g_1}$ on ∂G. We define the function $\widetilde{w}$ by $\widetilde{w} = w$ in G and $\widetilde{w} = 0$ in $\Omega \setminus G$. Then, $\widetilde{w} \in W_0^{1,p}(\Omega)$ and $\widetilde{w} = g_1$ sur Γ. Consequently, there exists $u = \widetilde{v} + \widetilde{w} \in W_0^{1,p}(\Omega)$ such that $u = g$ on Γ. So, γ is onto.

iii) Finally, it remains to show that Ker $\gamma = \overset{\circ}{W}{}_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W_0^{1,p}(\Omega)}}$. Let u be in $\overset{\circ}{W}{}_0^{1,p}(\Omega)$. Then, there exists a sequence $(\varphi_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ such that

$$\|\gamma(u - \varphi_\ell)\|_{W_0^{1-\frac{1}{p},p}(\Gamma)} \leq C\ \|u - \varphi_\ell\|_{W_0^{1,p}(\Omega)} \to 0.$$

But $\gamma\varphi_\ell = 0$ for any $\ell \in \mathbb{N}$ because $\varphi_\ell \in \mathcal{D}(\Omega)$, so, $\gamma u = 0$ in $W_0^{1-\frac{1}{p},p}(\Gamma)$ and $u \in$ Ker γ. Conversely, let u be in Ker γ, $u_i = \psi_i u$ for $i = 1, 2$. We have $u_2 = 0$ on $\mathbb{R}^{n-1}$ so $u_2 \in \overset{\circ}{W}{}_0^{1,p}(\mathbb{R}^n_+)$ (see [7]). Thus, there exists $(\varphi_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n_+) \subset \mathcal{D}(\Omega)$ such that $\|u_2 - \varphi_\ell\|_{W_0^{1,p}(\mathbb{R}^n_+)} \to 0$. Moreover, since, for any $\ell \in \mathbb{N}$, $u_2 = \varphi_\ell = 0$ in ω, we deduce that $\|u_2 - \varphi_\ell\|_{W_0^{1,p}(\Omega)} \to 0$, *i.e* $u_2 \in \overset{\circ}{W}{}_0^{1,p}(\Omega)$. With the same idea, $u_1 = 0$ on ∂G so $u_1 \in \overset{\circ}{W}{}^{1,p}(G)$ (see [10]). Thus, there exists $(\psi_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{D}(G) \subset \mathcal{D}(\Omega)$ such that $\|u_1 - \psi_\ell\|_{W^{1,p}(G)} \to 0$. Moreover, since for any $\ell \in \mathbb{N}$, $u_1 = \varphi_\ell = 0$ in $\Omega \setminus G$, we have $\|u_1 - \varphi_\ell\|_{W_0^{1,p}(\Omega)} \to 0$, *i.e* $u_1 \in \overset{\circ}{W}{}_0^{1,p}(\Omega)$. Consequently, $u = u_1 + u_2 \in \overset{\circ}{W}{}_0^{1,p}(\Omega)$ and Ker $\gamma = \overset{\circ}{W}{}_0^{1,p}(\Omega)$.

iv) To prove this lemma in the case of an aperture domain, we use the same kinds of arguments. First, for the continuity, we work like previously, in an unbounded domain and next in a bounded domain. For the surjectivity, we easily prove, using results in half-spaces, that we can find $u \in W_0^{1,p}(\Omega)$ such that $u = g_2$ on Γ and then, we study the bounded part like for the perturbed half-space. For the characterization of the kernel, the bounded part is again the same as the previous case and for the unbounded part, we find two sequences, the first one, C^∞ with compact support in $\mathbb{R}^n_+$ (so in Ω) and the second one in $\mathbb{R}^n_-$ (so in Ω) and we work with the sum of these two sequences. $\square$

For θ, any open subset sufficiently smooth of $\mathbb{R}^n$, we set

$$< .,. >_\theta = < .,. >_{W_0^{-1,p}(\theta), \overset{\circ}{W}{}_0^{1,p'}(\theta)},$$

and

$$< .,. >_{\partial\theta} = < .,. >_{W_0^{-\frac{1}{p},p}(\partial\theta), W_0^{1-\frac{1}{p'},p'}(\partial\theta)}.$$

In this article, C will denote a positive and real constant which may vary from line to line. We remind that we use only the basic weights and that we have supposed that Ω is of class $C^{1,1}$.

<u>*Remark*</u> : In this paper, we avoid the case $n = 2$. Indeed, when $n = 2$, we are not able to establish Lemma 2.1 (and consequently any result of this work) because of a condition of compatibility which is not satisfied.

2 Stokes system in a perturbed half-space

We remind that we suppose that $n \geq 3$. Let $p \in]1, +\infty[$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$. We consider here the problem $(\mathcal{S}_1)$: find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$(\mathcal{S}_1)\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma. \end{cases}$$

2.1 Case $p = 2$.

Thanks to Lemma 1.2, we notice first that there exists $\boldsymbol{u_g} \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that $\boldsymbol{u_g} = \boldsymbol{g}$ on Γ and satisfying

$$\|\boldsymbol{u_g}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C \, \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)}.$$

So, it is equivalent to solve the problem with homogeneous boudary conditions : for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and $h \in L^2(\Omega)$, find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ such that

$$(\mathcal{S}_{10})\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \end{cases}$$

Now, we want to establish Proposition 2.2 to lift the data for the divergence. For this, we use this preliminary lemma :

Lemma 2.1. *For any h in $L^2(\Omega)$, there exists $u \in W_0^{2,2}(\Omega)$ solution of*

$$\Delta u = h \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma. \tag{1}$$

Moreover, u satisfies

$$\|u\|_{W_0^{2,2}(\Omega)} \leq C \, \|h\|_{L^2(\Omega)},$$

where C is a real positive constant which depends only on Ω.

Proof- Let h be in $L^2(\Omega)$. We set $h_1 = \psi_1 h \in L^2(\Omega)$ and $h_2 = \psi_2 h \in L^2(\Omega)$.

i) By Theorem 3.1 in [3], there exists $v \in W_0^{2,2}(\mathbb{R}^n_+)$ such that

$$\Delta v = h_2 \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

and satisfying

$$\|v\|_{W_0^{2,2}(\mathbb{R}^n_+)} \leq C \, \|h\|_{L^2(\Omega)}.$$

We set for almost all $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$:

$$v_*(\boldsymbol{x'}, x_n) = v(\boldsymbol{x'}, x_n) \text{ if } x_n > 0, \quad v_*(\boldsymbol{x'}, x_n) = v(\boldsymbol{x'}, -x_n) \text{ if } x_n < 0.$$

It is clear that $v_* \in W_0^{2,2}(\mathbb{R}^n)$ and that $\|v_*\|_{W_0^{2,2}(\mathbb{R}^n)} \leq\ C\ \|h\|_{L^2(\Omega)}$. Moreover, we easily show that

$$\Delta v_* = h_2 \text{ in } \Omega, \tag{2}$$

and that $\dfrac{\partial v_*}{\partial \boldsymbol{n}} \in W_0^{\frac{1}{2},2}(\Gamma)$ has its support included in S.

ii) Now, we want to find $w \in W_0^{2,2}(\Omega)$ solution of

$$\Delta w = h_1 \text{ in } \Omega \quad \text{and} \quad \frac{\partial w}{\partial \boldsymbol{n}} = -\frac{\partial v_*}{\partial \boldsymbol{n}} \text{ on } \Gamma, \tag{3}$$

satisfying

$$\|w\|_{W_0^{2,2}(\Omega)} \leq\ C\ \|h\|_{L^2(\Omega)}. \tag{4}$$

Since $h_1 \in L^2(\Omega)$ has a compact support, we have $h_1 \in W_0^{-1,2}(\Omega)$ and it is reasonable to search first a solution $w \in W_0^{1,2}(\Omega)$. For this, we observe that (3) is equivalent to the following variational formulation : find $w \in W_0^{1,2}(\Omega)$ such that for any $z \in W_0^{1,2}(\Omega)$

$$(\mathcal{FV}) \qquad \int_\Omega \nabla w \cdot \nabla z\ d\boldsymbol{x} = \int_\Omega h_1 z\ d\boldsymbol{x} - < \frac{\partial v_*}{\partial \boldsymbol{n}}, z >_\Gamma .$$

Then, applying the theorem of Lax-Milgram in $(W_0^{1,2}(\Omega), \|\cdot\|_{W_0^{1,2}(\Omega)})$ (since $n > 2$, the coercivity is satisfied by the point **i)** of Theorem 1.1), we deduce that there exists a unique solution of $(\mathcal{FV})$.

Next, we prove that w is in $W_0^{2,2}(\Omega)$. For this, we set $w_1 = \psi_1 w \in W_0^{1,2}(\Omega)$ and $w_2 = \psi_2 w \in W_0^{1,2}(\Omega)$. Since supp $w_1 \subset G$ and $\Delta w = h_1 \in L^2(\Omega)$, we have

$$\Delta w_1 = w \Delta \psi_1 + 2 \nabla \psi_1 \cdot \nabla w + \psi_1 h_1\ \in L^2(G),$$

and since $\psi_1 = \dfrac{\partial \psi_1}{\partial \boldsymbol{n}} = 0$ on $\partial G \cap \mathbb{R}^n_+$ and $\dfrac{\partial w}{\partial \boldsymbol{n}} = -\dfrac{\partial v_*}{\partial \boldsymbol{n}} \in H^{\frac{1}{2}}(\partial G \cap \Gamma)$, we have

$$\frac{\partial w_1}{\partial \boldsymbol{n}} = \psi_1 \frac{\partial w}{\partial \boldsymbol{n}} + \frac{\partial \psi_1}{\partial \boldsymbol{n}} w\ \in H^{\frac{1}{2}}(\partial G).$$

Thanks to regularity results in bounded domains (see [21] when the boundary is very smooth and the technique of Grisvard [16] for the extension to a $C^{1,1}$ boundary), we deduce from this that $w_1 \in H^2(G)$ and since supp $w_1 \subset G$, then $w_1 \in W_0^{2,2}(\Omega)$. Thus, in $\mathbb{R}^n_+$ $\Delta w_2 \in L^2(\mathbb{R}^n_+)$ and $\dfrac{\partial w_2}{\partial \boldsymbol{n}} \in W_0^{\frac{1}{2},2}(\mathbb{R}^{n-1})$. We easily conclude thanks to Corollary 3.3 of [3] that $w_2 \in W_0^{2,2}(\mathbb{R}^n_+)$ and since supp $w_2 \subset \mathbb{R}^n_+$, $w_2 \in W_0^{2,2}(\Omega)$. So, $w = w_1 + w_2 \in W_0^{2,2}(\Omega)$, satisfies (3) and the estimate (4). Finally, from (2) and (3), $u = v_* + w$ is solution of (1) and the corresponding estimate follows immediately. $\square$

Proposition 2.2. *We suppose here that Ω is only Lipschitz-continuous. So, for any $h \in L^2(\Omega)$, there exists $\boldsymbol{w} \in \overset{\circ}{\boldsymbol{W}}{}^{1,2}_0(\Omega)$ satisfying*

$$\operatorname{div} \boldsymbol{w} = h \text{ in } \Omega, \quad \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,2}_0(\Omega)} \leq C\ \|h\|_{L^2(\Omega)},$$

where C is a real positive constant depending only on Ω.

Proof- 1) In a first time, we suppose that Ω is of class $C^{1,1}$. Let h be in $L^2(\Omega)$. We know, thanks to the previous lemma, that there exists $\varphi \in W^{2,2}_0(\Omega)$ solution of (1) with the corresponding estimate. We set $\boldsymbol{v} = \nabla\varphi \in \boldsymbol{W}^{1,2}_0(\Omega)$, $\boldsymbol{g} = \boldsymbol{v}_{|\Gamma}$, $\boldsymbol{g_1} = \psi_1\boldsymbol{g}$, $\boldsymbol{g_2} = \psi_2\boldsymbol{g}$ and we notice that $\boldsymbol{g}$, $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ belong to $\boldsymbol{W}^{\frac{1}{2},2}_0(\Gamma)$.

i) First, we want to solve the following problem : find $\boldsymbol{t} \in \boldsymbol{W}^{1,2}_0(\Omega)$ such that

$$\operatorname{div} \boldsymbol{t} = 0 \text{ in } \Omega \quad \text{and} \quad \boldsymbol{t} = \boldsymbol{g_2} \text{ on } \Gamma. \tag{5}$$

We define the function $\widetilde{\boldsymbol{g_2}}$ by $\widetilde{\boldsymbol{g_2}} = \boldsymbol{g_2}$ on Σ and $\widetilde{\boldsymbol{g_2}} = \boldsymbol{0}$ on D. Noticing that supp $\boldsymbol{g_2} \subset \Sigma$, we have $\widetilde{\boldsymbol{g_2}} \in \boldsymbol{W}^{\frac{1}{2},2}_0(\mathbb{R}^{n-1})$. Moreover, we know, thanks to results in the half-space, that there exists $\boldsymbol{u} \in \boldsymbol{W}^{1,2}_0(\mathbb{R}^n_+)$ such that

$$\operatorname{div} \boldsymbol{u} = 0 \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad \boldsymbol{u} = \widetilde{\boldsymbol{g_2}} \text{ on } \mathbb{R}^{n-1}$$

and satisfying the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,2}_0(\mathbb{R}^n_+)} \leq\ C\ \|\widetilde{\boldsymbol{g_2}}\|_{\boldsymbol{W}^{\frac{1}{2},2}_0(\mathbb{R}^{n-1})} \leq\ C\ \|h\|_{L^2(\Omega)}.$$

We define the function $\boldsymbol{t}$ by $\boldsymbol{t} = \boldsymbol{u}$ in $\mathbb{R}^n_+$ and $\boldsymbol{t} = \boldsymbol{0}$ in $\overline{\omega}$. We easily check that $\boldsymbol{t} \in \boldsymbol{W}^{1,2}_0(\Omega)$ and that div $\boldsymbol{t} = 0$ in Ω. Thus, since on Σ, $\boldsymbol{t} = \boldsymbol{g_2}$ and on S, $\boldsymbol{t} = \boldsymbol{g_2} = \boldsymbol{0}$, we have established that $\boldsymbol{t} \in \boldsymbol{W}^{1,2}_0(\Omega)$ is solution of (5) and that we have the estimate

$$\|\boldsymbol{t}\|_{\boldsymbol{W}^{1,2}_0(\Omega)} \leq\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{\frac{1}{2},2}_0(\Gamma)} \leq\ C\ \|h\|_{L^2(\Omega)}.$$

ii) Now, we want to solve the following problem : find $\boldsymbol{z} \in \boldsymbol{W}^{1,2}_0(\Omega)$ such that

$$\operatorname{div} \boldsymbol{z} = 0 \text{ in } \Omega \quad \text{and} \quad \boldsymbol{z} = \boldsymbol{g_1} \text{ on } \Gamma. \tag{6}$$

We define the function $\widetilde{\boldsymbol{g_1}}$ by $\widetilde{\boldsymbol{g_1}} = \boldsymbol{g_1}$ on $\partial G \cap \Gamma$ and $\widetilde{\boldsymbol{g_1}} = \boldsymbol{0}$ on $\partial G \cap \mathbb{R}^n_+$. Noticing that supp $\boldsymbol{g_1} \subset \partial G \cap \Gamma$, we have $\widetilde{\boldsymbol{g_1}} \in \boldsymbol{H}^{\frac{1}{2}}(G)$ and we notice that

$$\int_{\partial G} \widetilde{\boldsymbol{g_1}} \cdot \boldsymbol{n}\ d\sigma = 0,$$

We deduce from this, thanks to results in bounded domain that there exists $\boldsymbol{u_0} \in \boldsymbol{H}^1(G)$ such that

$$\operatorname{div} \boldsymbol{u_0} = 0 \text{ in } G \quad \text{and} \quad \boldsymbol{u_0} = \widetilde{\boldsymbol{g_1}} \text{ on } \partial G$$

and satisfying the estimate

$$\|\boldsymbol{u_0}\|_{\boldsymbol{H}^1(G)} \leq\ C\ \|\widetilde{\boldsymbol{g_1}}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial G)} \leq\ C\ \|h\|_{L^2(\Omega)}.$$

We define the function $\boldsymbol{z}$ by $\boldsymbol{z} = \boldsymbol{u_0}$ in G and $\boldsymbol{z} = \boldsymbol{0}$ in $\Omega \setminus G$. We easily check that $\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega)$ and that div $\boldsymbol{z} = 0$ in Ω. Thus, we have established that $\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega)$ is solution of (6) and that we have the estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq \ C \ \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)} \leq \ C \ \|h\|_{L^2(\Omega)}.$$

So, $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{t} - \boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega)$ is solution of our problem and we have the estimate searched.

2) Now, we suppose that Ω is only Lipschitz-continuous. Then, using similar arguments than Theorem 3.2 in [14], we can prove that the result is also satisfied. $\square$

So to solve $(\mathcal{S}_{10})$, it is sufficient to solve the following problem $(\mathcal{S}_{100})$: find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of

$$(\mathcal{S}_{100}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \end{cases}$$

and, the study of this problem is exactly equivalent to the one of the end of Section 2 in [4]. In consequence, we have the following theorem :

Theorem 2.3. *For any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$, $h \in L^2(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)$ there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of (S_1). Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \ C \ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)}),$$

where C is a real positive constant which depends only on Ω.

2.2 Case $p \neq 2$.

First, we suppose that $p > 2$ and we want to study the kernel of the Stokes system. We set :

$$\mathcal{D}_0^p(\Omega) = \{(\boldsymbol{z}, \pi) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega), \ -\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ and div } \boldsymbol{z} = 0 \text{ in } \Omega\}.$$

We have the following result :

Theorem 2.4. *For each $p > 2$, the kernel $\mathcal{D}_0^p(\Omega)$ is reduced to $\{(\boldsymbol{0}, 0)\}$.*

Proof- Let $(\boldsymbol{z}, \pi)$ be in $\mathcal{D}_0^p(\Omega)$. We set $(\boldsymbol{z_1}, \pi_1) = (\psi_1 \boldsymbol{z}, \psi_1 \pi) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega)$. Since supp $(\boldsymbol{z_1}, \pi_1)$ is included in G which is bounded, we have

$$(\boldsymbol{z_1}, \pi_1) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega).$$

Now, we set $(\boldsymbol{z_2}, \pi_2) = (\psi_2 \boldsymbol{z}, \psi_2 \pi) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega)$ and

$$\begin{aligned} \boldsymbol{f} &= -\ \Delta \boldsymbol{z_2} + \nabla \pi_2 = \Delta \boldsymbol{z_1} - \nabla \pi_1 \in \boldsymbol{W}_0^{-1,2}(\Omega) \\ h &= \text{div } \boldsymbol{z_2} = -\ \text{div } \boldsymbol{z_1} \in L^2(\Omega). \end{aligned}$$

By [8], there exists $(\boldsymbol{s},\theta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{s} + \nabla\theta = \boldsymbol{f} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{s} = h \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{s} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1}. \tag{7}$$

But, noticing that on $\mathbb{R}^{n-1}$, $\boldsymbol{z_2} = \boldsymbol{0}$ (because $\boldsymbol{z_2} = \boldsymbol{0}$ on Σ and $\psi_2 = 0$ on D), it is obvious that $(\boldsymbol{z_2}, \pi_2) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ is solution of (7). So, $(\boldsymbol{w},\tau) = (\boldsymbol{s} - \boldsymbol{z_2}, \theta - \pi_2)$ satisfy

$$-\Delta \boldsymbol{w} + \nabla\tau = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

and we easily deduce that $(\boldsymbol{w},\tau) = (\boldsymbol{0}, 0)$ in $\mathbb{R}^n_+$ (see Lemma 3.1 in [4]). Thus $(\boldsymbol{z_2}, \pi_2) = (\boldsymbol{s},\theta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$ and since supp $(\boldsymbol{z_2}, \pi_2) \subset \mathbb{R}^n_+$, we deduce that $(\boldsymbol{z_2}, \pi_2) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. Finally, we conclude that $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$, which implies that $(\boldsymbol{z}, \pi) \in \mathcal{D}_0^2(\Omega)$. But, we have seen at Theorem 2.3 that when $p = 2$, (S_{10}) has a unique solution. Here, $(\boldsymbol{0}, 0)$ is solution, so we have our result. $\square$

Now, supposing that $p > 2$, we want to study the Stokes system with homogeneous boundary conditions, that is to say : let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$, we want to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}_{10})$. First, we establish the following lemma :

Lemma 2.5. *For each $p > 2$ and for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$ with a compact support in Ω, there exists a unique $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega)) \times (L^2(\Omega) \cap L^p(\Omega))$ solution of (S_{10}).*

Proof- Let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$ with a compact support in Ω. Then, since $p > 2$, we easily check that $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and $h \in L^2(\Omega)$ and we deduce from Theorem 2.3 that there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of (S_{10}). It remains to show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$. We set $(\boldsymbol{u_1}, \pi_1) = (\psi_1 \boldsymbol{u}, \psi_1 \pi) \in \mathring{\boldsymbol{W}}{}_0^{1,2}(\Omega) \times L^2(\Omega)$, it has a compact support included in G. Elementaries calculus show that we have

$$-\Delta \boldsymbol{u_1} + \nabla\pi_1 = \psi_1 \boldsymbol{f} + \boldsymbol{F}_1 \text{ in } G, \quad \text{div } \boldsymbol{u_1} = \psi_1 h + H_1 \text{ in } G, \quad \boldsymbol{u_1} = \boldsymbol{0} \text{ on } \partial G,$$

where

$$\boldsymbol{F}_1 = -(2\nabla \boldsymbol{u} \nabla\psi_1 + \boldsymbol{u}\Delta\psi_1) + \pi\nabla\psi_1 \in \boldsymbol{L}^2(G) \quad \text{and} \quad H_1 = \boldsymbol{u} \cdot \nabla\psi_1 \in \boldsymbol{H}^1(G).$$

Thanks to the Sobolev imbeddings, we have

$$(\boldsymbol{F}_{_1}, H_{_1}) \in \boldsymbol{W}^{-1,s}(G) \times L^s(G), \quad \forall 1 < s \leq 2*,$$

where $2* = \dfrac{2n}{n-2}$. So, if $p \leq 2*$, thanks to results in bounded domains (see [5]), we have

$$(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}^{1,p}(G) \times L^p(G) \tag{8}$$

and

$$\|\boldsymbol{u_1}\|_{\boldsymbol{W}^{1,p}(G)} + \|\pi_1\|_{L^p(G)} \leq \; C \; (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}).$$

Now, if $p > 2*$, we can show that $(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}^{1,2*}(G) \times L^{2*}(G)$ because $(\psi_1 \boldsymbol{f}, \psi_1 h) \in \boldsymbol{W}^{-1,2*}(G) \times L^{2*}(G)$. Thus, we have $(\boldsymbol{F}_1, H_1) \in \boldsymbol{L}^{2*}(G) \times W^{1,2*}(G)$

and we apply the same argument as previously with $2*$ instead of 2. Finally, starting again, we reach any value of $p > 2$. Thus, we have (8) in any case and since supp $(\boldsymbol{u_1}, \pi_1) \subset G$, we deduce that

$$(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega) \tag{9}$$

Now, we set $(\boldsymbol{u_2}, \pi_2) = (\psi_2 \boldsymbol{u}, \psi_2 \pi) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega) \times L^2(\Omega)$ and

$$\begin{aligned}
&\boldsymbol{f_2} = -\Delta \boldsymbol{u_2} + \nabla \pi_2 = \boldsymbol{f} - (-\Delta \boldsymbol{u_1} + \nabla \pi_1) \in \boldsymbol{W}_0^{-1,p}(\Omega),\\
&h_2 = \text{div } \boldsymbol{u_2} = h - \text{ div } \boldsymbol{u_1} \in L^p(\Omega).
\end{aligned}$$

By [8], there exists $(\boldsymbol{s}, \theta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{s} + \nabla \theta = \boldsymbol{f_2} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{s} = h_2 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{s} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

But, noticing that on $\mathbb{R}^{n-1}$, $\boldsymbol{u_2} = \boldsymbol{0}$ (because $\boldsymbol{u_2} = \boldsymbol{0}$ on Σ and $\psi_2 = 0$ on D), it is obvious that in $\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$, $(\boldsymbol{u_2}, \pi_2)$ is solution of the same problem that $(\boldsymbol{s}, \theta)$ satisfies. We use the same reasoning as in Theorem 2.4 to conclude that

$$(\boldsymbol{u_2}, \pi_2) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega) \tag{10}$$

Finally, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ by (9) and (10). $\square$

Now, we establish the following theorem :

Theorem 2.6. *For any $p > 2$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of*

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma. \tag{11}$$

Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C \, \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)},$$

where C is a real positive constant which depends only on p and Ω.

Proof- The uniqueness comes from Theorem 2.4. Now, let $\boldsymbol{g}$ be in $\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$. We set $\boldsymbol{g_1} = \psi_1 \boldsymbol{g}$ and $\boldsymbol{g_2} = \psi_2 \boldsymbol{g}$.

i) First, we want to find $(\boldsymbol{v}, \mu) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$-\Delta \boldsymbol{v} + \nabla \mu = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{g_1} \text{ on } \Gamma. \tag{12}$$

We notice that supp $\boldsymbol{g_1} \subset \partial G \cap \Gamma$. We define the function $\widetilde{\boldsymbol{g}}_{\boldsymbol{1}}$ by

$$\widetilde{\boldsymbol{g}}_{\boldsymbol{1}} = \boldsymbol{g_1} \text{ on } \partial G \cap \Gamma \quad \text{and} \quad \widetilde{\boldsymbol{g}}_{\boldsymbol{1}} = \boldsymbol{0} \text{ on } \partial G \cap \mathbb{R}^n_+.$$

We easily check that $\widetilde{\boldsymbol{g}}_{\boldsymbol{1}} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\partial G)$. Let ψ be in $\mathcal{D}(\mathbb{R}^n)$ with a compact support in G such that

$$\int_G \psi(\boldsymbol{x}) \, d\boldsymbol{x} = \int_{\partial G} \widetilde{\boldsymbol{g}}_{\boldsymbol{1}} \cdot \boldsymbol{n} \, d\sigma.$$

Thanks to this condition and results in bounded domains (see [5]), there exists $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(G) \times L^p(G)$ such that

$$-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ in } G, \quad \text{div } \boldsymbol{z} = \psi \text{ in } G, \quad \boldsymbol{z} = \widetilde{\boldsymbol{g}}_{\boldsymbol{1}} \text{ on } \partial G,$$

We denote again by $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ its extension by $(\boldsymbol{0}, 0)$ in Ω. Thus, $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ satisfies

$$-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{\sigma} \text{ in } \Omega, \quad \text{div } \boldsymbol{z} = \psi \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{g}_{\boldsymbol{1}} \text{ on } \Gamma,$$

where we notice that $\boldsymbol{\sigma} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ has a compact support in Ω. As ψ has a compact support too, we deduce from Lemma 2.5 that there exists $(\boldsymbol{t}, \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$-\Delta \boldsymbol{t} + \nabla \tau = -\boldsymbol{\sigma} \text{ in } \Omega, \quad \text{div } \boldsymbol{t} = -\psi \text{ in } \Omega, \quad \boldsymbol{t} = \boldsymbol{0} \text{ on } \Gamma,$$

Finally, $(\boldsymbol{v}, \mu) = (\boldsymbol{z} + \boldsymbol{t}, \pi + \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of (12).

ii) Now, we want to find $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \eta = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \Omega, \quad \boldsymbol{w} = \boldsymbol{g}_{\boldsymbol{2}} \text{ on } \Gamma. \tag{13}$$

For this, we notice that supp $\boldsymbol{g}_{\boldsymbol{2}} \subset \Sigma$. We define the function $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}}$ by $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} = \boldsymbol{g}_{\boldsymbol{2}}$ on Σ and $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} = \boldsymbol{0}$ on D. We easily check that $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$. Thanks to [8], there exists $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ such that

$$-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{z} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{z} = \widetilde{\boldsymbol{g}}_{\boldsymbol{2}} \text{ on } \mathbb{R}^{n-1}.$$

We denote again by $(\boldsymbol{z}, \pi)$ its extension by $(\boldsymbol{0}, 0)$ in Ω. So, $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ satisfies

$$-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \boldsymbol{z} = 0 \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{g}_{\boldsymbol{2}} \text{ on } \Gamma,$$

where $\boldsymbol{\xi}$ has a compact support in Ω. We deduce from Lemma 2.5 that there exists $(\boldsymbol{t}, \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$-\Delta \boldsymbol{t} + \nabla \tau = -\boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \boldsymbol{t} = 0 \text{ in } \Omega, \quad \boldsymbol{t} = \boldsymbol{0} \text{ on } \Gamma.$$

Finally, $(\boldsymbol{w}, \eta) = (\boldsymbol{z} + \boldsymbol{t}, \pi + \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of (13).

In consequence, $(\boldsymbol{u}, \pi) = (\boldsymbol{v} + \boldsymbol{w}, p + \eta) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of (11) and the estimate follows immediately. $\square$

Now, we can solve the problem with homogeneous boundary conditions in the case $p > 2$.

Theorem 2.7. *For any $p > 2$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}_{10})$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \; C\; (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}),$$

where C is a real positive constant which depends only on p and Ω.

Proof- The uniqueness comes from Theorem 2.4. Then, as a consequence of Theorem 1.1 ii), we know that there exists a tensor of the second order $F \in [L^p(\Omega)]^{n\times n}$ such that div $F = \boldsymbol{f}$. We extend F (respectively h) by 0 in $\mathbb{R}^n$, and we denote by $\widetilde{F}$ (respectively $\widetilde{h}$) this extension. Then, we set $\widetilde{\boldsymbol{f}} =$ div $\widetilde{F}$. We have $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n)$ and $\widetilde{h} \in L^p(\mathbb{R}^n)$. By [1], there exists $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \widetilde{\boldsymbol{f}} \text{ in } \mathbb{R}^n \quad \text{and} \quad \text{div } \boldsymbol{v} = \widetilde{h} \text{ in } \mathbb{R}^n,$$

satisfying the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^n)} + \|\eta\|_{L^p(\mathbb{R}^n)} \leq \ C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}).$$

We denote again by $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and $\eta \in L^p(\Omega)$ the restrictions of $\boldsymbol{v}$ and η to Ω. We have $\boldsymbol{v}_{|\Gamma} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, thus, thanks to Theorem 2.6, there exists $(\boldsymbol{w}, \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \Omega, \quad \boldsymbol{w} = -\boldsymbol{v}_{|\Gamma} \text{ on } \Gamma,$$

satisfying the estimate

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\tau\|_{L^p(\Omega)} \leq \ C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}).$$

Finally, $(\boldsymbol{u}, \pi) = (\boldsymbol{v} + \boldsymbol{w}, \eta + \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of $(\mathcal{S}_{10})$ and the estimate follows immediately. $\square$

Now, we suppose that p is such that $p < 2$. Thanks to the previous theorem, if we set

$$\begin{aligned} S : \mathring{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) &\longrightarrow \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega), \\ (\boldsymbol{u}, \pi) &\longrightarrow (-\Delta \boldsymbol{u} + \nabla \pi, -\text{div } \boldsymbol{u}), \end{aligned}$$

then, S is an isomorphism. So, by duality,

$$S^* : \mathring{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega) \longrightarrow \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega),$$

is an isomorphism too, and, as it is standard to check that $S^*(\boldsymbol{u}, \pi) = (-\Delta \boldsymbol{u} + \nabla \pi, -\text{div } \boldsymbol{u})$, we have Theorem 2.7 for any $p < 2$. $\square$

Finally, it remains to return to the general problem with $p \neq 2$ and nonhomogeneous boundary conditions. For this, like for the case $p = 2$, we show that there exists a function $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\boldsymbol{w} = \boldsymbol{g}$ in Γ. Then, we have just seen that there exists a unique $(\boldsymbol{v}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{v} + \nabla \pi = \boldsymbol{f} + \Delta \boldsymbol{w} \text{ in } \Omega, \quad \text{div } \boldsymbol{v} = h - \text{div } \boldsymbol{w} \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma.$$

In consequence, the function $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of $(\mathcal{S}_1)$ and we have the following theorem :

Theorem 2.8. *For any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}_1)$ and satisfying*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \ C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)}),$$

where C is a real positive constant which depends only on p and Ω.

2.3 Regularity, strong solutions and very weak solutions.

First, in this section, we are interested in the existence of strong solutions of the Stokes system $(\mathcal{S}_1)$, *i.e.* of solutions $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{2,p}_{\ell+1}(\Omega)\times W^{1,p}_{\ell+1}(\Omega)$. Here, we limit ourselves to the two cases $\ell=0$ and $\ell=-1$.

We give at the beginning a regularity result studying the case $\ell=0$. Indeed, we notice that in this case, we have the continuous injections $\boldsymbol{W}^{2,p}_1(\Omega)\subset \boldsymbol{W}^{1,p}_0(\Omega)$ and $W^{1,p}_1(\Omega)\subset L^p(\Omega)$. So, Theorem which follows shows that the generalized solution of Theorems 2.3 and 2.8, with a stronger hypothesis on the data, is in fact a strong solution.

Theorem 2.9. *For any $p>1$ such that $\dfrac{n}{p'}\neq 1$, and for any $\boldsymbol{f}\in \boldsymbol{W}^{0,p}_1(\Omega)$, $h\in W^{1,p}_1(\Omega)$ and $\boldsymbol{g}\in \boldsymbol{W}^{2-\frac{1}{p},p}_1(\Gamma)$, there exists a unique $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{2,p}_1(\Omega)\times W^{1,p}_1(\Omega)$ solution of the problem $(\mathcal{S}_1)$. Moreover, $(\boldsymbol{u},\pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}_1(\Omega)}+\|\pi\|_{W^{1,p}_1(\Omega)}\leq\ C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,p}_1(\Omega)}+\|h\|_{W^{1,p}_1(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-\frac{1}{p},p}_1(\Gamma)}),$$

where C is a real positive constant which depends only on p and Ω.

Proof- First, we want to solve the problem with homogeneous boundary conditions. For this, we notice that we have the continuous injections $\boldsymbol{W}^{0,p}_1(\Omega)\subset \boldsymbol{W}^{-1,p}_0(\Omega)$ because $\dfrac{n}{p'}\neq 1$ and $W^{1,p}_1(\Omega)\subset L^p(\Omega)$. Thus, thanks to Theorems 2.3 and 2.8, there exists a unique $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p}_0(\Omega)\times L^p(\Omega)$ solution of (S_{10}). Next, it remains to show that $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{2,p}_1(\Omega)\times W^{1,p}_1(\Omega)$. We set $(\boldsymbol{u_1},\pi_1)=(\psi_1\boldsymbol{u},\psi_1\pi)\in \mathring{\boldsymbol{W}}{}^{1,p}_0(\Omega)\times L^p(\Omega)$, it has a compact support included in G. Elementaries calculus show that we have

$$-\Delta\boldsymbol{u_1}+\nabla\pi_1=\psi_1\boldsymbol{f}+\boldsymbol{F}_1 \text{ in } G,\quad \operatorname{div}\boldsymbol{u_1}=\psi_1 h+H_1 \text{ in } G,\quad \boldsymbol{u_1}=\boldsymbol{0} \text{ on } \partial G,$$

where

$$\boldsymbol{F}_1=-(2\nabla\boldsymbol{u}\nabla\psi_1+\boldsymbol{u}\Delta\psi_1)+\pi\nabla\psi_1\in \boldsymbol{L}^p(G)\quad\text{and}\quad H_1=\boldsymbol{u}\cdot\nabla\psi_1\in \boldsymbol{W}^{1,p}(G).$$

So, using results in bounded domains, we have $(\boldsymbol{u_1},\pi_1)\in \boldsymbol{W}^{2,p}(G)\times \boldsymbol{W}^{1,p}(G)$ and since supp $(\boldsymbol{u_1},\pi_1)\subset G$,

$$(\boldsymbol{u_1},\pi_1)\in \boldsymbol{W}^{2,p}_1(\Omega)\times \boldsymbol{W}^{1,p}_1(\Omega). \tag{14}$$

Now, we set $(\boldsymbol{u_2},\pi_2)=(\psi_2\boldsymbol{u},\psi_2\pi)\in \mathring{\boldsymbol{W}}{}^{1,p}_0(\Omega)\times L^p(\Omega)$ and

$$\begin{aligned}&\boldsymbol{f_2}=-\,\Delta\boldsymbol{u_2}+\nabla\pi_2=\boldsymbol{f}-(-\Delta\boldsymbol{u_1}+\nabla\pi_1)\in \boldsymbol{W}^{0,p}_1(\Omega),\\ &h_2=\operatorname{div}\boldsymbol{u_2}=h-\ \operatorname{div}\boldsymbol{u_1}\in W^{1,p}_1(\Omega).\end{aligned}$$

By [8], there exists $(\boldsymbol{s},\theta)\in \boldsymbol{W}^{2,p}_1(\mathbb{R}^n_+)\times W^{1,p}_1(\mathbb{R}^n_+)$ solution of

$$-\Delta\boldsymbol{s}+\nabla\theta=\boldsymbol{f_2} \text{ in } \mathbb{R}^n_+,\quad \operatorname{div}\boldsymbol{s}=h_2 \text{ in } \mathbb{R}^n_+,\quad \boldsymbol{s}=\boldsymbol{0} \text{ on } \mathbb{R}^{n-1}, \tag{15}$$

But, noticing that $(\boldsymbol{u_2},\pi_2)$ is also solution of (15) and that, by Theorem 4.2 in [8], the solution of this problem is unique in $\boldsymbol{W}^{1,p}_0(\mathbb{R}^n_+)\times L^p(\mathbb{R}^n_+)$, we have

$(\boldsymbol{u_2}, \pi_2) = (\boldsymbol{s}, \theta) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^n_+) \times W_1^{1,p}(\mathbb{R}^n_+)$. The support of $(\boldsymbol{u_2}, \pi_2)$ being included in $\mathbb{R}^n_+$, we deduce that

$$(\boldsymbol{u_2}, \pi_2) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega) \tag{16}$$

So, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ by (14) and (16). Finally, thanks to Lemma 1.2, we come back to a problem with nonhomogeneous boundary conditions. □

Now, we examine the basic case $\ell = -1$, corresponding to $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and first, we study the kernel of such a problem. We set

$$\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega),\ -\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ in } \Omega,$$
$$\text{div } \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \boldsymbol{z} = \boldsymbol{0} \text{ on } \Gamma\}.$$

The characterization of this kernel is given by this proposition :

Proposition 2.10. *For each $p > 1$ such that $\dfrac{n}{p'} \neq 1$, we have the following statements : i) If $p < n$, $\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{0}, 0)\}$.*
ii) If $p \geq n$, $\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \eta(\boldsymbol{\lambda}) - \mu),\quad \boldsymbol{\lambda} \in (\mathbb{R}x_n)^{n-1} \times \{0\},\ \mu \in \mathbb{R}\}$ where $(\boldsymbol{v}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda})) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ is the unique solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{\lambda} \text{ on } \Gamma. \tag{17}$$

Proof- Let $(\boldsymbol{z}, \pi) \in \mathcal{S}_0^p(\Omega)$. We set $(\boldsymbol{z_i}, \pi_i) = (\psi_i \boldsymbol{z}, \psi_i \pi) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ (for $i = 1$ or 2). Since supp $(\boldsymbol{z_1}, \pi_1)$ is bounded, we have $(\boldsymbol{z_1}, \pi_1) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$. Now, we set

$$\boldsymbol{f_2} = -\Delta \boldsymbol{z_2} + \nabla \pi_2 = \Delta \boldsymbol{z_1} - \nabla \pi_1, \quad h_2 = \text{ div } \boldsymbol{z_2} = - \text{ div } \boldsymbol{z_1}.$$

We have $(\boldsymbol{f_2}, h_2) \in \boldsymbol{W}_1^{0,p}(\Omega) \times W_1^{1,p}(\Omega)$, so by Theorem 5.2 of [8], there exists $(\boldsymbol{s}, \theta) \in (\boldsymbol{W}_1^{2,p}(\mathbb{R}^n_+) \times W_1^{1,p}(\mathbb{R}^n_+)) \subset (\boldsymbol{W}_0^{2,p}(\mathbb{R}^n_+) \times W_0^{1,p}(\mathbb{R}^n_+))$ solution of

$$(\mathcal{S}_+) \quad -\Delta \boldsymbol{s} + \nabla \theta = \boldsymbol{f_2} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{s} = h_2 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{s} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1}.$$

Noticing that $(\boldsymbol{z_2}, \pi_2) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^n_+) \times W_0^{1,p}(\mathbb{R}^n_+)$ is solution of $(\mathcal{S}_+)$, we can deduce, using Theorem 5.6 in [8], the following results :

i) If $p < n$, the solution of $(\mathcal{S}_+)$ is unique in $\boldsymbol{W}_0^{2,p}(\mathbb{R}^n_+) \times W_0^{1,p}(\mathbb{R}^n_+)$, so $(\boldsymbol{z_2}, \pi_2) = (\boldsymbol{s}, \theta)$. Thus, as the support of $(\boldsymbol{z_2}, \pi_2)$ is included in $\mathbb{R}^n_+$, we have $(\boldsymbol{z_2}, \pi_2) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ and so $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$. Thanks to Theorem 2.9, we have necessarily $(\boldsymbol{z}, \pi) = (\boldsymbol{0}, 0)$.

ii) If $p \geq n$, there exists $\boldsymbol{\lambda} \in (\mathbb{R}x_n)^{n-1} \times \{0\}$ and $\mu \in \mathbb{R}$ such that $\boldsymbol{z_2} = \boldsymbol{s} - \boldsymbol{\lambda}$ and $\pi_2 = \theta - \mu$ in $\mathbb{R}^n_+$. We define $\boldsymbol{w}$ by $\boldsymbol{w} = \boldsymbol{s}$ in $\mathbb{R}^n_+$, $\boldsymbol{w} = \boldsymbol{\lambda}$ in ω and ξ by $\xi = \theta$ in $\mathbb{R}^n_+$, $\xi = \mu$ in ω. We easliy check that $(\boldsymbol{w}, \xi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ and that $\boldsymbol{z_2} = \boldsymbol{w} - \boldsymbol{\lambda}$ and $\pi_2 = \xi - \mu$ in Ω. Finally, we set $\boldsymbol{v} = \boldsymbol{z} + \boldsymbol{\lambda}$ and $\eta = \pi + \mu$ in Ω. Then, $(\boldsymbol{v}, \eta)$ is in $\boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ and is the unique (see Theorem 2.9) solution of (17). In consequence, we have the characterization of the kernel. □

We have the following result, corresponding to Theorem 2.9 :

Theorem 2.11. *For any $p > 1$ such that $\dfrac{n}{p'} \neq 1$, and for any $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $h \in W_0^{1,p}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)$, there exists a unique $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega))/\mathcal{S}_0^p(\Omega)$ solution of $(\mathcal{S}_1)$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\inf_{(\boldsymbol{z},\alpha)\in\mathcal{S}_0^p(\Omega)} (\|\boldsymbol{u} + \boldsymbol{z}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\pi + \alpha\|_{W_0^{1,p}(\Omega)})$$
$$\leq C\,(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)}),$$

where C is a real positive constant which depends only on p and Ω.

Proof- i) First, we solve the following problem : find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ solution of

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma.$$

For this, we set $\boldsymbol{g_i} = \psi_i \boldsymbol{g} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)$ (for $i = 1$ or 2) and we define the function $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}}$ by $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} = \boldsymbol{g_2}$ on Σ, $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} = \boldsymbol{0}$ on D. So $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})$ and by Theorem 5.6 of [8], there exists $(\boldsymbol{w}, \tau) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^n_+) \times W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w} = \widetilde{\boldsymbol{g}}_{\boldsymbol{2}} \text{ on } \mathbb{R}^{n-1}.$$

Now, we define for almost all $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$, the following functions $\boldsymbol{w^*}$ and τ_* by

$$\boldsymbol{w}^*(\boldsymbol{x'}, x_n) = \boldsymbol{w}(\boldsymbol{x'}, x_n) \text{ if } x_n > 0, \quad \boldsymbol{w}^*(\boldsymbol{x'}, x_n) = -\boldsymbol{w}(\boldsymbol{x'}, -x_n) \text{ if } x_n < 0$$

and

$$\tau_*(\boldsymbol{x'}, x_n) = \tau(\boldsymbol{x'}, x_n) \text{ if } x_n > 0, \quad \tau_*(\boldsymbol{x'}, x_n) = \tau(\boldsymbol{x'}, -x_n) \text{ if } x_n < 0.$$

Then, we set $\widetilde{\boldsymbol{w}} = \boldsymbol{w}$ in $\mathbb{R}^n_+$, $\widetilde{\boldsymbol{w}} = \boldsymbol{w^*}$ in ω and $\widetilde{\tau} = \tau$ in $\mathbb{R}^n_+$, $\widetilde{\tau} = \tau_*$ in ω. We easily check that $\widetilde{\boldsymbol{w}} \in \boldsymbol{W}_0^{2,p}(\Omega)$ and that $\widetilde{\tau} \in W_0^{1,p}(\Omega)$. Finally, we denote by $\boldsymbol{\mu} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)$ the trace of the function $\widetilde{\boldsymbol{w}}$ and we set

$$-\Delta \widetilde{\boldsymbol{w}} + \nabla \widetilde{\tau} = \boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \widetilde{\boldsymbol{w}} = \sigma \text{ in } \Omega.$$

The functions $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$ and $\sigma \in W_0^{1,p}(\Omega)$ have clearly a compact support, so, by Theorem 2.9, there exists $(\boldsymbol{t}, \beta) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ such that

$$-\Delta \boldsymbol{t} + \nabla \beta = -\boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \boldsymbol{t} = -\sigma \text{ in } \Omega, \quad \boldsymbol{t} = \boldsymbol{0} \text{ on } \Gamma,$$

The pair $(\boldsymbol{z}, \eta) = (\widetilde{\boldsymbol{w}} + \boldsymbol{t}, \widetilde{\tau} + \beta) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ satisfy

$$-\Delta \boldsymbol{z} + \nabla \eta = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{z} = 0 \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{\mu} \text{ on } \Gamma.$$

In a last step, noticing that on $\Sigma \cap B_0$, $\boldsymbol{\mu} = \boldsymbol{0}$ because $\boldsymbol{\mu} = \boldsymbol{g_2}$ on Σ, we can say that the function $\boldsymbol{\gamma}$, defined by $\boldsymbol{\gamma} = -\boldsymbol{\mu}$ on S and $\boldsymbol{\gamma} = \boldsymbol{0}$ on Σ, belongs to $\boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)$. Moreover, since $\boldsymbol{g_1}$ and $\boldsymbol{\gamma}$ have a compact support, they belong to the space $\boldsymbol{W}_1^{2-\frac{1}{p},p}(\Gamma)$. Thus, applying Theorem 2.9, there exists $(\boldsymbol{v}, p) \in (\boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)) \subset (\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega))$ such that

$$-\Delta \boldsymbol{v} + \nabla p = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{\gamma} + \boldsymbol{g_1} \text{ on } \Gamma.$$

Noticing that on Γ, $\boldsymbol{\mu}+\boldsymbol{\gamma}+\boldsymbol{g_1}=\boldsymbol{g}$, we conclude that $(\boldsymbol{u},\pi)=(\boldsymbol{z}+\boldsymbol{v},\eta+p)$ answers the question.

ii) We easily show that there exists extensions $\widetilde{\boldsymbol{f}}\in \boldsymbol{L}^p(\mathbb{R}^n)$ of $\boldsymbol{f}$ and $\widetilde{h}\in W_0^{1,p}(\mathbb{R}^n)$ of h in $\mathbb{R}^n$ and, by Theorem 3.10 of [2], there exists $(\boldsymbol{w},\eta)\in \boldsymbol{W}_0^{2,p}(\mathbb{R}^n)\times W_0^{1,p}(\mathbb{R}^n)$ solution of

$$-\Delta \boldsymbol{w}+\nabla\eta=\widetilde{\boldsymbol{f}} \text{ in } \mathbb{R}^n, \quad \operatorname{div} \boldsymbol{w}=\widetilde{h} \text{ in } \mathbb{R}^n.$$

Moreover, by **i)**, there exists $(\boldsymbol{z},\mu)\in \boldsymbol{W}_0^{2,p}(\Omega)\times W_0^{1,p}(\Omega)$ such that

$$-\Delta \boldsymbol{z}+\nabla\mu=\boldsymbol{0} \text{ in } \Omega, \quad \operatorname{div} \boldsymbol{z}=0 \text{ in } \Omega, \quad \boldsymbol{z}=\boldsymbol{g}-\boldsymbol{w}_{|\Gamma} \text{ on } \Gamma.$$

Thus, $(\boldsymbol{u},p)=(\boldsymbol{z}+\boldsymbol{w},\mu+\eta)$ is solution of our problem. $\square$

On the other hand, we want to study the case of very weak solutions *i.e.*, we study $(\mathcal{S}_1)$ with $\boldsymbol{f}=0$, $h=0$ and singular data on the boundary. We use previous results for strong solutions and we argue by duality. The proofs are exactly the same as [4]. We give the two following theorems :

Theorem 2.12. *For each $p>1$ such that $\dfrac{n}{p}\neq 1$ and for any $\boldsymbol{g}\in \boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\Gamma)$ satisfying*

$$\boldsymbol{g}\cdot\boldsymbol{n}=0 \quad \text{on } \Gamma, \tag{18}$$

there exists a unique $(\boldsymbol{u},\pi)\in \boldsymbol{W}_{-1}^{0,p}(\Omega)\times W_{-1}^{-1,p}(\Omega)$ solution of $(\mathcal{S}_1)$ with $\boldsymbol{f}=0$, $h=0$. Moreover, $(\boldsymbol{u},\pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_{-1}^{0,p}(\Omega)}+\|\pi\|_{W_{-1}^{-1,p}(\Omega)}\leq\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\Omega)},$$

where C is a real positive constant wich depends only on Ω and p.

Theorem 2.13. *For each $p>1$ such that $\dfrac{n}{p}\neq 1$ and for any $\boldsymbol{g}\in \boldsymbol{W}_0^{-\frac{1}{p},p}(\Gamma)$ satisfying (18) and the following condition if $p\leq\dfrac{n}{n-1}$: for any $(\boldsymbol{z},p)\in\mathcal{S}_0^{p'}(\Omega)$*

$$<\boldsymbol{g},\frac{\partial\boldsymbol{z}}{\partial\boldsymbol{n}}>_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\Gamma),\boldsymbol{W}_0^{\frac{1}{p},p'}(\Gamma)}=0,$$

there exists a unique $(\boldsymbol{u},\pi)\in \boldsymbol{L}^p(\Omega)\times W_0^{-1,p}(\Omega)$ solution of $(\mathcal{S}_1)$ with $\boldsymbol{f}=0$, $h=0$. Moreover, $(\boldsymbol{u},\pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}+\|\pi\|_{W_0^{-1,p}(\Omega)}\leq\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\Gamma)},$$

where C is a real positive constant wich depends only on Ω and p.

3 Stokes system in an aperture domain

We remind that we suppose that $n\geq 3$. For each $p\in\,]1,+\infty[$, we want to study the following problem $(\mathcal{S}_2)$: for any $\boldsymbol{f}\in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h\in L^p(\Omega)$, $\boldsymbol{g}\in$

$\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$ and $\alpha \in \mathbb{R}$, we want to find $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^p(\Omega_+)$, $\pi - a_- \in L^p(\Omega_-)$ and

$$(\mathcal{S}_2) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = \alpha. \end{cases}$$

We define the following spaces :

$$\boldsymbol{\mathcal{V}}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega), \ \text{div } \boldsymbol{v} = 0 \text{ in } \Omega\},$$

$$\boldsymbol{V_p}(\Omega) = \overline{\boldsymbol{\mathcal{V}}(\Omega)}^{\|\cdot\|_{\boldsymbol{W}_0^{1,p}(\Omega)}},$$

$$\hat{\boldsymbol{V}}_{\boldsymbol{p}}(\Omega) = \{\boldsymbol{v} \in \mathring{\boldsymbol{W}}{}_0^{1,p}(\Omega), \ \text{div } \boldsymbol{v} = 0 \text{ in } \Omega\}$$

and the polar of $\hat{\boldsymbol{V}}_{\boldsymbol{p}}$:

$$\hat{\boldsymbol{V}}^{\circ}_{\boldsymbol{p}}(\Omega) = \{\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega), \ \forall \boldsymbol{w} \in \hat{\boldsymbol{V}}_{\boldsymbol{p}}(\Omega), \ <\boldsymbol{f}, \boldsymbol{w}>_\Omega = 0\}.$$

We easily show that :

$$\forall \boldsymbol{u} \in \boldsymbol{V_p}(\Omega), \ \int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = 0. \tag{19}$$

Moreover, contrary to cases of exterior domains, half-space or perturbed half-space, we notice (see [17]) that, in an aperture domain, we have only the strict inclusion :

$$\boldsymbol{V_p}(\Omega) \subsetneq \hat{\boldsymbol{V}}_{\boldsymbol{p}}(\Omega). \tag{20}$$

3.1 Case $p = 2$.

Thanks to Lemma 1.2, there exists $\boldsymbol{u_g} \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that $\boldsymbol{u_g} = \boldsymbol{g}$ on Γ and satisfying

$$\|\boldsymbol{u_g}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq \ C \ \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)}.$$

Thus, we show that it is equivalent to study the problem with homogeneous boundary conditions.

Moreover, let h be in $L^2(\Omega)$. We remind that we can consider Ω_+ and Ω_- as perturbed half-spaces. So, thanks to Proposition 2.2, there exists $\boldsymbol{t} \in \mathring{\boldsymbol{W}}{}_0^{1,2}(\Omega_+)$ such that

$$\text{div } \boldsymbol{t} = h \text{ in } \Omega_+, \quad \text{and} \quad \|\boldsymbol{t}\|_{\boldsymbol{W}_0^{1,2}(\Omega_+)} \leq C \ \|h\|_{L^2(\Omega)},$$

and there exists $\boldsymbol{z} \in \mathring{\boldsymbol{W}}{}_0^{1,2}(\Omega_-)$ such that

$$\text{div } \boldsymbol{z} = h \text{ in } \Omega_-, \quad \text{and} \quad \|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,2}(\Omega_-)} \leq C \ \|h\|_{L^2(\Omega)}.$$

We define the function $\boldsymbol{w}$ by $\boldsymbol{w} = \boldsymbol{t}$ in Ω_+ and $\boldsymbol{w} = \boldsymbol{z}$ in Ω_-. We have $\boldsymbol{w} \in \mathring{\boldsymbol{W}}{}_0^{1,2}(\Omega)$ because $\boldsymbol{t} = \boldsymbol{z} = \boldsymbol{0}$ on the join M and we easily show that

$$\text{div } \boldsymbol{w} = h \text{ in } \Omega \quad \text{and} \quad \|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C \ \|h\|_{L^2(\Omega)}.$$

Thus, thanks to this result, it remains only to study the following problem : for $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and $\alpha \in \mathbb{R}$, find $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^2(\Omega_+)$, $\pi - a_- \in L^2(\Omega_-)$ and

$$(\mathcal{S}_{200}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = \alpha. \end{cases}$$

First, we establish the following lemma :

Lemma 3.1. *There exists $\boldsymbol{b} \in \hat{\boldsymbol{V}}_{\boldsymbol{2}}(\Omega)$ such that*

$$\int_M \boldsymbol{b} \cdot \boldsymbol{n} \, d\sigma = 1. \tag{21}$$

Proof- By Lemma 11 of Heywood in [17], there exists $\boldsymbol{b} \in \overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\|\nabla \cdot\|_{\boldsymbol{L}^2(\Omega)}}$ such that

$$\operatorname{div} \boldsymbol{b} = \boldsymbol{0} \text{ in } \Omega \quad \text{and} \quad \int_M \boldsymbol{b} \cdot \boldsymbol{n} \, d\sigma = 1.$$

Moreover, thanks to Lemma 2.1 of Farwig and Sohr in [12], we show that, in an aperture domain, $\overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\|\nabla \cdot\|_{L^2(\Omega)}} = \{\boldsymbol{u} \in \boldsymbol{L}^2_{loc}(\overline{\Omega}),\ \nabla \boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma\}$. So, it remains only to show that $\boldsymbol{b} \in \boldsymbol{W}_0^{1,2}(\Omega)$. Let us set, for all $i = 1, \ldots, n$, $\boldsymbol{s_i} = \nabla b_i \in \boldsymbol{L}^2(\Omega)$ and

$$\widetilde{\boldsymbol{s}}_{\boldsymbol{i}} = \boldsymbol{s_i} \text{ in } \Omega \quad \text{and} \quad \widetilde{\boldsymbol{s}}_{\boldsymbol{i}} = \boldsymbol{0} \text{ in } \mathbb{R}^n \setminus \Omega.$$

We have $\widetilde{\boldsymbol{s}}_{\boldsymbol{i}} \in \boldsymbol{L}^2(\mathbb{R}^n)$ and we check that for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \widetilde{\boldsymbol{s}}_{\boldsymbol{i}} \cdot \boldsymbol{\varphi} \, d\boldsymbol{x} = \int_\Omega \boldsymbol{s_i} \cdot \boldsymbol{\varphi} \, d\boldsymbol{x} = 0.$$

Moreover, thanks to Lemma 4.2 of Amrouche, Girault and Giroire in [6], $\boldsymbol{\mathcal{V}}(\mathbb{R}^n)$ is dense in $\boldsymbol{H_2}(\mathbb{R}^n) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathbb{R}^n),\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \mathbb{R}^n\}$. So, $\widetilde{\boldsymbol{s}}_{\boldsymbol{i}} \in \boldsymbol{L}^2(\mathbb{R}^n) \perp \boldsymbol{H_2}(\mathbb{R}^n)$. Then, thanks to Proposition 9.2 of [6], for each $i = 1, \ldots, n$, there exists $\widetilde{w}_i \in W_0^{1,2}(\mathbb{R}^n)$ such that $\nabla \widetilde{w}_i = \widetilde{\boldsymbol{s}}_{\boldsymbol{i}}$ in $\mathbb{R}^n$. We set $w_i \in W_0^{1,2}(\Omega)$ the restriction of $\widetilde{w}_i$ in Ω ; we have $\nabla w_i = \boldsymbol{s}_i$ *i.e.* $\nabla w_i = \nabla b_i$ in Ω. So, Ω being connected, there exists a real constant $K_i \in \mathbb{R}^n$ such that $w_i = b_i + K_i \in W_0^{1,2}(\Omega)$. Thus, since $b_i = 0$ on Γ, $w_i = K_i$ on Γ. Moreover, we notice that $\nabla \widetilde{w}_i = \boldsymbol{0}$ in $\mathbb{R}^n \setminus \overline{\Omega}$, so $\widetilde{w}_i$ is constant in each of the two infinite and connected components Θ_j $(j = 1, 2)$ of $\mathbb{R}^n \setminus \overline{\Omega}$. As $\widetilde{w}_i \in W_0^{1,2}(\Theta_j)$ and that constants are not in this space, we deduce that $\widetilde{w}_i = 0$ in $\Theta_1 \cup \Theta_2$. Finally, reminding that $w_i = K_i$ on Γ and that $\widetilde{w}_i \in W_0^{1,2}(\mathbb{R}^n)$, we conclude that $K_i = 0$. Thus, $b_i \in W_0^{1,2}(\Omega)$ for any $i = 1, \ldots, n$. $\square$

Now, for any $\boldsymbol{w} \in \boldsymbol{V_2}(\Omega)$, we define the following bilinear and continuous application $\boldsymbol{T}$ by

$$\boldsymbol{T}\boldsymbol{w} = < \boldsymbol{f}, \boldsymbol{w} >_\Omega - \alpha \int_\Omega \nabla \boldsymbol{b} : \nabla \boldsymbol{w} \, d\boldsymbol{x}.$$

We apply the theorem of Lax-Milgram in $(\boldsymbol{V_2}(\Omega), \| \cdot \|_{\boldsymbol{W}_0^{1,2}(\Omega)})$ to conclude that there exists a unique $\boldsymbol{v} \in \boldsymbol{V_2}(\Omega)$ such that

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w} \, d\boldsymbol{x} = < \boldsymbol{f}, \boldsymbol{w} >_\Omega - \alpha \int_\Omega \nabla \boldsymbol{b} : \nabla \boldsymbol{w} \, d\boldsymbol{x},$$

(we notice that we have the coercivity thanks to the point ii) of Theorem 1.1 since $\boldsymbol{V_2}(\Omega) \subset \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)$). Then, setting $\boldsymbol{u} = \boldsymbol{v} + \alpha \boldsymbol{b} \in \hat{\boldsymbol{V}}_{\boldsymbol{2}}(\Omega)$, we have for any $\boldsymbol{w} \in \boldsymbol{V_2}(\Omega)$:

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{w} \; d\boldsymbol{x} = < \boldsymbol{f}, \boldsymbol{w} >_\Omega$$

and, by (19) and (21),

$$\int_M \boldsymbol{u} \cdot \boldsymbol{n} \; d\sigma = \alpha. \tag{22}$$

Then, let Ω' be a connected open bounded subset of Ω. If $\boldsymbol{w} \in \boldsymbol{V_2}(\Omega')$, we easily show that

$$\int_{\Omega'} \nabla \boldsymbol{u} : \nabla \boldsymbol{w} \; d\boldsymbol{x} = < \boldsymbol{f}, \boldsymbol{w} >_{\Omega'} .$$

Now, let $\boldsymbol{w}$ be in $\boldsymbol{H}_0^1(\Omega')$. We define the linear continuous form $\boldsymbol{\mathcal{F}}$ by

$$\boldsymbol{\mathcal{F}}(\boldsymbol{w}) = -\int_{\Omega'} \nabla \boldsymbol{u} : \nabla \boldsymbol{w} \; d\boldsymbol{x} + < \boldsymbol{f}, \boldsymbol{w} >_{\Omega'} .$$

We have $\boldsymbol{\mathcal{F}} \in \boldsymbol{H}^{-1}(\Omega')$, $\boldsymbol{\mathcal{F}}$ is equal to zero on $\boldsymbol{V_2}(\Omega')$ and consequently on $\boldsymbol{\mathcal{V}}(\Omega')$. We apply a result established by Girault and Raviart in bounded domain (see [15]) to deduce from this that there exists $p \in L^2(\Omega')$, unique up to an additive constant, such that

$$\nabla p = \boldsymbol{\mathcal{F}} \text{ in } \Omega'. \tag{23}$$

This permits us to prove the following lemma :

Lemma 3.2. *There exists $\pi \in L^2_{loc}(\overline{\Omega})$ such that for any $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$, we have*

$$\int_\Omega \pi \; div \; \boldsymbol{\psi} \; d\boldsymbol{x} = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{\psi} \; d\boldsymbol{x} - < \boldsymbol{f}, \boldsymbol{\psi} >_\Omega \tag{24}$$

Proof- Let $(B_m)_{m \in \mathbb{N}^*}$ an increasing sequence of open balls included in $\mathbb{R}^n$. We set, for any $m \geq 1$, $\Omega_m = B_m \cap \Omega$. For any $m \geq 1$, we know, thanks to (23), that there exists $p_m \in L^2(\Omega_m)$ such that $\nabla p_m = \boldsymbol{\mathcal{F}_m}$ in Ω_m, where $\boldsymbol{\mathcal{F}_m}$ is defined, for any $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega_m)$, by

$$\boldsymbol{\mathcal{F}_m}(\boldsymbol{w}) = -\int_{\Omega_m} \nabla \boldsymbol{u} : \nabla \boldsymbol{w} \; d\boldsymbol{x} + < \boldsymbol{f}, \boldsymbol{w} >_{\Omega_m} .$$

Moreover, we easily notice that in Ω_m, we have $\boldsymbol{\mathcal{F}_m} = \boldsymbol{\mathcal{F}_{m+1}}$ which implies that, for any $m \geq 1$, $\nabla p_m = \nabla p_{m+1}$ in Ω_m. As each Ω_m is connected, we deduce that each p_m is unique up to an additive constant, constant that we can choose in order to have $p_m = p_{m+1}$ in Ω_m. Thus, starting again, we construct a function π defined by :

$$\forall \; m \geq 1, \; \pi = p_m \text{ in } \Omega_m.$$

Because of the definition of the space $L^2_{loc}(\overline{\Omega})$, it becomes obvious that $\pi \in L^2_{loc}(\overline{\Omega})$. Now, let $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$, then, there exists $m \in \mathbb{N}^*$ such that supp $\boldsymbol{\psi} \subset \Omega_m$. Since $\pi = p_m$ in Ω_m and $\boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega_m)$, we have

$$\int_{\Omega_m} \pi \text{ div } \boldsymbol{\psi} \; d\boldsymbol{x} = \int_{\Omega_m} \nabla \boldsymbol{u} : \nabla \boldsymbol{\psi} \; d\boldsymbol{x} - < \boldsymbol{f}, \boldsymbol{\psi} >_{\Omega_m}$$

and consequently (24). $\square$

Thus, we have found $\boldsymbol{u} \in \hat{\boldsymbol{V}}_{\boldsymbol{2}}(\Omega)$ and $\pi \in L^2_{loc}(\overline{\Omega})$ such that $-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}$ in Ω. It remains us to find two real constants a_+ and a_- such that $\pi - a_+ \in L^2(\Omega_+)$ and $\pi - a_- \in L^2(\Omega_-)$. Let $\boldsymbol{\varphi}$ be in $\mathring{\boldsymbol{W}}{}^{1,2}_0(\Omega_+)$. We define $\boldsymbol{\mathcal{F}}_+ \in \boldsymbol{W}_0^{-1,2}(\Omega_+)$ by

$$\boldsymbol{\mathcal{F}}_+(\boldsymbol{\varphi}) = -\int_{\Omega_+} \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} \, d\boldsymbol{x} + < \boldsymbol{f}, \boldsymbol{\varphi} >_{\Omega_+} = -\int_{\Omega_+} \pi \text{ div } \boldsymbol{\varphi} \, d\boldsymbol{x}.$$

We notice that $\boldsymbol{\mathcal{F}}_+$ is equal to zero on $\hat{\boldsymbol{V}}_{\boldsymbol{2}}(\Omega_+)$, so $\boldsymbol{\mathcal{F}}_+ \in \hat{\boldsymbol{V}}^{\circ}_{\boldsymbol{2}}(\Omega_+)$. Moreover, considering Ω_+ as a perturbed half-space, we establish (see the previous section or [4]), that there exists $\pi_+ \in L^2(\Omega_+)$ such that $\nabla \pi_+ = \boldsymbol{\mathcal{F}}_+$. But, in $\Omega_+ \subset \Omega$, we have $\boldsymbol{\mathcal{F}}_+ = \nabla \pi$. So, $\nabla \pi = \nabla \pi_+$ and there exists a real constant a_+ such that $\pi - a_+ = \pi_+ \in L^2(\Omega_+)$. Then, we may proceed with the same reasoning for Ω_-.

Finally, we define $\overline{\pi} \in L^2(\Omega)$ by

$$\pi - a_+ \ \in \Omega_+, \quad \text{and} \quad \pi - a_- \ \in \Omega_-,$$

and γ_∞ by

$$\gamma_\infty = a_+ - a_-,$$

and we easily check that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\overline{\pi}\|_{L^p(\Omega)} + |\gamma_\infty| \leq \ C \ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + |\alpha|).$$

Thus, we have solved the problem (S_{200}) and consequently, we have the following theorem :

Theorem 3.3. *For any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$, $h \in L^2(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)$ and $\alpha \in \mathbb{R}$, there exists $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^2(\Omega_+)$, $\pi - a_- \in L^2(\Omega_-)$ and solution of $(\mathcal{S}_2)$. Moreover $\boldsymbol{u}$ is unique, π, a_+ and a_- are unique up to an additive and common constant and it holds*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \|\overline{\pi}\|_{L^2(\Omega)} + |\gamma_\infty| \leq \ C \ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)} + |\alpha|),$$

where C is a real positive constant which depends only on Ω.

Proof- It remains to prove that $\boldsymbol{u}$ is unique and that π, a_+ and a_- are unique up to an additive and common constant. We set :

$$\begin{aligned}\mathcal{B}^2_0(\Omega) = \{(\boldsymbol{z}, \pi, a_+, a_-) \in \mathring{\boldsymbol{W}}{}^{1,2}_0(\Omega) \times L^2_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}, \text{ with } \overline{\pi} \in L^2(\Omega), \\ -\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ and div } \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \int_M \boldsymbol{z} \cdot \boldsymbol{n} \, d\sigma = 0\}.\end{aligned}$$

Let $(\boldsymbol{u}, \pi, a_+, a_-) \in \mathcal{B}^2_0(\Omega)$. For any $\boldsymbol{v} \in \mathring{\boldsymbol{W}}{}^{1,2}_0(\Omega)$, we define the linear and continuous application $\boldsymbol{\ell} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ by

$$\boldsymbol{\ell}(\boldsymbol{v}) = \int_M \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma.$$

Let $\varphi \in \mathcal{D}(\Omega)$. Then, reminding that $\boldsymbol{n}$ is the unit normal vector on M directed to Ω_-, we have

$$
\begin{aligned}
&< -\Delta \boldsymbol{u} + \nabla \pi, \varphi >_\Omega \, = \, < -\Delta \boldsymbol{u}, \varphi >_\Omega - \int_\Omega \pi \text{ div } \varphi \; d\boldsymbol{x} \\
&= < -\Delta \boldsymbol{u}, \varphi >_\Omega - \int_\Omega \overline{\pi} \text{ div } \varphi \; d\boldsymbol{x} - \; a_+ \int_{\Omega_+} \text{div } \varphi \; d\boldsymbol{x} - \; a_- \int_{\Omega_-} \text{div } \varphi \; d\boldsymbol{x} \\
&=< -\Delta \boldsymbol{u} + \nabla \overline{\pi}, \varphi >_\Omega - \, a_+ \int_{\partial\Omega_+} \varphi \cdot \boldsymbol{n} \; d\boldsymbol{x} + \; a_- \int_{\partial\Omega_-} \varphi \cdot \boldsymbol{n} \; d\boldsymbol{x} \\
&= < -\Delta \boldsymbol{u} + \nabla \overline{\pi}, \varphi >_\Omega - \, (a_+ - a_-) \int_M \varphi \cdot \boldsymbol{n} \; d\boldsymbol{x} \\
&= < -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}, \varphi >_\Omega,
\end{aligned}
$$

i.e. $\mathbf{0} = -\Delta \boldsymbol{u} + \nabla \pi = -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}$ in Ω. Now, let $\boldsymbol{v}$ be in $\overset{\circ}{\boldsymbol{W}}{}^{1,2}_{0}(\Omega)$, we have $< -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}, \boldsymbol{v} >_\Omega = 0$ and so

$$
\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \; d\boldsymbol{x} - \int_\Omega \overline{\pi} \text{ div } \boldsymbol{v} \; d\boldsymbol{x} - \gamma_\infty \int_M \boldsymbol{v} \cdot \boldsymbol{n} \; d\sigma = 0.
$$

But $\boldsymbol{u} \in \overset{\circ}{\boldsymbol{W}}{}^{1,2}_{0}(\Omega)$, div $\boldsymbol{u} = 0$ in Ω and $\int_M \boldsymbol{u} \cdot \boldsymbol{n} \; d\sigma = 0$. Thus, $\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} = 0$ which implies that $\boldsymbol{u}$ is a constant vector which is equal to zero because $\boldsymbol{u} = \mathbf{0}$ on Γ. Consequently, π is constant in Ω. So $\pi - a_+ \in L^2(\Omega_+)$ is constant and since Ω_+ is not bounded, $\pi = a_+$ on Ω_+. With the same reasoning, we establish that $\pi = a_-$ on Ω_- and since π is constant in Ω, we have $\pi = a_+ = a_-$ in Ω and our result. $\square$

3.2 Case $p \neq 2$.

First, we suppose that $p > 2$ and we study the kernel of te Stokes system. We set :

$$
\begin{aligned}
\mathcal{B}^p_0(\Omega) = \{(\boldsymbol{z}, \pi, a_+, a_-) \in \overset{\circ}{\boldsymbol{W}}{}^{1,p}_{0}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}, \text{ with } \overline{\pi} \in L^p(\Omega), \\
- \Delta \boldsymbol{z} + \nabla \pi = \mathbf{0} \text{ and div } \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \int_M \boldsymbol{z} \cdot \boldsymbol{n} \; d\sigma = 0\}.
\end{aligned}
$$

Theorem 3.4. *We have* $\mathcal{B}^p_0(\Omega) = \{\lambda(\mathbf{0}, 1, 1, 1), \lambda \in \mathbb{R}\}$.

Proof- Let $(\boldsymbol{z}, \pi, a_+, a_-)$ be in $\mathcal{B}^p_0(\Omega)$ and $(\boldsymbol{z_1}, \overline{\pi}_1) = (\psi_1 \boldsymbol{z}, \psi_1 \overline{\pi}) \in \overset{\circ}{\boldsymbol{W}}{}^{1,p}_{0}(\Omega) \times L^p(\Omega)$. Since supp $(\boldsymbol{z_1}, \overline{\pi}_1) \subset G$ which is bounded, we have $(\boldsymbol{z_1}, \overline{\pi}_1) \in \boldsymbol{W}^{1,2}_0(\Omega) \times L^2(\Omega)$. In $\mathbb{R}^n_+$, we notice that $\overline{\pi} = \pi - a_+$, so $\nabla \overline{\pi} = \nabla \pi$ which implies that $-\Delta \boldsymbol{z} + \nabla \overline{\pi} = \mathbf{0}$ in $\mathbb{R}^n_+$. We set $(\boldsymbol{z_2}, \overline{\pi}_2) = (\psi_2 \boldsymbol{z}, \psi_2 \overline{\pi}) \in \overset{\circ}{\boldsymbol{W}}{}^{1,p}_{0}(\Omega) \times L^p(\Omega)$ and

$$
\begin{aligned}
&\boldsymbol{f_+} = - \Delta \boldsymbol{z_2} + \nabla \overline{\pi}_2 = \Delta \boldsymbol{z_1} - \nabla \overline{\pi}_1) \in \boldsymbol{W}^{-1,2}_0(\mathbb{R}^n_+), \\
&h_+ = \text{div } \boldsymbol{z_2} = - \text{ div } \boldsymbol{z_1} \in L^2(\mathbb{R}^n_+).
\end{aligned}
$$

We may proceed with the same reasoning as in Theorem 2.4 to obtain that $(\boldsymbol{z_2}, \overline{\pi}_2) \in \boldsymbol{W}^{1,2}_0(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$. Now, working on $\mathbb{R}^n_-$ instead of $\mathbb{R}^n_+$ (even if we move the origin to a distance equal to d), we obtain too that $(\boldsymbol{z_2}, \overline{\pi}_2) \in$

$\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_{-d}) \times L^2(\mathbb{R}^n_{-d})$. Finally, since supp $(\boldsymbol{z_2}, \overline{\pi}_2) \subset \mathbb{R}^n_+ \cup \mathbb{R}^n_{-d}$, we conclude that $(\boldsymbol{z_2}, \overline{\pi}_2) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ and so $(\boldsymbol{z}, \overline{\pi}) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ too. Moreover, it is obvious that $\pi \in L^2_{loc}(\overline{\Omega})$ because $\pi \in L^p_{loc}(\overline{\Omega})$. Thus, we conclude that $(\boldsymbol{z}, \pi, a_+, a_-) \in \mathcal{B}_0^2(\Omega)$. $\square$

Now, supposing again that $p > 2$, we want to study the Stokes system with homogeneous boundary conditions, that is to say : for $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\alpha \in \mathbb{R}$, we want to find $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^p(\Omega_+)$, $\pi - a_- \in L^p(\Omega_-)$ and

$$(\mathcal{S}_{20}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = \alpha. \end{cases}$$

First, we establish the following lemma :

Lemma 3.5. *For each $p > 2$, for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$ with a compact support in Ω and for any $\alpha \in \mathbb{R}$, there exists $(\boldsymbol{u}, \pi, a_+, a_-) \in (\boldsymbol{W}_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ solution of (S_{20}) with*

$$\pi - a_+ \in L^p(\Omega_+) \cap L^2(\Omega_+) \quad \text{and} \quad \pi - a_- \in L^p(\Omega_-) \cap L^2(\Omega_-).$$

Moreover $\boldsymbol{u}$ is unique and π, a_+ and a_- are unique up to an additive and common constant.

Proof- Let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$ with a compact support in Ω and $\alpha \in \mathbb{R}$. Then, since $p > 2$, we easily check that $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and $h \in L^2(\Omega)$ and we deduce from Theorem 3.3 that there exists $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ solution of (S_{20}) where $\boldsymbol{u}$ is unique and π, a_+ and a_- are unique up to an additive and common constant. It remains to show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega})$. For any $\boldsymbol{v} \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega)$, we define the linear and continuous application $\boldsymbol{\ell} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ by $\boldsymbol{\ell}(\boldsymbol{v}) = \displaystyle\int_M \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma$, and we recall that $-\Delta \boldsymbol{u} + \nabla \pi = -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}$ in Ω. We set, for $i = 1, 2$, $(\boldsymbol{u_i}, \overline{\pi}_i) = (\psi_i \boldsymbol{u}, \psi_i \overline{\pi}) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega) \times L^2(\Omega)$ and we notice that $(u_1, \overline{\pi}_1)$ has a compact support included in G. Elementaries calculus show that we have

$$\begin{cases} -\Delta \boldsymbol{u_1} + \nabla \overline{\pi}_1 = \psi_1 \boldsymbol{f} + \boldsymbol{F}_1 + \psi_1 \gamma_\infty \boldsymbol{\ell} & \text{in } G, \\ \text{div } \boldsymbol{u_1} = \psi_1 h + H_1 & \text{in } G, \\ \boldsymbol{u_1} = \boldsymbol{0} & \text{on } \partial G, \end{cases}$$

where

$$\boldsymbol{F}_1 = -(2 \nabla \boldsymbol{u} \nabla \psi_1 + \boldsymbol{u} \Delta \psi_1) + \overline{\pi} \nabla \psi_1 \in \boldsymbol{L}^2(G), \quad \text{and} \quad H_1 = \boldsymbol{u} \cdot \nabla \psi_1 \in \boldsymbol{H}^1(G).$$

Noticing that the support of $\boldsymbol{\ell}$, subset of M, is compact and using the same reasoning as Lemma 2.5 for the perturbed half-space, we conclude that we have $(\boldsymbol{u_1}, \overline{\pi}_1) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ and here again, we do like Lemma 2.5 to obtain that $(\boldsymbol{u_2}, \overline{\pi}_2) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$. Thus $(\boldsymbol{u}, \overline{\pi}) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ and the estimate follows immediately. Finally, we easily deduce from this, since $\overline{\pi} \in L^p(\Omega)$, that $\pi \in L^p_{loc}(\overline{\Omega})$ and we have our result. $\square$

Now, we establish the following theorem :

Theorem 3.6. *For any $p > 2$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, there exists a unique $(\boldsymbol{u}, \pi, a_+, a_-) \in (\boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R})/\mathcal{B}_0^p(\Omega)$ with $\pi - a_+ \in L^p(\Omega_+)$ and $\pi - a_- \in L^p(\Omega_-)$ solution of*

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = \alpha, \end{cases}$$

and satisfying

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\overline{\pi}\|_{L^p(\Omega)} + |\gamma_\infty| \leq \; C \; (\|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)} + |\alpha|),$$

where C is a real positive constant which depends only on p and Ω.

Proof- The idea is the same as Theorem 2.6. First, we use results in bounded domains to say that there exists $(\boldsymbol{z}, \theta) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{z} + \nabla \theta = \boldsymbol{\sigma} \text{ in } \Omega, \quad \text{div } \boldsymbol{z} = \psi \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{g_1} \text{ on } \Gamma,$$

where $\boldsymbol{\sigma} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $\psi \in L^p(\Omega)$ have a compact support. Using Lemma 3.5, then there exists $(\boldsymbol{t}, \tau, a'_+, a'_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} -\Delta \boldsymbol{t} + \nabla \tau = -\boldsymbol{\sigma} & \text{in } \Omega, \quad \text{div } \boldsymbol{t} = -\psi \quad \text{in } \Omega, \\ \boldsymbol{t} = \boldsymbol{0} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{t} \cdot \boldsymbol{n} \, d\sigma = \frac{1}{2}\alpha - \int_M \boldsymbol{z} \cdot \boldsymbol{n} \, d\sigma, \end{cases}$$

with $\tau - a'_+ \in L^p(\Omega_+)$ and $\tau - a'_- \in L^p(\Omega_-)$. Noticing that $\theta \in L^p(\Omega) \subset L^p_{loc}(\overline{\Omega})$, we deduce that $(\boldsymbol{v} = \boldsymbol{z} + \boldsymbol{t}, \mu = \theta + \tau, a'_+, a'_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \mu = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{g_1} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma = \frac{1}{2}\alpha, \end{cases}$$

with $\mu - a'_+ \in L^p(\Omega_+)$ and $\mu - a'_- \in L^p(\Omega_-)$. Next, we follow again the same ideas as Theorem 2.6. First, we use results in $\mathbb{R}^n_+$ and in $\mathbb{R}^n_{-d}$ and we extend by $(\boldsymbol{0}, 0)$ in Ω. Then, summing the two found pairs, we construct $(\boldsymbol{r}, \alpha) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{r} + \nabla \alpha = \boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \boldsymbol{r} = 0 \text{ in } \Omega, \quad \boldsymbol{r} = \boldsymbol{g_2} \text{ on } \Gamma.$$

Then, using like previously Lemma 3.5, we are able to find $(\boldsymbol{w}, \eta, a''_+, a''_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \eta = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \quad \text{in } \Omega, \\ \boldsymbol{w} = \boldsymbol{g_2} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{w} \cdot \boldsymbol{n} \, d\sigma = \frac{1}{2}\alpha, \end{cases}$$

with $\eta - a''_+ \in L^p(\Omega_+)$ and $\eta - a''_- \in L^p(\Omega_-)$. Finally $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi = \mu + \eta, a_+ = a'_+ + a''_+, a_- = a'_- + a''_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of our problem and the estimate follows immediately. $\square$

Theorem 3.7. *For any $p > 2$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$, there exists $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ solution of $(\mathcal{S}_{20})$. Moreover, $\boldsymbol{u}$ is unique, π , a_+ and a_- are unique up to an additive and common constant and we have*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\overline{\pi}\|_{L^p(\Omega)} + |\gamma_\infty| \leq \ C \ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + |\alpha|),$$

where C is a real positive constant which depends only on p and Ω.

Proof- The uniqueness comes from Theorem 3.4. Then, there exists, as a consequence of Theorem 1.1 ii), a tensor of the second order $F \in [L^p(\Omega)]^{n\times n}$ such that div $F = \boldsymbol{f}$. We extend F (respectively h) by 0 in $\mathbb{R}^n$, and we denote by $\widetilde{F}$ (respectively $\widetilde{h}$) this extension. Then, we set $\widetilde{\boldsymbol{f}} = \text{div } \widetilde{F}$. We have $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n)$ and $\widetilde{h} \in L^p(\mathbb{R}^n)$. By [1], there exists $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \widetilde{\boldsymbol{f}} \text{ in } \mathbb{R}^n \quad \text{and} \quad \text{div } \boldsymbol{v} = \widetilde{h} \text{ in } \mathbb{R}^n.$$

We denote again by $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and $\eta \in L^p(\Omega) \subset L^p_{loc}(\overline{\Omega})$ the restrictions of $\boldsymbol{v}$ and η to Ω. We have $\boldsymbol{v}_{|\Gamma} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, thus, thanks to Theorem 3.6, there exists $(\boldsymbol{w}, \tau, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ solution of

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \quad \text{in } \Omega, \\ \boldsymbol{w} = -\boldsymbol{v}_{|\Gamma} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{w} \cdot \boldsymbol{n} \, d\sigma = \alpha - \int_M \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma, \end{cases}$$

with $\tau - a_+ \in L^p(\Omega_+)$ and $\tau - a_- \in L^p(\Omega_-)$. Finally, $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi = \eta + \tau, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of $(\mathcal{S}_{20})$ and the estimate follows immediately. $\square$

Now, we suppose that p is such that $p < 2$ and we want to solve $(\mathcal{S}_{20})$. Since $p < 2$, its dual exponent p' satisfies $p' > 2$. So, if $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p'}(\Omega)$, $h \in L^{p'}(\Omega)$ and $\alpha \in \mathbb{R}$, there exists, thanks to Theorem 3.7, $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p'}(\Omega) \times L^{p'}_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = \alpha, \end{cases}$$

with $\pi - a_+ \in L^{p'}(\Omega_+)$ and $\pi - a_- \in L^{p'}(\Omega_-)$. We easily notice that this is equivalent to say that, for any $(\boldsymbol{f}, h, \alpha) \in \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R}$, the following problem $(\overline{\mathcal{S}}_{20})$

$$-\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell} = \boldsymbol{f} \text{ in } \Omega, \quad \text{div } \boldsymbol{u} = h \text{ in } \Omega, \quad \boldsymbol{\ell}(\boldsymbol{u}) = \alpha,$$

possesses a unique solution $(\boldsymbol{u}, \overline{\pi}, \gamma_\infty) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R}$ (the uniqueness comes from the definition of $\mathcal{B}_0^{p'}(\Omega)$). So, the mapping

$$\begin{aligned} S : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R} &\to \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R} \\ (\boldsymbol{u}, \overline{\pi}, \gamma_\infty) &\mapsto (-\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}, -\text{div } \boldsymbol{u}, -\boldsymbol{\ell}(\boldsymbol{u})) \end{aligned}$$

is an isomorphism. Furthermore, we have, for any $(\boldsymbol{u}, \overline{\pi}, \gamma_\infty) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R}$ and $(\boldsymbol{v}, \overline{\eta}, \theta_\infty) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega) \times \mathbb{R}$,

$$
\begin{aligned}
&< -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}, \boldsymbol{v} >_\Omega - \int_\Omega \operatorname{div} \boldsymbol{u}\, \overline{\eta}\, d\boldsymbol{x} \ - \theta_\infty \boldsymbol{\ell}(\boldsymbol{u}) \\
&= < -\Delta \boldsymbol{v} + \nabla \overline{\eta} - \theta_\infty \boldsymbol{\ell}, \boldsymbol{u} >_\Omega - \int_\Omega \operatorname{div} \boldsymbol{v}\, \overline{\pi}\, d\boldsymbol{x} \ - \gamma_\infty \boldsymbol{\ell}(\boldsymbol{v}).
\end{aligned}
$$

Thus, by duality

$$
\begin{aligned}
S^* : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega) \times \mathbb{R} &\to \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega) \times \mathbb{R} \\
(\boldsymbol{v}, \overline{\eta}, \theta_\infty) &\mapsto (-\Delta \boldsymbol{v} + \nabla \overline{\eta} - \theta_\infty \boldsymbol{\ell}, -\operatorname{div} \boldsymbol{v}, -\boldsymbol{\ell}(\boldsymbol{v}))
\end{aligned}
$$

is also an isomorphism, *i.e.*, when $p < 2$, there exists a unique $(\boldsymbol{v}, \overline{\eta}, \theta_\infty) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega) \times \mathbb{R}$ solution of $(\overline{\mathcal{S}}_{20})$. Finally, it remains to return to the problem $(\mathcal{S}_{20})$. For this, let c_+ and c_- be two constants such that $\theta_\infty = c_+ - c_-$. We set $\eta = \overline{\eta} + c_+$ in Ω_+, and $\eta = \overline{\eta} + c_-$ in Ω_-, and we easily show that

$$
-\Delta \boldsymbol{v} + \nabla \eta = -\Delta \boldsymbol{v} + \nabla \overline{\eta} - \theta_\infty \boldsymbol{\ell} \text{ in } \Omega.
$$

Thus, $(\boldsymbol{v}, \eta, c_+, c_-) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of $(\mathcal{S}_{20})$. For the kernel, let $(\boldsymbol{w}, \mu, k_+, k_-) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ be an other solution of $(\mathcal{S}_{20})$. So, setting $\overline{\mu} = \mu - k_+$ in Ω_+, $\overline{\mu} = \mu - k_-$ in Ω_-, and $\alpha_\infty = k_+ - k_-$, we easily check that $(\boldsymbol{w}, \overline{\mu}, \alpha_\infty)$ is solution of $(\overline{\mathcal{S}}_{20})$, problem which admits a unique solution. Consequently, $\boldsymbol{w} = \boldsymbol{v}$, $\overline{\mu} = \overline{\eta}$ and $\alpha_\infty = \theta_\infty$ and so, there exists $\lambda \in \mathbb{R}$ such that $k_+ = c_+ + \lambda$, and $k_- = c_- + \lambda$. We easily deduce from this that $\eta = \mu - \lambda$ and thus, the kernel of the problem when $p < 2$ is again $\mathcal{B}_0^p(\Omega)$.

Finally, it remains to return to the general problem when $p \neq 2$. Thanks to Lemma 1.2, there exists $\boldsymbol{u_g} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\boldsymbol{u_g} = \boldsymbol{g}$ on Γ and satisfying

$$
\|\boldsymbol{u_g}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C \, \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)},
$$

and thanks to previous results, we have seen that there exists a unique $(\boldsymbol{v}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^p(\Omega_+)$, $\pi - a_- \in L^p(\Omega_-)$ and

$$
\begin{cases}
-\Delta \boldsymbol{v} + \nabla \pi = \boldsymbol{f} - \Delta \boldsymbol{u_g} & \text{in } \Omega, \quad \operatorname{div} \boldsymbol{v} = h - \operatorname{div} \boldsymbol{u_g} \quad \text{in } \Omega, \\
\boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma = \alpha - \int_M \boldsymbol{u_g} \cdot \boldsymbol{n} \, d\sigma.
\end{cases}
$$

Finally, the function $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{u_g}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of $(\mathcal{S}_2)$ and the estimate follows immediately. In consequence, we have the following theorem :

Theorem 3.8. *For any $p > 1$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$ and $\alpha \in \mathbb{R}$, there exists $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^p(\Omega_+)$, $\pi - a_- \in L^p(\Omega_-)$ and solution of $(\mathcal{S}_2)$. Moreover $\boldsymbol{u}$ is unique, π, a_+ and a_- are unique up to an additive and common constant and it holds*

$$
\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\overline{\pi}\|_{L^p(\Omega)} + |\gamma_\infty| \leq C \, (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)} + |\alpha|),
$$

where C is a real positive constant which depends only on p and Ω.

3.3 Regularity result

Here, we want to give a regularity result for the problem (S_2). For this, we define the space :

$$D_1^p(\Omega) = \{\pi \in L^p_{loc}(\overline{\Omega}), \rho\nabla\pi \in L^p(\Omega)\}.$$

We have the following regularity theorem :

Theorem 3.9. *For any $p > 1$ satisfying $\dfrac{n}{p'} \neq 1$ and for any $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\Omega)$, $h \in W_1^{1,p}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}_1^{2-\frac{1}{p},p}(\Gamma)$ and $\alpha \in \mathbb{R}$, there exists a unique $(\boldsymbol{u}, \pi, a_+, a_-) \in (\boldsymbol{W}_1^{2,p}(\Omega) \times D_1^p(\Omega) \times \mathbb{R} \times \mathbb{R})/\mathcal{B}_0^p(\Omega)$ such that $\pi - a_+ \in W_1^{1,p}(\Omega_+)$, $\pi - a_- \in W_1^{1,p}(\Omega_-)$ solution of $(\mathcal{S}_2)$. Moreover $\boldsymbol{u}$ is unique and π, a_+ and a_- are unique up to an additive and common constant.*

Proof- Thanks to Lemma 1.2, it is easily to show that it is sufficient to solve the problem with $\boldsymbol{g} = \boldsymbol{0}$. Now, we notice that we have the continuous injections $\boldsymbol{W}_1^{0,p}(\Omega) \subset \boldsymbol{W}_0^{-1,p}(\Omega)$ because $\dfrac{n}{p'} \neq 1$ and $W_1^{1,p}(\Omega) \subset L^p(\Omega)$. Thus, thanks to Theorems 2.3 and 3.5, there exists $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ solution of $(\mathcal{S}_{20})$ and such that $\pi - a_+ \in L^p(\Omega_+)$, and $\pi - a_- \in L^p(\Omega_-)$. Then, it remains to show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times D_1^p(\Omega)$ and that $\pi - a_+ \in W_1^{1,p}(\Omega_+)$, and $\pi - a_- \in W_1^{1,p}(\Omega_-)$. We set $(\boldsymbol{u_i}, \pi_i) = (\psi_i \boldsymbol{u}, \psi_i \pi)$, for $i = 1, 2$. An elementary calculation shows that we have

$$\begin{cases} -\Delta \boldsymbol{u_1} + \nabla \pi_1 = \psi_1 \boldsymbol{f} + \boldsymbol{F}_1 & \text{in } G, \\ \text{div } \boldsymbol{u_1} = \psi_1 h + H_1 & \text{in } G, \\ \boldsymbol{u_1} = \boldsymbol{0} & \text{on } \partial G, \end{cases}$$

where

$$\boldsymbol{F}_1 = -(2\nabla\boldsymbol{u}\nabla\psi_1 + \boldsymbol{u}\Delta\psi_1) + \pi\nabla\psi_1 \in \boldsymbol{L}^p(G), \quad \text{and} \quad H_1 = \boldsymbol{u}\cdot\nabla\psi_1 \in \boldsymbol{W}^{1,p}(G).$$

Thanks to results in bounded domains and since supp $(\boldsymbol{u_1}, \pi_1)$ is included in G, we have

$$(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega).$$

Now, we define the function $\overline{\pi_2}$ by

$$\overline{\pi_2} = \pi_2 - a_+ \text{ in } \Omega_+, \quad \overline{\pi_2} = \pi_2 - a_- \text{ in } \Omega_-,$$

and we easily check that $\overline{\pi_2} \in L^p(\Omega)$. Now, considering the half-space $\mathbb{R}^n_+$ and noticing that $\nabla\overline{\pi_2} = \nabla\pi_2$ in $\mathbb{R}^n_+$, we define

$$\begin{aligned} \boldsymbol{f_2} &= -\,\Delta\boldsymbol{u_2} + \nabla\overline{\pi_2} = \boldsymbol{f} + \Delta\boldsymbol{u_1} - \nabla\pi_1 \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^n_+), \\ h_2 &= \text{div } \boldsymbol{u_2} = h - \text{ div } \boldsymbol{u_1} \in W_1^{1,p}(\mathbb{R}^n_+). \end{aligned}$$

By [8], there exists $(\boldsymbol{s}, \theta) \in (\boldsymbol{W}_1^{2,p}(\mathbb{R}^n_+) \times W_1^{1,p}(\mathbb{R}^n_+)) \subset (\boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+))$ solution of

$$-\Delta\boldsymbol{s} + \nabla\theta = \boldsymbol{f_2} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{s} = h_2 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{s} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

and since in $\boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ there is a unique solution of this problem, we have

$$(\boldsymbol{u_2}, \overline{\pi_2}) = (\boldsymbol{s}, \theta) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^n_+) \times W_1^{1,p}(\mathbb{R}^n_+).$$

We use the same reasoning in $\mathbb{R}^n_{-d}$ and since supp $(\boldsymbol{u_2}, \overline{\pi_2})$ is included in $\mathbb{R}^n_+ \cup \mathbb{R}^n_{-d}$, we have

$$(\boldsymbol{u_2}, \overline{\pi_2}) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega).$$

Finally, we deduce from this that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times D_1^p(\Omega)$ and that $\pi - a_+ \in W_1^{1,p}(\Omega_+)$ and $\pi - a_- \in W_1^{1,p}(\Omega_-)$. $\square$

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