An elementary proof of an inequality of Maz'ya involving L^1-vector fields

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Abstract

We give a short elementary proof of the inequality

$$||D(-\Delta)^{-1}\mathbf{f}||_{L^q(|x|^{n(q-1)-q}dx)} \leq c(||\mathbf{f}||_{L^1}+||\nabla(-\Delta)^{-1}\, div\, \mathbf{f}||_{L^1}), \mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n), 1 \leq q < \frac{n}{n-1},$$

essentially established by Maz'ya [4].

1 Introduction

For any $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$, we denote by $\mathbf{u} := (-\Delta)^{-1}\mathbf{f}$ the Newtonian (logarithmic for $n = 2$) potential of $\mathbf{f}$:

$$\mathbf{u}(x) = \int_{\mathbb{R}^n} \Gamma(x-y)\mathbf{f}(y)\, dy,$$

where Γ is the fundamental solution of $-\Delta$:

$$\Gamma(x) := \begin{cases} \dfrac{-1}{2\pi} \ln |x| & \text{if} \quad n = 2 \\ \dfrac{1}{|S^{n-1}|(n-2)} \dfrac{1}{|x|^{n-2}} & \text{if} \quad n \geq 3. \end{cases}$$

One of the main results in [3] states that

$$||D\mathbf{u}||_{L^{n'}(\mathbb{R}^n)} \leq C\,(||\mathbf{f}||_{L^1} + ||div\, \mathbf{f}||_{W^{-2,n'}}) \qquad (1)$$

where $n' = n/(n-1)$.

In [4], Maz'ya established the following family of estimates related to (1)

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Theorem 0 *Let* $1 \leq q < n'$ *and* $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. *Assume that* $\int_{\mathbb{R}^n} \mathbf{f} = \mathbf{0}$. *Let* $h := div\, \mathbf{f}$.
i)] If $q > 1$ *and* $\nabla(-\Delta)^{-1}h \in L^1$, *then*

$$||D\mathbf{u}||_{L^q(|x|^{n(q-1)-q}dx)} \leq c\left(||\mathbf{f}||_{L^1} + ||\nabla(-\Delta)^{-1}h||_{L^1}\right).$$

ii)] If $q = 1$ *and* $(-\Delta)^{-1/2}h \in \mathscr{H}^1$ *(where* $\mathscr{H}^1$ *denotes the Hardy space), then*

$$||D\mathbf{u}||_{L^1(|x|^{-1}dx)} \leq c\left(||\mathbf{f}||_{L^1} + ||(-\Delta)^{-1/2}h||_{\mathscr{H}^1}\right).$$

These estimates partly solve Open Problem 1 in [3].

The aim of this note is to unify the two statements of Theorem 0 and to present a proof both shorter and more elementary than the original one in [4]. Our result is

Theorem 1 *Let* $1 \leq q < n'$ *and* $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. *Let* $h := div\, \mathbf{f}$. *If* $\nabla(-\Delta)^{-1}h \in L^1$, *then*

$$||D\mathbf{u}||_{L^q(|x|^{n(q-1)-q}dx)} \leq c\left(||\mathbf{f}||_{L^1} + ||\nabla(-\Delta)^{-1}h||_{L^1}\right).$$

Remarks. *i)* In Theorem 0, it is required that $\int_{\mathbb{R}^n} \mathbf{f} = \mathbf{0}$. In fact, this equality is implied by the assumptions $\mathbf{f} \in L^1$ and $\nabla(-\Delta)^{-1}h \in L^1$.
ii) If $(-\Delta)^{-1/2}h \in H$, then $\nabla(-\Delta)^{-1}h \in L^1$, but the converse is false. Thus, when $q = 1$, Theorem 1 requires a weaker assumption than Theorem 0.

We start by proving *i)*. The Fourier transform of $\partial_j(-\Delta)^{-1}h$ is

$$F_j(\xi) = -\sum_k \frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}_k(\xi) = -\sum_k \frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}_k(0) + o(1) \text{ as } \xi \to 0.$$

The continuity of F_j at the origin implies $\widehat{f}_j(0) = 0$, i. e., $\int f_j = 0$.
We next briefly justify *ii)*. If $g := (-\Delta)^{-1/2}h \in \mathscr{H}^1$, then the Riesz transforms of g satisfy $R_j g \in L^1$, $1 \leq j \leq n$, so that $\nabla(-\Delta)^{-1}h = \imath(R_1 g, ..., R_n g) \in L^1$. In order to see that the converse is false, pick a temperate distribution g such that $R_j g \in L^1$, $1 \leq j \leq n$, but $g \notin L^1$. Such a g exists, see [5], 6.16, p. 184 and the references therein. If $\mathbf{f} := -\imath(R_1 g, \ldots, R_n g) \in L^1$, then $(-\Delta)^{-1/2} div\, \mathbf{f} = g \notin L^1$, while $\nabla(-\Delta)^{-1} div\, \mathbf{f} = -\mathbf{f} \in L^1$.

2 Proof of Theorem 1

Let $\rho_0 \in C_c^\infty(\mathbb{R}^+)$ be such that $0 \leq \rho_0 \leq 1$ and

$$\rho_0(r) = \begin{cases} 1 & \text{if } r \leq 1/4 \\ 0 & \text{if } r \geq 1/2 \end{cases}.$$

We introduce $\rho(y,x) = \rho_0(|y|/|x|)$ for $(y,x) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. For $1 \le k \le n$, we have

$$\partial_{x_k}\mathbf{u}(x) = c_n \int_{\mathbb{R}^n} \frac{x_k - y_k}{|x-y|^n}\mathbf{f}(y)\,dy = I_1(x) + I_2(x),$$

where

$$I_1(x) = c_n \int_{\mathbb{R}^n} \rho(y,x)\frac{x_k - y_k}{|x-y|^n}\mathbf{f}(y)\,dy\ ,\ I_2(x) = c_n \int_{\mathbb{R}^n} (1-\rho(y,x))\frac{x_k - y_k}{|x-y|^n}\mathbf{f}(y)\,dy.$$

We estimate $||I_2||_{L^q(|x|^{n(q-1)-q}dx)}$ using the following straightforward consequence of Hölder's inequality

$$\left\| x \mapsto \int f(y) g(x,y)\,dy \right\|_{L^q} \le \|f\|_{L^1} \sup_y \|g(\cdot,y)\|_{L^q(dx)}. \tag{2}$$

We have

$$|I_2(x)| \le c \int_{|y|\ge |x|/4} |\mathbf{f}(y)| \frac{dy}{|x-y|^{n-1}}$$

so that, by (2),

$$||I_2||_{L^q(|x|^{n(q-1)-q}dx)} \le c \int |\mathbf{f}(y)|\,dy \sup_{y\ne 0} \left\{ \int_{|x|\le 4|y|} \frac{|x|^{n(q-1)-q}}{|x-y|^{(n-1)q}} \right\}^{1/q}.$$

The quantity $\displaystyle\int_{|x|\le 4|y|} \frac{|x|^{n(q-1)-q}}{|x-y|^{(n-1)q}}\,dx$ is finite and does not depend on $y \ne 0$ (since it depends only on the norm of y and is homogeneous of degree 0). This implies that

$$||I_2||_{L^q(|x|^{n(q-1)-q}dx)} \le c \int |\mathbf{f}(y)|\,dy.$$

In order to estimate $||I_1||_{L^q(|x|^{n(q-1)-q}dx)}$, we note that for $|y| \le |x|/2$ we have

$$\left| \frac{x_k - y_k}{|x-y|^n} - \frac{x_k}{|x|^n} \right| \le c\frac{|y|}{|x|^n}.$$

Thus

$$|I_1(x)| \le c \left\{ \frac{1}{|x|^n} \int_{|y|\le |x|/2} |\mathbf{f}(y)||y|\,dy + J(x) \right\}$$

where

$$J(x) := \frac{1}{|x|^{n-1}} \left| \int_{\mathbb{R}^n} \rho(y,x)\mathbf{f}(y)\,dy \right|.$$

Using (2), we obtain

$$||I_1||_{L^q(|x|^{n(q-1)-q}dx)} \le c \int |\mathbf{f}(y)||y|\,dy \sup_{y\ne 0} \left\{ \int_{|x|\ge 2|y|} \frac{dx}{|x|^{n+q}} \right\}^{1/q}$$

$$+c||J||_{L^q(|x|^{n(q-1)-q}dx)} \le c||\mathbf{f}||_{L^1} + c||J||_{L^q(|x|^{n(q-1)-q}dx)}.$$

It then remains to estimate $||J||_{L^q(|x|^{n(q-1)-q}dx)}$.

To start with, we asume, in addition to the hypotheses of Theorem 1, that $\mathbf{f} \in C^\infty$. Then we have

$$0 = \int_{\mathbb{R}^n} div\ (y_1 \rho(y,x)\mathbf{f}(y))\ dy$$

$$= \int_{\mathbb{R}^n} \left[\rho(y,x) f_1(y) + \frac{y_1}{|y||x|} \rho_0' \left(\frac{|y|}{|x|} \right) \sum_i y_i f_i(y) + y_1 \rho(y,x) div\, \mathbf{f}(y) \right] dy.$$

Thus

$$\left| \int_{\mathbb{R}^n} \rho(y,x) f_1(y)\, dy \right| \leq c \int_{|y| \leq |x|/2} \frac{|y|}{|x|} |\mathbf{f}(y)|\, dy$$
$$+ \left| \int_{\mathbb{R}^n} y_1 \rho(y,x)\, div\, \mathbf{f}(y)\, dy \right|. \quad (3)$$

We claim that, with $h = div\, \mathbf{f}$, we have

$$\left| \int_{\mathbb{R}^n} y_1 \rho(y,x)\, div\, \mathbf{f}(y)\, dy \right| \leq c \int_{|y| \leq |x|/2} \frac{|y|}{|x|} |\nabla(-\Delta)^{-1} h(y)|\, dy. \quad (4)$$

Indeed, let $\mathbf{k} := \nabla(y_1 \rho(y,x))$ and

$$\mathbf{l} := \left(-\frac{\partial}{\partial y_2}(y_2 \rho(y,x)),\ \frac{\partial}{\partial y_1}(y_2 \rho(y,x)),\ 0, \ldots, 0 \right).$$

Then $div\, \mathbf{k} = \Delta(y_1 \rho(y,x))$, $div\, \mathbf{l} = 0$ and

$$(\mathbf{k}+\mathbf{l})_i = \begin{cases} y_1 \dfrac{\partial}{\partial y_1} \rho(y,x) - y_2 \dfrac{\partial}{\partial y_2} \rho(y,x) & \text{if } \ i = 1, \\ \\ y_1 \dfrac{\partial}{\partial y_2} \rho(y,x) + y_2 \dfrac{\partial}{\partial y_1} \rho(y,x) & \text{if } \ i = 2, \\ \\ y_1 \dfrac{\partial}{\partial y_i} \rho(y,x) \ \text{ if } \ i \geq 3. \end{cases}$$

Thus $|\mathbf{k}+\mathbf{l}| \leq c \dfrac{|y|}{|x|} \mathbf{1}_{|y| \leq |x|/2}$.

Therefore,

$$\int_{\mathbb{R}^n} y_1 \rho(y,x)\, div\, \mathbf{f}(y)\, dy = -\int_{\mathbb{R}^n} div\, (\mathbf{k}+\mathbf{l})(-\Delta)^{-1}(div\, \mathbf{f})$$
$$= \int_{\mathbb{R}^n} (\mathbf{k}+\mathbf{l}) \nabla(-\Delta)^{-1}(div\, \mathbf{f}).$$

By taking the absolute values in the above identity, we find that (4) holds under the additional assumption that $\mathbf{f}$ is smooth. The general case follows by noting that $(-\Delta)^{-1}\, div$ and $\nabla(-\Delta)^{-1}\, div$ commute with convolution of vector fields with a scalar mollifier. Applying (4) to $\mathbf{f} * \rho_\varepsilon$, where ρ is a compactly supported mollifier, and letting $\varepsilon \to 0$, we find that (4) holds for all $\mathbf{f}$.

Now, (3) and (4) imply that

$$\left| \int_{\mathbb{R}^n} \rho(y,x) f_1(y)\, dy \right| \le c \int_{|y|\le|x|/2} \frac{|y|}{|x|} \left(|\mathbf{f}(y)| + |\nabla(-\Delta)^{-1} h(y)| \right) dy.$$

The same is true with $\mathbf{f}$ instead of f_1 on the left hand side. Using again (2), it follows that

$$||J||_{L^q(|x|^{n(q-1)-q}dx)} \le c \left\{ ||\mathbf{f}||_{L^1} + ||\nabla(-\Delta)^{-1}h||_{L^1} \right\}. \qquad (5)$$

This completes the proof of Theorem 1.

□

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