# Nonparametric estimation for pure jump irregularly sampled or noisy Lévy processes

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# NONPARAMETRIC ESTIMATION FOR PURE JUMP IRREGULARLY SAMPLED OR NOISY LÉVY PROCESSES

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Abstract. In this paper, we study nonparametric estimation of the Lévy density for pure jump Lévy processes. We consider $n$ discrete time observations that may be irregularly sampled or possibly corrupted by a small noise independent of the main process. The case of non noisy observations with regular sampling interval has been studied by the authors in previous works which are the benchmark for the extensions proposed here. We study first the case of a regular sampling interval and noisy data, then the case of irregular sampling for non noisy data. In each case, non adaptive and adaptive estimators are proposed and risk bounds are derived. October 13, 2009

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## 1. Introduction

Recently, Lévy processes, *i.e.* processes with stationary independent increments, have become of common use in modeling financial data (see *e.g.* Eberlein and Keller (1995), Barndorff-Nielsen and Shephard (2001), Cont and Tankov (2004)). Statistical inference for such processes has been the subject of many recent contributions which, for the major part, focus on nonparametric estimation, as the parametric approach is rather difficult (see e.g. Figueroa-López (2009) and the references therein).

The distribution of a Lévy process $(L_t, t \geq 0)$ is completely specified by the characteristic function $\psi_t$ of the random variable $L_t$, given by the Lévy-Kintchine formula (see *e.g.* Bertoin (1996) or Sato (1999)). This is why nonparametric inference for Lévy processes is often based on the relation between the characteristic function $\psi_t$ and the characteristic triple (drift, Gaussian component, Lévy measure) of the process (see *e.g.* Watteel and Kulperberg (2003), Jongbloed and van der Meulen (2006), van Es *et al.* (2007), Neumann and Reiss (2009), Gugushvili (2009)).

In the simpler case where $(L_t)$ is of pure jump type, with finite variation on compact sets, nonparametric estimation of the Lévy measure is investigated in Comte and Genon-Catalot (2008, 2009) for discretely observed real-valued Lévy processes. More precisely, the present authors consider a Lévy process whose characteristic function has the form:

$$\phi_{L_t}(u) := \psi_t(u) = \mathbb{E}(\exp iuL_t) = \exp\left(t\int_{\mathbb{R}}(e^{iux}-1)n(x)dx\right), \quad (1)$$

where the Lévy density $n(x)$ satisfies

(H1) $\int_{\mathbb{R}} |x|n(x)dx < \infty.$

Nonparametric estimation of the function

$$g(x) = xn(x)$$

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is studied based on a discrete observation of the sample path with regular sampling interval $\Delta$. The statistical procedure relies on the i.i.d. sample $(L_{k\Delta} - L_{(k-1)\Delta},\ k = 1, \dots, n)$ with common characteristic function $\psi_\Delta(u)$. Due to (H1), $\mathbb{E}|L_\Delta| < \infty$ and using (1), the following relation holds:

$$-i\psi'_\Delta(u) = \mathbb{E}[L_\Delta e^{iuL_\Delta}] = g^*(u)\Delta\psi_\Delta(u), \tag{2}$$

where $g^*(u) = \int e^{iux}g(x)dx$ is the Fourier transform of $g$. Relation (2) suggests to build first an empirical estimator $\hat{g}^*$ of $g^*$ and then deduce an estimator of $g$ by Fourier inversion. For integrability purpose, we introduce a cutoff parameter $m$, and define:

$$\hat{g}_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-ixu}\hat{g}^*(u)du. \tag{3}$$

Then, a data-driven procedure must be found to select the appropriate cut-off parameter $\hat{m}$ leading to an adaptive estimator $\hat{g}_{\hat{m}}$. This strategy is developed in Comte and Genon-Catalot (2008) for low frequency data, *i.e.* when $\Delta$ is kept fixed and $n \to +\infty$, and in Comte and Genon-Catalot (2009) for high frequency data, *i.e.* $\Delta = \Delta_n \to 0$ while $n\Delta_n \to +\infty$.

Our aim in this paper is to show that the method can be extended to the case of noisy observations or irregular sampling interval. We consider $(L_t, t \geq 0)$ a Lévy process with characteristic function of the form (1) and Lévy density satisfying (H1). We focus on the estimation of $g$. At $n$ discrete instants $0 < t_1 < \dots < t_n$, we have at disposal noisy observations

$$U_k = L_{t_k} + \delta\varepsilon_k, \quad k = 1, \dots, n, \tag{4}$$

where $\delta > 0$ is a small parameter and $(\varepsilon_k)$ is a sequence of i.i.d. centered random variables with unit variance, independent of the process $(L_t)$. The statistical procedure and the Fourier strategy are adapted to the random variables:

$$V_k = U_k - U_{k-1} = Z_k + \delta\eta_k \tag{5}$$

with

$$Z_k = L_{t_k} - L_{t_{k-1}}, \quad \eta_k = \varepsilon_k - \varepsilon_{k-1}. \tag{6}$$

In Section 2, we present some notations and preliminary assumptions. Sections 3-4-5 are devoted to study the $L^2$-risks of the estimators (3) for a fixed cutoff parameter $m$, with $\hat{g}$ defined according to observations (5)-(6). The estimation method based on (3) generates a systematic bias which is common to all contexts of observations. In Section 3, the order of decay of the systematic bias is studied according to classes of regularity for the unknown function $g$ and some examples are investigated. Section 4 concerns the case where the sampling interval $t_k - t_{k-1} = \Delta$ is regular. For the sake of clarity, we recall the results of Comte and Genon-Catalot (2008, 2009) which deal with the non noisy case. Difficulties appearing due to the presence of the noise are discussed. Section 5 is devoted to the case of irregular sampling interval. In this case, for simplicity, we only study non noisy observations ($\delta = 0$). Two estimating strategies are proposed. Conditions on the sampling scheme $\{t_k = t_{k,n},\ k = 1, \dots, n\}$ are given generalizing the case where $\Delta_k = t_k - t_{k-1} = \Delta \to 0$ an $n\Delta \to +\infty$. Section 6 deals with the random cutoff case and adaptive estimation. Section 7 concludes the paper. Proofs are given in Section 8, except for the adaptive procedures. Indeed, the latter proofs are technical and lengthy. Full details may be found in Comte and Genon-Catalot (2008, 2009) for the regular sampling case with non noisy observations. In the Appendix, some auxiliary results are given.

## 2. Notations and preliminary assumptions.

For any complex valued function $h$ belonging to $\mathbb{L}^1(\mathbb{R})$, we denote by $h^*$ its Fourier transform defined as $h^*(u) = \int e^{iux}h(x)dx$. For integrable and square integrable functions $h, h_1, h_2$ we denote by $\|h\|$, $< h_1, h_2 >$, $h_1 \star h_2$ the quantities

$$\|h\| = \int |h(x)|^2 dx, \quad < h_1, h_2 >= \int h_1(x)\bar{h}_2(x)dx,$$

and

$$h_1 \star h_2(x) = \int h_1(y)\bar{h}_2(x-y)dy.$$

We recall that

$$(h^*)^*(x) = 2\pi h(-x), \quad < h_1, h_2 >= (2\pi)^{-1} < h_1^*, h_2^* > .$$

For a random variable $Y$, we denote by $\phi_Y(u) = \mathbb{E}(e^{iuY})$ its characteristic function. When $Y$ has finite expectation, we set

$$\theta_Y(u) = \mathbb{E}(Ye^{iuY}) = -i\phi_Y'(u). \tag{7}$$

As described in the introduction, in all settings of observations, we first propose an estimator $\hat{g}^*$ of $g^*$, and then deduce a collection of estimators $(\hat{g}_m)$ of $g$ depending on a cutoff parameter $m$. Each $\hat{g}_m$ is given by (3). Additional assumptions on $g$ are required:

(H2)($p$) For $p$ integer, $\int_{\mathbb{R}} |x|^{p-1}|g(x)|dx = \int_{\mathbb{R}} |x|^p n(x)dx < \infty$.
(H3) The function $g$ belongs to $\mathbb{L}^2(\mathbb{R})$.

Assumptions (H1) and (H2)($p$) are moment assumptions for the random variables $Z_k$. Under (H1), (H2)($p$) for $p > 1$ implies (H2)($k$) for $k \le p$. The required value of $p$ is given in each proposition or theorem.

For the risk bound computation, we define

$$g_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-ixu} g^*(u)du.$$

Notice that the Fourier transform of $g_m$ is given by $g_m^*(u) = g^*(u)1_{[-\pi m,\pi m]}(u)$ and analogously (see 3)) the Fourier transform of the estimator $\hat{g}_m$ is $\hat{g}_m^*(u) = \hat{g}^*(u)1_{[-\pi m,\pi m]}(u)$. To compute the $L^2$-risk of the estimator $\hat{g}_m$, the basic relation is the following:

$$\begin{aligned}
\|g - \hat{g}_m\|^2 &= \frac{1}{2\pi}\|g^* - g_m^* + g_m^* - \hat{g}_m^*\|^2 \\
&= \frac{1}{2\pi}\int_{|u|\ge \pi m} |g^*(u)|^2 du + \frac{1}{2\pi}\int_{-\pi m}^{\pi m} |\hat{g}^*(u) - g^*(u)|^2 du \\
&= \|g - g_m\|^2 + \|g_m - \hat{g}_m\|^2.
\end{aligned}$$

The $L^2$-orthogonality of the two terms is due to the disjoint supports of their Fourier transforms. The term $\|g_m - \hat{g}_m\|^2$ is a variance term, which increases with $m$ with a rate depending on $\hat{g}^*$. Whatever the estimator $\hat{g}^*$, there appears a common systematic square bias term produced by the method, which decreases as $m$ increases, given by

$$\|g - g_m\|^2 = \frac{1}{2\pi}\int_{|u|\ge \pi m} |g^*(u)|^2 du. \tag{8}$$

The order of this bias term is evaluated by considering classes of regularities for the function $g$ expressed in terms of $g^*$. Since the study of this term is common to all cases investigated here, we detail it first and give examples.

## 3. Systematic bias on examples.

Suppose $g$ belongs to the Sobolev class

$$\mathcal{C}(a,L)=\left\{g\in(\mathbb{L}^1\cap\mathbb{L}^2)(\mathbb{R}),\ \int(1+u^2)^a|g^*(u)|^2du\le L\right\}.$$

In that case,

$$\|g-g_m\|^2=\frac{1}{2\pi}\int_{|u|\ge\pi m}|g^*(u)|^2du\le\frac{L}{2\pi}(\pi m)^{-2a}.$$

If $g$ is analytic, *i.e.* belongs to a class

$$\mathcal{A}(\gamma,Q)=\{f,\ \int(e^{\gamma x}+e^{-\gamma x})^2|f^*(x)|^2dx\le Q\},\tag{9}$$

then (8) has order $O(\exp(-\gamma m))$.

Let us now look at examples of Lévy processes for which we can compute the order of (8). The last line of the tables below are useless now and to be read after Section 4.

**Example 1.** Compound Poisson processes.

Let $L_t=\sum_{i=1}^{N_t}Y_i$, where $(N_t)$ is a Poisson process with constant intensity $c$ and $(Y_i)$ is a sequence of i.i.d. random variables with density $f$ independent of the process $(N_t)$. Then, $(L_t)$ is a Lévy process with characteristic function (1) with $n(x)=cf(x)$. Since $f$ is integrable,

$$1\ge|\psi_t(u)|\ge\exp\left(-2ct\int_{\mathbb{R}}f(x)dx\right).$$

As $f$ is any density and $g(x)=cxf(x)$, any type of rate can be obtained. We summarize in Table 1 the bias orders obtained for several choices of $f$. Note that the specific problem of decompounding for known $c$ is studied in van Es *et al.* (2007).

| Density $f$ | Gaussian $\mathcal{N}(0,1)$ | Exponential $\mathcal{E}(1)$ | Uniform $\mathcal{U}([0,1])$ |
|---|---|---|---|
| $g(x)(=cxf(x))=$ | $cxe^{-x^2/2}/\sqrt{2\pi}$ | $cxe^{-x}\mathbf{1}_{\mathbb{R}^+}(x)$ | $cx\mathbf{1}_{[0,1]}(x)$ |
| $g^*(u)=$ | $ciue^{-u^2/2}$ | $c/(1-iu)^2$ | $c\dfrac{e^{iu}-1-iue^{iu}}{u^2}$ |
| $\int_{\lvert u\rvert\ge\pi m}\lvert g^*(u)\rvert^2du=$ | $O(me^{-\pi^2m^2})$ | $O(m^{-3})$ | $O(m^{-1})$ |
| $\int_{\lvert u\rvert\le\pi m}u^2\lvert g^*(u)\rvert^2du=$ | $O(1)$ | $O(1)$ | $O(m)$ |

Table 1. Bias order in three compound Poisson examples.

**Example 2.** The Lévy gamma process. Let $\alpha>0,\beta>0$. The Lévy Gamma process $(L_t)$ with parameters $(\beta,\alpha)$ is a subordinator such that, for all $t>0$, $L_t$ has distribution Gamma with parameters $(\beta t,\alpha)$, *i.e.* has density:

$$\frac{\alpha^{\beta t}}{\Gamma(\beta t)}x^{\beta t-1}e^{-\alpha x}1_{x\ge0}.\tag{10}$$

The characteristic function of $L_t$ is equal to:

$$\psi_t(u)=\mathbb{E}(e^{iuL_t})=\left(\frac{\alpha}{\alpha-iu}\right)^{\beta t}.\tag{11}$$

The Lévy density is $n(x) = \beta x^{-1}e^{-\alpha x}\mathbf{I}_{\{x>0\}}$ so that $g(x) = \beta e^{-\alpha x}\mathbf{I}_{\{x>0\}}$ satisfies our assumptions. We have: $g^*(u) = \beta/(\alpha - iu)$. Table 2 gives the bias orders.

**Example 2. (continued)** More generally, we consider the Lévy process $(L_t)$ with parameters $(\omega, \beta, c)$ and Lévy density

$$n(x) = cx^{\omega-1/2}x^{-1}e^{-\beta x}1_{x>0}.$$

For $\omega > 1/2$, $\int_0^{+\infty} n(x)dx < +\infty$, and we recover compound Poisson processes. For $0 < \omega \leq 1/2$, $\int_0^{+\infty} n(x)dx = +\infty$ and $g(x) = xn(x)$ belongs to $\mathbb{L}^2(\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R})$. This includes the case $\omega = 1/2$ of the Lévy Gamma process. We have:

$$g^*(u) = c\frac{\Gamma(\omega + 1/2)}{(\beta - iu)^{\omega+1/2}}.$$

Moreover

$$|\psi_t(u)| = \exp\left(-c\frac{t\Gamma(\omega + 1/2)}{1/2 - \omega}[(\beta^2 + u^2)^{-(\omega-1/2)/2} - \beta^{-(\omega-1/2)}]\right).$$

**Example 3.** The variance Gamma stochastic volatility model. This model was introduced by Madan and Seneta (1990).

Let $(W_t)$ be a Brownian motion, and let $(V_t)$ be an increasing Lévy process (subordinator), independent of $(W_t)$. Assume that the observed process is

$$L_t = W_{V_t}.$$

We have

$$\psi_t(u) = \mathbb{E}(e^{iuL_t}) = \mathbb{E}(e^{-\frac{u^2}{2}V_t}) = \left(\frac{\alpha}{\alpha + \frac{u^2}{2}}\right)^{t\beta}.$$

The Lévy measure of $(L_t)$ is equal to:

$$n_L(x) = \beta(2\alpha)^{1/4})|x|^{-1}\exp\left(-(2\alpha)^{1/2}|x|\right).$$

Setting $\tilde{\alpha} = (2\alpha)^{1/2}$, $\tilde{\beta} = \beta(2\alpha)^{1/4}$,

$$g(x) = \tilde{\beta}\exp(-\tilde{\alpha}x)\mathbf{I}_{x\geq 0} - \tilde{\beta}\exp(\tilde{\alpha}x)\mathbf{I}_{x<0} \Rightarrow g^*(x) = \frac{2i\tilde{\alpha}\tilde{\beta}x}{\tilde{\alpha}^2 + x^2}.$$

**Example 3 (continued).** The variance Gamma stochastic volatility model is a special case of bilateral Gamma process (see Küchler and Tappe (2008), Comte and Genon-Catalot (2008)). Consider the Lévy process $L_t$ with characteristic function

$$\psi_t(u) = \left(\frac{\alpha}{\alpha - iu}\right)^{\beta t}\left(\frac{\alpha'}{\alpha' + iu}\right)^{\beta' t}$$

and Lévy density

$$n(x) = |x|^{-1}(\beta e^{-\alpha x}\mathbf{I}_{(0,+\infty)}(x) + \beta' e^{-\alpha|x|}\mathbf{I}_{(-\infty,0)}(x)).$$

Bias orders are given in Table 2.

| Process | Example 2 | Ex.2 (continued) $\omega \in ]0,1/2[$ | Example 3 (continued) |
|---|---|---|---|
| $g^*(u) =$ | $\dfrac{\beta}{\alpha - iu}$ | $c\dfrac{\Gamma(\omega+1/2)}{(\beta - iu)^{\omega+1/2}}$ | $\dfrac{\beta}{\alpha - iu} - \dfrac{\beta'}{\alpha' - iu}$ |
| $\int_{\|u\|\geq \pi m} \|g^*(u)\|^2 du =$ | $O(1/m)$ | $O(1/m^{2\omega})$ | $O(1/m)$ |
| $\int_{\|u\|\leq \pi m} u^2 \|g^*(u)\|^2 du =$ | $O(m)$ | $O(m^{2-2\omega})$ | $O(m)$ |

TABLE 2. Bias order in examples 2, 2 (continued), 3 (continued).

## 4. ESTIMATORS IN THE CASE OF A REGULAR SAMPLING INTERVAL AND A FIXED CUT-OFF PARAMETER.

In this section, we build estimators of $g$ based on observations (5)-(6) with $t_k = k\Delta$ for $k = 1, \dots, n$. We separate the case of low frequency data ($\Delta$ fixed) and the case of high frequency data ($\Delta = \Delta(n)$ tends to 0 as $n$ tends to infinity while $n\Delta(n)$ tends to infinity). To have a better understanding of the definitions of the estimators, we first recall what was done in the case of non noisy observations ($\delta = 0$).

4.1. **Low frequency (LF).** We start with no noise as is done in Comte and Genon-Catalot (2008). Assume that $V_k = Z_k = L_{k\Delta} - L_{(k-1)\Delta}$, for $k = 1, \dots, n$. These r.v. are i.i.d. with common characteristic function $\psi_\Delta(u)$ (see (1)) satisfying under (H1):

$$g^*(u) = \frac{\theta_Z(u)}{\Delta \psi_\Delta(u)}, \tag{12}$$

with $\theta_Z(u) = \mathbb{E}(Z_1 e^{iuZ_1})$ (see (2) and (7)). To estimate $g^*$, we replace the numerator and the denominator above by empirical counterparts. Since the empirical estimator of the denominator may be null, we truncate it as in Neumann (1997) and Neumann and Reiss (2009). This gives the following low frequency (LF) estimator of $g^*$:

$$\hat{g}^*_{LF}(u) = \hat{\theta}_Z(u)/(\Delta \tilde{\psi}_\Delta(u)), \tag{13}$$

where

$$\hat{\theta}_Z(u) = \frac{1}{n}\sum_{k=1}^n Z_k e^{iuZ_k}, \tag{14}$$

$$\frac{1}{\tilde{\psi}_\Delta(u)} = \frac{1}{\hat{\psi}_\Delta(u)} \mathbf{1}_{|\hat{\psi}_\Delta(u)| > \kappa_\psi n^{-1/2}}, \quad \hat{\psi}_\Delta(u) = \frac{1}{n}\sum_{k=1}^n e^{iuZ_k}, \tag{15}$$

and $\kappa_\psi$ is a constant (that can be equal to one). Then, we build the estimator denoted by $\hat{g}_{m,LF}$ by formula (3) with $\hat{g}^* = \hat{g}^*_{LF}$.

Now, we turn to the noisy case. When $\delta \neq 0$, the r.v. $(V_k)$ given by (5) are identically distributed with common characteristic function

$$\phi_V(u) = \mathbb{E}(e^{iuV_1}) = \psi_\Delta(u)\phi_\eta(\delta u). \tag{16}$$

where the characteristic function of the $\eta_k$'s satisfies $\phi_\eta(u) = |\phi_\varepsilon(u)|^2$ and $\phi_\varepsilon$ is the common characteristic function of the $\varepsilon_k$'s. The question is how small must be $\delta$ for the procedure with

$\delta = 0$ to be still correct? Derivating (16), we get

$$\theta_V(u) = \mathbb{E}(V_k e^{iuV_k}) = \theta_Z(u)\phi_\eta(\delta u) + \delta\theta_\eta(\delta u)\psi_\Delta(u). \tag{17}$$

Thus, using (12), we find

$$\frac{\theta_V(u)}{\phi_V(u)} = \Delta g^*(u) + \delta\frac{\theta_\eta(\delta u)}{\phi_\eta(\delta u)}. \tag{18}$$

To build the new estimator of $g^*$, we set

$$\hat{\theta}_V(u) = \frac{1}{n}\sum_{k=1}^{n} V_k e^{iuV_k}, \quad \hat{\phi}_V(u) = \frac{2}{n}\sum_{k=1}^{[n/2]} e^{iuV_{2k}}. \tag{19}$$

Note that, since the $V_k$'s are no more independent because of the presence of the $\eta_k$'s, we construct $\hat{\phi}_V$ using only even indices $2k$ to maintain a sum of independent random variables. This point is useful in the proofs. Thus, we set, for $\kappa_V$ a constant (that can be taken equal to one):

$$\frac{1}{\tilde{\phi}_V(u)} = \frac{1}{\hat{\phi}_V(u)}\mathbb{I}_{|\hat{\phi}_V(u)|>\kappa_V n^{-1/2}}. \tag{20}$$

We propose the estimator

$$\hat{g}^*_{LFN}(u) = \hat{\theta}_V(u)/(\Delta\tilde{\phi}_V(u)),$$

The estimator of $g$, denoted by $\hat{g}_{m,LFN}$, is given by formula (3) with $\hat{g}^* = \hat{g}^*_{LFN}$:

$$\hat{g}_{m,LFN}(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-ixu}\hat{g}^*_{LFN}(u)du.$$

To obtain a risk bound for the estimators $\hat{g}_{m,LF}$ and $\hat{g}_{m,LFN}$, additional assumptions concerning $\psi_\Delta$, $g$ and the noise distribution are required.

(H4) There exist constants $c_\psi, C_\psi$ and $\beta \geq 0$ such that
$\forall x \in \mathbb{R},$ we have $c_\psi(1+u^2)^{-\Delta\beta/2} \leq |\psi_\Delta(u)| \leq C_\psi(1+u^2)^{-\Delta\beta/2}$.

(H5) There exists some positive $a$ such that $\int |g^*(u)|^2(1+u^2)^a du < +\infty$.

(H6) There exists some positive $c_0$ such that, $\forall u \in [-\pi,\pi]$, $|\phi_\eta(u)| \geq 1/c_0$.

(H7) $\mathbb{E}(\varepsilon_1^2) < \infty$ and $\mathbb{E}(\varepsilon_1) = 0$.

Note that we give separate assumptions on $\psi_\Delta$ and on $g$ since there may be no relation at all between $g^*$ and $\psi_\Delta$. For instance, in Example 1, $\beta = 0$ and $g^*$ can have any order of regularity. Assumption (H4) is used to compute rates of convergence for $\mathbb{L}_2$-risks. Exponential rates for $\psi_\Delta$ could also be considered (see Example 2 (continued)). Note that the assumptions on the noise distribution (H6)-(H7) are rather weak.

**Proposition 4.1.**

- *Under Assumptions* (H1)-(H2)(4)-(H3)*, for all m:*

$$\mathbb{E}(\|\hat{g}_{m,LF} - g\|^2) \leq \|g - g_m\|^2 + K\frac{\mathbb{E}^{1/2}(Z_1^4)\int_{-\pi m}^{\pi m} du/|\psi_\Delta(u)|^2}{n\Delta^2}, \tag{21}$$

*where $K$ is a constant.*

- *Under Assumptions* (H1)-(H2)(4)-(H3)*, for all m,*

$$\mathbb{E}(\|\hat{g}_{m,LFN} - g\|^2) \leq \|g - g_m\|^2 + K\frac{\mathbb{E}^{1/2}(V_1^4)\int_{-\pi m}^{\pi m} du/|\phi_V(u)|^2}{n\Delta^2} + 2\frac{\delta^2}{\Delta^2}\int_{-\pi m}^{\pi m} |\frac{\theta_\eta(\delta u)}{\phi_\eta(\delta u)}|^2 du, \tag{22}$$

*where $K$ is a positive constant.*

The first bound is proved in Comte and Genon-Catalot (2008). The second one is proved below (see Section 8). We will require $\delta$ to be small enough for the last term to be negligible when computing rates. This is obtained in the following corollary:

**Corollary 4.1.** *Under Assumptions* (H1)-(H2)(4)-(H3) *and* (H6)-(H7). *Then, for all $m \leq n$ and $\delta \leq 1/n$,*

$$\mathbb{E}(\|\hat{g}_{m,LFN} - g\|^2) \leq \|g - g_m\|^2 + Kc_0^2 \mathbb{E}^{1/2}(V_1^4) \frac{1}{n\Delta^2} \int_{-\pi m}^{\pi m} \frac{du}{|\psi_\Delta(u)|^2} + 2\frac{c_0^2 \mathbb{E}(\eta_1^2)}{n\Delta^2}. \tag{23}$$

**Remark 4.1.** *If $\theta_\eta(u)/\phi_\eta(u)$ is square-integrable on $\mathbb{R}$, we have:*

$$\delta^2 \int_{-\pi m}^{\pi m} |\frac{\theta_\eta(\delta u)}{\phi_\eta(\delta u)}|^2 du \leq \delta \int_{\mathbb{R}} |\frac{\theta_\eta(u)}{\phi_\eta(u)}|^2 du \propto \delta.$$

*The last noise-related term in inequality (22) is negligible for $\delta \leq 1/n$.*

Hence, if the noise level is small enough, its presence will not affect the rates of the risk bound. More precisely, suppose that $g$ belongs to the Sobolev class $\mathcal{C}(a, L)$. From Section 3, we have $\|g - g_m\|^2 = O(m^{-2a})$. Under (H4), the second term in (21)-(23), a variance term of the estimator, satisfies:

$$\frac{\int_{-\pi m}^{\pi m} du/|\psi_\Delta(u)|^2}{n\Delta} = O\left(\frac{m^{2\beta\Delta+1}}{n\Delta}\right).$$

The best compromise between the first and the second term in the risk bounds yields that the optimal choice for $m$ is $m = O((n\Delta)^{1/(2\beta\Delta+2a+1)})$. The resulting rate for the risk is $O((n\Delta)^{-2a/(2\beta\Delta+2a+1)})$. It is worth noting that the sampling interval $\Delta$ explicitly appears in the exponent of the rate. Therefore, for positive $\beta$, the rate is worse for large $\Delta$ than for small $\Delta$. Thus we can state the following result as a consequence of Proposition 4.1 and Corollary 4.1:

**Corollary 4.2.** *Assume that $g \in \mathcal{C}(a, L)$. Under assumptions* (H1)-(H2)(4)-(H3)-(H4)-(H6)-(H7)*, then*

$$\mathbb{E}(\|\hat{g}_{m,LFN} - g\|^2) = O(n\Delta)^{-2a/(2\beta\Delta+2a+1)}) \text{ when } m = O((n\Delta)^{1/(2\beta\Delta+2a+1)}).$$

*The same holds for $\hat{g}_{m,LF}$ without assumptions* (H6)-(H7).

We can illustrate these rates through the examples described in Section 3. The results are also summarized in Tables 1, 2, 3, 4.

**Example 1.** Compound Poisson processes.

In this case, $\beta = 0$. If $g$ belongs to the Sobolev class $\mathcal{C}(a, L)$, the upper bound of $L^2$-risk is of order $O((n\Delta)^{-2a/(2a+1)})$.

If $g$ is analytic, i.e. belongs to a class given by (9), then the bias satisfies $\|g - g_m\|^2 = O(e^{-2\gamma\pi m})$. Choosing $m = O(\ln(n\Delta))$, we obtain that the risk is of order $O(\ln(n\Delta)/(n\Delta))$.

**Example 2.** The Levy Gamma process.

We have $\int_{|u|\geq \pi m} |g^*(u)|^2 dx = O(m^{-1})$ and $\int_{[-\pi m, \pi m]} du/|\psi_\Delta(u)|^2 = O(m^{2\beta\Delta+1})$. The resulting rate is of order $(n\Delta)^{-1/(2\beta\Delta+2)}$ for a choice of $m$ of order $O((n\Delta)^{1/(2\beta\Delta+2)})$.

**Example 2, continued.** For the process described in example 2 (continued), $\int_{[-\pi m, \pi m]} du/|\psi_\Delta(u)|^2 = O(m^{\omega+1/2} \exp(\kappa m^{1/2-\omega}))$ and $\int_{|u|\geq \pi m} |g^*(u)|^2 dx = O(m^{-2\omega})$. In this case, choosing $\kappa m^{1/2-\omega} =$

$\ln(n\Delta)/2$ gives the rate $[\ln(n\Delta)]^{-2\omega}$ which is thus very slow. This case does not satisfy (H4) (which is not required for the general risk bound).

**Example 3.** For the Bilateral Gamma process with $(\beta,\alpha)=(\beta',\alpha')$, we have

$$\psi_\Delta(u)=\frac{\alpha^{\beta\Delta}}{(\alpha^2+u^2)^{\beta\Delta}},\quad g^*(u)=\frac{2i\beta\alpha u}{\alpha^2+u^2}.$$

Therefore $\int_{|u|\geq \pi m}|g^*(u)|^2dx=O(m^{-1})$ and $\int_{[-\pi m,\pi m]}du/|\psi_\Delta(u)|^2=O(m^{4\beta\Delta+1})$. The resulting rate is of order $(n\Delta)^{-1/(4\beta\Delta+2)}$ for a choice of $m$ of order $O((n\Delta)^{1/(4\beta\Delta+2)})$.

As can be seen from these examples, the relevant choice of $m$ depends on the unknown function, in particular on its smoothness. A model selection procedure that proposes a data driven criterion to select $m$ is presented in Section 6.

**Example of distributions for the noise.** Let us now give examples of noise distributions and the corresponding functions $\phi_\eta(u)=|\phi_\varepsilon|^2$ (to study (H6)) and $\theta_\eta(u)/\phi_\eta(u)$ (in relation with Remark 4.1).

• Ordinary smooth case: $\phi_\eta(u)=c/(1+u^2)^{\gamma/2}$ ($\gamma>0$) satisfies (H6) and

$$|\frac{\theta_\eta(u)}{\phi_\eta(u)}|=2\gamma|u|/(1+u^2)$$

is square integrable on $\mathbb{R}$.

Examples of such type of densities for the noise $\eta_k$ are given by Laplace densities (where the density of $\eta_k$ is $f_\eta(x)=(1/2)e^{|x|}$) or more generally, symmetrized Gamma densities.

• Super smooth case: $\phi_\eta(u)=c_1\exp(-c_2(1+u^2)^{c_3/2})$, $c_i>0$, $i=1,2$, $0<c_3\leq 2$, satisfies (H6) and

$$|\theta_\eta(u)/\phi_\eta(u)|=c_2c_3\frac{|u|}{(1+u^2)^{1-c_3/2}}.$$

The order of $|\theta_\eta(u)/\phi_\eta(u)|$ for large $u$ is $O(|u|^{c_3-1})$, that is $O(|u|)$ in the Gaussian case ($c_3=2$). Therefore, it is not square integrable.

4.2. **High frequency data (HF).** Now, the asymptotic setting is that $\Delta=\Delta(n)$ tends to 0 and $n\Delta$ tends to infinity. For simplicity, we omit the dependence on $n$ in the sampling interval $\Delta$. However, the benchmark for rates is now evaluated in terms of $n\Delta$, the total length time interval where observations are considered.

We start by defining the estimators in the case of non noisy observations ($\delta=0$). Since $\Delta$ is small, $\psi_\Delta(u)$ is close to 1 and we need not estimate it (see 12). Therefore, we construct the estimator $\hat{g}^*_{HF}$ of $g^*$ by simply setting

$$\hat{g}^*_{HF}(u)=\frac{\hat{\theta}_Z(u)}{\Delta},$$

with $\hat{\theta}_Z(u)$ given in (14). Then, as before $\hat{g}_{m,HF}(x)$ is given by (3) with $\hat{g}^*=\hat{g}^*_{HF}$. In this case, the integral (3) can be explicitly computed and yields an explicit formula for the estimator of $g$:

$$\hat{g}_{m,HF}(x)=\frac{1}{2\pi}\int_{-\pi m}^{\pi m}e^{-iux}\hat{g}^*_{HF}(u)du=\frac{m}{n\Delta}\sum_{k=1}^{n}Z_k\varphi(m(Z_k-x)),$$

where

$$\varphi(x) = \frac{\sin(\pi x)}{\pi x}, \quad (\text{with } \varphi(0) = 1). \tag{24}$$

Consequently, the estimator is easy to compute and appears as a kernel estimator with kernel $\varphi$ and bandwidth $1/m$.

Now, we look at the noisy observations and recall (see (16)) that $\phi_V(u) = \mathbb{E}(e^{iuV_1}) = \psi_\Delta(u)\phi_\eta(\delta u)$. Derivating, we get (see (17)): $\theta_V(u) = \Delta\psi_\Delta(u)g^*(u)\phi_\eta(\delta u) + \delta\theta_\eta(\delta u)\psi_\Delta(u)$.

Since both $\Delta$ and $\delta$ are now small, both $\psi_\Delta(u)$ and $\phi_\eta(\delta u)$ are close to 1. Thus, we propose the estimator $\hat{g}^*_{HFN}$ of $g^*$ given by:

$$\hat{g}^*_{HFN}(u) = \hat{\theta}_V(u)/\Delta, \tag{25}$$

with $\hat{\theta}_V(u)$ defined in (19). As previously, we define the estimator of $g$, $\hat{g}_{m,HFN}(x)$, using $\hat{g}^*_{HFN}$ and (3). Again, explicit integration is possible and yields:

$$\hat{g}_{m,HFN}(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-iux}\hat{g}^*_{HFN}(u)du = \frac{m}{n\Delta}\sum_{k=1}^{n} V_k\varphi(m(V_k - x)),$$

Now, we can prove the following result:

**Proposition 4.2.** *Assume that* (H2)*(2)-* (H3) *hold.*

- *For all positive $m$,*

$$\mathbb{E}(\|\hat{g}_{m,HF} - g\|^2) \leq \|g - g_m\|^2 + [\mathbb{E}(Z_1^2/\Delta)]\frac{m}{n\Delta} + \frac{\|g\|_1^2}{\pi}\Delta^2\int_{-\pi m}^{\pi m} u^2|g^*(u)|^2du.$$

- *Assume, moreover, that $H := \int |\theta_\eta(v)|^2dv < +\infty$. Then, for all $m$,*

$$\begin{aligned}\mathbb{E}(\|\hat{g}_{m,HFN} - g\|^2) \quad &\leq \quad \|g - g_m\|^2 + 12[\mathbb{E}(Z_1^2/\Delta) + (\delta^2/\Delta)\mathbb{E}(\eta_1^2)]\frac{m}{n\Delta} \\ &+\frac{3}{\pi}\Delta^2\|g\|_1^2\int_{-\pi m}^{\pi m} u^2|g^*(u)|^2du + \frac{3}{\pi}\delta^4\mathbb{E}^2(\eta_1^2)\int_{-\pi m}^{\pi m} u^4|g^*(u)|^2du + \frac{3}{2\pi}H\frac{\delta}{\Delta^2}. \end{aligned} \tag{26}$$

The first inequality is proved in Genon-Catalot and Comte (2009). The second one is proved below (see Section 8). In the high frequency framework, $\mathbb{E}(Z_1^2/\Delta)$ is bounded under (H2)(2). This is due to the fact that:

$$\mathbb{E}(Z_1^2) = \Delta m_2 + \Delta^2 m_1^2,$$

where $m_l = \int x^l n(x)dx$ is well defined for $l = 1, 2$ (see the Appendix).

Let us look at the last two terms of (26). Choosing $\delta = \Delta^4$ and assuming that $m \leq n\Delta$ and $n\Delta^3 \leq 1$ yields

$$\delta^4\int_{-\pi m}^{\pi m} u^4|g^*(u)|^2du \leq \Delta^{16}(\pi m)^2\int_{-\pi m}^{\pi m} u^2|g^*(u)|^2du \leq \pi^2\Delta^{12}\int_{-\pi m}^{\pi m} u^2|g^*(u)|^2du.$$

So, we can state:

**Corollary 4.3.** *Assume that assumptions* (H1), (H2)(2) *and* (H3) *are satisfied and that $H := \int |\theta_\eta(v)|^2dv < +\infty$. If in addition, $\delta = \Delta^4$, $n\Delta^3 \leq 1$ and $m \leq n\Delta$, then*

$$\begin{aligned}\mathbb{E}(\|\hat{g}_{m,HFN} - g\|^2) \quad &\leq \quad \|g - g_m\|^2 + 12\pi[\mathbb{E}(Z_1^2/\Delta) + \Delta^5\mathbb{E}(\eta_1^2)]\frac{m}{n\Delta} \\ &+C\Delta^2\|g\|_1^2\int_{-\pi m}^{\pi m} u^2|g^*(u)|^2du,\end{aligned}$$

*where $C$ is a constant depending on $H$, $\mathbb{E}(\eta_1^2)$.*

**Remark 4.2.** For $\delta = \Delta^3$ and $n\Delta^2 \le 1$, we get
$$\delta/\Delta^2 = \Delta \le \frac{1}{n\Delta},$$
which would also make the last term in (26) negligible.

Let us discuss the rates obtained for the risk. The compromise between bias and variance term correspond to the compromise between $\|g - g_m\|^2$ and $m/(n\Delta)$. If $g$ belongs to the Sobolev class $\mathcal{C}(a, L)$, the optimal choice for $m$ is $m = O((n\Delta)^{1/2a+1})$ and yields the rate $O((n\Delta)^{2a/2a+1})$. The other terms must be negligible.

The term containing $\int_{-\pi m}^{\pi m} u^2|g^*(u)|^2du$ is evaluated in Table 1 and 2 (last line) on the examples. In the general case where $g$ belongs to $\mathcal{C}(a, L)$, then
$$\Delta^2 \int_{-\pi m}^{\pi m} u^2|g^*(u)|^2du = \Delta^2 O(m^{2(1-a)^+}).$$
We restrict the choice of $m$ to $m \le n\Delta$. Therefore, if $a \ge 1$, we get the constraint $n\Delta^3 = O(1)$. If $a \in (0, 1)$, we need
$$\Delta^2 m^{2(1-a)^+} = O(m^{-2a}).$$
This holds under the condition $n\Delta^2 = O(1)$.

**Proposition 4.3.** *Assume that* (H1)-(H2)*(2)*-(H3) *hold and that* $H := \int |\theta_\eta(v)|^2dv < +\infty$. *Assume that $g$ belongs to $\mathcal{C}(a, L)$. If $n \to +\infty$, $\Delta \to 0$, $n\Delta^2 \le 1$ and $\delta = \Delta^4$, we have, for* $m = O((n\Delta)^{1/(2a+1)})$,
$$\mathbb{E}(\|\hat{g}_{m,HFN} - g\|^2) \le O((n\Delta)^{-2a/(2a+1)}).$$
*If $a \ge 1$, then it is enough to have $n\Delta^3 = O(1)$ (instead of $n\Delta^2 \le 1$).*

The rates corresponding to the different examples described in Section 3 are given in Tables 3 and 4. In the cases LF-LFN, the rates are to be read as functions of $n$. In the cases HF-HFN, the rates are measured as functions of $n\Delta$.

| Density $f$ | Gaussian $\mathcal{N}(0,1)$ | Exponential $\mathcal{E}(1)$ | Uniform $\mathcal{U}([0,1])$ |
|---|---|---|---|
| $g(x)(= cxf(x)) =$ | $cxe^{-x^2/2}/\sqrt{2\pi}$ | $cxe^{-x}\mathbf{1}_{\mathbb{R}^+}(x)$ | $cx\mathbf{1}_{[0,1]}(x)$ |
| Optimal $m =$ | $m = \sqrt{\ln(n\Delta)}/\pi$ | $m = O((n\Delta)^{1/4})$ | $m = O((n\Delta)^{1/2})$ |
| Rate $=$ | $O(\frac{\sqrt{\ln(n\Delta)}}{n\Delta})$ | $O((n\Delta)^{-3/4})$ | $O((n\Delta)^{-1/2})$ |

TABLE 3. Choice of $m$ and rates in three compound Poisson examples (Cases LF-LFN, set $\Delta = 1$ and HF-HFN, $\Delta$ small).

Let us give examples of noise distribution and study of the condition $H = \int |\theta(v)|^2dv < +\infty$.
• Ordinary smooth case: $\phi_\eta(u) = c/(1 + u^2)^{\gamma/2}$ $(\gamma > 0)$ gives
$$|\theta_\eta(u)| = \gamma|u|/(1 + u^2)^{1+\gamma/2}.$$
thus $\theta_\eta$ is square integrable on $\mathbb{R}$.
Examples of such type of densities for $\eta_k$ are given by Laplace distributions (with density $(1/2)e^{-|x|}$) or more generally, symmetrized Gamma densities.

| Process | Example 2 | Ex.2 (continued) $\omega \in ]0, 1/2[$ | Example 3 (continued) |
|---|---|---|---|
| $g^*(u) =$ | $\dfrac{\beta}{\alpha - iu}$ | $c\dfrac{\Gamma(\omega+1/2)}{(\beta - iu)^{\omega+1/2}}$ | $\dfrac{\beta}{\alpha - iu} - \dfrac{\beta'}{\alpha' - iu}$ |
| Optimal $m =$ | $O((n\Delta)^{1/2})$ | $O((n\Delta)^{1/(2\omega+1)})$ | $O((n\Delta)^{1/2})$ |
| Rate in case HF-HFN | $O((n\Delta)^{-1/2})$ | $O((n\Delta)^{-2\omega/(2\omega+1)})$ | $O((n\Delta)^{-1/2})$ |
| Rate in case LF-LFN | $O((n\Delta)^{-1/(2\beta\Delta+1)})$ | $O([\ln(n\Delta)]^{-2\omega})$ | $O((n\Delta)^{-1/(4\beta\Delta+1)})$ |

Table 4. Choice of $m$ and rates in examples 2, 2 (continued), 3 (continued), cases LF-LFN and HF-HFN.

• Super smooth case: $\phi_\eta(u) = c_1 \exp(-c_2(1+u^2)^{c_3/2})$, $c_i > 0$, $i = 1, 2$, $0 < c_3 \leq 2$, gives

$$|\theta_\eta(u)| = c_1 c_2 c_3 u(1+u^2)^{c_3/2-1} \exp(-c_2(1+u^2)^{c_3/2}).$$

This implies $\int |\theta_\eta(v)|^2 dv < +\infty$.

## 5. Estimators in the case of irregular sampling and fixed cutoff parameter.

Here, we study the extension of the high frequency setting to irregular sampling. For simplicity, we only study the non noisy observations case. Hence, we consider observations $Z_k, k = 1, \dots, n$ which are independent, but not identically distributed. The r.v. $Z_k$ has characteristic function $\psi_{\Delta_k}$.

Let $\bar{D}$ and $\bar{\Delta}$ be defined by

$$\bar{D}^{-1} = \frac{1}{n}\sum_{k=1}^{n} \Delta_k^{-1}, \quad \bar{\Delta} = \frac{1}{n}\sum_{k=1}^{n} \Delta_k. \tag{27}$$

Clearly we have

$$1 = \left(\frac{1}{n}\sum_{k=1}^{n}\sqrt{\Delta_k}\frac{1}{\sqrt{\Delta_k}}\right)^2 \leq \frac{1}{n}\sum_{k=1}^{n}\Delta_k \frac{1}{n}\sum_{k=1}^{n}\frac{1}{\Delta_k}.$$

Therefore, $\bar{\Delta}/\bar{D} \geq 1$ or $\bar{\Delta} \geq \bar{D}$.

We assume that $\bar{\Delta}$ tends to 0 and $n\bar{\Delta} = t_n$ tends to infinity. Hence, $\bar{D}$ tends to 0. Moreover, in one the two strategies, we assume further that $n\bar{D}$ tends to infinity.

Two different strategies can be used to build an estimator of $g^*$. Derivating $\psi_{\Delta_k}$ yields the relation:

$$\Delta_k \psi_{\Delta_k}(u) g^*(u) = \theta_{Z_k}(u)$$

with $\theta_{Z_k}(u) = \mathbb{E}(Z_k e^{iuZ_k})$ (compare with (12)). The first strategy follows from writing

$$\frac{1}{n}\left(\sum_{k=1}^{n}\psi_{\Delta_k}(u)\right) g^*(u) = \frac{1}{n}\sum_{k=1}^{n}\Delta_k^{-1}\theta_{Z_k}(u).$$

The second strategy comes from

$$\frac{\sum_{k=1}^n \Delta_k \psi_{\Delta_k}(u)}{\sum_{k=1}^n \Delta_k} g^*(u) = \frac{\sum_{k=1}^n \theta_{Z_k}(u)}{\sum_{k=1}^n \Delta_k}.$$

For both approaches, the coefficient of $g^*(u)$ is close to 1 and need not be estimated.

5.1. **First strategy.** To estimate $g^*$, we propose first to define

$$\hat{g}^*_{IR1}(u) = \frac{1}{n}\sum_{k=1}^n \Delta_k^{-1} Z_k e^{iuZ_k}, \tag{28}$$

and with (3),

$$\hat{g}_{m,IR1}(x) = \frac{m}{n}\sum_{k=1}^n \Delta_k^{-1} Z_k \varphi(m(x-Z_k)).$$

**Proposition 5.1.** *Assume that assumptions* (H1), (H2)(2) *and* (H3) *are fulfilled. Then*

$$\mathbb{E}(\|\hat{g}_{m,IR1}-g\|^2) \leq \|g-g_m\|^2 + 2(m_2+m_1^2)\frac{m}{n\bar{D}} + 2\bar{\Delta}^2\|g\|_1^2\left(\int_{-\pi m}^{\pi m} u^2|g^*(u)|^2 du\right), \tag{29}$$

*where* $m_\ell = \int x^{\ell-1}g(x)dx$, $\ell = 1,2$. *Moreover, if* $\bar{\Delta}^2 = O(1/(n\bar{D}))$ *and* $g$ *belongs to* $\mathcal{C}(a,L)$, *for* $m = (n\bar{D})^{1/(2a+1)}$,

$$\mathbb{E}(\|\hat{g}_{m,IR1}-g\|^2) = O((n\bar{D})^{-2a/(2a+1)}).$$

This result is proved in Section 8 and also uses the Appendix.

5.2. **Second strategy.** To estimate $g^*$, we propose secondly to define

$$\hat{g}^*_{IR2}(u) = (n\bar{\Delta})^{-1}\sum_{k=1}^n Z_k e^{iuZ_k}, \tag{30}$$

and by (3),

$$\hat{g}_{m,IR2}(x) = \frac{m}{n\bar{\Delta}}\sum_{k=1}^n Z_k \varphi(m(x-Z_k)).$$

Now, we can prove the following result:

**Proposition 5.2.** *Assume that assumptions* (H1), (H2)(2) *and* (H3) *are fulfilled. Then*

$$\mathbb{E}(\|\hat{g}_{m,IR2}-g\|^2) \leq \|g-g_m\|^2 + 2(m_2+m_1^2)\frac{m}{n\bar{\Delta}} + 2\|g\|_1^2\left(\int_{-\pi m}^{\pi m} u^2|g^*(u)|^2 du\right)\frac{\bar{\Delta}_2^2}{(\bar{\Delta})^2}, \tag{31}$$

*where* $\bar{\Delta}_2 = (1/n)\sum_{k=1}^n \Delta_k^2$.
*Moreover, if* $\bar{\Delta}_2^2/\bar{\Delta}^2 = O(1/(n\bar{\Delta}))$ *and* $g$ *belongs to* $\mathcal{C}(a,L)$, *for* $m = (n\bar{\Delta})^{1/(2a+1)}$,

$$\mathbb{E}(\|\hat{g}_{m,IR2}-g\|^2) = O((n\bar{\Delta})^{-2a/(2a+1)}).$$

Since $1/(n\bar{\Delta}) \leq 1/(n\bar{D})$, the variance term in strategy 2 is smaller. If $g \in \mathcal{C}(a,L)$, the rate for strategy 1 is $O((n\bar{D})^{-2a/(2a+1)})$ , whereas the rate for strategy 2 is $O((n\bar{\Delta})^{-2a/(2a+1)})$. The latter rate is thus always of lower order. Strategy 2 should therefore be preferred.

Concerning the residual term, $(\bar{\Delta})^2 \leq \bar{\Delta}_2$. Hence, $\bar{\Delta}_2^2/(\bar{\Delta})^2 \geq (\bar{\Delta})^2$. Therefore, we search for some examples of $\Delta_k$ satisfying $(\bar{\Delta}_2)^2/(\bar{\Delta})^2 \leq 1/(n\bar{\Delta})$:

- $\Delta_k = 1/k$, $\bar{\Delta} = O(\ln(n)/n)$, $\bar{\Delta}_2 = O(1/n)$ and thus $\bar{\Delta}_2^2/(\bar{\Delta})^2 = O(1/\ln^2(n)) = O(1/(n\bar{\Delta})^2)$.

- $\Delta_k = 1/\sqrt{k}$, $\bar{\Delta} = O(1/\sqrt{n})$, $\bar{\Delta}_2 = O(\ln(n)/n)$ and thus $(\bar{\Delta}_2)^2/(\bar{\Delta})^2 = O(\ln^2(n)/n) = o(1/(n\bar{\Delta}))$.
- $\Delta_k = 1/k^\beta$, $1/2 < \beta < 1$, $\bar{\Delta} = O(n^{-\beta})$, $\bar{\Delta}_2 = O(1/n)$ and thus $\bar{\Delta}_2^2/(\bar{\Delta})^2 = O(n^{-2(1-\beta)})$ which is $o(1/(n\bar{\Delta}))$ as $1/(n\bar{\Delta}) = 1/n^{1-\beta}$.

On the opposite,

- $\Delta_k = 1/k^\beta$, $0 < \beta < 1/2$, $\bar{\Delta} = O(n^{-\beta})$, $\bar{\Delta}_2 = O(n^{-2\beta})$ and thus $\bar{\Delta}_2^2/(\bar{\Delta})^2 = O(n^{-2\beta})$ which is not $O(1/(n\bar{\Delta})) = O(1/n^{1-\beta})$.
- $\Delta_k = 1/k^\beta$, with $\beta > 1$ is such that $n\bar{\Delta} = O(1)$ does not tend to infinity.

Thus the admissible values of this form are $\Delta_k = k^{-\beta}$ with $\beta \in [1/2, 1]$.

## 6. Model selection and adaptive estimator.

As shown previously, there is an optimal choice of the cutoff parameter which realizes the best compromise between the square bias and the variance terms in the risk bounds (see Tables 3, 4, Corollary 4.2, Propositions 4.3, 5.1, 5.2).

The aim of the model selection procedure is to propose a data driven value $\hat{m}$ of the cutoff parameter which realizes automatically the bias-variance compromise. Recall that the risk is decomposed into

$$\mathbb{E}(\|\hat{g}_m - g\|^2) = \|g - g_m\|^2 + \mathbb{E}(\|g_m - \hat{g}_m\|^2),$$

where $\|g - g_m\|^2$ is the bias term and $\mathbb{E}(\|g_m - \hat{g}_m\|^2)$ is the variance term.

Using Parseval's Equality, it is easy to see that $\|g - g_m\|^2 = \|g\|^2 - \|g_m\|^2$. Therefore, the bias term is estimated, up to the constant $\|g\|^2$, by an estimation of $\|g_m\|^2$, which is taken as $\|\hat{g}_m\|^2$, where $\hat{g}_m$ is an estimate of $g$. We introduce a penalty function $\text{pen}(m)$ which estimates the variance term $\mathbb{E}(\|g_m - \hat{g}_m\|^2)$. Actually, we only estimate its highest order term.

We define the criterion:

$$\hat{m} = \arg\min_{m \in \mathcal{M}_n} (-\|\hat{g}_m\|^2 + \text{pen}(m))$$

where $\mathcal{M}_n = \{1, \dots, m_n\}$ with $m_n \le n\Delta$ and

$$\text{pen}(m) \propto \text{ estimator of the highest order term in the variance.}$$

The low frequency (LF) case is rather difficult and treated in Comte and Genon-Catalot (2008), see Theorem 4.2 therein. The low frequency with noise (LFN) case may be studied analogously.

The high frequency case is simpler and we detail it. When the sampling is regular, we set

$$\hat{m}_{HFN} = \arg\min_{m \in \mathcal{M}_n} (-\|\hat{g}_{m,HFN}\|^2 + \text{pen}_{HFN}(m)),$$

where $\mathcal{M}_n = \{1, \dots, m_n\}$ with $m_n \le n\Delta$ and

$$\text{pen}_{HFN}(m) = \kappa \left( \frac{1}{n\Delta} \sum_{k=1}^{n} V_k^2 \right) \frac{m}{n\Delta}.$$

We shall denote by

$$\hat{\hat{g}}_{HFN} = \hat{g}_{\hat{m}_{HFN},HFN}.$$

Note that

$$\mathbb{E}(\text{pen}_{HFN}(m)) = \kappa(\mathbb{E}(V_1^2)/\Delta)\frac{m}{n\Delta} = \kappa \frac{\mathbb{E}(Z_1^2) + \delta^2\mathbb{E}(\eta_1^2)}{\Delta} \frac{m}{n\Delta}.$$

Compare with bound (26).
The constant $\kappa$ here is a numerical value that helps to avoid under-penalization.
The following oracle-type result is obtained

**Theorem 6.1.** *Assume that* (H2)(8)-(H3) *are fulfilled and* $\int x^2 g^2(x)dx < +\infty$*, that* $n$ *is large and* $\Delta$ *is small with* $n\Delta$ *tends to infinity when* $n$ *tends to infinity. Assume in addition*

$$\int |\theta_\eta(v)|^2 dv < +\infty, \quad \delta = \Delta^4 \quad \textit{and} \quad n\Delta^3 \leq 1. \tag{32}$$

*Then there exists a universal constant* $\kappa$ *such that*

$$\begin{aligned}\mathbb{E}(\|g - \hat{\hat{g}}_{HFN}\|^2) \quad &\leq \quad C \inf_{m\in\{1,\ldots,m_n\}} \left(\|g - g_m\|^2 + \mathbb{E}(\mathrm{pen}_{HFN}(m)\right) \\ &\qquad + \frac{C'\Delta^2}{2\pi}\int_{-\pi m_n}^{\pi m_n} u^2|g^*(u)|^2 du + \frac{C" \ln^2(n\Delta)}{n\Delta}\end{aligned}$$

*where* $C, C', C"$ *are constants.*

Note that (H3) and $\int x^2 g^2(x)dx < +\infty$ imply (H1) since:

$$\|g\|_1^2 = (\int |g(x)|dx)^2 \leq \int (1+x^2)g^2(x)dx \int \frac{dx}{1+x^2}.$$

Theorem 6.1 is proved for the case HF (case $\delta = 0$, see Theorem 3.1 in Comte and Genon-Catalot (2009)). In this case, the conditions given in (32) are not required. The extension to the noisy case follows the same line and is not detailed here. The proof of the result relies on Talagrand deviation inequality (see Talagrand (1996) or Klein and rio (2005)) and Rosenthal type bounds on higher moments of the observed process.

The constant $\kappa$ may be increased. Actually, it is calibrated by numerical simulations (see Comte and Genon-Catalot (2009)).

If under the assumptions of Theorem 6.1, $g$ belongs to a class of regularity $\mathcal{C}(a, L)$, with unknown $a$ and $L$, the estimator is automatically such that

$$\mathbb{E}(\|g - \hat{\hat{g}}_{HFN}\|^2) \leq C\left[(n\Delta)^{-2a/(2a+1)} + \Delta^2 m_n^{2(1-a)_+} + \frac{C" \ln^2(n\Delta)}{n\Delta}\right].$$

Then, even if $a$ is unknown,

$$\mathbb{E}(\|g - \hat{g}_{\hat{m}}\|^2) = O((n\Delta)^{-2a/(2a+1)}).$$

In the case of irregular sampling with strategy 2,

$$\mathrm{pen}_{IR2}(m) = \kappa\left(\frac{1}{n\bar{\Delta}}\sum_{k=1}^n Z_k^2\right)\frac{m}{n\bar{\Delta}}.$$

Then

$$\mathbb{E}(\mathrm{pen}_{IR2}(m)) = \kappa(m_2 + \frac{\bar{\Delta}_2}{\bar{\Delta}}m_1^2)\frac{m}{n\bar{\Delta}},$$

which must be compared to (31). Let us denote by

$$\hat{\hat{g}}_{IR2} = \hat{g}_{\hat{m}_{IR2},IR2}$$

the corresponding adaptive estimator. Then the result of Comte and Genon-Catalot (2009) can be extended as follows:

**Theorem 6.2.** *Assume that* (H2)(8)-(H3) *are fulfilled and* $\int x^2 g^2(x)dx < +\infty$*, that* $n$ *is large and* $\bar{\Delta}$ *is small with* $n\bar{\Delta}$ *tends to infinity when* $n$ *tends to infinity. Then there exists a universal* *constant $\kappa$ such that*

$$\begin{aligned}\mathbb{E}(\|g-\hat{\hat{g}}_{IR2}\|^2) &\leq C \inf_{m\in\{1,\dots,m_n\}} \left(\|g-g_m\|^2+\mathbb{E}(\mathrm{pen}_{IR2}(m)\right) \\ &\qquad + \frac{C'(\bar\Delta_2)^2}{(\bar\Delta)^2}\int_{-\pi m_n}^{\pi m_n} u^2|g^*(u)|^2du + \frac{C\text{"}\ln^2(n\bar\Delta)}{n\bar\Delta}\end{aligned}$$

*where $C, C', C$" are constants.*

If under the assumptions of Theorem 6.2, $g$ belongs to a class of regularity $\mathcal{C}(a,L)$, with unknown $a$ and $L$, the estimator is automatically such that

$$\mathbb{E}(\|g-\hat{\hat{g}}_{IR2}\|^2) \leq C\left[(n\bar\Delta)^{-2a/(2a+1)} + \frac{(\bar\Delta_2)^2}{(\bar\Delta)^2}m_n^{2(1-a)_+} + \frac{\ln^2(n\bar\Delta)}{n\bar\Delta}\right].$$

Then, if $n\bar\Delta^2 \leq 1$ and $\bar\Delta_2^2 \leq C\bar\Delta^4$ (with $C>1$),

$$\mathbb{E}(\|g-\hat{g}_{\hat m}\|^2) = O((n\bar\Delta)^{-2a/(2a+1)}),$$

even if $a$ is unknown.

## 7. Concluding remarks

In this paper, the nonparametric estimation of the Lévy density $n(.)$ of a pure jump Lévy process is investigated under assumption (H1). This is done through the estimation of $g(x) = xn(x)$. Several kinds of observations are considered: discrete observations with regular sampling interval corrupted by a small noise, or irregular sampling interval. The methods developped in Comte and Genon-Catalot (2008, 2009) are extended and yield an adaptive estimator of $g$. Our estimators reach the minimax risk bounds of Figuroa-López (2009). The numerical implementation for high frequency data is performed in Comte and Genon-Catalot (2009) and illustrates the fact that the method works on simulated data.

## 8. Proofs

8.1. **Proof of Proposition 4.1 and Corollary 4.1.** The result in (23) follows from

$$\|g-\hat g_{m,LFN}\|^2 = \frac{1}{2\pi}\int_{|u|\geq\pi m}|g^*(u)|^2du + \frac{1}{2\pi}\int_{-\pi m}^{\pi m}|\hat g^*_{LFN}(u)-g^*(u)|^2dx, \tag{33}$$

and

$$\hat g^*_{LFN}(u)-g^*(u) = (\hat g^*_{LFN}(u) - \frac{\theta_V(u)}{\Delta\phi_V(u)}) + (\frac{\theta_V(u)}{\Delta\phi_V(u)} - g^*(u)).$$

The first term above can be studied as in Comte and Genon-Catalot (2008). The definition of $\hat\phi_V$ and $\tilde\phi_V$ allows us to extend the result of Neumann (1997) and Neumann and Reiss (2007) (see Lemma 4.1 of Comte and Genon-Catalot (2008)). For the second one, it follows from (18) that

$$\frac{\theta_V(u)}{\Delta\phi_V(u)} - g^*(u) = (\delta/\Delta)\frac{\theta_\eta(\delta u)}{\Delta\phi_\eta(\delta u)}.$$

This implies (22).
Next, Inequality (23) follows from (H6), (H7) and $\delta|u| \leq \pi m\delta \leq \pi$ which imply

$$1/|\phi_V(u)|^2 \leq c_0^2/|\psi_\Delta(u)|^2$$

and

$$|\frac{\theta_\eta(\delta u)}{\Delta\phi_\eta(\delta u)}|^2 \leq c_0^2\mathbb{E}(\eta_1^2).$$

□.

8.2. **Proof of Proposition 4.2.** We start as in the previous cases and get

$$\|g-\hat{g}_{m,HFN}\|^2 \quad = \quad \|g-g_m\|^2+\frac{1}{2\pi}\int_{-\pi m}^{\pi m}|\hat{g}^*_{HFN}(u)-g^*(u)|^2du.$$

Now, using (17), the following decomposition appears:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi m}^{\pi m}|\hat{g}^*_{HFN}(u)-g^*(u)|^2du \quad &\leq \quad \frac{3}{2\pi\Delta^2}\int_{-\pi m}^{\pi m}|\hat{\theta}_V(u)-\theta_V(u)|^2du\\ &+\frac{3}{2\pi}\int_{-\pi m}^{\pi m}|g^*(u)(\psi_\Delta(u)\phi_\eta(\delta u)-1)|^2du\\ &+\frac{3\delta^2}{2\pi\Delta^2}\int_{-\pi m}^{\pi m}|\theta_\eta(\delta u)\psi_\Delta(u)|^2du\end{aligned}$$

It is easy to see that

$$\mathbb{E}(|\hat{\theta}_V(u)-\theta_V(u)|^2)=\mathbb{E}\{|\frac{1}{n}\sum_{k=1}^n[V_ke^{iuV_k}-\mathbb{E}(V_ke^{iuV_k})]|^2\}\leq 4\frac{\mathbb{E}(V_1^2)}{n}.$$

Therefore

$$\frac{2}{\Delta^2}\mathbb{E}\left(\int_{-\pi m}^{\pi m}|\hat{\theta}_V(u)-\theta_V(u)|^2du\right)\leq\frac{4\pi m}{n\Delta}(\mathbb{E}(Z_1^2/\Delta)+(\delta^2/\Delta)\mathbb{E}(\eta_1^2)).$$

Our first constraint here is $\delta^2=O(\Delta)$.

For the second term, note that $|\psi_\Delta(u)-1|\leq\Delta|u|\|g\|_1$ and, with a second order development using $\mathbb{E}(\eta)=0$,

$$|\phi_\eta(\delta u)-1|\leq\frac{u^2\delta^2}{2}\mathbb{E}(\eta^2).$$

Thus

$$\begin{aligned}\int_{-\pi m}^{\pi m}|g^*(u)(\psi_\Delta(u)\phi_\eta(\delta u)-1)|^2du \quad &\leq \quad 2\int_{-\pi m}^{\pi m}|g^*(u)|^2|\psi_\Delta(u)-1|^2|\phi_\eta(\delta u)|^2du\\ &+2\int_{-\pi m}^{\pi m}|g^*(u)|^2|\phi_\eta(\delta u)-1|^2du\\ &\leq \quad 2\Delta^2\|g\|_1^2\int_{-\pi m}^{\pi m}u^2|g^*(u)|^2du+2\delta^4\mathbb{E}^2(\eta^2)\int_{-\pi m}^{\pi m}u^4|g^*(u)|^2du\end{aligned}$$

Lastly,

$$\left(\frac{\delta}{\Delta}\right)^2\int_{-\pi m}^{\pi m}|\psi_\Delta(u)\theta_\eta(\delta u)|^2du\leq\frac{\delta}{\Delta^2}\int|\theta_\eta(v)|^2dv.$$

Gathering all terms implies inequality (26) and gives the result. □

8.3. **Proof of Proposition 5.1.** As announced, we start from

$$\|g-\hat{g}_{m,IR1}\|^2=\|g-g_m\|^2+\frac{1}{2\pi}\int_{-\pi m}^{\pi m}|\hat{g}^*_{IR1}(u)-g^*(u)|^2dx. \tag{34}$$

The first term is the standard bias term already evaluated. We study the second one. We can note that

$$\mathbb{E}(\hat{g}^*_{IR1}(u)-g^*(u))=\left[\frac{1}{n}\sum_{k=1}^n(\psi_{\Delta_k}(u)-1)\right]g^*(u).$$

Moreover, a first order development implies that (see the Appendix)

$$|\psi_{\Delta_k}(u) - 1| \leq |u|\Delta_k \|g\|_1. \tag{35}$$

Thus,

$$\begin{aligned}\int_{-\pi m}^{\pi m} |\hat{g}^*_{IR1}(u) - g^*(u)|^2 du &\leq 2\int_{-\pi m}^{\pi m} |\hat{g}^*_{IR1}(u) - \mathbb{E}(\hat{g}^*_{IR1}(u))|^2 du \\ &\quad +2\int_{-\pi m}^{\pi m} |g^*(u)|^2 |\frac{1}{n}\sum_{k=1}^{n}(\psi_{\Delta_k}(u) - 1)|^2 du.\end{aligned}$$

It is easy to see that

$$\mathbb{E}(|\hat{g}^*_{IR1}(u) - \mathbb{E}(\hat{g}^*_{IR1}(u))|^2) = \mathbb{E}\{|\frac{1}{n}\sum_{k=1}^{n}[\Delta_k^{-1} Z_k e^{iuZ_k} - \mathbb{E}(\Delta_k^{-1} Z_k e^{iuZ_k})]|^2\} \leq \frac{1}{n^2}\sum_{k=1}^{n}\Delta_k^{-2}\mathbb{E}(Z_k^2).$$

Therefore, using Proposition 9.1, we find

$$\mathbb{E}\left(\int_{-\pi m}^{\pi m} |\hat{g}^*_{IR1}(u) - g^*(u)|^2 du\right) \leq \frac{2\pi m}{n}(m_2/\bar{D} + m_1^2) \leq \frac{2\pi m(m_2 + m_1^2)}{n\bar{D}}.$$

On the other hand,

$$\int_{-\pi m}^{\pi m} |g^*(u)|^2 |\frac{1}{n}\sum_{k=1}^{n}(\psi_{\Delta_k}(u) - 1)|^2 du \leq 2\|g\|_1^2 \bar{\Delta}^2 \int_{-\pi m}^{\pi m} u^2 |g^*(u)|^2 du.$$

Gathering the bounds implies (29). □

8.4. **Proof of Proposition 5.2.** We start by (34) as above and we study the second term as well. We can note that

$$\mathbb{E}(\hat{g}^*_{IR2}(u)) = (n\bar{\Delta})^{-1}(\sum_{k=1}^{n}\Delta_k \psi_{\Delta_k}(u)) g^*(u).$$

Thus,

$$\begin{aligned}\int_{-\pi m}^{\pi m} |\hat{g}^*_{IR2}(u) - g^*(u)|^2 du &\leq 2\int_{-\pi m}^{\pi m} |\hat{g}^*_{IR2}(u) - \mathbb{E}(\hat{g}^*_{IR2}(u))|^2 du \\ &\quad +2\int_{-\pi m}^{\pi m} (\bar{\Delta})^{-2} |g^*(u)|^2 |\frac{1}{n}\sum_{k=1}^{n}\Delta_k(\psi_{\Delta_k}(u) - 1)|^2 du\end{aligned}$$

It is easy to see that

$$\mathbb{E}\{|\frac{1}{n}\sum_{k=1}^{n}[Z_k e^{iuZ_k} - \mathbb{E}(Z_k e^{iuZ_k})]|^2\} \leq \frac{1}{n^2}\sum_{k=1}^{n}\mathbb{E}(Z_k^2) = \frac{1}{n}(m_2\bar{\Delta} + m_1^2\bar{\Delta}_2).$$

Therefore

$$2\int_{-\pi m}^{\pi m} \mathbb{E}\left(|\hat{g}^*_{IR2}(u) - \mathbb{E}(\hat{g}^*_{IR2}(u))|^2\right) du \leq \frac{4\pi m}{n\bar{\Delta}}(m_2 + (\bar{\Delta}_2/\bar{\Delta})m_1^2) \leq \frac{4\pi m(m_2 + m_1^2)}{n\bar{\Delta}}.$$

On the other hand, using (35) again yields

$$\frac{2}{(\bar{\Delta})^2}\int_{-\pi m}^{\pi m} |g^*(u)|^2 |\frac{1}{n}\sum_{k=1}^{n}\Delta_k(\psi_{\Delta}(u) - 1)|^2 du \leq \frac{2\|g\|_1^2\bar{\Delta}_2^2}{(\bar{\Delta})^2}\int_{-\pi m}^{\pi m} u^2 |g^*(u)|^2 du.$$

Gathering the bounds implies (31). □

## 9. Appendix.

The random variables $(Z_k), k = 0, \ldots, n$ are independent, with characteristic function equal to $\psi_{\Delta_k}(u) := \mathbb{E}(e^{iuZ_k})$, with, according to (1):

$$\psi_{\Delta_k}(u) = \exp\left(\Delta_k \int (e^{iux} - 1)n(x)dx\right). \tag{36}$$

By derivation under (H1), we have

$$\psi'_{\Delta_k}(u) = i\Delta_k \psi_k(u) g^*(u) = i\mathbb{E}(Z_k e^{iuZ_k}), \tag{37}$$

The following bounds hold under (H1):

$$|\psi_{\Delta_k}(u) - 1| \leq |u|\Delta_k \|g\|_1, \tag{38}$$

$$|\Delta_k^{-1}\mathbb{E}(Z_k e^{iuZ_k}) - g^*(u)| \leq |u|\Delta_k \|g\|_1^2. \tag{39}$$

The first bound follows from the Taylor formula and the second is deduced from the first one.

The following result is a straightforward generalization of Proposition 2.2 in Comte and Genon-Catalot (2009):

**Proposition 9.1.** *If $\int |x|^p n(x)dx < +\infty$, then $\mathbb{E}(|Z_k|^p) < +\infty$. Moreover, let us define*

$$m_\ell = \int_{\mathbb{R}} x^{\ell-1} g(x)dx = \int_{\mathbb{R}} x^\ell n(x)dx.$$

*Then $\mathbb{E}(Z_k) = \Delta_k m_1$, $\mathbb{E}(Z_k^2) = \Delta_k m_2 + \Delta_k^2 m_1$, and for $\ell = 2, \ldots, p$,*

$$\mathbb{E}(Z_k^\ell) = \Delta_k m_\ell + \sum_{j=2}^{\ell} \Delta_k^j c_j,$$

*where the $c_j$'s are explicit functions of the $m_j$'s, for $j \leq \ell$.*
*Lastly, $\mathbb{E}|Z_k| \leq 2\Delta_k |g|_1$.*

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