Optimal multiple stopping time problem

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June 9, 2010

Abstract

We study the optimal multiple stopping time problem defined for each stopping time S by $v(S) = \operatorname*{ess\,sup}_{\tau_1,\cdots,\tau_d \geq S} \mathrm{E}[\psi(\tau_1,\cdots,\tau_d)\,|\,\mathcal{F}_S]$.

The key point is the construction of a *new reward* ϕ such that the value function $v(S)$ satisfies also $v(S) = \operatorname{ess\,sup}_{\theta\geq S} E[\phi(\theta)\,|\,\mathcal{F}_S]$. This new reward ϕ is not a right continuous adapted process as in the classical case but a family of random variables. For such a reward, we prove a new existence result of optimal stopping times under weaker assumptions than in the classical case. This result is used to prove the existence of optimal multiple stopping times for $v(S)$ by a constructive method. Moreover, under strong regularity assumptions on ψ, we show that the new reward ϕ can be aggregated by a progressive process. This leads to different applications in particular in finance to American options with multiple exercise times.

Key words: Optimal stopping, optimal multiple stopping, aggregation, swing options, American options.

AMS subject classifications: Primary 60G40; secondary 60G07, 28B20, 62L15

Introduction

Our present work on the optimal multiple stopping time problem consists, following the optimal one stopping time problem, in proving the existence of the

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maximal reward, finding necessary or sufficient conditions for the existence of optimal stopping times, and giving a method to compute these optimal stopping times.

The results are well known in the case of the optimal one stopping time problem. Consider a *reward* given by a right continuous left limited (RCLL) positive adapted process $(\phi_t)_{\,0\leq t\leq T}$ on $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{\,0\leq t\leq T}, P)$, $\mathbb{F}$ satisfying the usual conditions, and look for the maximal reward

$$v(0) = \sup\{\, E[\phi_\tau],\ \tau \in T_0 \,\},$$

where $T \in \,]0, \infty[$ is the fixed time horizon and where T_0 is the set of stopping times θ smaller than T. From now on, the process $(\phi_t)_{\,0\leq t\leq T}$ will be denoted by (ϕ_t). In order to compute $v(0)$ we introduce for each $S \in T_0$ the *value function* $v(S) = \text{ess sup}\{\, E[\phi_\tau \,|\, \mathcal{F}_S], \tau \in T_S \,\}$, where T_S is the set of stopping times in T_0 greater than S. The value function is given by a family of r.v. $\{\, v(S), S \in T_0 \,\}$. By using the right continuity of the reward (ϕ_t), it can be shown that there exists an adapted process (v_t) which *aggregates* the family of r.v. $\{\, v(S), S \in T_0 \,\}$ that is such that $v_S = v(S)$ a.s. for each $S \in T_0$. This process is the *Snell envelope* of (ϕ_t), that is the smallest supermartingale process that dominates ϕ. Moreover, when the reward (ϕ_t) is continuous, the stopping time defined trajectorially by

$$\overline{\theta}(S) = \inf\{\, t \geq S,\ v_t = \phi_t \,\}$$

is optimal. For details, one is referred to El Karoui (1981), Karatzas and Shreve (1998) or Peskir and Shiryaev (2006).

In the present work, we show that computing the value function for the optimal multiple stopping time problem

$$v(S) = \text{ess sup}\{\, E[\psi(\tau_1, \cdots, \tau_d) \,|\, \mathcal{F}_S],\ \tau_1, \cdots, \tau_d \in T_S \,\},$$

reduces to computing the value function for an optimal one stopping time problem

$$u(S) = \text{ess sup}\{\, E[\phi(\theta) \,|\, \mathcal{F}_S],\ \theta \in T_S \,\},$$

where the *new reward* ϕ is no longer a RCLL process but a family $\{\, \phi(\theta), \theta \in T_0 \,\}$ of positive random variables which satisfies some compatibility properties. For this new optimal one stopping time problem with a reward $\{\, \phi(\theta), \theta \in T_0 \,\}$, we show that the minimal optimal stopping time for the value function $u(S)$ is no longer given by a hitting time of processes but by the essential infimum

$$\theta^*(S) := \text{ess inf}\{\, \theta \in T_S\,,\ u(\theta) = \phi(\theta) \text{ a.s.} \,\}.$$

This method also presents the advantage that it does no longer require some aggregation results that need stronger hypotheses and whose proofs are rather technical.

By using the reduction property $v(S) = u(S)$ a.s., we give a method to construct by induction optimal stopping times $(\tau_1^*, \cdots, \tau_d^*)$ for $v(S)$, which are also defined as essential infima, in terms of *nested* optimal one stopping time problems.

Some examples of optimal multiple stopping time problems have been studied in different mathematical fields. In finance, this type of problem appears for instance in the study of *swing options* (e.g. Carmona and Touzi (2008), Carmona and Dayanik (2008)) in the case of ordered stopping times. In the non ordered case, some optimal multiple stopping time problems appear as a useful mathematical tool to establish some large deviations estimations (see Kobylanski and Rouy (1998)). Further applications can be imagined for example in finance and insurance (see Kobylanski et al. (2010)). In a work in preparation (see Kobylanski and Quenez (2010)), the markovian case will be studied in details and some applications will be presented.

The paper is organized as follows. In section 1, we revisit the optimal one stopping time problem for admissible families. We prove the existence of optimal stopping times when the family ϕ is right and left continuous in expectation along stopping times. We also characterize the minimal optimal stopping times. In section 2, we solve the optimal double stopping time problem. Under quite weak assumption, we show the existence of a pair of optimal stopping times and we give a construction of those optimal stopping times. In section 3, we generalize the results obtained in section 2 to the optimal d-stopping times problem. Also, we study the simpler case of a symmetric reward. In this case, the problem clearly reduces to ordered stopping times and our general characterization of the optimal multiple stopping problem in terms of the *nested* optimal one stopping time problems straightforwardly reduces to a sequence of optimal one stopping time problems defined by backward induction. We apply these results to *swing options* and in this particular case, our results correspond to those of Carmona and Dayanik (2008). In the last section, we prove some aggregation results, and we characterize the optimal stopping times in terms of hitting times of processes.

Let $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a probability space where $T \in]0, \infty[$ is the fixed time horizon and where $(\mathcal{F}_t)_{0 \leq t \leq T}$ is a filtration satisfying the usual conditions of right continuity and augmentation by the null sets of $\mathcal{F} = \mathcal{F}_T$. We suppose that $\mathcal{F}_0$ contains only sets of probability 0 or 1. We denote by T_0 the collection of stopping times of $\mathbb{F}$ with values in $[0, T]$. More generally, for any stopping time S, we denote by T_S the class of stopping times $\theta \in T_0$ with $S \leq \theta$ a.s. .

We use the following notation: for real valued random variables X and X_n, $n \in \mathbb{N}$, the notation "$X_n \uparrow X$" stands for "the sequence (X_n) is nondecreasing

and converges to X a.s.".

1 The optimal one stopping time problem revisited

We first recall some classical results on the optimal one stopping time problem.

1.1 Classical results

The following classical results, namely the supermartingale property of the value function, the optimality criterium and the right continuity in expectation of the value function are well known (see El Karoui (1981) or Karatzas and Shreve (1998) or Peskir and Shiryaev (2006)). They are very important tools in optimal stopping theory and they will be often used in this paper in the (non usual) case of a reward given by an admissible family of random variables defined as follows:

Definition 1.1. *A family of random variables* $\{\,\phi(\theta),\ \theta \in T_0\}$ *is said to be* admissible *if it satisfies the following conditions*

1. *for all* $\theta \in T_0$ $\phi(\theta)$ *is a* $\mathcal{F}_\theta$*-measurable positive random variable (r.v.),*
2. *for all* $\theta, \theta' \in T_0$, $\phi(\theta) = \phi(\theta')$ *a.s. on* $\{\,\theta = \theta'\,\}$.

Remark 1.1. *Let* (ϕ_t) *be a progressive process. The family defined by* $\phi(\theta) = \phi_\theta$ *is admissible.*

Note also that the definition of admissible families corresponds to the notion of T_0*-systems introduced by El Karoui (1981).*

For the convenience of the reader, we recall the definition of the essential supremum and its main properties in the Appendix A1.

Suppose the *reward* is given by an admissible family $\{\phi(\theta),\ \theta \in T_0\}$.
The *value function at time* S, where $S \in T_0$, is given by

$$v(S) = \text{ess sup}_{\,\theta \in T_S} E[\phi(\theta)\,|\,\mathcal{F}_S]\,. \tag{1.1}$$

Proposition 1.1. (Admissibility of the value function) *The value function that is the family of r.v.* $\{\,v(S),\ S \in T_0\,\}$ *defined by (1.1) is an admissible family.*

Proof: Property 1 of admissibility for $\{\,v(S),\ S \in T_0\,\}$ follows from the existence of the essential supremum (see Appendix Theorem A.1).
Take $S, S' \in T_0$ and let $A = \{\,S = S'\,\}$. For each $\theta \in T_S$ put $\theta_A = \theta \mathbf{1}_A + T\mathbf{1}_{A^c}$. As $A \in \mathcal{F}_S \cap \mathcal{F}_{S'}$, one has a.s. on A, $E[\phi(\theta)\,|\,\mathcal{F}_S] = E[\phi(\theta_A)\,|\,\mathcal{F}_S] = E[\phi(\theta_A)\,|\,\mathcal{F}_{S'}] \leq v(S')$, hence taking the essential supremum over $\theta \in T_S$ one has $v(S) \leq v(S')$ a.s. and by symmetry of S and S', we have shown property 2 of admissibility. □

Proposition 1.2. *There exists a sequence of stopping times* $(\theta^n)_{n\in\mathbb{N}}$ *with* θ^n *in* T_S *such that*

$$E[\phi(\theta^n)\,|\,\mathcal{F}_S] \quad \uparrow \quad v(S) \quad a.s..$$

PROOF: For each $S \in T_0$, one can show that the set $\{\, E[\phi(\theta)\,|\,\mathcal{F}_S],\ \theta \in T_S \,\}$ is closed under pairwise maximization. Indeed, let $\theta, \theta' \in T_0$, and let $A = \{\, E[\phi(\theta')\,|\,\mathcal{F}_S] \leq E[\phi(\theta)\,|\,\mathcal{F}_S] \,\}$. One has $A \in \mathcal{F}_S$. Put $\tau = \theta \mathbf{1}_A + \theta' \mathbf{1}_{A^c}$, τ is a stopping time. It is easy to check that $E[\phi(\tau)\,|\,\mathcal{F}_S] = E[\phi(\theta)\,|\,\mathcal{F}_S] \vee E[\phi(\theta')\,|\,\mathcal{F}_S]$. The result follows by a classical result (see Appendix Theorem A.1). □

Recall that for each fixed $S \in T_0$, an admissible family $\{\, h(\theta),\ \theta \in T_S \,\}$ is said to be a *supermartingale system* (resp. a *martingale system*) if for any $\theta, \theta' \in T_0$ such that $\theta \geq \theta'$ a.s. ,

$$E[h(\theta)\,|\,\mathcal{F}_{\theta'}] \leq h(\theta') \quad \text{a.s.}\,, \qquad (\text{resp.} \quad E[h(\theta)\,|\,\mathcal{F}_{\theta'}] = h(\theta') \quad \text{a.s.}\,.)$$

Proposition 1.3.

- *The value function* $\{\, v(S),\ S \in T_0 \,\}$ *is a supermartingale system.*
- *Futhermore, it is characterized as the* Snell envelope system *associated with* $\{\, \phi(S),\ S \in T_0 \,\}$*, that is the smallest supermartingale system which is greater (a.s.) than* $\{\, \phi(S),\ S \in T_0 \,\}$.

PROOF: Let us prove the first part. Fix $S \geq S'$ a.s.. By Proposition 1.2, there exists an optimizing sequence (θ^n) for $v(S)$. By the monotone convergence theorem, $E[v(S)\,|\,\mathcal{F}_{S'}] = \lim_{n\to\infty} E[\phi(\theta^n)\,|\,\mathcal{F}_{S'}]$ a.s.. Now, for each n, since $\theta^n \geq S'$ a.s. , we have $E[\phi(\theta^n)\,|\,\mathcal{F}_{S'}] \leq v(S')$ a.s.. Hence, $E[v(S)\,|\,\mathcal{F}_{S'}] \leq v(S')$ a.s. , which gives the supermartingale property of the value function.

Let us prove the second part. Let $\{v'(S),\ S \in T_0\}$ be a supermartingale system such that for each $\theta \in T_0$, $v'(\theta) \geq \phi(\theta)$ a.s.. Fix $S \in T_0$. By the properties of v', for all $\theta \in T_S$, $v'(S) \geq E[v'(\theta)\,|\,\mathcal{F}_S] \geq E[\phi(\theta)\,|\,\mathcal{F}_S]$ a.s. .Taking the supremum over $\theta \in T_S$, we have $v'(S) \geq v(S)$ a.s.. □

Remark 1.2. *One can easily show that the supermartingale property of the value function (first point of Proposition 1.3) is equivalent to the* dynamic programming principle *that is, for each* S, $S' \in T_0$ *with* $S \leq S'$ *a.s.*

$$v(S) = \text{ess sup}_{\theta \in T_{S,\,S'}} E[v(\theta)\,|\,\mathcal{F}_S] \quad a.s.\,, \tag{1.2}$$

where $T_{S,\,S'}$ *is the set of stopping times* θ *in* T_0 *such that* $S \leq \theta \leq S'$ *a.s.*.

Recall now the following Bellman optimality criterium (see for instance El Karoui (1981)):

Proposition 1.4. (Optimality criterium) *Fix $S \in T_0$ and let $\theta^* \in T_S$ such that $E[\phi(\theta^*)] < \infty$. The two following assertions are equivalent*

1. *θ^* is S-optimal for $v(S)$, that is*

$$v(S) = E[\phi(\theta^*) \,|\, \mathcal{F}_S] \quad a.s.\,. \tag{1.3}$$

2. *The following equalities hold*

$$v(\theta^*) = \phi(\theta^*) \quad a.s.\,, \quad \text{and} \quad E[v(S)] = E[v(\theta^*)].$$

3. *The following equality holds*

$$E[v(S)] = E[\phi(\theta^*)].$$

Remark 1.3. *Note that since the value function is a supermartingale system, equality $E[v(S)] = E[v(\theta^*)]$ is equivalent to the fact that the family $\{\, v(\theta), \theta \in T_{S,\theta^*} \,\}$ is a martingale system.*

Proof: Let us show that 1) implies 2). Suppose 1) is satisfied. Since the value function v is a supermartingale system greater that ϕ, we have clearly

$$v(S) \geq E[v(\theta^*) \,|\, \mathcal{F}_S] \geq E[\phi(\theta^*) \,|\, \mathcal{F}_S] \text{ a.s.}\,.$$

Since equality (1.3) holds, this implies that the previous inequalities are actually equalities.
In particular, $E[v(\theta^*) \,|\, \mathcal{F}_S] = E[\phi(\theta^*) \,|\, \mathcal{F}_S]$ a.s. but as inequality $v(\theta^*) \geq \phi(\theta^*)$ holds a.s., and as $E[\phi(\theta^*)] < \infty$, we have $v(\theta^*) = \phi(\theta^*)$ a.s..
Moreover, $v(S) = E[v(\theta^*) \,|\, \mathcal{F}_S]$ a.s. which gives $E[v(S)] = E[v(\theta^*)]$. Hence, 2) is satisfied.

Clearly, 2) implies 3). It remains to show that 3) implies 1).

Suppose that 3) is satisfied. Since $v(S) \geq E[\phi(\theta^*) \,|\, \mathcal{F}_S]$ a.s., this gives $v(S) = E[\phi(\theta^*) \,|\, \mathcal{F}_S]$ a.s.. Hence, 1) is safisfied. □

Remark 1.4. *It is clear that*

$$E[v(S)] = \sup_{\theta \in T_S} E[\phi(\theta)]. \tag{1.4}$$

By 3) of Proposition 1.4, a stopping time $\theta^* \in T_S$ such that $E[\phi(\theta^*)] < \infty$ is S-optimal for $v(S)$ if and only if it is optimal for the optimal stopping time problem (1.4), that is

$$\sup_{\theta \in T_S} E[\phi(\theta)] = E[\phi(\theta^*)].$$

We now give a regularity result on v (see lemma 2.13 in El Karoui (1981)). Let us first introduce the following definition.

Definition 1.2. *An admissible integrable family* $\{\phi(\theta),\ \theta \in T_0\}$ *is said to be* right (resp. left) continuous along stopping times in expectation (RCE (resp. LCE)) *if for any* $\theta \in T_0$ *and for any sequence* $(\theta_n)_{n\in\mathbb{N}}$ *of stopping times such that* $\theta_n \downarrow \theta$ *a.s. (resp.* $\theta_n \uparrow \theta$ *a.s.) one has* $E[\phi(\theta)] = \lim_{n\to\infty} E[\phi(\theta_n)]$.

Remark 1.5. *If* (ϕ_t) *is a continuous adapted process such that* $E[\sup_{t\in[0,T]} \phi_t] < \infty$, *then the family defined by* $\phi(\theta) = \phi_\theta$ *is clearly RCE and LCE. Also, if* (ϕ_t) *is a RCLL adapted process such that its jumps are totally inaccessible (which is the case for Poisson processes), then the family defined by* $\phi(\theta) = \phi_\theta$ *is clearly RCE and even LCE.*

Proposition 1.5. *Let* $\{\phi(\theta),\ \theta \in T_0\}$ *be an admissible family which is RCE. Then, the family* $\{v(S),\ S \in T_0\}$ *is RCE.*

Proof: Since $\{v(S),\ S \in T_0\}$ is a supermartingale system, the function $S \mapsto E[v(S)]$ is a nonincreasing function of stopping times. Suppose it is not RCE at $S \in T_0$. If $E[v(S)] < \infty$, there exists a constant $\alpha > 0$ and a sequence of stopping times $(S_n)_{n\in\mathbb{N}}$ such that $S_n \downarrow S$ a.s. and such that

$$\lim_{n\to\infty} \uparrow E[v(S_n)] + \alpha \leq E[v(S)]. \tag{1.5}$$

Now, recall that $E[v(S)] = \sup_{\theta\in T_S} E[\phi(\theta)]$ (see (1.4)). Hence, there exists $\theta' \in T_S$ such that

$$\sup_{n\in\mathbb{N}} \sup_{\theta\in T_{S_n}} E[\phi(\theta)] + \frac{\alpha}{2} \leq E[\phi(\theta')].$$

Hence for all $n \in \mathbb{N}$, $E[\phi(\theta' \vee S_n)] + \frac{\alpha}{2} \leq E[\phi(\theta')]$. As $\theta' \vee S_n \downarrow \theta'$ a.s. we obtain, by taking the limit when $n \to \infty$ and by using the RCE property of ϕ that

$$E[\phi(\theta')] + \frac{\alpha}{2} \leq E[\phi(\theta')],$$

which gives the expected contradiction in the case $E[v(S)] < \infty$.
Otherwise, we have instead of (1.5), $\lim_{n\to\infty} \uparrow E[v(S_n)] \leq C$ for some constant $C > 0$ and similar arguments as in the finite case lead to a contradiction as well. □

1.2 New results

We will now give a new result which generalizes the classical existence result of an optimal stopping time stated in the case of a reward process to the case of a reward family of random variables.

Theorem 1.1. (Existence of optimal stopping times)
Let $\{\phi(\theta), \theta \in T_0\}$ *be an admissible family that satisfies the integrability condition* $E[\text{ess sup}_{\theta\in T_0} \phi(\theta)] < \infty$ *and which is RCE and LCE along stopping times.*

Then, for each $S \in T_0$, there exists an optimal stopping time for $v(S)$. Moreover the random variable defined by

$$\theta^*(S) := \text{ess inf}\{\, \theta \in T_S\,,\, v(\theta) = \phi(\theta)\ \textit{a.s.}\,\} \tag{1.6}$$

is the minimal optimal stopping time for $v(S)$.

Let us emphasize that in this theorem, the optimal stopping time $\theta^*(S)$ is not defined trajectorially but as an essential infimum of random variables. In the classical case that is, when the reward is given by an adapted RCLL process, recall that the minimal optimal stopping time is given by the random variable $\overline{\theta}(S)$ defined trajectorially by

$$\overline{\theta}(S) = \inf\{\, t \geq S,\ v_t = \phi_t\,\}\,.$$

The definition of $\theta^*(S)$ as an essential infimum allows to relax the assumption on the regularity of the reward. More precisely, whereas in the previous works (quoted in the introduction) the reward was given by a RCLL and LCE process, in our setting, the reward is given by a RCE and LCE family of r.v.. The idea of the proof is classical: we use an approximation method introduced by Maingueneau (1978) but our setting allows to simplify and shorten the proof.

PROOF: the proof will be divided in two parts.
Part I: In this part, we will prove the existence of an optimal stopping time.

Fix $S \in T_0$. We begin by constructing a family of stopping times (see Maingueneau (1978) or El Karoui (1981)). For $\lambda \in\,]0,1[$, define the $\mathcal{F}_S$-measurable random variable $\theta^\lambda(S)$ by

$$\theta^\lambda(S) := \text{ess inf}\{\, \theta \in T_S\,,\, \lambda v(\theta) \leq \phi(\theta)\ \text{a.s.}\,\}. \tag{1.7}$$

The following lemma holds:

Lemma 1.1. *The stopping time $\theta^\lambda(S)$ is a $(1-\lambda)$-optimal stopping time for*

$$E[v(S)] = \sup_{\theta \in T_S} E[\phi(\theta)], \tag{1.8}$$

that is

$$\lambda E[v(S)] \ \leq\ E[\phi(\theta^\lambda(S))]. \tag{1.9}$$

Suppose now that we have proved Lemma 1.1.
Since $\lambda \mapsto \theta^\lambda(S)$ is non decreasing, for $S \in T_0$, the following stopping time

$$\hat{\theta}(S) := \lim_{\lambda \uparrow 1} \uparrow \theta^\lambda(S)\,, \tag{1.10}$$

is well defined. Let us show that $\hat{\theta}(S)$ is optimal for $v(S)$.
By letting $\lambda \uparrow 1$ in the above inequality (1.9), and since ϕ is LCE, we easily

derive that $E[v(S)] = E[\phi(\hat{\theta}(S))]$. Consequently, by the optimality criterium 3) of Proposition 1.4 , $\hat{\theta}(S)$ is S-optimal for $v(S)$. This ends part **I**.

Part II: Let us now prove that $\theta^*(S) = \hat{\theta}(S)$ a.s. , where $\theta^*(S)$ is defined by (1.6), and that it is the minimal optimal stopping time for $v(S)$.

For each $S \in T_0$, the set $\mathbb{T}_S = \{\, \theta \in T_S \,,\, v(\theta) = \phi(\theta) \text{ a.s.} \,\}$ is not empty (since T belongs to $\mathbb{T}_S$) and is closed under pairwise minimization. Hence there exists a sequence $(\theta_n)_{n\in\mathbb{N}}$ of stopping times in $\mathbb{T}_S$ such that $\theta_n \downarrow \theta^*(S)$ a.s.. Consequently, $\theta^*(S)$ is a stopping time.

Let θ be an optimal stopping time for $v(S)$. By the optimality criterium (Proposition 1.4) and since by assumption $E[\phi(\theta)] < \infty$, we have $v(\theta) = \phi(\theta)$ a.s. and hence

$$\theta^*(S) \leq \text{ess inf}\{\, \theta \in T_0,\ \theta \text{ optimal for } v(S) \,\} \ \text{ a.s. } .$$

Now for each $\lambda < 1$ the stopping time $\theta^\lambda(S)$ defined by (1.7) satisfies clearly $\theta^\lambda(S) \leq \theta^*(S)$ a.s.. Passing to the limit when $\lambda \uparrow 1$ we obtain $\hat{\theta}(S) \leq \theta^*(S)$. As $\hat{\theta}(S)$ is optimal for $v(S)$, this implies that $\hat{\theta}(S) \geq \text{ess inf}\{\ \theta \in T_0,\ \theta \text{ optimal for } v(S) \,\}$ a.s.. Hence

$$\theta^*(S) = \hat{\theta}(S) = \text{ess inf}\{\, \theta \in T_0,\ \theta \text{ optimal for } v(S) \,\} \quad \text{a.s.}\,,$$

which gives the desired result. The proof of Theorem 1.1 is ended. □

It remains now to prove Lemma 1.1.

Proof of Lemma 1.1: We have to prove inequality (1.9). This will be done by the following steps.

Step 1: Fix $\lambda \in]0,1[$. It is easy to check that the set $\mathbb{T}_S^\lambda = \{\, \theta \in T_S \,,\, \lambda v(\theta) \leq \phi(\theta) \text{ a.s.} \,\}$ is non empty (since $T \in \mathbb{T}_S^\lambda$) and closed by pairwise minimization. By Theorem A.1 in the Appendix, there exists a sequence (θ^n) in $\mathbb{T}_S$ such that $\theta^n \downarrow \theta^\lambda(S)$ a.s.. Therefore $\theta^\lambda(S)$ is a stopping time and $\theta^\lambda(S) \geq S$ a.s.. Moreover we have $\lambda v(\theta^n) \leq \phi(\theta^n)$ a.s. for all n. Taking expectation and using the RCE properties of v and ϕ we obtain

$$\lambda E[v(\theta^\lambda(S)])] \leq E[\phi(\theta^\lambda(S))] \,. \tag{1.11}$$

Step 2: Let us show that for each $\lambda \in]0,1[$ and for each $S \in T_0$,

$$v(S) = E[v(\theta^\lambda(S)) \,|\, \mathcal{F}_S] \quad \text{a.s.}\,. \tag{1.12}$$

Let us define for each $S \in T_0$, the random variable $J(S) = E[v(\theta^\lambda(S)) \,|\, \mathcal{F}_S]$. Step 2 amounts to show that $J(S) = v(S)$ a.s..

Since $\{\, v(S), S \in T_0 \,\}$ is a supermartingale system and since $\theta^\lambda(S) \geq S$ a.s. , we have that

$$J(S) = E[v(\theta^\lambda(S)) \,|\, \mathcal{F}_S] \leq v(S) \quad \text{a.s.}\,.$$

It remains to show the reverse inequality.
Step 2 a: Let us show that the family $\{ J(S),\ S \in T_0 \}$ is a supermartingale system.

Let $S, S^{'} \in T_0$ such that $S^{'} \geq S$ a.s.. As $\theta^{\lambda}(S^{'}) \geq \theta^{\lambda}(S) \geq S$ a.s., one has

$$E[J(S^{'}) \,|\, \mathcal{F}_S] = E[v(\theta^{\lambda}(S^{'})) \,|\, \mathcal{F}_S] = E\left[E[v(\theta^{\lambda}(S^{'})) \,|\, \mathcal{F}_{\theta^{\lambda}(S)}] \,|\, \mathcal{F}_S\right] \quad \text{a.s.}.$$

Now, since $\{ v(S),\ S \in T_0 \}$ is a supermartingale system, $E[v(\theta^{\lambda}(S^{'})) \,|\, \mathcal{F}_{\theta^{\lambda}(S)}] \leq v(\theta^{\lambda}(S))$ a.s.. Consequently,

$$E[J(S^{'}) \,|\, \mathcal{F}_S] \leq E[v(\theta^{\lambda}(S)) \,|\, \mathcal{F}_S] = J(S) \quad \text{a.s.}.$$

Step 2 b: Let us show that for each $S \in T_0$, and each $\lambda \in]0,1[$,

$$\lambda v(S) + (1-\lambda) J(S) \geq \phi(S) \quad \text{a.s.}.$$

Fix $S \in T_0$ and $\lambda \in]0,1[$.
On $\{ \lambda v(S) \leq \phi(S) \}$, we have $\theta^{\lambda}(S) = S$ a.s.. Hence, on $\{ \lambda v(S) \leq \phi(S) \}$, $J(S) = E[v(\theta^{\lambda}(S)) \,|\, \mathcal{F}_S] = E[v(S) \,|\, \mathcal{F}_S] = v(S)$, and therefore

$$\lambda v(S) + (1-\lambda) J(S) = v(S) \geq \phi(S) \quad \text{a.s.}.$$

Furthermore, on$\{ \lambda v(S) > \phi(S) \}$, as $J(S)$ is nonnegative, we have

$$\lambda v(S) + (1-\lambda) J(S) \geq \lambda v(S) \geq \phi(S) \quad \text{a.s.},$$

and the proof of Step 2 b is complete.

Now, the family $\{\lambda v(S) + (1-\lambda) J(S),\ S \in T_0\}$ is a supermartingale system by convex combination of two supermartingale systems. Hence, as the value function $\{ v(S),\ S \in T_0 \}$ is characterized as the smallest supermartingale system which dominates $\{ \phi(S),\ S \in T_0 \}$, we derive that for each $S \in T_0$,

$$\lambda v(S) + (1-\lambda) J(S) \geq v(S) \quad \text{a.s.}.$$

Now, by the integrability assumption made on ϕ, we have $v(S) < \infty$ a.s.. Hence, we have $J(S) \geq v(S)$ a.s.. Consequently, for each $S \in T_0$, $J(S) = v(S)$ a.s., which ends step 2.

Finally, Step 1 (inequality (1.11)) and Step 2 (equality (1.12)) give

$$\lambda E[v(S)] = \lambda E[v(\theta^{\lambda}(S))] \leq E[\phi(\theta^{\lambda}(S))],$$

In orther terms, $\theta^{\lambda}(S)$ is a $(1-\lambda)$-optimal stopping time for (1.8), which ends the proof of Lemma 1.1. $\square$

Remark 1.6. *Recall that in the previous works (see for example Karatzas and Shreve (1998) Proposition D.10 and Theorem D.12) the proof of the existence of optimal stopping times requires to aggregate the value function and thus to use some fine aggregation results such as Proposition 4.1. In our work, since we only work with families of r.v., we do not need any aggregation techniques which simplifies and shorten the proof.*

Under some regularity assumptions on the reward, we can show that the value function family is left continuous along stopping times. More precisely,

Proposition 1.6. *Suppose that the admissible family* $\{\, \phi(\theta),\ \theta \in T_0 \,\}$ *is LCE and RCE and satisfies the integrability condition* $E[\operatorname{ess\,sup}_{\theta \in T_0} \phi(\theta)] < \infty$.
Then, the value function $\{\, v(S),\ S \in T_0 \,\}$ *defined by (1.1) is LCE.*

Proof : Let $S \in T_0$ and let (S_n) be a sequence of stopping times such that $S_n \uparrow S$ a.s.. Note that by the supermartingale property of v we have

$$E[v(S_n)] \geq E[v(S)]. \tag{1.13}$$

Now by Theorem 1.1, the stopping time $\theta^*(S_n)$ defined by (1.6) is optimal for $v(S_n)$. Moreover, it is clear that $(\theta^*(S_n))_n$ is a nondecreasing sequence of stopping times dominated by $\theta^*(S)$.

Let us define $\overline{\theta} = \lim_{n\to\infty} \uparrow \theta^*(S_n)$. Note that $\overline{\theta}$ is a stopping time. Also, as for each n, $\theta^*(S_n) \geq S_n$ a.s., it follows that $\overline{\theta} \geq S$ a.s.. Therefore, since ϕ is LCE,

$$E[v(S)] \geq E[\phi(\overline{\theta})] = \lim_{n\to\infty} E[\phi(\theta^*(S_n))] = \lim_{n\to\infty} E[v(S_n)].$$

This together with (1.13) gives $E[v(S)] = \lim_{n\to\infty} E[v(S_n)]$. □

Remark 1.7. *In this proof, we have also proved that* $\overline{\theta}$ *is optimal for* $v(S)$. *Hence, by the optimality criterium,* $v(\overline{\theta}) = \phi(\overline{\theta})$ *a.s. which implies that* $\overline{\theta} \geq \theta^*(S)$ *a.s. Moreover, since for each* n, $\theta^*(S_n) \leq \theta^*(S)$ *a.s., by letting* n *tend to* ∞, *we have clearly that* $\overline{\theta} \leq \theta^*(S)$ *a.s.. Hence,* $\overline{\theta} = \lim_{n\to\infty} \uparrow \theta^*(S_n) = \theta^*(S)$ *a.s.. Thus, we have also shown that the map* $S \mapsto \theta^*(S)$ *is left continuous along stopping times.*

2 The optimal double stopping time problem

2.1 Definition and first properties of the value function

We consider now the optimal double stopping time problem. We introduce the following definitions

Definition 2.1. *The family* $\{\psi(\theta,S),\ \theta,S\in T_0\}$ *is a* biadmissible family *if it satisfies*

1. *for all* $\theta,S\in T_0$, $\psi(\theta,S)$ *is a* $\mathcal{F}_{\theta\vee S}$*-measurable positive r.v.,*
2. *for all* $\theta,\theta',S,S'\in T_0$, $\psi(\theta,S)=\psi(\theta',S')$ *a.s. on* $\{\theta=\theta'\}\cap\{S=S'\}$.

Remark 2.1. *Let* Ψ *be a biprocess that is a function*

$$\Psi:\ [0,T]^2\times\Omega\to\mathbb{R}^+;\ (t,s,\omega)\mapsto\Psi_{t,s}(\omega)$$

such that for almost surely ω*, the map* $(t,s)\mapsto\Psi_{t,s}(\omega)$ *is* right continuous *(that is* $\Psi_{t,s}=\lim_{(t',s')\to(t^+,s^+)}\Psi_{t',s'}$ *) and for each* $(t,s)\in[0,T]^2$, $\Psi_{t,s}$ *is* $\mathcal{F}_{t\vee s}$*-measurable . In this case, the family* $\{\psi(\theta,S),\ \theta,S\in T_0\}$ *defined by*

$$\psi(\theta,S)(\omega):=\Psi_{\theta(\omega),S(\omega)}(\omega)$$

is clearly biadmissible.

For a biadmissible family $\{\psi(\theta,S),\ \theta,S\in T_0\}$ let us consider the value function associated with the reward family $\{\psi(\theta,S),\ \theta,S\in T_0\}$

$$v(S)=\operatorname{ess\,sup}_{\tau_1,\tau_2\in T_S}E[\psi(\tau_1,\tau_2)\,|\,\mathcal{F}_S]\tag{2.1}$$

As in the case of one stopping time problem, we have the following properties:

Proposition 2.1. *Let* $\{\psi(\theta,S),\ \theta,S\in T_0\}$ *be a biadmissible family of r.v. , then the following properties hold:*

1. *The family* $\{v(S),\ S\in T_0\}$ *is an admissible family of r.v. .*
2. *For each* $S\in T_0$*, there exists a sequence of pairs of stopping times* $((\tau_1^n,\tau_2^n))_{n\in\mathbb{N}}$ *in* $T_S\times T_S$ *such that* $\{E[\psi(\tau_1^n,\tau_2^n)\,|\,\mathcal{F}_S]\}_{n\in\mathbb{N}}$ *is nondecreasing and a.s.*

$$E[\psi(\tau_1^n,\tau_2^n)\,|\,\mathcal{F}_S]\uparrow v(S).$$

3. *The family of r.v.* $\{v(S),\ S\in T_0\}$ *is a supermartingale system. In other words, it satisfies the dynamic programming principle (1.2).*

Proof: 1) As in the case of one stopping time, Property 1 of admissibility for $\{v(S),\ S\in T_0\}$ follows from the existence of the essential supremum.
Take $S,S'\in T_0$ and put $A=\{S=S'\}$, and for each $\tau_1,\tau_2\in T_S$ put $\tau_1^A=\tau_1\mathbf{1}_A+T\mathbf{1}_{A^c}$ and $\tau_2^A=\tau_2\mathbf{1}_A+T\mathbf{1}_{A^c}$. As $A\in\mathcal{F}_S\cap\mathcal{F}_{S'}$, one has a.s. on A, $E[\psi(\tau_1,\tau_2)\,|\,\mathcal{F}_S]=E[\psi(\tau_1^A,\tau_2^A)\,|\,\mathcal{F}_S]=E[\psi(\tau_1^A,\tau_2^A)\,|\,\mathcal{F}_{S'}]\le v(S')$. Hence, taking the essential supremum over $\tau_1,\tau_2\in T_S$, we have $v(S)\le v(S')$ a.s. and

by symmetry, we have shown Property 2 of admissibility. Hence, the family $\{ v(S),\ S \in T_0 \}$ is an admissible family of r.v..

The proofs of 2) and 3) can be easily adapted from the proofs of Proposition 1.2 and Proposition 1.3. □

Following the case of one stopping time, we now give some regularity results on the value function.

Definition 2.2. *A biadmissible family* $\{ \psi(\theta, S),\ \theta, S \in T_0 \}$ *is said to be* right continuous along stopping times in expectation (RCE) *if for any* $\theta, S \in T_0$ *and for any sequences* $(\theta_n)_{n\in\mathbb{N}} \in T_0$ *and* $(S_n)_{n\in\mathbb{N}} \in T_0$ *such that* $\theta_n \downarrow \theta$ *and* $S_n \downarrow S$ *a.s. , one has* $E[\psi(\theta, S)] = \lim_{n\to\infty} E[\psi(\theta_n, S_n)]$.

Proposition 2.2. *Suppose that the biadmissible family* $\{ \psi(\theta, S),\ \theta, S \in T_0 \}$ *is RCE. Then, the family* $\{ v(S),\ S \in T_0 \}$ *defined by (2.1) is RCE.*

Proof: The proof follows the proof of Proposition 1.5. □

2.2 Reduction to an optimal single stopping time problem

In this section, we will show that the optimal double stopping time problem (2.1) can be reduced to an optimal single stopping time problem associated with a *new reward family*.
More precisely, for each stopping time $\theta \in T_S$, let us introduce the two $\mathcal{F}_\theta$-measurable random variables

$$u_1(\theta) = \operatorname{ess\,sup}_{\tau_1 \in T_\theta} E[\psi(\tau_1, \theta) \,|\, \mathcal{F}_\theta],\ \ u_2(\theta) = \operatorname{ess\,sup}_{\tau_2 \in T_\theta} E[\psi(\theta, \tau_2) \,|\, \mathcal{F}_\theta]. \tag{2.2}$$

Note that since $\{ \psi(\theta, S),\ \theta, S \in T_0 \}$ is biadmissible, for each fixed $\theta \in T_0$, the families $\{ \psi(\tau_1, \theta), \tau_1 \in T_0 \}$ and $\{ \psi(\theta, \tau_2), \tau_2 \in T_0 \}$ are admissible. Hence, by Proposition 1.1, the families $\{ u_1(\theta),\ \theta \in T_S \}$ and $\{ u_2(\theta),\ \theta \in T_S \}$ are admissible. Put

$$\phi(\theta) = \max[u_1(\theta), u_2(\theta)]. \tag{2.3}$$

The family $\{ \phi(\theta),\ \theta \in T_S \}$, which is called the *new reward family*, is also clearly admissible. Consider the value function associated with the new reward

$$u(S) = \operatorname{ess\,sup}_{\theta \in T_S} E[\phi(\theta) \,|\, \mathcal{F}_S] \quad \text{a.s. .} \tag{2.4}$$

Theorem 2.1. (Reduction) *Suppose that* $\{ \psi(\theta, S),\ \theta, S \in T_0 \}$ *is a biadmissible family. For each stopping time* S*, consider* $v(S)$ *defined by (2.1) and* $u(S)$ *defined by (2.2), (2.3), (2.4), then*

$$v(S) = u(S) \quad a.s. .$$

PROOF : Let S be a stopping time.
Step 1: First, let us show that $v(S) \leq u(S)$ a.s..

Let $\tau_1, \tau_2 \in T_S$. Put $A = \{ \tau_1 \leq \tau_2 \}$. As A is in $\mathcal{F}_{\tau_1} \cap \mathcal{F}_{\tau_2}$,

$$E[\psi(\tau_1, \tau_2) \,|\, \mathcal{F}_S] = E[\mathbf{1}_A E[\psi(\tau_1, \tau_2) \,|\, \mathcal{F}_{\tau_1}] \,|\, \mathcal{F}_S] + E[\mathbf{1}_{A^c} E[\psi(\tau_1, \tau_2) \,|\, \mathcal{F}_{\tau_2}] \,|\, \mathcal{F}_S].$$

By noticing that on A one has $E[\psi(\tau_1, \tau_2) \,|\, \mathcal{F}_{\tau_1}] \leq u_2(\tau_1) \leq \phi(\tau_1 \wedge \tau_2)$ a.s., and similarly on A^c one has $E[\psi(\tau_1, \tau_2) \,|\, \mathcal{F}_{\tau_2}] \leq u_1(\tau_2) \leq \phi(\tau_1 \wedge \tau_2)$ a.s., we get

$$E[\psi(\tau_1, \tau_2) \,|\, \mathcal{F}_S] \leq E[\phi(\tau_1 \wedge \tau_2) \,|\, \mathcal{F}_S] \leq u(S) \quad \text{a.s.}\,.$$

By taking the supremum over τ_1 and τ_2 in T_S we obtain step 1.
Step 2: Let us show now that $v(S) \geq u(S)$ a.s..

We have clearly $v(S) \geq \text{ess sup}_{\tau_2 \in T_S} E[\psi(S, \tau_2) \,|\, \mathcal{F}_S] = u_2(S)$ a.s.. By similar arguments, $v(S) \geq u_1(S)$ a.s. and consequently,

$$v(S) \geq \max[u_1(S), u_2(S)] = \phi(S) \quad \text{a.s.}\,.$$

Thus, $\{ v(S),\ S \in T_0 \}$ is a supermartingale system which is greater than $\{ \phi(S),\ S \in T_0 \}$. Now, by Proposition 1.3, $\{ u(S),\ S \in T_0 \}$ is the smallest supermartingale system which is greater than $\{ \phi(S),\ S \in T_0 \}$. Consequently, step 2 follows, which ends the proof. □

Note that the reduction to an optimal one stopping time problem associated with a new reward will be the key property used to construct optimal multiple stopping times and to establish an existence result of optimal multiple stopping times (see sections 2.3 to 2.5).

2.3 Properties of optimal stopping times

In this section, we are given a biadmissible family $\{\psi(\theta, S),\ \theta, S \in T_0\}$ such that $E[\text{ess sup}_{\theta, S \in T_0} \psi(\theta, S)] < \infty$.

Proposition 2.3. (A necessary condition of optimality) *Let S be a stopping time and consider the value function $v(S)$ defined by (2.1), for all $\theta \in T_S$, $u_1(\theta), u_2(\theta)$ defined by (2.2), $\phi(\theta)$ defined by (2.3) and $u(S)$ defined by (2.4). Suppose that the pair (τ_1^*, τ_2^*) is optimal for $v(S)$, and put $A = \{ \tau_1^* \leq \tau_2^* \}$ then*

1. *$\tau_1^* \wedge \tau_2^*$ is optimal for $u(S)$,*
2. *τ_2^* is optimal for $u_2(\tau_1^*)$ a.s. on A,*
3. *τ_1^* is optimal for $u_1(\tau_2^*)$ a.s. on A^c.*

Moreover $A = \{ \tau_1^* \leq \tau_2^* \} \subset B = \{ u_1(\tau_1^* \wedge \tau_2^*) \leq u_2(\tau_1^* \wedge \tau_2^*) \}$.

PROOF : Let $S \in T_0$, and suppose the pair of stopping times (τ_1^*, τ_2^*) is optimal for $v(S)$. As $u(S) = v(S)$ a.s., we obtain equality in step 1 of the proof of Theorem 2.1. More precisely,

$$\begin{aligned}
&v(S) = E[\psi(\tau_1^*, \tau_2^*) \,|\, \mathcal{F}_S] = E[\phi(\tau_1^* \wedge \tau_2^*) \,|\, \mathcal{F}_S] = u(S) \quad \text{a.s.}, \\
&E[\psi(\tau_1^*, \tau_2^*) \,|\, \mathcal{F}_{\tau_1^*}] = u_2(\tau_1^*) = u_2(\tau_1^* \wedge \tau_2^*) = \phi(\tau_1^* \wedge \tau_2^*) \ \text{ a.s. on } A, \\
&E[\psi(\tau_1^*, \tau_2^*) \,|\, \mathcal{F}_{\tau_2^*}] = u_1(\tau_2^*) = u_1(\tau_1^* \wedge \tau_2^*) = \phi(\tau_1^* \wedge \tau_2^*) \quad \text{ a.s. on } A^c,
\end{aligned}$$

which easily leads to 1), 2), 3) and $A \subset B$. $\square$

Remark 2.2. *Note that in general for a pair (τ_1^*, τ_2^*) of optimal stopping times for $v(S)$ the inclusion $A \subset B$ is strict. Indeed if $\psi \equiv 0$, then $v = u = u_1 = u_2 = \phi = 0$, and all pairs of stopping times are optimal. Consider $\tau_1^* = T, \tau_2^* = 0$. In this case, $A = \emptyset$ and $B = \Omega$.*

We now give a sufficient condition of optimality.

Proposition 2.4. (Construction of optimal stopping times) *Under the notation of Proposition 2.3, suppose that*

1. *θ^* is optimal for $u(S)$,*
2. *θ_2^* is optimal for $u_2(\theta^*)$,*
3. *θ_1^* is optimal for $u_1(\theta^*)$.*

and put $B = \{\ u_1(\theta^*) \le u_2(\theta^*)\ \}$, then the pair of stopping times (τ_1^*, τ_2^*) defined by

$$\tau_1^* = \theta^* \mathbf{1}_B + \theta_1^* \mathbf{1}_{B^c}, \quad \tau_2^* = \theta_2^* \mathbf{1}_B + \theta^* \mathbf{1}_{B^c}, \tag{2.5}$$

is optimal for $v(S)$.
Moreover, $\tau_1^* \wedge \tau_2^* = \theta^*$ and $B = \{\, \tau_1^* \le \tau_2^* \,\}$.

PROOF: Let θ^* be an optimal stopping time for $u(S)$ that is $u(S) = E[\phi(\theta^*) \,|\, \mathcal{F}_S]$ a.s.. Let θ_1^* be an optimal stopping time for $u_1(\theta^*)$ ($u_1(\theta^*) = E[\psi(\theta_1^*, \theta^*) \,|\mathcal{F}_{\theta^*}]$ a.s.) and let θ_2^* be an optimal stopping time for $u_2(\theta^*)$ ($u_2(\theta^*) = E[\psi(\theta^*, \theta_2^*) \,|\mathcal{F}_{\theta^*}]$ a.s.). We introduce the set $B = \{\ u_1(\theta^*) \le u_2(\theta^*)\ \}$. Note that B is in $\mathcal{F}_{\theta^*}$. Let τ_1^*, τ_2^* be the stopping times defined by (2.5). We have clearly the inclusion

$$B \subset \{\, \tau_1^* \le \tau_2^* \,\}. \tag{2.6}$$

Since $u(S) = E[\phi(\theta^*) \,|\, \mathcal{F}_S]$ and since $\phi(\theta^*) = \max[u_1(\theta^*), u_2(\theta^*)]$, we have

$$u(S) \quad = \quad E[\mathbf{1}_B u_2(\theta^*) + \mathbf{1}_{B^c} u_1(\theta^*) \,|\, \mathcal{F}_S].$$

The optimality of θ_1^* and θ_2^* gives that a.s.,

$$\begin{aligned}
u(S) \quad &= \quad E[\mathbf{1}_B \psi(\theta^*, \theta_2^*) + \mathbf{1}_{B^c} \psi(\theta_1^*, \theta^*) \,|\, \mathcal{F}_S] \\
&= \quad E[\mathbf{1}_B \psi(\tau_1^*, \tau_2^*) + \mathbf{1}_{B^c} \psi(\tau_1^*, \tau_2^*) \,|\, \mathcal{F}_S] = E[\psi(\tau_1^*, \tau_2^*) \,|\, \mathcal{F}_S]\,.
\end{aligned}$$

As $u(S) = v(S)$ a.s. , the pair of stopping times (τ_1^*, τ_2^*) is S-optimal for $v(S)$. By Proposition 2.3, we have $\{\, \tau_1^* \leq \tau_2^* \,\} \subset B$. Hence, by (2.6), $B = \{\, \tau_1^* \leq \tau_2^* \,\}$. □

Remark 2.3. *Proposition 2.4 still holds true if condition 2. holds true on the set B, and condition 3. holds true on the set B^c.*

Note that by Remark 2.2, we do not have a characterization of optimal pairs of stopping times. However, it is possible to give a characterization of *minimal optimal* stopping times in a particular sense (see the Appendix B).

2.4 Regularity of the new reward

Before studying the problem of existence of optimal stopping times, we have to state some regularity properties of the new reward family $\{\phi(\theta),\ \theta \in T_0\}$.

Let us introduce the following definition,

Definition 2.3. *A biadmissible family $\{\, \psi(\theta, S),\ \theta, S \in T_0 \,\}$ is said to be* uniformly right (resp. left) continuous in expectation along stopping times (URCE (resp. ULCE)) *if $E[\text{ess sup}_{\,\theta, S \in T_0} \psi(\theta, S)] < \infty$ and if, for each $\theta, S \in T_0$ and for each sequence of stopping times $(S_n)_{n\in\mathbb{N}}$ such that $S_n \downarrow S$ a.s. (resp. $S_n \uparrow S$ a.s.)*

$$\begin{aligned} & \lim_{n\to\infty} E\left[\operatorname*{ess\,sup}_{\theta\in T_0} |\psi(\theta, S) - \psi(\theta, S_n)|\right] = 0 \\ \textit{and} \quad & \lim_{n\to\infty} E\left[\operatorname*{ess\,sup}_{\theta\in T_0} |\psi(S, \theta) - \psi(S_n, \theta)|\right] = 0. \end{aligned}$$

The following right continuity property holds true for the new reward family:

Theorem 2.2. *Suppose that the biadmissible family $\{\, \psi(\theta, S),\ \theta, S \in T_0 \,\}$ is URCE and ULCE. Then, the family $\{\, \phi(S),\ S \in T_0 \,\}$ defined by (2.3) is RCE and LCE.*

Proof: As $\phi(\theta) = \max[u_1(\theta), u_2(\theta)]$, it is sufficient to show the RCE and LCE properties for the family $\{\, u_1(\theta),\ \theta \in T_0 \,\}$.
Let us introduce the following value function for each $S, \theta \in T_0$,

$$U_1(\theta, S) = \text{ess sup}_{\tau_1 \in T_\theta} E[\psi(\tau_1, S) \,|\, \mathcal{F}_\theta] \quad \text{a.s.}\,. \tag{2.7}$$

As for all $\theta \in T_0$,

$$u_1(\theta) = U_1(\theta, \theta) \quad \text{a.s.}\,,$$

it is sufficient to prove that $\{\, U_1(\theta, S), \theta, S \in T_0 \,\}$ is RCE (resp. LCE) that is, if $\theta, S \in T_0$ and $(\theta_n)_n$, $(S_n)_n$ in T_0 are such that $\theta_n \downarrow \theta$ and $S_n \downarrow S$ a.s. (resp.

$\theta_n \uparrow \theta$ and $S_n \uparrow S$ a.s..), then $\lim_{n\to\infty} E[U_1(\theta_n, S_n)] = E[U_1(\theta, S)]$. Now, we have

$$|E[U_1(\theta,S)] - E[U_1(\theta_n,S_n)]| \quad \leq \\ \underbrace{|E[U_1(\theta,S)] - E[U_1(\theta_n,S)]|}_{\text{(I)}} + \underbrace{|E[U_1(\theta_n,S)] - E[U_1(\theta_n,S_n)]|}_{\text{(II)}}.$$

Let us show that (I) tends to 0 as $n \to \infty$. For each $S \in T_0$, $\{\, \psi(\theta, S),\ \theta \in T_0 \,\}$ is an admissible family of positive r.v. which is RCE and LCE. By Proposition 1.5 (resp. by Proposition 1.6), the value function $\{U_1(\theta, S), \theta \in T_0\}$ is RCE (resp. LCE). It follows that (I) converges to 0 as n tends to ∞.

Let us show that (II) tends to 0 as $n \to \infty$. By definition of the value function $U_1(.,.)$ (2.7), it follows that

$$|E[U_1(\theta_n,S)] \quad - \quad E[U_1(\theta_n,S_n)]| \leq E\left(\text{ess sup}_{\tau\in T_0}|\psi(\tau,S) - \psi(\tau,S_n)|\right)$$

which converges to 0 since $\{\, \psi(\theta, S),\ \theta, S \in T_0 \,\}$ is URCE (resp. ULCE). The proof of Theorem 2.2 is ended. □

Lemma 2.1. *Suppose that* $E[\text{ess sup}_{\theta,S\in T_0}\psi(\theta,S)^p] < \infty$ *for some* $p > 1$. *Then,* $E[\text{ess sup}_{\theta\in T_0}\phi(\theta)^p] < \infty$.

Proof: We have $u_1(\theta) \leq E[\text{ess sup}_{\tau,S\in T_0} \psi(\tau,S) \,|\, \mathcal{F}_\theta]$ a.s.. Hence, by martingale moments inequalities, $E[\text{ess sup}_{\theta\in T_0} u_1(\theta)^p] \leq C_p\, E[\text{ess sup}_{\tau,S\in T_0} \psi(\tau,S)^p]$ where $C_p > 0$. The same arguments hold for u_2. The result clearly follows. □

Corollary 2.1. *Suppose that* $E[\text{ess sup}_{\theta,S\in T_0}\psi(\theta,S)^p] < \infty$ *for some* $p > 1$. *Under the same hypothesis as Theorem 2.2, the family* $\{\, v(S),\ S \in T_0 \,\}$ *defined by (2.1) is LCE.*

Proof: It follows from the fact that $v(S) = u(S)$ a.s. (Theorem 2.1) where $\{\, u(S),\ S \in T_0 \,\}$ is the value function family associated with the new reward $\{\, \phi(S),\ S \in T_0 \,\}$. Moreover by Lemma 2.1 one has $E[\text{ess sup}_{S\in T_0}\phi(S)] < \infty$. Applying Proposition 1.6 we obtain the required LCE property. □

We will now turn to the problem of existence of optimal stopping times.

2.5 Existence of optimal stopping times

Let $\{\, \psi(\theta, S),\ \theta, S \in T_0 \,\}$ be a biadmissible family which is URCE and ULCE. Suppose that there exists $p > 1$ such that $E[\text{ess sup}_{\theta,S\in T_0}\psi(\theta,S)^p] < \infty$.
By Theorem 2.2 the admissible family of positive r.v. $\{\, \phi(\theta),\ \theta \in T_0 \,\}$ defined by (2.3) is RCE and LCE. By Theorem 1.1, the stopping time

$$\theta^* = \text{ess inf}\{\, \theta \in T_S\ ,\ u(\theta) = \phi(\theta) \,\text{a.s.}\,\},$$

is optimal for $u(S)$ $(= v(S))$, that is

$$u(S) = \text{ess sup}_{\theta \in T_S} E[\phi(\theta) \,|\, \mathcal{F}_S] = E[\phi(\theta^*) \,|\, \mathcal{F}_S] \quad \text{a.s.}\,.$$

Moreover, the families $\{\, \psi(\theta, \theta^*),\ \theta \in T_{\theta^*} \,\}$ and $\{\, \psi(\theta^*, \theta),\ \theta \in T_{\theta^*} \,\}$ are admissible and are RCE and LCE. Consider the following optimal stopping time problems defined for each $S \in T_{\theta^*}$:

$$v_1(S) = \text{ess} \sup_{\theta \in T_S} E[\psi(\theta, \theta^*) \,|\, \mathcal{F}_S] \text{ and } v_2(S) = \text{ess} \sup_{\theta \in T_S} E[\psi(\theta^*, \theta) \,|\, \mathcal{F}_S].$$

By Theorem 1.1, the stopping times θ_1^* and θ_2^* defined by $\theta_1^* = \text{ess inf}\{\, \theta \in T_{\theta^*}\ ,\ v_1(\theta) = \psi(\theta, \theta^*) \,\text{a.s.}\, \}$ and $\theta_2^* = \text{ess inf}\{\, \theta \in T_{\theta^*}\ ,\ v_2(\theta) = \psi(\theta^*, \theta) \,\text{a.s.}\, \}$ are optimal stopping times for $v_1(\theta^*)$ and $v_2(\theta^*)$. Note that $v_1(\theta^*) = u_1(\theta^*)$ and $v_2(\theta^*) = u_2(\theta^*)$ a.s..
Let τ_1^* and τ_2^* be the stopping times defined by

$$\tau_1^* = \theta^* \mathbf{1}_B + \theta_1^* \mathbf{1}_{B^c}, \qquad \tau_2^* = \theta^* \mathbf{1}_{B^c} + \theta_2^* \mathbf{1}_B\,, \tag{2.8}$$

where $B = \{\, u^1(\theta^*) \le u^2(\theta^*) \,\}$. By Proposition 2.4, the pair (τ_1^*, τ_2^*) is optimal for $v(S)$. Consequently, we have proved the following theorem

Theorem 2.3. (Existence of an optimal pair of stopping times)
Let $\{\, \psi(\theta, S),\ \theta, S \in T_0 \,\}$ be a biadmissible family which is URCE and ULCE. Suppose that there exists $p > 1$ such that $E[\text{ess sup}_{\theta, S \in T_0} \psi(\theta, S)^p] < \infty$.
Then, the pair of stopping times (τ_1^*, τ_2^*) defined by (2.8) is optimal for $v(S)$ defined by (2.1).

Remark 2.4. *Note that, since θ^*, θ_1^*, θ_2^* are minimal optimal, by the Appendix B, (τ_1^*, τ_2^*) is* minimal optimal *for $v(S)$ (in the sense defined in the Appendix B).*

3 The optimal d-stopping time problem

Let $d \in \mathbb{N}$, $d \ge 2$. In this section, we show that computing the value function for the d-optimal multiple stopping time problem

$$v(S) = \text{ess sup}\{\, E[\psi(\tau_1, \cdots, \tau_d) \,|\, \mathcal{F}_S],\ \tau_1, \cdots, \tau_d \in T_S \,\}\,,$$

reduces to computing the value function for an optimal one stopping time problem that is

$$v(S) = \text{ess sup}\{\, E[\phi(\theta) \,|\, \mathcal{F}_S],\ \theta \in T_S \,\} \text{ a.s.}$$

for a *new reward* ϕ. This new reward is expressed in terms of $d-1$-optimal multiple stopping time problems. Hence, by induction, the initial d-optimal multiple stopping time problem can be reduced to *nested* optimal one stopping time problems.

3.1 Definition and first properties of the value function

Definition 3.1. *We say that the family of random variables* $\{\psi(\theta),\ \theta \in T_0^d\}$ *is a* d-admissible family *if it satisfies the following conditions*

1. *for all* $\theta=(\theta_1,\ldots,\theta_d) \in T_0^d$, $\psi(\theta)$ *is a* $\mathcal{F}_{\theta_1 \vee \ldots \vee \theta_d}$ *measurable positive r.v.,*
2. *for all* $\theta, \theta' \in T_0^d$, $\quad \psi(\theta) = \psi(\theta')$ *a.s. on* $\{\theta = \theta'\}$.

For each stopping time $S \in T_0$, we consider the value function associated with the reward $\{\psi(\theta),\ \theta \in T_0^d\}$

$$v(S) = \operatorname{ess\,sup}_{\tau \in T_S^d} E[\psi(\tau) \,|\, \mathcal{F}_S]. \tag{3.1}$$

As in the optimal two stopping times problem, the value function satisfies the following properties:

Proposition 3.1. *Let* $\{\psi(\theta),\ \theta \in T_0^d\}$ *be a* d*-admissible family of r.v. , then the following properties hold:*

1. *The family* $\{v(S),\ S \in T_0\}$ *is an admissible family of r.v. .*
2. *For each* $S \in T_0$*, there exists a sequence of stopping times* $(\theta^n)_{n\in\mathbb{N}}$ *in* T_S^d *such that the sequence* $\{E[\psi(\theta^n) \,|\, \mathcal{F}_S]\}_{n\in\mathbb{N}}$ *is nondecreasing and such that* $v(S) = \lim_{n\to\infty} \uparrow E[\psi(\theta^n) \,|\, \mathcal{F}_S]$ *a.s. .*
3. *The family of r.v.* $\{v(S),\ S \in T_0\}$ *defined by (3.1) is a supermartingale system.*

The proof is an easy generalization of the optimal two stopping problem (Proposition 2.1).

Following the case with one or two stopping times, we now state the following result on the regularity of the value function.

Proposition 3.2. *Suppose that the* d*-admissible family* $\{\psi(\theta),\ \theta \in T_0^d\}$ *is RCE and that* $E[\operatorname{ess\,sup}_{\theta\in T_0^d} \psi(\theta)] < \infty$*. Then, the family* $\{v(S),\ S \in T_0\}$ *defined by (3.1) is RCE.*

The definition of RCE and the proof of this property are easily derived from the one or two stopping times case (see Definition 2.2 and Proposition 2.2).

3.2 Reduction to an optimal single stopping time problem

The optimal d-stopping time problem (3.1) can be expressed in terms of an optimal single stopping time problem as follows.

For $i = 1, \cdots, d$ and for $\theta \in T_0$ consider the random variable

$$u^{(i)}(\theta) = \operatorname*{ess\ sup}_{\tau_1,\cdots,\tau_{i-1},\tau_{i+1},\cdots,\tau_d \in T_\theta^{d-1}} E[\psi(\tau_1, \cdots, \tau_{i-1}, \theta, \tau_{i+1}, \cdots, \tau_d) \,|\, \mathcal{F}_\theta] \tag{3.2}$$

Note that this notation is adapted to the d-dimensional case.
In the two dimensional case ($d = 2$), we have

$$u^{(1)}(\theta) = \text{ess sup}_{\tau_2 \in T_\theta} E[\psi(\theta, \tau_2) \,|\, \mathcal{F}_\theta] = u_2(\theta) \quad \text{a.s.}$$

and

$$u^{(2)}(\theta) = \text{ess sup}_{\tau_1 \in T_\theta} E[\psi(\tau_1, \theta) \,|\, \mathcal{F}_\theta] = u_1(\theta) \quad \text{a.s.}\,,$$

by definition of $u_1(\theta)$ and $u_2(\theta)$(see (2.2)). Thus, the notation in the two dimensional case was different but more adapted to that simpler case.
For each $\theta \in T_0$, define the $\mathcal{F}_\theta$-measurable random variable called *new reward*

$$\phi(\theta) = \max[u^{(1)}(\theta), \cdots, u^{(d)}(\theta)], \tag{3.3}$$

and for each stopping time S, define the $\mathcal{F}_S$-measurable variable

$$u(S) = \text{ess sup}_{\theta \in T_S} E[\phi(\theta) \,|\, \mathcal{F}_S]. \tag{3.4}$$

Theorem 3.1. (Reduction) *Let $\{\, \psi(\theta), \theta \in T_0^d \,\}$ be a d-admissible family of r.v., and for each stopping time S, consider $v(S)$ defined by (3.1) and $u(S)$ defined by (3.2), (3.3), (3.4), then*

$$v(S) = u(S) \quad a.s..$$

PROOF :
Step 1: Let us prove that for all $S \in T_0$, $\quad v(S) \leq u(S) \quad$ a.s..

Let S be a stopping time and let $\tau = (\tau_1, \cdots, \tau_d) \in T_S^d$. There exists $(A_i)_{i=1,\cdots,d}$ with $\Omega = \cup_i A_i$, $A_i \cap A_j =$ for $i \neq j$ and such that $\tau_1 \wedge \cdots \wedge \tau_d = \tau_i$ a.s. on A_i and A_i are in $\mathcal{F}_{\tau_1 \wedge \cdots \wedge \tau_d}$ for $i = 1, \cdots, d$ (For $d = 2$, one can take $A_1 = \{\tau_1 \leq \tau_2\}$ and $A_2 = A_1^c$). One has

$$E[\psi(\tau) \,|\, \mathcal{F}_S] = \sum_{i=1}^{d} E[\mathbf{1}_{A_i} E[\psi(\tau) \,|\, \mathcal{F}_{\tau_i}] \,|\, \mathcal{F}_S].$$

By noticing that on A_i one has a.s. $\quad E[\psi(\tau) \,|\, \mathcal{F}_{\tau_i}] \leq u^{(i)}(\tau_i) \leq \phi(\tau_i) = \phi(\tau_1 \wedge \cdots \wedge \tau_d)$, we get $E[\psi(\tau) \,|\, \mathcal{F}_S] \leq E[\phi(\tau_1 \wedge \cdots \wedge \tau_d) \,|\, \mathcal{F}_S] \leq u(S) \quad$ a.s.. By taking the supremum over $\tau = (\tau_1, \cdots, \tau_d)$ we obtain step 1.

Step 2: Let us show that for all $S \in T_0$, $\quad v(S) \geq u(S) \quad$ a.s..

This follows from the fact that $\{\, v(S),\ S \in T_0 \,\}$ is a supermartingale system greater than $\{\, \phi(S),\ S \in T_0 \,\}$ and that $\{\, u(S),\ S \in T_0 \,\}$ is the smallest supermartingale system of this class. $\square$

Note that the new reward is expressed in terms of $d-1$-optimal multiple stopping time problems. Hence, by induction, the initial d-optimal multiple stopping time problem can be reduced to *nested* optimal one stopping time problems. In the case of a symmetric reward, the problem reduces to ordered stopping times and the *nested* optimal one stopping time problems simply reduces to a sequence of optimal one stopping time problems defined by backward induction (see section 3.6 and the application to *swing options*).

3.3 Properties of optimal stopping times in the d-stopping time problem

Let $\{\psi(\theta),\ \theta \in T_0^d\}$ be a d-admissible family such that $E[\text{ess sup}_{\theta\in T_0^d} \psi(\theta)] < \infty$. Let us introduce the following notation. For $i = 1, \cdots, d$ and for $\theta \in T_0$ and for $\tau_1, \cdots, \tau_{d-1}$ in T_0, consider the random variable

$$\psi^{(i)}(\tau_1, \cdots, \tau_{d-1}, \theta) = \psi(\tau_1, \cdots, \tau_{i-1}, \theta, \tau_i, \cdots, \tau_{d-1}). \tag{3.5}$$

Using this notation, note that for each $i = 1, \cdots, d$ the value function $u^{(i)}$ defined above (3.2) can be written

$$u^{(i)}(\theta) = \text{ess sup}_{\tau\in T_\theta^{d-1}} E[\psi^{(i)}(\tau, \theta) \,|\, \mathcal{F}_\theta] \tag{3.6}$$

Proposition 3.3. (Construction of optimal stopping times) *Suppose that*

1. *there exists an optimal stopping time θ^* for $u(S)$,*
2. *for $i = 1, \cdots, d$, there exist $(\theta_1^{(i)*}, \cdots, \theta_{i-1}^{(i)*}, \theta_{i+1}^{(i)*}, \cdots, \theta_d^{(i)*}) = \theta^{(i)*}$ in T_θ^{d-1} such that $u^{(i)}(\theta^*) = E[\psi^{(i)}(\theta^{(i)*}, \theta^*) \,|\, \mathcal{F}_{\theta^*}]$.*

Let $(B_i)_{i=1,\cdots,d}$ with $\Omega = \cup_i B_i$, $B_i \cap B_j =$ for $i \neq j$ and such that such that $\phi(\theta^*) = u^{(i)}(\theta^*)$ a.s. on B_i, and B_i is $\mathcal{F}_{\theta^*}$ measurable for $i = 1, \cdots, d$. Put

$$\tau_j^* = \theta^* \mathbf{1}_{B_j} + \sum_{i\neq j, i=1}^{d} \theta_j^{(i)*} \mathbf{1}_{B_i}, \tag{3.7}$$

then $(\tau_1^*, \cdots, \tau_d^*)$ is optimal for $v(S)$, and $\tau_1^* \wedge \cdots \wedge \tau_d^* = \theta^*$.

Proof: It is clear that $\tau_1^* \wedge \cdots \wedge \tau_d^* = \theta^*$ and a.s. ,

$$\begin{aligned}
u(S) &= E[\phi(\theta^*) \,|\, \mathcal{F}_S] = \sum_{i=1}^{d} E[\mathbf{1}_{B_i} u^{(i)}(\theta^*) \,|\, \mathcal{F}_S] \\
&= \sum_{i=1}^{d} E[\mathbf{1}_{B_i} E[\psi^{(i)}(\theta^{(i)*}, \theta^*) \,|\, \mathcal{F}_{\theta^*}] \,|\, \mathcal{F}_S] \\
&= \sum_{i=1}^{d} E[\mathbf{1}_{B_i} E[\psi(\theta_1^{(i)*}, \cdots, \theta_{i-1}^{(i)*}, \theta^*, \theta_{i+1}^{(i)*}, \cdots, \theta_d^{(i)*}) \,|\, \mathcal{F}_{\theta^*}] \,|\, \mathcal{F}_S] \\
&= E[\psi(\tau_1^*, \cdots, \tau_{i-1}^*, \tau_i^*, \tau_{i+1}^*, \cdots, \tau_d^*) \,|\, \mathcal{F}_S] \leq v(S) = u(S). \quad \square
\end{aligned}$$

Remark 3.1. *As in the bidimensional case, one can easily derive a necessary condition for obtaining optimal stopping times. Moreover, for an adapted partial order relation on $\mathbb{R}^d$, one can also derive a characterization of minimal optimal d-stopping times. This result is given in the Appendix B.2.*

Before studying the existence of an optimal d-stopping time for $v(S)$, we will study the regularity properties of the new reward $\{ \phi(\theta),\ \theta \in T_0 \}$ defined by (3.3).

3.4 Regularity of the new reward

Let us introduce the following definition of uniform continuity.

Definition 3.2. *A d-admissible family $\{ \psi(\theta),\ \theta \in T_0^d \}$ is said to be* uniformly right (resp. left) continuous along stopping times in expectation (URCE (resp. ULCE)) *if $E[\text{ess sup}_{\theta \in T_0^d} \psi(\theta)^p] < \infty$ for some $p > 1$, and for each $i = 1, \cdots, d$ and for each $S \in T_0$ and for each sequence of stopping times $(S_n)_{n \in \mathbb{N}}$ such that $S_n \downarrow S$ a.s. (resp. $S_n \uparrow S$ a.s.)*

$$\lim_{n \to \infty} E\left[\operatorname*{ess\ sup}_{\theta \in T_0^{d-1}} |\psi^{(i)}(\theta, S_n) - \psi^{(i)}(\theta, S)| \right] = 0 \quad a.s..$$

Proposition 3.4. *Let $\{ \psi(\theta),\ \theta \in T_0^d \}$ be a d-admissible family which is URCE and ULCE, then the family of positive r.v. $\{ \phi(S),\ S \in T_0 \}$ defined by (3.3) is RCE and LCE.*

Proof: The proof uses an induction argument. For $d = 1$ and $d = 2$, the result has already been shown. Fix $d \geq 1$ and suppose by induction that the property holds for any d-admissible family URCE and ULCE. Let $\{ \psi(\theta), \theta \in T_S^{d+1} \}$ be a $d+1$-admissible family URCE and ULCE. As $\phi(\theta) = \max[u^{(1)}(\theta), \cdots, u^{(d+1)}(\theta)]$, it is sufficient to show the RCE and LCE properties for the family $\{ u^{(i)}(\theta),\ \theta \in T_0 \}$ for all $i = 1, \cdots, d+1$.
Let us introduce the following value function for each $S, \theta \in T_0$,

$$U^{(i)}(\theta, S) = \text{ess sup}_{\tau \in T_\theta^d} E[\psi^{(i)}(\tau, S) \,|\, \mathcal{F}_\theta] \quad \text{a.s.}. \tag{3.8}$$

As for all $\theta \in T_0$,

$$u^{(i)}(\theta) = U^{(i)}(\theta, \theta) \quad \text{a.s.},$$

it is sufficient to prove that the biadmissible family $\{ U^{(i)}(\theta, S),\ \theta, S \in T_0 \}$ is RCE and LCE (as in the bidimensional case).
Let $\theta, S \in T_0$ and $(\theta_n)_n$ and $(S_n)_n$ be monotonic sequences of stopping times that converge to θ and S a.s.. We have,

$$E[|U^{(i)}(\theta, S) \ - \ U^{(i)}(\theta_n, S_n)|] \leq \underbrace{E[|U^{(i)}(\theta, S) - U^{(i)}(\theta_n, S)|]}_{\text{(I)}} + \underbrace{E[|U^{(i)}(\theta_n, S) - U^{(i)}(\theta_n, S_n)|]}_{\text{(II)}}.$$

Let us show that (I) tends to 0 as $n \to \infty$. For each $S \in T_0$, as $\{ \psi^{(i)}(\tau, S),\ \tau \in T_0^d \}$ is a d-admissible family of positive r.v. which is URCE and ULCE and $\{ U^{(i)}(\theta, S),\ \theta \in T_0 \}$ is the corresponding value function family. By the induction assumption, this family is RCE and LCE. Hence (I) converges a.s. to 0 as n tends to ∞ when (θ_n) is monotonic.

Let us show now that (II) tends to 0 as $n \to \infty$. By definition of the value function $U^{(i)}(.,.)$ (3.8), it follows that

$$E[|U^{(i)}(\theta_n, S) - U^{(i)}(\theta_n, S_n)|] \leq E[\text{ess sup}_{\theta \in T_0^d} |\psi^{(i)}(\theta, S) - \psi^{(i)}(\theta, S_n)]$$

and the right hand side tends to 0 by the URCE and ULCE properties of ψ. □

3.5 Existence of optimal stopping times

By Theorem 1.1, the regularity properties of the new reward will ensure the existence of an optimal stopping time $\theta^* \in T_0$ for $u(S)$. By Proposition 3.3, this will allow to show by induction the existence of an optimal stopping time for $v(S)$.

Theorem 3.2. (Existence of optimal stopping times) *Let $\{ \psi(\theta),\ \theta \in T_0^d \}$ be a d-admissible family of positive r.v. which is URCE and ULCE, then there exists $\tau^* \in T_S^d$ optimal for $v(S)$, that is such that*

$$v(S) = \text{ess sup}_{\tau \in T_S^d} E[\psi(\tau)\,|\mathcal{F}_S\,] = E[\psi(\tau*)\,|\mathcal{F}_S\,]\,.$$

Proof of Theorem 3.2: The result is proved by induction on d.
For $d = 1$ it is Theorem 1.1. Suppose now $d \geq 1$, and suppose by induction that for all d-admissible families URCE and ULCE, optimal d-stopping times do exist. Let $\{\psi(\theta)\,, \theta \in T_S^{d+1}\}$ be a $d+1$-admissible family URCE and ULCE. The existence of an optimal $d+1$-stopping times for the associated value function $v(S)$ will be derived by applying Proposition 3.3. Now, by Proposition 3.4, the new reward family $\{\phi(\theta),\ \theta \in T_0\}$ is LCE and RCE. By Theorem 1.1, there exists an optimal stopping time θ^* for $u(S)$. Thus, we have proved that condition 1) of Proposition 3.3 is satisfied.

Note now that for $i = 1, \cdots, d+1$, the d-admissible families $\{\psi^{(i)}(\theta, \theta^*),\ \theta \in T_0^d\}$ are URCE and ULCE. Thus by the induction hypothesis, for each $\theta \in T_0$, there exists an optimal $\theta^{*(i)} \in T_{\theta^*}^d$ for the value function $U^{(i)}(\theta^*, \theta^*)$ defined by (3.8). Noticing that $U^{(i)}(\theta^*, \theta^*) = u^{(i)}(\theta^*)$, we have proved that condition 2) of Proposition 3.3 is satisfied. Applying now Proposition 3.3, the result follows. □

3.6 Symmetric case

Suppose that $\psi(\tau_1,\ldots,\tau_d)$ is symmetric with respect to $(\tau_1,\ldots,\tau_d)$ that is

$$\psi(\tau_1,\ldots,\tau_d) = \psi(\tau_{\sigma(1)},\ldots,\tau_{\sigma(d)})$$

for each permutation σ of $\{1,...,d\}$. By symmetry we can suppose that $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_d$, that is the value function $v(S)$ coincide with

$$v_d(S) = \operatorname*{ess\ sup}_{(\tau_1,\cdots,\tau_{\rm d})\in\mathcal{S}_{\rm S}^{\rm d}} \mathrm{E}[\psi(\tau_1,\cdots,\tau_{\rm d})\,|\,\mathcal{F}_{\rm S}]\,,$$

where $\mathcal{S}_S^d = \{\tau_1,\cdots,\tau_d \in T_S \text{ s.t. } \tau_1 \leq \tau_2 \leq \ldots \leq \tau_d\}$. It follows that the value functions $u^{(i)}(\theta)$ and the new reward $\phi(\theta)$ coincide and are simply given for each $\theta \in T_0$ by the following random variable:

$$\phi_1(\theta) = \operatorname*{ess\ sup}_{(\tau_2,\tau_3,\cdots,\tau_{\rm d})\in\mathcal{S}_\theta^{\rm d-1}} \mathrm{E}[\psi(\theta,\tau_2,\cdots,\tau_{\rm d})\,|\,\mathcal{F}_\theta].$$

The reduction property can be written as follows:

$$v(S) = \text{ess sup}_{\theta\in T_S} E[\phi_1(\theta)\,|\,\mathcal{F}_S].$$

Then, we consider the value function $\phi_1(\theta_1)$. The associated new reward is given for θ_1, θ_2 such that $S \leq \theta_1 \leq \theta_2$ by

$$\phi_2(\theta_1,\theta_2) = \operatorname*{ess\ sup}_{(\tau_3,\cdots,\tau_{\rm d})\in\mathcal{S}_{\theta_2}^{\rm d-2}} \mathrm{E}[\psi(\theta_1,\theta_2,\tau_3,\cdots,\tau_{\rm d})\,|\,\mathcal{F}_{\theta_2}].$$

Again, the reduction property gives:

$$\phi_1(\theta_1) = \text{ess sup}_{\theta\in T_{\theta_1}} E[\phi_2(\theta_1,\theta_2)\,|\,\mathcal{F}_{\theta_1}]. \tag{3.9}$$

Then, we consider the value function $\phi_2(\theta_1,\theta_2)$ and so on. Thus, by forward induction, we define the new rewards ϕ_i for $i = 1,2,\ldots,d-1$ by

$$\phi_i(\theta_1,\cdots,\theta_i) = \operatorname*{ess\ sup}_{(\tau_{\rm i+1},\cdots,\tau_{\rm d})\in\mathcal{S}_{\theta_{\rm i}}^{\rm d-i}} \mathrm{E}[\psi(\theta_1,\cdots,\theta_{\rm i},\tau_{\rm i+1},\cdots,\tau_{\rm d})\,|\,\mathcal{F}_{\theta_{\rm i}}],$$

for each $(\theta_1,\cdots,\theta_i) \in \mathcal{S}_S^i$. The reduction property gives:

$$\phi_i(\theta_1,\cdots,\theta_i) = \text{ess sup}_{\theta_{i+1}\in T_{\theta_i}} E[\phi_{i+1}(\theta_1,\cdots,\theta_i,\theta_{i+1})\,|\,\mathcal{F}_{\theta_i}]\,. \tag{3.10}$$

Note that for $i = d-1$,

$$\phi_{d-1}(\theta_1,\cdots,\theta_{d-1}) = \text{ess sup}_{\theta_d\in T_{\theta_{d-1}}} E[\Psi(\theta_1,\cdots,\theta_{d-1},\theta_d)\,|\,\mathcal{F}_{\theta_{d-1}}]\,, \tag{3.11}$$

for each $(\theta_1,\cdots,\theta_{d-1}) \in \mathcal{S}_S^{d-1}$.
Hence, we now can define by backward induction $\phi_{d-1}(\theta_1,\cdots,\theta_{d-1})$ by (3.11) and then $\phi_{d-2}(\theta_1,\cdots,\theta_{d-2})$, $\cdots$, $\phi_2(\theta_1,\theta_2)$, $\phi_1(\theta_1)$ by the induction formula (3.10). Consequently, we have the following characterization of the value function and a construction of a multiple optimal stopping time (which are rather intuitive):

Proposition 3.5. *Let*

- *Let* $\{\psi(\theta), \theta \in T_0^d\}$ *be a symmetric d-admissible family of r.v. , and for each stopping time* S*, consider the associated value function* $v(S)$.
  Let ϕ_i*, for* $i = d-1,\ d-2, \cdots,\ 2,\ 1$ *be defined by backward induction as follows:*
  $\phi_{d-1}(\theta_1, \cdots, \theta_{d-1})$ *is given by (3.11) for each* $(\theta_1, \cdots, \theta_{d-1}) \in \mathcal{S}_S^{d-1}$*. Also, for* $i = d-2, \cdots,\ 2,\ 1$ *and for each* $(\theta_1, \cdots, \theta_i) \in \mathcal{S}_S^i$*,* $\phi_i(\theta_1, \cdots, \theta_i)$ *is given in function of* ϕ_{i+1} *by backward induction formula (3.10).*
  Then, the value function satisfies

$$v(S) = \text{ess sup}_{\theta \in T_S} E[\phi_1(\theta) \,|\, \mathcal{F}_S]. \tag{3.12}$$

- *Suppose that* $\{\psi(\theta),\ \theta \in T_0^d\}$ *is URCE and ULCE. Let* θ_1^* *be an optimal stopping time for* $v(S)$ *given by (3.12), let* θ_2^* *be an optimal stopping time for* $\phi_1(\theta_1^*)$ *given by (3.9), and for* $i = 2,\ 3, , \cdots, d-1$*, let* θ_{i+1}^* *be an optimal stopping time for* $\phi_i(\theta_1^*, \cdots, \theta_i^*)$ *given by (3.10).*
  Then, $(\theta_1^*, \cdots, \theta_d^*)$ *is a multiple optimal stopping time for* $v(S)$.

Some simple examples

First, consider the very simple additive case: suppose that the reward is given by

$$\psi(\tau_1, \ldots, \tau_d) = Y_{\tau_1} + Y_{\tau_2} + \cdots + Y_{\tau_d}, \tag{3.13}$$

where (Y_t) is a RCLL non negative adapted process such that $\limsup_{t \to +\infty} Y_t \leq Y_\infty$. Then, we obviously have that $v(S) = d\, v^1(S)$, where $v^1(S)$ is the value function of the one optimal stopping time problem associated with reward Y. Also, if θ_1^* is an optimal stopping time for $v_1(S)$, then $(\theta_1^*, \cdots, \theta_1^*)$ is optimal for $v(S)$.

Application to swing options

Let us now consider the more interesting additive case of *swing options*: suppose that $T = +\infty$ and that the reward is still given by (3.13) but the stopping times are separated by a fixed amount of time $\delta > 0$ (sometimes called "refracting time"). In this case, the value function is given by

$$v(S) = \text{ess sup}\{\, E[\psi(\tau_1, \cdots, \tau_d) \,|\, \mathcal{F}_S],\ (\tau_1, \cdots, \tau_d) \in \mathcal{S}_S^d \,\},$$

where $\mathcal{S}_S^d = \{\tau_1, \cdots, \tau_d \in T_S \text{ s.t. } \tau_i \in T_{\tau_{i-1}+\delta},\ 2 \leq i \leq d-1\}$. Then all the previous properties still hold. Again, the ϕ_i satisfy the following induction equality:

$$\phi_i(\theta_1, \cdots, \theta_i) = \text{ess sup}_{\theta_{i+1} \in T_{\theta_i + \delta}} E[\phi_{i+1}(\theta_1, \cdots, \theta_i, \theta_{i+1}) \,|\, \mathcal{F}_{\theta_i}].$$

Then, one can easily derive that $\phi_{d-1}(\theta_1, \theta_2, \cdots, \theta_{d-1}) = Y_{\theta_1} + \cdots + Y_{\theta_{d-1}} + Z_{d-1}(\theta_{d-1})$ where

$$Z_{d-1}(\theta_{d-1}) = \text{ess sup}_{\tau \in T_{\theta_{d-1}+\delta}} E[Y_\tau \,|\, \mathcal{F}_{\theta_{d-1}}] \,.$$

$\phi_{d-2}(\theta_1, \cdots, \theta_{d-2}) = Y_{\theta_1} + \cdots + Y_{\theta_{d-2}} + Z_{d-2}(\theta_{d-2})$ where

$$Z_{d-2}(\theta_{d-2}) = \text{ess sup}_{\tau \in T_{\theta_{d-2}+\delta}} E[Y_\tau + Z_{d-1}(\tau) \,|\, \mathcal{F}_{\theta_{d-2}}]$$

and so on. Hence for $i = 1, 2, , \cdots, d-2$, $\phi_i(\theta_1, \cdots, \theta_i) = Y_{\theta_1} + \cdots + Y_{\theta_i} + Z_i(\theta_i)$, where

$$Z_i(\theta_i) = \text{ess sup}_{\tau \in T_{\theta_i+\delta}} E[Y_\tau + Z_{i+1}(\tau) \,|\, \mathcal{F}_{\theta_i}].$$

The value function satisfies

$$v(S) = \text{ess sup}_{\theta \in T_S} E[Z_1(\theta) \,|\, \mathcal{F}_S]. \tag{3.14}$$

This corresponds to Proposition 3.2 of Carmona and Dayanik (2008).

Also, if θ_1^* is an optimal stopping time for $v(S)$ given by (3.14) and if for $i = 1, 2, , \cdots, d-1$, θ_{i+1}^* is an optimal stopping time for $Z_i(\theta_i^*)$, then $(\theta_1^*, \cdots, \theta_d^*)$ is a multiple optimal stopping time for $v(S)$.
This corresponds to Proposition 5.4 of Carmona and Dayanik (2008).

Note that the multiplicative case can be solved similarly. Further applications to American options with multiple exercise times are studied in Kobylanski and Quenez (2010).

4 Aggregation and multiple optimal stopping times

As explained in the introduction, in previous works on the one optimal stopping time problem, the reward is given by a RCLL postive adapted process (ϕ_t). Moreover, when the reward (ϕ_t) is continuous, a S-optimal stopping time is given by

$$\overline{\theta}(S) = \inf\{\, t \geq S,\ v_t = \phi_t \,\}\,, \tag{4.1}$$

which corresponds to the first hitting time after S of 0 by the RCLL adapted process $(v_t - \phi_t)$. This formulation is very important since it gives a simple and efficient method to compute an optimal stopping time.

In the two dimensional case, instead of considering a reward process, it is quite natural to suppose that the reward is given by a biprocess $(\Psi_{t,s})_{(t,s)\in[0,T]^2}$ such that a.s., the map $(t,s) \mapsto \Psi_{t,s}$ is continuous and for each $(t,s) \in [0,T]^2$, $\Psi_{t,s}$ is $\mathcal{F}_{t\vee s}$-measurable (see Remark 2.1).

We would like to construct some optimal stopping times by using hitting times of processes. By the existence and construction properties of optimal

stopping times given in Theorem 2.3, we are led to construct θ^*, θ_1^*, θ_2^* as hitting times of processes. Since Ψ is a continuous biprocess, there is no problem for θ_1^*, θ_2^*. But for θ^*, we need to aggregate the new reward $\{ \phi(\theta),\ \theta \in T_0 \}$, which requires new aggregation results. These results hold under stronger assumptions on the reward than those made in the previous existence theorem (Theorem 2.3).

4.1 Some general aggregation results

4.1.1 Aggregation of a supermartingale system

Recall the classical result of aggregation of a supermartingale system (El Karoui (1981)):

Proposition 4.1. *Let $\{ h(S),\ S \in T_0 \}$ be a supermartingale system which is RCE, then there exists a RCLL adapted process (h_t) which* aggregates *the family $\{ h(S),\ S \in T_0 \}$ that is, for each $S \in T_0$, $h_S = h(S)$ a.s.,*

This lemma relies on a well known result (see for example El Karoui (1981) or Theorem 3.13 in Karatzas and Shreve (1994); for details, see the proof in section 4.4).

Classicaly, the above Proposition 4.1 is used to aggregate the value function of the one stopping time problem. However, it cannot be applied to the new reward since it is no longer a supermartingale system. Thus, we will now state a new result of aggregation.

4.1.2 A new result of aggregation of an admissible family

Let us introduce the following right continuous property for admissible families.

Definition 4.1. *An admissible family $\{ \phi(\theta),\ \theta \in T_0 \}$ is said to be* right continuous along stopping times (RC) *if for any $\theta \in T_0$ and for any sequence $(\theta_n)_{n\in\mathbb{N}}$ of stopping times such that $\theta_n \downarrow \theta$ a.s. one has $\phi(\theta) = \lim_{n\to\infty} \phi(\theta_n)$ a.s..*

We state the following result

Theorem 4.1. *Suppose that the admissible family of positive r.v. $\{ \phi(\theta), \theta \in T_0 \}$ is right continuous along stopping times, then there exists a progressive process (ϕ_t) such that for each $\theta \in T_0$, $\phi_\theta = \phi(\theta)$ a.s., and such that there exists a nonincreasing sequence of right continuous processes $(\phi_t^n)_{n\in\mathbb{N}}$ such that for each $(\omega, t) \in \Omega \times [0,T]$, $\lim_{n\to\infty} \phi_t^n(\omega) = \phi_t(\omega)$.*

PROOF: see section 4.4 . □

4.2 The optimal stopping problem

First, recall the following classical result (El Karoui (1981)).

Proposition 4.2. (Aggregation of the value function)
Let $\{\, \phi(\theta),\, \theta \in T_0\}$ *be an admissible family of r.v. which is RCE. Suppose that* $E[\text{ess sup}_{\theta\in T_0}\phi(\theta)] < \infty$.
Then there exists a RCLL supermartingale (v_t) *which aggregates the family* $\{\, v(S),\ S \in T_0\,\}$ *defined by (1.1) that is, for each stopping time* S, $v(S) = v_S \quad a.s\,.$

Proof of Proposition 4.2: The family $\{\, v(S),\ S \in T_0\,\}$ is a supermartingale system (Proposition 1.3), and has the RCE property (Proposition 1.5). The result clearly follows by applying the aggregation property of supermartingale systems (Proposition 4.1). □

Theorem 4.2. *Suppose the reward is given by a RC and LCE admissible family* $\{\, \phi(\theta), \theta \in T_0\,\}$ *such that* $E[\text{ess sup}_{\theta\in T_0}\phi(\theta)] < \infty$.
Let (ϕ_t) *be a progressive process that aggregates this family given by Theorem 4.1. Let* $\{\, v(S),\ S \in T_0\,\}$ *be the family of value function defined by (1.1), and let* (v_t) *be a RCLL adapted process that aggregates the family* $\{\, v(S),\ S \in T_0\,\}$. *The random variable defined by*

$$\overline{\theta}(S) = \inf\{\, t \geq S, v_t = \phi_t\,\}. \tag{4.2}$$

is an optimal stopping time for $v(S)$ (i.e $v(S) = E[\phi(\overline{\theta}(S)) \,|\, \mathcal{F}_S]$).

As for Theorem 1.1, the proof relies on the construction of a family of stopping times that are approximatively optimal. The details, which require some fine techniques of the general theory of processes are given in section 4.4.

Remark 4.1. *In the case of a RCLL reward process supposed to be LCE, the above theorem corresponds to the classical existence result (see El Karoui (1981) and Karatzas and Shreve (1998)).*

Proposition 4.3. *Under the same assumptions as in the previous theorem, for each* $S \in T_0$, $\overline{\theta}(S)$ *is the minimal optimal stopping time for* v_S. *In other words ,*

$$\overline{\theta}(S) = \theta^*(S) \ a.s..$$

Proof : Fix $S \in T_0$. Suppose that there exists $\theta^* \in T_S$ which is optimal for v_S and such that $P(\{\theta^* < \overline{\theta}(S)\}) > 0$.
Now, by definition of $\overline{\theta}(S)$, for all $t \in [S, \overline{\theta}(S)[$, we have $v_t > \phi_t$.
Hence, for each $\omega \in \{\theta^* < \overline{\theta}(S)\}$, we have $v_{\theta^*}(\omega) > \phi_{\theta^*}(\omega)$. Hence,

$$P(\{v_{\theta^*} > \phi_{\theta^*}\}) \geq P(\{\theta^* < \overline{\theta}(S)\}) > 0,$$

which contradicts the optimality criterium 1.4. □

4.3 The optimal multiple stopping time problem

For simplicity, we study only the case when $d = 2$. We will now prove that the minimal optimal pair of stopping times (τ_1^*, τ_2^*) defined by (2.8) can also be given in terms of *hitting times*. In order to do this, we need first to aggregate the value function and the new reward.

4.3.1 Aggregation of the value function

Proposition 4.4. *Suppose the reward is given by a RCE biadmissible family* $\{\psi(\theta, S),\ \theta, S \in T_0\}$ *such that* $E[\text{ess sup}_{\theta, S \in T_0} \psi(\theta, S)] < \infty$.
Then, there exists a supermartingale (v_t) *with RCLL paths that aggregates the family* $\{v(S),\ S \in T_0\}$ *defined by (2.1), that is such for each* $S \in T_0$, $v(S) = v_S$ *a.s.* .

Proof of Proposition 4.4: The RCE property of $\{v(S),\ S \in T_0\}$ shown in Proposition 2.2, together with the supermartingale property (Proposition 2.1(3)) give, by Proposition 4.1, the desired result. □

4.3.2 Aggregation of the new reward

We will now study the aggregation problem of the new reward family $\{\phi(\theta),\ \theta \in T_0\}$.
Let us introduce the following definition,

Definition 4.2. *A biadmissible family* $\{\psi(\theta, S),\ \theta, S \in T_0\}$ *is said to be* uniformly right continuous along stopping times (URC) *if for each nonincreasing sequence of stopping times* $(S_n)_{n\in\mathbb{N}}$ *in* T_S *which converges a.s. to a stopping time* $S \in T_0$,

$$\lim_{n\to\infty} \left[\text{ess sup}_{\theta \in T_S} \{|\psi(\theta, S_n) - \psi(\theta, S)|\}\right] = 0 \quad a.s.,$$
$$\textit{and} \quad \lim_{n\to\infty} \left[\text{ess sup}_{\theta \in T_S} \{|\psi(S_n, \theta) - \psi(S, \theta)|\}\right] = 0 \quad a.s..$$

The following right continuity property holds true for the new reward family:

Theorem 4.3. *Suppose that the admissible family of positive r.v.* $\{\psi(\theta, S), \theta, S \in T_0\}$ *is URC, then the family of positive r.v.* $\{\phi(S),\ S \in T_0\}$ *defined by (2.3) is RC.*

Proof : As $\phi(\theta) = \max[u_1(\theta), u_2(\theta)]$, it is sufficient to show the RC property for the family $\{u_1(\theta),\ \theta \in T_0\}$.
Now, for that for all $\theta \in T_0$, $u_1(\theta) = U_1(\theta, \theta)$ a.s., where

$$U_1(\theta, S) = \text{ess sup}_{\tau_1 \in T_\theta} E[\psi(\tau_1, S) \,|\, \mathcal{F}_\theta] \quad \text{a.s.}. \tag{4.3}$$

Hence, it is sufficient to prove that $\{ U_1(\theta, S),\ \theta, S \in T_0\}$ is RC.
Let $\theta, S \in T_0$ and $(\theta_n)_n$ and $(S_n)_n$ be nonincreasing sequences of stopping times in T_O that converge to θ and S a.s.. We have,

$$|U_1(\theta, S) - U_1(\theta_n, S_n)| \leq \underbrace{|U_1(\theta, S) - U_1(\theta_n, S)|}_{\text{(I)}} + \underbrace{|U_1(\theta_n, S) - U_1(\theta_n, S_n)|}_{\text{(II)}}.$$

(I) tends to 0 as $n \to \infty$

For each $S \in T_0$, as $\{ \psi(\theta, S),\ \theta \in T_0 \}$ is an admissible family of postive r.v. which is RC, Proposition 4.4 gives the existence of a RCLL adapted processes $(U_t^{1,S})$ such that for each stopping time $\theta \in T_0$

$$U_\theta^{1,S} = U_1(\theta, S) \quad \text{a.s.}. \tag{4.4}$$

(I) can be rewritten $|U_1(\theta, S) - U_1(\theta_n, S)| = |U_\theta^{1,S} - U_{\theta_n}^{1,S}|$ a.s. which converges a.s. to 0 as n tends to ∞ by the right continuity of the process $(U_t^{1,\theta})$.

(II) tends to 0 as $n \to \infty$

By definition of the value function $U_1(.,.)$ (4.3), it follows that

$$\begin{aligned} |U_1(\theta_n, S) - U_1(\theta_n, S_n)| &\leq E\left(\text{ess sup}_{\tau_1 \in T_{\theta_n}} |\psi(\tau_1, S) - \psi(\tau_1, S_n)| \mid \mathcal{F}_{\theta_n}\right) \\ &\leq E\left(Z_m \mid \mathcal{F}_{\theta_n}\right) \quad \text{a.s.}, \end{aligned}$$

for any $n \geq m$, where $Z_m := \sup\limits_{r \geq m} \left\{ \text{ess sup}_{\tau \in T_0} |\psi(\tau, S_r) - \psi(\tau, S)| \right\}$, and where $(E(Z_m \mid \mathcal{F}_t))_{t \geq 0}$ is a RCLL version of the conditional expectation. Hence, by the right continuity of this process, for each fixed $m \in \mathbb{N}$, the sequence of r.v. $(E(Z_m \mid \mathcal{F}_{\theta_n}))_{n \in \mathbb{N}}$ converges a.s. to $E(Z_m \mid \mathcal{F}_\theta)$ as n tends to ∞. It follows that for each $m \in \mathbb{N}$,

$$\limsup_{n \to \infty} |U_1(\theta_n, S) - U_1(\theta_n, S_n)| \leq E\left(Z_m \mid \mathcal{F}_\theta\right) \quad \text{a.s.}. \tag{4.5}$$

Now, the sequence $(Z_m)_{m \in \mathbb{N}}$ converges a.s. to 0 and

$$|Z_m| \leq 2\, \text{ess sup}_{\theta, S \in T_0} |\psi(\theta, S)| \quad \text{a.s.}.$$

Note that the second member of this inequality is integrable. By the Lebesgue theorem for the conditional expectation, $E\left(Z_m \mid \mathcal{F}_\theta\right)$ converges to 0 in L^1 as m tends to ∞. The sequence $(Z_m)_{m \in \mathbb{N}}$ is decreasing. It follows that the sequence $\{ E\left(Z_m \mid \mathcal{F}_\theta\right) \}_{m \in \mathbb{N}}$ is also decreasing and hence converges a.s.. Since this sequence converges to 0 in L^1, its limit is also 0 almost surely. By letting m tend to ∞ in (4.5), we obtain

$$\limsup_{n \to \infty} |U_1(\theta_n, S) - U_1(\theta_n, S_n)| \leq 0 \quad \text{a.s.}.$$

The proof of Theorem 4.3 is ended. □

Corollary 4.1. (Aggregation of the new reward) *Under the same hypothesis as Theorem 4.3, there exists some progressive right continuous adapted processes* (ϕ_t) *which aggregates the family* $\{\ \phi(\theta),\ \theta \in T_0\ \}$ *that is,* $\phi_\theta = \phi(\theta)$ *a.s. for each* $\theta \in T_0$*, and such that there exists a decreasing sequence of right continuous adapted processes* $(\phi_t^n)_{n\in\mathbb{N}}$ *that converges to* (ϕ_t)*.*

PROOF: It follows from the right continuity of the new reward (Theorem 4.3) which we can aggregate (Theorem 4.1). □

Remark 4.2. *For the optimal d-stopping time problem, the same result holds for URC d-admissible families* $\{\psi(\theta)\ \theta \in T_0^d\}$ *i.e families that satisfy*

$$\lim_{n\to\infty} \text{ess sup}_{\,\theta\in T_0} |\psi^{(i)}(\theta, S) - \psi^{(i)}(\theta, S_n)| = 0$$

for $i = 1, \cdots, d,\ \theta, S \in T_0$ *and* (S_n) *sequences in* T_0 *such that* $S_n \downarrow S$ *a.s..*

The proof is strictly the same with $U_1(\theta, S)$ *replaced by* $U^{(i)}(\theta, S)$ *for* $\theta, S \in T_0$ *and* $\psi(\tau, S)$ *with* $\tau, S \in T_0$ *replaced by* $\psi^{(i)}(\tau, S)$ *with* $\tau \in T_0^{d-1}$ *and* $S \in T_0$*.*

4.3.3 Optimal multiple stopping times as hitting times of processes

As before, for the sake of simplicity, we suppose that $d = 2$. Suppose that $\{\psi(\theta, S),\ \theta, S \in T_0\}$ is an URC and ULCE biadmissible family such that $E[\text{ess sup}_{\,\theta,S\in T_0}\psi(\theta, S)^p] < \infty$ for some $p > 1$. Let $\{\phi(\theta),\ \theta \in T_0\}$ be the new reward family. By Theorem 2.2, this family is LCE. Futhermore, by Theorem 4.3, this family is RC. Let (ϕ_t) be the progressive process that aggregates this family given by Theorem 4.1. Let (u_t) be a RCLL process that aggregates the value function associated with (ϕ_t). By Theorem 4.2 and Proposition 4.3, the stopping time

$$\theta^* = \inf\{t \geq S\ , u_t = \phi_t\}$$

is optimal for $u(S)$.

The family $\{\psi(\theta, \theta^*)\ \theta \in T_{\theta^*}\}$ is admissible, RC and LCE. Let (ψ_t^1) be the progressive process that aggregates this family given by Theorem 4.1. Let (v_t^1) be a RCLL process that aggregates the value function associated with (ψ_t^1). By Theorem 4.2, the stopping time $\theta_1^* = \inf\{t \geq \theta^*\ ,\ v_t^1 = \psi_t^1\}$ is optimal for $v_{\theta^*}^1$ and $v_{\theta^*}^1 = u^1(\theta^*)$.

The family $\{\psi(\theta^*, \theta)\ \theta \in T_{\theta^*}\}$ is admissible, RC and LCE. Let (ψ_t^2) be the progressive process that aggregates this family given by Theorem 4.1. Let (v_t^2) be a RCLL process that aggregates the value function associated with (ψ_t^2). By Theorem 4.2, the stopping time $\theta_2^* = \inf\{t \geq \theta^*\ ,\ v_t^2 = \psi_t^2\}$ is optimal for $v_{\theta^*}^2$, and $v_{\theta^*}^2 = u_2(\theta^*)$.

By Proposition 2.4, the pair of stopping times (τ_1^*, τ_2^*) defined by

$$\tau_1^* = \theta^* \mathbf{1}_B + \theta_1^* \mathbf{1}_{B^c}; \quad \tau_2^* = \theta_2^* \mathbf{1}_B + \theta^* \mathbf{1}_{B^c}, \tag{4.6}$$

where $B = \{\ u_1(\theta^*) \leq u_2(\theta^*)\ \} = \{\ v_{\theta^*}^1 \leq u_{\theta^*}^2\ \}$, is optimal for $v(S)$.

Theorem 4.4. *Let* $\{ \psi(\theta, S),\ \theta, S \in T_0 \}$ *be a biadmissible family which is URC and ULCE. Suppose that* $E[\text{ess sup}_{\theta, S \in T_0} \psi(\theta, S)^p] < \infty$ *for some* $p > 1$. *Then, the pair of stopping times* (τ_1^*, τ_2^*) *defined by (4.6) is optimal for* $v(S)$.

Note that the above construction of (τ_1^*, τ_2^*) as hitting times of processes requires stronger assumptions on the reward than those made in Theorem 2.3. Futhermore, let us emphasize that it also requires some new aggregation results (Theorem 4.1 and Theorem 4.2).

4.4 Proofs of Proposition 4.1, Theorem 4.1 and Theorem 4.2

We now give the proofs of Proposition 4.1, Theorem 4.1 and Theorem 4.2.

First, we give the short proof of the classical Proposition 4.1 which we recall here (for the reader's comfort).

Proposition 4.1 *Let* $\{ h(S),\ S \in T_0 \}$ *be a supermartingale system which is right continuous along stopping times in expectation, then there exists a RCLL adapted process* (h_t) *which* aggregates *the family* $\{ h(S),\ S \in T_0 \}$, *that is* $h_S = h(S)$ *a.s.*.

Proof: Let us consider the process $(h(t))_{0 \leq t \leq T}$. It is a supermartingale and the function $t \mapsto E(h(t))$ is right continuous. By classical results (see Theorem 3.13 in Karatzas and Shreve (1994)), there exists a RCLL supermartingale $(h_t)_{0 \leq t \leq T}$ such that for each $t \in [0, T]$, $h_t = h(t)$ a.s.. Then, it is clear that for each dyadic stopping time $S \in T_0$, $h_S = h(S)$ a.s. (for details, see step 2 of the proof of Theorem 1.1). This implies that

$$E[h_S] = E[h(S)]. \tag{4.7}$$

Since the process $(h_t)_{0 \leq t \leq T}$ is RCLL and since the family $\{h(S), S \in T_0\}$ is right continuous in expectation, equality (4.7) still holds for any stopping time $S \in T_0$. Then, it remains to show that $h_S = h(S)$ a.s. but this is classical. Let $A \in \mathcal{F}_S$ and define $S_A = S\mathbf{1}_A + T\mathbf{1}_{A^c}$. Since S_A is a stopping time, $E[h_{S_A}] = E[h(S_A)]$. Since $h_T = h(T)$ a.s., it gives that $E[h_S \mathbf{1}_A] = E[h(S)\mathbf{1}_A]$, which gives the desired result. □

We now give the proof of Theorem 4.1.

Theorem 4.1 *Suppose that the admissible family of positive r.v.* $\{ \phi(\theta), \theta \in T_0 \}$ *is right continuous along stopping times, then there exists a progressive process* (ϕ_t) *such that for each* $\theta \in T_0$, $\phi_\theta = \phi(\theta)$ *a.s.*, *and such that there exists a nonincreasing sequence of right continuous processes* $(\phi_t^n)_{n \in \mathbb{N}}$ *such that for each* $(\omega, t) \in \Omega \times [0, T]$, $\lim_{n \to \infty} \phi_t^n(\omega) = \phi_t(\omega)$.

PROOF: For each $n \in \mathbb{N}^*$, let us define a process $(\phi_t^n)_{t\geq 0}$ that is a function of (ω, t) by

$$\phi_t^n(\omega) = \sup_{s\in\mathbb{D}\cap]t, \frac{[2^n t]+1}{2^n}[} \phi(s \wedge T), \tag{4.8}$$

for each $(\omega, t) \in \Omega \times [0, T]$ where $\mathbb{D}$ is the set of dyadic rationals.

For each $t \in [0, T]$ and for each $\varepsilon > \frac{1}{2^n}$, the process (ϕ_t^n) is $(\mathcal{F}_{t+\varepsilon})$-adapted and, for each $\omega \in \Omega$, the function $t \mapsto \phi_t^n(\omega)$ is right continuous. Hence, the process (ϕ_t^n) is also $(\mathcal{F}_{t+\varepsilon})$-progressive. Moreover, the sequence $(\phi_t^n)_{n\in\mathbb{N}^*}$ is decreasing. Let ϕ_t be its limit *i.e* for each $(\omega, t) \in \Omega \times [0, T]$,

$$\phi_t(\omega) = \lim_{n\to\infty} \phi_t^n(\omega) \quad .$$

It follows that for each $\varepsilon > 0$, the process (ϕ_t) is $(\mathcal{F}_{t+\varepsilon})$-progressive. Thus, (ϕ_t) is $(\mathcal{F}_{t^+})$-progressive and consequently $(\mathcal{F}_t)$-progressive since $\mathcal{F}_{t^+} = \mathcal{F}_t$.

Step 1: Fix $\theta \in T_0$. Let us show that $\phi_\theta \leq \phi(\theta)$ a.s..

Let us suppose by contradiction that the above inequality does not hold. Then there exists $\varepsilon > 0$ such that the set $A = \{\phi(\theta) \leq \phi_\theta - \varepsilon\}$ satisfies $P(A) > 0$. Fix $n \in N$. We have for all $\omega \in A$ that $\phi(\theta)(\omega) \leq \phi^n_{\theta(\omega)}(\omega) - \varepsilon$, where $\phi^n_{\theta(\omega)}(\omega)$ is defined by (4.8) with t replaced by $\theta(\omega)$.

By definition of ϕ^n there exists $t \in]\theta(\omega), \frac{[2^n\theta(\omega)]+1}{2^n}[\cap\mathbb{D}$ such that

$$\phi(\theta)(\omega) \leq \phi(t)(\omega) - \frac{\varepsilon}{2}.$$

We introduce the following subset of $[0, T] \times \Omega$:

$$\overline{A}_n = \left\{ (t, \omega)\,,\ t \in]\theta(\omega), \frac{[2^n\theta(\omega)] + 1}{2^n}[\cap\mathbb{D} \text{ and } \phi(\theta)(\omega) \leq \phi(t)(\omega) - \frac{\varepsilon}{2} \right\}.$$

First, note that $\overline{A}_n$ is optional. Indeed, we have $\overline{A}_n = \cup_{t\in\mathbb{D}}\{t\} \times B_{n,t}$, where

$$B_{n,t} = \left\{ \theta < t < \frac{[2^n\theta] + 1}{2^n} \right\} \cap \left\{ \phi(\theta) \leq \phi(t) - \frac{\varepsilon}{2} \right\}$$

and the process $(\omega, t) \mapsto \mathbf{1}_{B_{n,t}}(\omega)$ is optional since θ and $\frac{[2^n\theta]+1}{2^n}$ are stopping times and since $\{\, \phi(\theta), \theta \in T_0 \,\}$ is admissible. Also, A is included in $\pi(\overline{A}_n)$, the projection of $\overline{A}_n$ on Ω, that is

$$A \subset \pi(\overline{A}_n) = \left\{\, \omega \in \Omega, \quad \exists t \in [0, T] \,\text{s.t.}(t, \omega) \in \overline{A}_n \,\right\}.$$

Hence, by a section Theorem (see Chap. IV Dellacherie and Meyer (1977)), there exists a dyadic stopping time T_n such that for each ω in $\{T_n < \infty\}$, $(T_n(\omega), \omega) \in \overline{A}_n$ and

$$P(T_n < \infty) \geq P(\pi(\overline{A}_n)) - \frac{P(A)}{2^{n+1}} \geq P(A) - \frac{P(A)}{2^{n+1}}.$$

Hence, for all ω in $\{T_n < \infty\}$

$$\phi(\theta)(\omega) \leq \phi(T_n(\omega)) - \frac{\varepsilon}{2} \quad \text{and} \quad T_n(\omega) \in]\theta(\omega), \frac{[2^n\theta(\omega)]+1}{2^n}[\cap\mathbb{D}.$$

Note that

$$P(\cap_{n\geq 1}\{T_n < \infty\}) \geq P(A) - \left(\sum_{n\geq 1}\frac{P(A)}{2^{n+1}}\right) \geq \frac{P(A)}{2} > 0.$$

Put $\overline{T}_n = T_1 \wedge \cdots \wedge T_n$. One has $\overline{T}_n \downarrow \theta$ and $\phi(\theta) \leq \phi(\overline{T}_n) - \frac{\varepsilon}{2}$ for each n on $\cap_{n\geq 1}\{T_n < \infty\}$. By letting n tend to ∞ in this inequality, since $\{\, \phi(\theta), \theta \in T_0 \,\}$ is right-continuous along stopping times, we derive that $\phi(\theta) \leq \phi(\theta) - \frac{\varepsilon}{2}$ a.s. on $\cap_{n\geq 1}\{T_n < \infty\}$ which gives the desired contradiction.

Step 2: Fix $\theta \in T_0$. Let us show that $\phi(\theta) \leq \phi_\theta$ a.s..

Put $T^n = \frac{[2^n\theta]+1}{2^n}$. The sequence (T^n) is a nonincreasing sequence of stopping times such that $T^n \downarrow \theta$. Moreover, note that since the family $\{\, \phi(\theta), \theta \in T_0 \,\}$ is admissible, for each $d \in \mathbb{D}$, for almost every $\omega \in \{T^{n+1} = d\}$, $\phi(T^{n+1})(\omega) = \phi(d)(\omega)$. Now, we have $T^{n+1} \in]\theta, T^n[\cap\mathbb{D}$. Also, for each $\omega \in \Omega$ and each $d \in]\theta(\omega), T^n(\omega)[\cap\mathbb{D}$,

$$\phi(d)(\omega) \leq \sup_{s\in]\theta(\omega),T^n(\omega)[\cap\mathbb{D}} \phi(s)(\omega) = \phi^n_{\theta(\omega)}(\omega),$$

where the last equality follows by definition of $\phi^n_{\theta(\omega)}(\omega)$ (see (4.8) with t replaced by $\theta(\omega)$). Hence,

$$\phi(T^{n+1}) \leq \phi^n_\theta \quad \text{a.s.}.$$

Letting n tend to ∞, by using the right continuous property of $\{\, \phi(\theta), \theta \in T_0 \,\}$ along stopping times and the convergence of $\phi^n_{\theta(\omega)}(\omega)$ to $\phi_{\theta(\omega)}(\omega)$ for each ω, we derive that $\phi(\theta) \leq \phi_\theta$ a.s.. $\square$

We now give the proof of Theorem 4.2.

Theorem 4.2 *$\overline{\theta}(S) = \inf\{t \geq S\,,\ v_t = \phi_t\}$ is an optimal stopping time for v_S.*

PROOF: We begin by constructing a family of stopping times that are approximatively optimal. For $\lambda \in]0,1[$, define the stopping time

$$\overline{\theta}^\lambda(S) := \inf\{\, t \geq S\,,\ \lambda v_t \leq \phi_t \,\} \wedge T\,. \tag{4.9}$$

The proof follows exactly the proof of theorem 1.1 except for step1, which corresponds to the following lemma:

Lemma 4.1. *For each $S \in T_0$ and $\lambda \in]0,1[$,*

$$\lambda v_{\overline{\theta}^\lambda(S)} \leq \phi_{\overline{\theta}^\lambda(S)} \quad \textit{a.s.}. \tag{4.10}$$

By the same arguments as in the proof of Theorem 1.1, $\overline{\theta}^{\lambda}(S)$ is nondecreasing with respect to λ and converges as $\lambda \uparrow 1$ to an optimal stopping time which coincides with $\overline{\theta}(S)$ a.s.. $\square$

PROOF OF LEMMA (4.1): To simplify notation, $\overline{\theta}^{\lambda}(S)$ will be denoted by $\overline{\theta}^{\lambda}$. For the sake of simplicity, without loss of generality, we suppose that $t \mapsto v_t(\omega)$ is RCLL for each $\omega \in \Omega$.
Fix $\omega \in \Omega$. In the following, we only use simple analytic arguments.
By definition of $\overline{\theta}^{\lambda}(\omega)$ (1.7), for each $n \in \mathbb{N}^*$, there exists $t \in [\overline{\theta}^{\lambda}(\omega), \overline{\theta}^{\lambda}(\omega) + \frac{1}{n}[$ such that $\lambda v_t(\omega) \leq \phi_t(\omega)$.
Note also that for each $m \in \mathbb{N}^*$, $\phi_t(\omega) \leq \phi_t^m(\omega)$.
Fix now $m \in \mathbb{N}^*$ and fix $\alpha > 0$.
By the right continuity of $t \mapsto v_t(\omega)$ and $t \mapsto \phi_t^m(\omega)$, there exists $t_n^m(\omega) \in \mathbb{D} \cap [\overline{\theta}^{\lambda}(\omega), \overline{\theta}^{\lambda}(\omega) + \frac{1}{n}[$ such that

$$\lambda v_{t_n^m(\omega)}(\omega) \leq \phi^m_{t_n^m(\omega)}(\omega) + \alpha \tag{4.11}$$

Note that $\lim_{n\to\infty} t_n^m(\omega) = \overline{\theta}^{\lambda}(\omega)$ and $t_n^m(\omega) \geq \overline{\theta}^{\lambda}(\omega)$ for any n.
Again, by using the right continuity of $t \mapsto v_t(\omega)$ and $t \mapsto \phi_t^m(\omega)$ and by letting n tend to ∞ in (4.11), we derive that

$$\lambda v_{\overline{\theta}^{\lambda}(\omega)}(\omega) \leq \phi^m_{\overline{\theta}^{\lambda}(\omega)}(\omega) + \alpha\,,$$

and this inequality holds for each $\alpha > 0$, for each $m \in \mathbb{N}^*$ and for each $\omega \in \Omega$. By letting m tend to ∞ and α tend to 0, we derive that for each $\omega \in \Omega$, $\lambda v_{\overline{\theta}^{\lambda}(\omega)}(\omega) \leq \phi_{\overline{\theta}^{\lambda}(\omega)}(\omega)$, which ends the proof of the lemma. $\square$

A

We recall the classical following theorem (see for example Karatzas and Shreve (1998), Neveu (1975)).

Theorem A.1. (Essential supremum) *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{X}$ be a non empty family of positive r.v. defined on $(\Omega, \mathcal{F}, P)$. There exists a r.v. X^* satisfying*

1. *for all $X \in \mathcal{X}$, $X \leq X^*$ a.s.,*
2. *if Y is a r.v. satisfying $X \leq Y$ a.s. for all $X \in \mathcal{X}$, then $X^* \leq Y$ a.s..*

This r.v., which, is unique a.s., is called the essential supremum of $\mathcal{X}$ *and is denoted* ess sup $\mathcal{X}$.

Furthermore, if $\mathcal{X}$ is closed under pairwise maximization (i.e $X, Y \in \mathcal{X}$ *implies $X \vee Y \in \mathcal{X}$), then there is a nondecreasing sequence $\{ Z_n \}_{n\in\mathbb{N}}$ of r.v. in $\mathcal{X}$ satisfying $X^* = \lim_{n\to\infty} Z_n$ a.s..*

B

B.1 Characterization of minimal optimal two stopping times

In order to give a characterization of *minimal optimal* stopping times, we introduce the following partial order relation on $\mathbb{R}^2$: $(a,b) \prec (a',b')$ if and only if

$$[(a \wedge b < a' \wedge b') \text{ or } (a \wedge b = a' \wedge b' \text{ and } a \leq a' \text{ and } b \leq b')].$$

Note that although the minimum of two elements of $\mathbb{R}^2$ is not defined, the infimum, that is the greatest minorant of the couple, does exist and $\inf[(a,b),(a',b')] = \mathbf{1}_{\{a\wedge b<a'\wedge b'\}}(a,b) + \mathbf{1}_{\{a'\wedge b'<a\wedge b\}}(a',b') + \mathbf{1}_{\{a\wedge b=a'\wedge b'\}}(a\wedge a', b\wedge b')$.

Note also that if (τ_1^*,τ_2^*), $(\tau_1^{'},\tau_2^{'}) \in T_0 \times T_0$ are optimal for $v(S)$, then the infimum of the couple $\inf[(\tau_1^*,\tau_2^*),(\tau_1^{'},\tau_2^{'})]$ in the sense of the relation $\prec$ a.s. is optimal for $v(S)$.

One can show that the two following assertions are equivalent:

1. A pair $(\tau_1^*,\tau_2^*) \in T_0 \times T_0$ is *minimal optimal* for $v(S)$
   (*i.e* is the minimum for the order $\prec$ a.s. of the set $\left\{(\tau_1^*,\tau_2^*) \in T_S^2\right.$, $\left. v(S) = E[\psi(\tau_1^*,\tau_2^*)\,|\,\mathcal{F}_S]\right\}$)
   and $\theta^* = \tau_1^* \wedge \tau_2^*$ and θ_1^*, $\theta_2^* \in T_0$ are such that $\theta_2^* = \tau_2^*$ on $\{\tau_1^* < \tau_2^*\}$ and $\theta_1^* = \tau_1^*$ on $\{\tau_1^* > \tau_2^*\}$.

2. (a) $\theta^* \in T_0$ is minimal optimal for $u(S)$,
   (b) $\theta_2^* \in T_0$ is minimal optimal for $u_2(\theta^*)$ on $\{u_1(\theta^*) < u_2(\theta^*)\}$,
   (c) $\theta_1^* \in T_0$ is minimal optimal for $u_1(\theta^*)$ on $\{u_2(\theta^*) < u_1(\theta^*)\}$,
   and $\tau_1^* = \theta^* \mathbf{1}_{\{u_1(\theta^*)\leq u_2(\theta^*)\}} + \theta_1^* \mathbf{1}_{\{u_1(\theta^*)>u_2(\theta^*)\}}$, $\tau_2^* = \theta^* \mathbf{1}_{\{u_2(\theta^*)\leq u_1(\theta^*)\}}$ $+\theta_2^* \mathbf{1}_{\{u_2(\theta^*)>u_1(\theta^*)\}}$.

B.2 Characterization of minimal optimal d-stopping times

Consider the following partial order relation $\prec_d$ on $\mathbb{R}^d$ defined by induction in the following way:
for $d = 1$, $\quad \forall a, a' \in \mathbb{R} \quad a \prec_1 a' \quad$ if and only if $\quad a \leq a'$,
and,
for $d > 1$,

$$\forall (a_1,\cdots,a_d),(a_1',\cdots,a_d') \in \mathbb{R}^d \quad (a_1,\cdots,a_d) \prec_d (a_1',\cdots,a_d') \text{ if and only if}$$

$$\text{either} \quad a_1 \wedge \cdots \wedge a_d < a_1' \wedge \cdots \wedge a_d' \,,$$

$$\text{or} \quad \left\{ \begin{array}{l} a_1 \wedge \cdots \wedge a_d = a_1' \wedge \cdots \wedge a_d' \,, \quad \text{and for } i = 1,\cdots d, \\ a_i = a_1 \wedge \cdots \wedge a_d \implies \left\{ \begin{array}{l} a_i' = a_1' \wedge \cdots \wedge a_d' \quad \text{and} \\ (a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_d) \prec_{d-1} \\ \qquad (a_1',\cdots,a_{i-1}',a_{i+1}',\cdots,a_d'). \end{array} \right. \end{array} \right.$$

Note that for $d = 2$ the order relation $\prec_2$ is the order relation $\prec$ defined above.

One can show that a d-stopping time $(\tau_1, \cdots, \tau_d)$ is the d-minimal optimal stopping time for $v(S)$, that is is minimal for the order $\prec_d$ in the set $\{\, \tau \in T_S^d \,,\ v(S) = E[\psi(\tau) \,|\, \mathcal{F}_S] \,\}$ if and only if

1. $\theta^* = \tau_1 \wedge \cdots \wedge \tau_d$ is minimal optimal for $u(S)$,
2. for $i = 1, \cdots, d$ $\theta^{*(i)} = \tau_i \in T_S^{d-1}$ is the $d-1$-minimal optimal stopping time for $u^{(i)}(\theta^*)$ on the set $\{u^{(i)}(\theta^*) \geq \vee_{k \neq i} u^{(k)}(\theta^*)\}$.

Acknowledgements

The authors thank Gilles Pagès and the anonymous referee for their relevant remarks and suggestions.

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