Rigorous and heuristic treatment of sensitive singular perturbations arising in elliptic shells

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Résumé

We consider singular perturbations of elliptic systems depending on a parameter ε such that, for $\varepsilon = 0$ the boundary conditions are not adapted to the equation (they do not satisfy the Shapiro - Lopatinskii condition). The limit holds only in very abstract spaces out of distribution theory involving complexification and non-local phenomena. This system appears in the thin shell theory when the middle surface is elliptic and the shell is fixed on a part of the boundary and free on the rest. We use a heuristic reasoning applying some simplifications which allow to reduce the original problem in a domain to another problem on its boundary. The novelty of this work is that we consider systems of partial differential equations while in our previous work we were dealing with single equations.

1 Introduction

This paper is devoted to a very singular kind of perturbation problems arising in thin shell theory. Up to our knowledge, it is disjoint of relevant and well known contributions of V. Mazya on perturbation of domains and multistructures for elliptic problems including the Navier - Stokes system ([12], [11], [13]), as the pathological feature of our problem is concerned with ill-posedness of the limit problem, generating singularities out of the distribution space. So, it may be considered as a contribution to enlarge perturbation theory of Mazya. More precisely, the main purpose of this paper is to generalize the previous work done on equations, see [7], [14] to systems of partial differential equations. The motivation for studying that kind of problems comes from the shell theory. It appears that when the middle surface is elliptic (both

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principal curvatures have same sign) and is fixed on a part Γ_0 of the boundary and free on the rest Γ_1, the "limit problem" as the thickness ε tends to zero is elliptic, with boundary conditions satisfying Shapiro - Lopatinskii (SL hereafter) on Γ_0 but not satisfying it on Γ_1. In other words, the "limit problem" for $\varepsilon = 0$ is highly ill-posed. This pathological behavior arises only as $\varepsilon = 0$. In fact, for $\varepsilon > 0$ the problem is "classical".

In such kind of situations, the limit problem has no solution within classical theory of partial differential equations, which uses distribution theory. It is sometimes possible to prove the convergence of the solutions u^ε towards some limit u^0, but this "limit solution" and the topology of the convergence are concerned with abstract spaces not included in the distribution space.

The variational problem we are interested in is :

$$\left\{ \begin{array}{l} \text{Find } u^\varepsilon \in V \text{ such that, } \forall v \in V \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v \rangle, \end{array} \right. \tag{1.1}$$

or, equivalently, the minimization in V of the functional

$$a(u, u) + \varepsilon^2 b(u, u) - 2\langle f, u \rangle,$$

where $f \in V'$ is given and the brackets denote the duality between V' and V.

This is the Koiter model of shells, ε denoting the relative thickness. The corresponding energy space V is a classical Sobolev space.

The limit boundary partial differential system associated with (1.1) when $\varepsilon = 0$ is elliptic and ill-posed.

Let us consider formally the variationnal problem of the membrane problem (i.e. $\varepsilon = 0$) :

$$\left\{ \begin{array}{l} \text{Find } u \in V_a \text{ such that, } \forall v \in V_a \\ a(u, v) = \langle f, v \rangle, \end{array} \right. \tag{1.2}$$

where V_a is the abstract completion of the "Koiter space" V with the norm $\|v\|_a = a(v, v)^{1/2}$, it is to be noted that the elements of V_a are not necessary distributions. The term "sensitive" originates from the fact that this latter problem is unstable. Very small and smooth variations of f (even in $\mathcal{D}(\Omega)$) induce modifications of the solution which are large and singular (out of the distribution space).

The plan of the article is as follows. After recalling the Koiter shell model (Section 2), we recall the definitions of ellipticity and the Schapiro-Lopatinskii condition for systems elliptic in the Douglis-Nirenberg sense (Section 3). In Section 4, we study four systems of partial differential equations which are involved in our study of shell theory. These systems are the rigidity system, the membrane tension system, the membrane system and the Koiter shell system.

In section 5, we study a sensitive perturbation problem arising in Koiter linear shell theory and we briefly recall some abstract convergence results. In Section 6, we report the heuristic procedure of [7]. In this latter article, we addressed a model problem including a variational structure, somewhat analogous to the shell problem studied here, but simpler, as concerning an equation instead of a system. It is shown that the limit problem involves in particular an elliptic Cauchy problem. This problem was handled in both a rigorous (very abstract) framework and using a heuristic procedure for exhibiting the structure of the solutions with very small ε. The reasons why the solution goes out of the distibution space as ε goes to 0 are then evident. The heuristic procedure is very much analogous to the method of construction of a parametrix in elliptic problems [20], [8] :

-Only principal (with higher differentiation order) terms are taken into account.

-Locally, the coefficients are considered to be constant, their values being frozen at the corresponding points.

-After Fourier transform ($x \to \xi$), terms with small ξ are neglected with respect to those with larger ξ (which amounts to taking into account singular parts of the solutions while neglecting smoother ones). We note that this approximation, aside with the two previous ones, lead to some kind of "local Fourier transform" which we shall use freely in the sequel.

Another important feature of the heuristics is a previous drastic restriction of the space where the variational problem is handled. In order to search for the minimum of energy, we only take into account functions such that the energy of the limit problem is very small. This is done using a boundary layer method within the previous approximations, i.e. for large $|\xi|$. This leads to an approximate simpler formulation of the problem for small ε, where it is apparent that the limit problem involves a smoothing operator and cannot have a solution within distribution theory.

Notations are standard. We denote

$$\partial_k = \frac{\partial}{\partial x_k}, \quad k = 1, 2, \tag{1.3}$$

and

$$D_k = -i\frac{\partial}{\partial x_k}, \ \ k = 1, 2 \text{ and } D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}, \ \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2. \tag{1.4}$$

Moreover, the definition of the Sobolev space $H^s(\Gamma)$, $s \in \mathbb{R}$, where Γ is a one dimensional compact manifold is classical using a partition of unity and local mappings.

The inner product and the duality products associated with a space V and its dual V' will be denoted by $(.,.)$ and $\langle .,. \rangle$ respectively.

The usual convention of summation of repeated indices is used. Greek and latin indices will belong to the sets $\{1, 2\}$ and $\{1, 2, 3\}$ respectively.

2 Generalities on the Koiter shell model

Let Ω be a bounded open set of $\mathbb{R}^2$ with smooth boundary Γ. Let $\mathcal{E}^3$ be the euclidean space referred to the othonormal frame (O, e_1, e_2, e_3). We consider the shell theory in the framework of the Koiter theory and more precisely the mathematical framework of this linear theory. The middle surface S of the shell is the image in $\mathcal{E}^3$ of Ω for the map

$$\varphi \ : \ (y^1, y^2) \in \overline{\Omega} \to \varphi(y) \in \mathcal{E}^3.$$

The two tangent vectors of S at any point y are given by :

$$a_\alpha = \partial_\alpha \varphi, \ \alpha \in \{1, 2\},$$

where ∂_α denotes the differentiation with respect to y^α, while the unit normal vector is :

$$a_3 = \frac{a_1 \wedge a_2}{\|a_1 \wedge a_2\|}.$$

For simplicity, we omitted y in the previous notation $(a_\alpha(y))$.

The middle surface S is assumed to be smooth ($\mathcal{C}^\infty$) and we may consider in a neighbourhood of it a system of "normal coordinates" y^1, y^2, y^3, when y^3 is the normal distance to S. More precisely we consider a shell of constant thickness ε, i.e. it is the set

$$C = \{M \in \mathcal{E}^3, \ M = \varphi(y^1, y^2) + y^3 a_3, \ (y^1, y^2) \in \Omega, -\frac{1}{2}\varepsilon < y^3 < \frac{1}{2}\varepsilon\}.$$

Under these conditions, let $u = u(y^1, y^2)$ be the displacement vector of the middle surface of the shell. In the linear theory of shells, which is our framework here, the displacement vector is assumed to describe the first order term of the mathematical expression as the thickness ε is small, see [4, 18].

Remark 1. In the sequel smooth should be understood in the sense of $\mathcal{C}^\infty$.

Remark 2. We consider here the case where the surface is defined by only one chart but this could be easily generalized to the case of several charts (atlas).

More precisely, since we consider the case where u is supposed to be small, the Koiter theory is described in terms of the *deformation tensor* (or strain tensor) $\gamma_{\alpha\beta}$ of the middle surface :

$$\gamma_{\alpha\beta} = \frac{1}{2}(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})$$

and the *change of curvature tensor* $\rho_{\alpha\beta}$:

$$\rho_{\alpha\beta} = \tilde{b}_{\alpha\beta} - b_{\alpha\beta}.$$

In the previous definitions, the expressions $a_{\alpha\beta}$ (resp. $\tilde{a}_{\alpha\beta}$) denote the coefficients of the first fundamental form of the middle surface before (resp. after) deformation :

$$a_{\alpha\beta} = a_\alpha \cdot a_\beta = \partial_\alpha \varphi \cdot \partial_\beta \varphi,$$

and $b_{\alpha\beta}$ (resp. $\tilde{b}_{\alpha\beta}$) the coefficients of the second fundamental form accounting for the curvatures before (resp. after) deformation :

$$b_{\alpha\beta} = -a_\alpha \cdot \partial_\beta a_3 = a_3 \cdot \partial_\beta a_\alpha = a_3 \cdot \partial_\alpha a_\beta = b_{\beta\alpha},$$

due to the fact that $a_\alpha \cdot a_3 = 0$.

The dual basis a^i is defined by

$$a_i \cdot a^j = \delta_i^j,$$

where δ denotes the Kronecker symbol. The contravariant components a^{ij} of the metric tensor are :

$$a^{ij} = a^i \cdot a^j,$$

and a_{ij} are used to write covariant components of vectors and tensors in the usual way. Finally, the tensors γ and ρ take the form :

$$\begin{aligned}
\gamma_{\beta\alpha}(u) &= \gamma_{\alpha\beta}(u) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}u_3, && (2.1)\\
\rho_{\alpha\beta}(u) &= u_{3|\alpha\beta} + b^\lambda_{\beta|\alpha}u_\lambda + b^\lambda_\beta u_{\lambda|\alpha} + b^\lambda_\alpha u_{\lambda|\beta} - b^\lambda_\alpha b_{\lambda\beta}u_3, && (2.2)
\end{aligned}$$

where $\partial_\alpha a_3 = b^\gamma_\alpha a_\gamma$, $b^\beta_\alpha = a^{\beta\sigma} b_{\alpha\sigma}$, $_{.|\alpha}$ denotes the *covariant differentiation* which is defined by

$$\left\{ \begin{array}{l} u_{\alpha|\beta} = \partial_\beta u_\alpha - \Gamma^\lambda_{\alpha\beta} u_\lambda \\ u_{3|\beta} = \partial_\beta u_3, \end{array} \right. \tag{2.3}$$

and

$$\left\{ \begin{array}{l} b^\lambda_{\alpha|\beta} = \partial_\alpha b^\lambda_\beta + \Gamma^\lambda_{\alpha\nu} b^\nu_\beta - \Gamma^\nu_{\beta\alpha} b^\lambda_\nu = b^\lambda_{\beta|\alpha} \\ u_{3|\alpha\beta} = \partial_{\alpha\beta} u_3 - \Gamma^\lambda_{\alpha\beta}\partial_\lambda u_3, \end{array} \right. \tag{2.4}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the surface

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = a^\alpha \cdot \partial_\beta a_\gamma = a^\alpha \cdot \partial_\gamma a_\beta.$$

Let us now define the energy of the shell in the Koiter framework. It consists of two bilinear forms a and b : a corresponds to a *membrane strain energy* and b is a *bending energy* (which acts as a perturbation term). More precisely, a is defined by

$$a(u,v) = \int_S A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u)\gamma_{\alpha\beta}(\overline{v})\ \mathrm{d}s, \tag{2.5}$$

where $A^{\alpha\beta\lambda\mu}$ are the *membrane rigidity* coefficients which we assume to be smooth on Ω. Moreover, we assume that some symmetry holds

$$A^{\alpha\beta\lambda\mu} = A^{\lambda\mu\alpha\beta} = A^{\mu\lambda\alpha\beta}. \tag{2.6}$$

Defining the *membrane stress tensors* by

$$T^{\alpha\beta}(u) = A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u), \tag{2.7}$$

using the symmetry of γ, we immediately see that

$$T^{\alpha\beta}(u) = T^{\beta\alpha}(u), \tag{2.8}$$

and

$$a(u,v) = \int_S T^{\alpha\beta}(u)\gamma_{\alpha\beta}(\overline{v})\ \mathrm{d}s = \int_S \gamma_{\alpha\beta}(u)T^{\alpha\beta}(\overline{v})\ \mathrm{d}s. \tag{2.9}$$

Furthermore, we assume that a coercivity condition holds uniformly on the surface :

$$A^{\alpha\beta\lambda\mu}\xi_{\alpha\beta}\xi_{\lambda\mu} \geq C\|\xi\|^2,\ C > 0. \tag{2.10}$$

Remark 3. It is to be noticed that there are two different symmetries on A : the first one $A^{\alpha\beta\lambda\mu} = A^{\lambda\mu\alpha\beta}$ is necessary to exchange u and v in (2.9) while the second $A^{\lambda\mu\alpha\beta} = A^{\mu\lambda\alpha\beta}$ is used to obtain (2.8) but is not necessary in order to obtain (2.9) since we could use the symmetry of γ.

Analogously, we define the bilinear form b which corresponds to the bending energy of the shell and which will act as a perturbation term :

$$b(u,v) = \int_S B^{\alpha\beta\lambda\mu} \rho_{\lambda\mu}(u)\rho_{\alpha\beta}(\overline{v})\ \mathrm{d}s, \tag{2.11}$$

where $B^{\alpha\beta\lambda\mu}$ are the *bending rigidity* coefficients which we assume to be smooth on Ω and to have the same properties (2.6) and (2.10) as A, namely

$$B^{\alpha\beta\lambda\mu} = B^{\lambda\mu\alpha\beta} = B^{\mu\lambda\alpha\beta}, \tag{2.12}$$

and

$$B^{\alpha\beta\lambda\mu}\xi_{\alpha\beta}\xi_{\lambda\mu} \geq C\|\xi\|^2 \tag{2.13}$$

uniformly on the surface.

Similarly to a we can write

$$b(u,v) = \int_S M^{\alpha\beta}(u)\rho_{\alpha\beta}(\overline{v})\ \mathrm{d}s, \tag{2.14}$$

where the *bending stress tensors* are

$$M^{\alpha\beta}(u) = B^{\alpha\beta\lambda\mu}\rho_{\lambda\mu}(u). \tag{2.15}$$

In this work, we will restrict ourselves to the case of elliptic surface, i.e. we will always assume that the coefficients $b_{\alpha\beta}$ are such that

$$b_{11}b_{22} - b_{12}^2 > 0 \text{ uniformly on } S \text{ and } b_{11} > 0. \tag{2.16}$$

Let us finish this introduction by topoligical considerations, the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ is assumed to be smooth (i.e. of class $\mathcal{C}^\infty$) in the variable $y = (y^1, y^2)$, where Γ_0 and Γ_1 are disjoint ; they are one-dimensional compact smooth manifolds without boundary, then diffeomorphic to the unit circle.

We consider the following variational problem (which has possibly only a formal sense)

$$\left\{ \begin{array}{l} \text{Find } u^\varepsilon \in V \text{ such that, } \forall v \in V \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v \rangle, \end{array} \right. \tag{2.17}$$

with a and b defined by (2.5) and (2.14) where the space V is the "energy space" with the essential boundary conditions on Γ_0

$$V = \{v;\ v_\alpha \in H^1(\Omega),\ v_3 \in H^2(\Omega);\ v_{|\Gamma_0} = 0 \text{ in the sense of trace}\}. \tag{2.18}$$

Remark 4. The essential boundary conditions on Γ_0 (2.18) corresponds to the case of the fixed boundary of the shell. Other boundary conditions could have been considered such as :

$$V = \{v;\ v_\alpha \in H^1(\Omega),\ v_3 \in H^2(\Omega);\ v_{|\Gamma_0} = 0,\ \partial_\nu v_{3|\Gamma_0} = 0 \text{ in the sense of trace}\}, \tag{2.19}$$

where ν is the normal to Γ_0 (i.e. the normal to the boundary which lies in the tangent plane), which corresponds to the clamped case.

The following Lemma was obtained by Bernardou and Ciarlet see [4].

Lemma 2.1. *The bilinear form $a + b$ is coercive on V.*

We shall denote by V' the dual space of V. Here dual is obviously understood in the abstract sense of the space of continuous linear functionals on V. In order to make explicit computations in terms of equation and boundary conditions, we shall often take f as a "function" defined on Ω, in the space

$$\begin{aligned}&\{f \in H^{-1}(\Omega;\mathbb{R}) \times H^{-1}(\Omega;\mathbb{R}) \times H^{-2}(\Omega;\mathbb{R});\\ &f \text{ "smooth" in a neighbourhood of } \Gamma_1\} \subset V',\end{aligned} \tag{2.20}$$

where "smooth" means allowing classical integration by parts. Obviously other choices for f are possible.

Moreover, we immediately obtain the following result.

Proposition 2.2. *For $\varepsilon > 0$ and for f in V', the variational problem (2.17) is of Lax-Milgram type and it is a self-adjoint problem which has a coerciveness constant larger than $c\varepsilon^2$, with $c > 0$.*

Remark 5. It is to be noticed that the coerciveness of the previous problem disapears when $\varepsilon = 0$.

3 The ellipticity of systems and the Shapiro-Lopatinskii condition

In this section, we recall some classical results on the linear boundary value problems for elliptic systems in the sense of Douglis and Nirenberg [6]. We begin with the definition of ellipticity for systems, then we recall the Shapiro-Lopatinskii condition. This latter condition states which boundary conditions are well suited in order to have well posed problems for elliptic systems. We then recall in what sense an elliptic system with Shapiro-Lopatinskii condition is "well-behaved".

For brevity, from now on we will denote SL the Shapiro-Lopatinskii condition.

3.1 Elliptic systems in the sense of Douglis and Nirenberg [6]

In this work, we shall deal with systems of l ($l = 3$ or $l = 6$) equations with 3 unknowns (noted here u_1, u_2, u_3) defined on an open set $\Omega \subset \mathbb{R}^2$ with smooth boundary, which has the form :

$$l_{kj} u_j = f_k, \ k = 1, \dots, l, \tag{3.1}$$

or equivently $L\mathbf{u} = f$. The coefficients $l_{kj}(x, D)$ with $D = (D_1, D_2)$ and $D_l = -i\frac{\partial}{\partial x_l}$, $l \in \{1, 2\}$, are linear differential operators with real smooth coefficients. In our systems (3.1), the highest order of differentiation is different for the three unknowns and depends on the equation. A way to take into account such differences between the various equations and unknowns is to define integer indices (s_1, s_2, s_3) attached to the equations and integer indices (t_1, t_2, t_3) attached to the unknowns (see Douglis and Nirenberg [6]) so that the "higher order terms" (which will be called "principal terms") are in equation j the terms where each unknown "k" appears by its derivative of order $s_k + t_j$. More precisely, the integers (s_k, t_j) are such that

$$\begin{cases} \text{if } s_k + t_j \geq 0, \ \text{the order of } l_{kj} \text{ is less or equal to } s_k + t_j, \\ \text{if } s_k + t_j < 0, \ l_{kj} \text{ is equal to zero.} \end{cases}$$

The *principal part* l'_{kj} of l_{kj} is obtained by keeping the terms of order $s_k + t_j$ if $s_k + t_j \geq 0$ and by taking $l'_{kj} = 0$ if $s_k + t_j < 0$. The matrix $L'(x, \xi)$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, obtained by substituting ξ_α for D_α in l'_{kj}, is called the *principal symbol of the system*. Since l'_{kj} are homogeneous of order $s_k + t_j$ with respect to ξ_α, the determinant of the matrix $L'(x, \xi)$, denoted $\mathrm{D}(x, \xi)$, is homogeneous of degree $\Sigma_k s_k + \Sigma_j t_j$.

Definition 3.1. The system (3.1) is *elliptic in the sense of Douglis and Nirenberg* at the point $x \in \Omega$ if and only if

$$\mathrm{D}(x, \xi) \neq 0, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}. \tag{3.2}$$

Remark 6. Since the coefficients are assumed to be real, the function $\mathrm{D}(x, \xi)$ for an elliptic system is even in ξ of order $2m$ with

$$\Sigma_k s_k + \Sigma_j t_j = 2m.$$

Remark 7. The definition of the indices s_j and t_k for a system is slightly ambiguous. Indeed the result is exactly the same after adding an integer n to the indices s_j and substracting n from the t_k.

Remark 8. Let $x_0 \in \Omega$ be such that the system (3.1) is not elliptic, then there exists a $\xi \in \mathbb{R}^2 \setminus \{0\}$ such that $\mathrm{D}(x_0, \xi) = 0$. In such a case the system $L'(x_0, D)u = 0$, with frozen coefficients at x_0 admits a solution of the form $u(x) = ve^{i\xi x}$, with $v \in \mathbb{R}^3 \setminus \{0\}$.

Remark 9. Moreover, throughout this paper, *ellipticity* will be understood in the sequel as uniform, i.e. there exists a positive constant A such that

$$A^{-1}\Sigma_\alpha |\xi_\alpha|^2 \leq |\det L'(x, \xi)| \leq A \Sigma_\alpha |\xi_\alpha|^2,$$

for all $x \in \Omega$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$.

3.2 Shapiro-Lopatinskii conditions for elliptic systems in the sense of Douglis and Nirenberg [6]

From now on, for simplicity, we will say that a system is *elliptic* when it is elliptic in the sense of Douglis and Nirenberg [6].

Let l_{kj} (L) be an elliptic system of order $2m$ with principal part l'_{kj} (L') and let m boundary conditions be given by :

$$b_{kj}u_j = g_k,\ k \in \{1,..,m\},$$

where $b_{kj}(x,D)$ are differential operators with smooth coefficients. Let us define the integers r_k (indices of the boundary conditions, $k = 1,..,m$) such that

$$\begin{cases} \text{if } r_k + t_j \geq 0, \text{ the order of } b_{kj} \text{ is less or equal to } r_k + t_j \\ \text{if } r_k + t_j < 0, \ b_{kj} \text{ is equal to zero.} \end{cases}$$

The principal part b'_{kj} is b_{kj} if $r_k + t_j \geq 0$ and zero otherwise.

Assume that the smooth real coefficients are defined in $\overline{\Omega}$.

Let $x_0 \in \Gamma$, we assume that L' is elliptic at x_0. Usually, see [1] and [8] for instance, the SL condition at x_0 is defined via a local diffeomrophism sending a neighbourhood of x_0 in Ω into a neighbourhood of the origin in a half-plane. For ulterior computations, it is worth-while to take a special diffeomorphism which amounts to taking locally cartesian coordinates x_1, x_2, respectively, tangent and (inwards) normal to the boundary at x_0. We then consider only the principal parts of the equations and of the boundary conditions frozen at x_0. Next, we consider the corresponding boundary value problem obtained by formal tangential Fourier transform (i.e. $D_1 \to \xi_1$, with $\xi_1 \in \mathbb{R}$ and $u \to \tilde{u}$) which amounts to the following algebraic conditions :

$$\begin{cases} \tilde{l}'_{kj}(x_0, \xi_1, D_2)\tilde{u} = 0 \text{ for } x_2 > 0 \\ \tilde{b}'_{kj}(x_0, \xi_1, D_2)\tilde{u} = \tilde{g}_j \text{ for } x_2 = 0, \end{cases} \tag{3.3}$$

$j, k \in \{1, ..., m\}$, see [7] Sec. 3.2 for details, if necessary.

The problem (3.3) involves a system of ordinary differential equations with constant coefficients of the variable $x_2 \in \mathbb{R}^+$ and m boundary conditions at $x_2 = 0$, whose solutions are classically a linear combination of terms of the form :

$$\tilde{u}(\xi_1, x_2) = \begin{cases} ve^{i\xi_2 x_2},\ v \in \mathbb{C}^3 \\ P(x_2)e^{i\xi_2 x_2},\ \text{where } P \text{ is a polynomial, in the case of Jordan block.} \end{cases} \tag{3.4}$$

Recalling that the system L is elliptic, it follows that the imaginary part of ξ_2 does not vanish. Furthermore, there are m solutions ξ_2 of $D(x_0, \xi_1, \xi_2) = 0$ with positive imaginary part that we denote ξ_2^+ (and m with negative imaginary part denoted ξ_2^-).

We then try to solve (3.3) using only linear combinations of the m solutions of the form (3.4) for the m roots ξ_2^+ (i.e. exponentially decreasing towards the domain).

Definition 3.2. The SL condition is satisfied at $x_0 \in \Gamma$ if one of the following equivalent conditions holds :

1. The solution of the previous problem is defined uniquely.
2. Zero is the only solution of the homogeneous (i.e. with $g_j = 0$) previous problem.

Remark 10. The two conditions (which are equivalent) of the previous definition are clearly equivalent to the non annulation of the determinant of the corresponding algebraic "system".

Remark 11. The reason for defining the SL condition amounts to the possibilty of solving the problem in a half plane via tangential Fourier transform. The reason for not considering the ξ_2^- roots is that, for $x_2 > 0$, they should give exponentially growing Fourier transforms in $x_1 \to \xi_1$, which are not allowed in distribution theory (note that ξ_1 and ξ_2 are proportional as $D(\xi)$ is homogeneous).

The verification of the SL condition is often tricky. In some situations, we can use equivalent expressions which are simpler to treat. More precisely, define the function u by $u(x_1, x_2) = \tilde{u}(\xi_1, x_2)e^{i|\xi_1|x_1}$, with $\tilde{u}(\xi_1, x_2) = ve^{i\xi_2^+ x_2}$ (or expressed as exponential polynomial in the case of Jordan block), it is an exponentially decreasing function in the direction inwards the domain (when $x_2 \to +\infty$), it is also a periodic function in the tangential direction x_1 and it satisfies

$$\begin{cases} \tilde{l}'_{kj}(x_0, D_1, D_2)u = 0 \text{ for } x_2 > 0 \\ \tilde{b}'_{kj}(x_0, D_1, D_2)u = g_j \text{ for } x_2 = 0, \end{cases} \tag{3.5}$$

$j, k \in \{1, ..., m\}$. The following proposition is very useful in the case where ellipticity is linked with positive energy integrals obtained by integrating by parts. For instance, we have :

Proposition 3.3. *Consider the homogeneous problem associated with (3.5) (i.e. taking $g_j = 0$) for $x_0 \in \Gamma$. If any solution u, which is periodic in the tangential direction x_1 and exponentially decreasing in the direction x_2 inwards the domain, is zero, then the SL condition is satisfied.*

Remark 12. In order to have well-posed problems for elliptic systems, boundary conditions satisfying the SL condition should be prescribed at any points of the boundary. Their number is half the total order of the system.

Remark 13. The specific boundary conditions may differ from a point to another on the boundary. In particular, each connected component of the boundary may have its own set of boundary conditions. Otherwise, local changes of boundary conditions (as well as non-smoothness of the boundary) induces local singularities. **A changer**

3.3 Some results for "well posed" elliptic systems

Let us now consider a boundary value problem formed by an elliptic system with boundary conditions satisfying the SL condition. In what sense is this problem "well-behaved"? The obvious example of an eigenvalue problem, even for an equation shows that uniqueness is only ensured up to the kernel formed by the eigenvectors associated with the zero eigenvalue, whereas existence involves compatibility conditions (orthogonality to the kernel of the adjoint problem). The general results are those of Agmon, Douglis and Nirenberg [1].

First, let us recall the definition of a Fredholm operator.

Definition 3.4. Let E and F be two Hilbert spaces and A an operator (closed with dense domain in E) from E into F. We say that A is a Fredholm operator if and only if the following three conditions hold :

1. Ker(A) is of finite dimension,
2. R(A) is closed,
3. R(A) is of finite codimension.

The operator A is also said to be an index operator, the index is defined as dim Ker(A) − codim R(A).

Let us consider an elliptic system of order $2m$ whose coefficients are smooth :

$$\begin{cases} l_{kj}u_j = f_k, \ j,k \in \{1,...,l\} \text{ in } \Omega \\ b_{hj}u_j = g_h, \ h \in \{1,...,m\} \text{ on } \partial\Omega, \end{cases} \tag{3.6}$$

whose indices associated with unknowns, equations and boundary conditions are respectively t_j, s_j, r_j. Let ρ be a "big enough" real number, called regularity index. Consider operator (3.6) as a linear operator from the space E to the space F defined by :

$$E = \Pi_{j=1}^{l} H^{\rho+t_j}(\Omega), \ F = \Pi_{j=1}^{l} H^{\rho-s_j}(\Omega) \times \Pi_{j=1}^{m} H^{\rho-r_j-\frac{1}{2}}(\partial\Omega). \tag{3.7}$$

The real ρ is chosen in order to give a sense to the traces which are involved, i.e. it is such that $\rho - r_j - 1/2 > 0$ for $j \in \{1,...,m\}$.

The following result is the main result of the theory of Agmon, Douglis and Nirenberg :

Theorem 3.5 (Agmon, Douglis and Nirenberg [1])**.** *Let Ω be a bounded open set with smooth boundary Γ. Let us consider an elliptic system with boundary conditions satisfying the SL condition everywhere on Γ. Assume that the coefficients of the system are smooth and that u, f and g satisfy (3.6). Then the following estimate holds true :*

$$\|u\|_E \le C(\|(f,g)\|_F + \|u\|_{(L^2(\Omega))^l}), \tag{3.8}$$

where C does not depend on u, f, g. Moreover, the operator defined by (3.6) from the space E to the space F, given by (3.7), is a Fredholm operator, for all value of ρ such that $\rho - r_j - 1/2 > 0$ for $j \in \{1,...,m\}$. Furthermore, the dimension of the kernel and the dimension of the subspace orthogonal to the range do not depend on ρ. The kernel is composed of smooth functions.

Remark 14. The previous theorem means that in general existence and uniqueness of the solution only hold up to a finite number of compatibility conditions for f and g and existence of the solution holds up to a finite dimension kernel. More precise properties need specific properties of the system.

Remark 15. For all values of ρ, the kernel formed by the eigenvectors corresponding to the eigenvalue 0 is of finite dimension and is composed of smooth functions, independent of ρ (in $\mathcal{C}^\infty(\overline{\Omega})$).

Remark 16. Denote A the operator defined by (3.6) in the spaces E and F. Let us consider the case where dim Ker$(A) > 0$ and define the inverse B of A as a closed operator from R(A) to $E/\mathrm{Ker}(A)$, we have that

$$\|\tilde{u}\|_{E/\mathrm{Ker}(A)} \leq C\|(f,g)\|_F, \tag{3.9}$$

where $\tilde{u}$ is an element of the equivalence class of u.

The element $\tilde{u}$ can also be viewed as an element of the orthogonal of Ker(A) in E, which is identified with $E/\mathrm{Ker}(A)$. In such a case, there exists a unique $(\tilde{u},\hat{u}) \in E/\mathrm{Ker}(A) \times \mathrm{Ker}(A)$ such that

$$u = \tilde{u} + \hat{u}.$$

Since Ker(A) is of finite dimension, all the norms are equivalent and we can choose for $\hat{u}$ a norm in a space $H^{-\nu}$ with ν very big. Therefore, inequality (3.8) can be rewritten as

$$\|u\|_E \leq C(\|\tilde{u}\|_{E/\mathrm{Ker}(A)} + \|\hat{u}\|_{H^{-\nu}}) \leq C(\|\tilde{u}\|_{E/\mathrm{Ker}(A)} + \|u\|_{H^{-\nu}}), \tag{3.10}$$

for ν big enough such that $E \subset H^{-\nu}$. Recalling (3.9), we then deduce that

$$\|u\|_E \leq C(\|(f,g)\|_F + \|\hat{u}\|_{H^{-\nu}}). \tag{3.11}$$

Moreover, the norm in $H^{-\nu}$ may be replaced by a seminorm, provided it is a norm on Ker(A).

Remark 17. In the case where dim Ker$(A) = 0$, the inverse B of the operator A is well defined on $R(A)$. It is a closed operator, hence it is bounded and the following estimate holds :

$$\|u\|_E \leq C\|(f,g)\|_F. \tag{3.12}$$

4 Study of four systems involved in shell theory

In this section, we study four systems, denoted by *rigidity* system, *membrane tension* system, *membrane* system and *Koiter shell* system, which will appear in the sequel. We prove that these four systems satisfy the ellipticity condition and we study some boundary conditions. It is to be noticed that the boundary conditions may be different on Γ_0 and Γ_1 which are supposed to be disjoints.

Let us recall the situation : Ω is a connected bounded open set of $\mathbb{R}^2$ with $\mathcal{C}^\infty$ boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. The middle surface S of the shell is the image in $\mathcal{E}^3$ of Ω for the map

$$\varphi \ : \ (y^1, y^2) \in \overline{\Omega} \to \varphi(y) \in \mathcal{E}^3.$$

We assume that the ellipticity assumption of the surface holds :

$$b_{11}b_{22} - b_{12}^2 > 0 \text{ uniformly on } \Omega.$$

4.1 The rigidity system

Let us begin with the *rigidity* system defined by $\gamma_{\alpha\beta}(u)$:

$$\begin{cases} \gamma_{11}(u) := \partial_1 u_1 - \Gamma_{11}^{\alpha} u_\alpha - b_{11} u_3 \\ \gamma_{22}(u) := \partial_2 u_2 - \Gamma_{22}^{\alpha} u_\alpha - b_{22} u_3 \\ \gamma_{12}(u) := \frac{1}{2}(\partial_2 u_1 + \partial_1 u_2) - \Gamma_{12}^{\alpha} u_\alpha - b_{12} u_3. \end{cases} \tag{4.1}$$

Clearly u_α and u_3 play very different roles as u_α appears with derivatives whereas u_3 only appears without. Therefore take $(1,1,0)$ as the indices of the unknowns (u_1, u_2, u_3) and $(0,0,0)$ as equation indices in the order $(\gamma_{11}, \gamma_{22}, \gamma_{12})$. The principal system is obtained by substituting 0 for $\Gamma_{\lambda\mu}^{\alpha}$ but keeping $b_{\lambda\mu}$.

Lemma 4.1. *Do to the ellipticity assumption of the surface (2.16), the rigidity system γ is elliptic of total order 2 on Ω.*

Démonstration. Substitute $-i\xi_\alpha$ for ∂_α in the principal system, we obtain a system whose determinant is $D(x,\xi) = 2b_{12}\xi_1\xi_2 - b_{22}\xi_1^2 - b_{11}\xi_2^2$, hence due to the ellipticity hypothesis (2.16), for all $x \in \Omega$, we have

$$D(x,\xi) > 0.$$

□

4.1.1 Cauchy boundary conditions

It is classical that the Cauchy problem associated with elliptic system is not well posed in the sense that it does not enjoy existence, uniqueness and stability of solutions. Nevertheless, the Cauchy problem associated with the rigidity system will be involved in the sequel and we study it now. In particular, we shall need the following uniqueness theorem for solutions $u \in H^1 \times H^1 \times L^2$.

Lemma 4.2. *Under the ellipticity assumption of the surface (2.16), the system $\gamma_{\alpha\beta}(u) = 0$ on Ω with the boundary conditions $u_1 = u_2 = 0$ on a part of the boundary (of positive measure) admits a unique solution which is $u = 0$.*

Démonstration. Let us assume that $v \in H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$ is such that $\gamma_{\alpha\beta}(v) = 0$ and $v_1 = v_2 = 0$ on a part of the boundary. Thanks to the ellipticity hypothesis (2.16), we know that $b_{11} \neq 0$ on $\overline{\Omega}$. We can eliminate v_3 from the first and third equations ($\gamma_{11}(v) = 0$ and $\gamma_{22}(v) = 0$) of the system γ. This yields the system of two equations for two unknowns (v_1, v_2) :

$$\begin{cases} 0 = \partial_2 v_2 - \Gamma_{22}^{\alpha} v_\alpha - \frac{b_{22}}{b_{11}}(\partial_1 v_1 - \Gamma_{11}^{\alpha} v_\alpha) \\ 0 = \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) - \Gamma_{12}^{\alpha} v_\alpha - \frac{b_{12}}{b_{11}}(\partial_1 v_1 - \Gamma_{11}^{\alpha} v_\alpha). \end{cases} \tag{4.2}$$

The eliminated unknown being then given by :

$$v_3 = \frac{1}{b_{11}}(\partial_1 v_1 - \Gamma_{11}^{\alpha} v_\alpha).$$

The problem then reduces to the uniqueness in the class $H^1(\Omega)$ of (v_1, v_2) satisfying

$$\left\{\begin{array}{l} \partial_1 v_1 - b_{11} v_3 = 0 \\ \partial_2 v_2 - b_{22} v_3 = 0 \\ \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) - b_{12} v_3 = 0, \end{array}\right. \tag{4.3}$$

with $v_1 = v_2 = 0$ on a part of the boundary. This problem is more or less classical. Under analyticity hypotheses about the coefficients and the boundary, the uniqueness follows from Holmgren local uniqueness theorem and analytic continuation (as u_1, u_2 are in this case analytic inside Ω). Under the $\mathcal{C}^\infty$ hypotheses adopted here, uniqueness follows from theory of pseudo-analytic functions. There are two nearly equivalent theories of such functions attached to the names of L. Bers (see for instance supplement of chapter IV of [5], written by Bers himself) and I.N. Vekua see [21].

Let (v_1, v_2) be a solution of (4.3) vanishing on a part Γ of the boundary. Let $(\tilde{v}_1, \tilde{v}_2)$ be an extension of (v_1, v_2) with values zero to an extended domain across Γ. Classically $(\tilde{v}_1, \tilde{v}_2)$ satisfies the same system (4.3) on the extended domain and, according to interior regularity theory for elliptic systems, is of class $\mathcal{C}^\infty$ inside it. The function $\tilde{w} = \tilde{v}_1 + i\tilde{v}_2$ is pseudo-analytic, of class $\mathcal{C}^\infty$ and vanishes on the outer region of the extended domain. We then use either theorem 3.5 of [21], p. 146, which gives directly the uniqueness or the representation theorem of [5] p. 379. In this case, $\tilde{w}(z)$ admits the expression (here $z = x_1 + ix_2$) :

$$\tilde{w}(z) = e^{\delta(z)} f(z),$$

where $f(z)$ is analytic and $\delta(z)$ is continuous. As $e^{\delta(z)}$ vanishes nowhere, the uniqueness follows. □

Remark 18. Strictly speaking, the evoked theorems of pseudo-analytic functions apply to systems with principal part of the canonical form

$$\left\{\begin{array}{l} \partial_1 v_1 - \partial_2 v_2 = ... \\ \partial_2 v_1 + \partial_1 v_2 = ..., \end{array}\right. \tag{4.4}$$

so that the classical reduction to this form (see for instance [5] p. 169-170) should be previously considered. But obviously, this does not modify the $\mathcal{C}^\infty$ regularity inside the domain.

Let us make several comments about this uniqueness result.

Remark 19. This result, known as the infinitesimal rigidity of the surface, does not depend on the curvilinear coordinates.

Remark 20. The key ingredients of the previous uniqueness result are a uniqueness theorem for the Cauchy problem for elliptic systems of two equations of order 1. It is not based upon a coercivity assumption for an elliptic system. But we know that the Cauchy problem for elliptic systems is precarious in the sense that it does not enjoy existence, uniqueness and stability of solutions. This means that such a system could lead to instability in the sense that there could exist v_1, v_2, v_3 very "big" in usual spaces such that $\gamma_{\alpha\beta}(v)$ are very "small".

4.1.2 Boundary value problems for the *rigidity* system

From now on, we will consider the frame (O, a_1, a_2, a_3) to be orthonormal on the boundary and such that $u_t = (u_1, 0, 0)$ and $u_n = (0, u_2, 0)$, where u_t denotes the component of u in the tangential direction to the boundary and u_n is the component of u in the normal direction to the boundary and in the tangent plane. This point which is not absolutely necessary, implies a special local parametrization.

Lemma 4.3. *The boundary condition $u_1 = g$ satisfies the SL condition for the system γ.*

Démonstration. We take as index of the boundary condition $r = -1$. Let x_0 belong to Γ. As explained in Section 3.2, using a partition of unity, local mappings, with axes y_1 tangential and y_2 inwards Γ, dropping lower order differential terms, we obtain a new system :

$$\text{For } y_2 > 0, \begin{cases} \partial_1 u_1 - b_{11} u_3 = 0 \\ \partial_2 u_2 - b_{22} u_3 = 0 \\ \frac{1}{2}(\partial_2 u_1 + \partial_1 u_2) - b_{12} u_3 = 0. \end{cases} \tag{4.5}$$

We look for solutions which are exponentially decreasing when $y_2 \to +\infty$ of the form :

$$u(y_1, y_2) = U e^{i\zeta y_2 + i\xi_1 y_1}, \ \xi_1 \in \mathbb{R} \setminus \{0\},$$

with $U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \in \mathbb{C}^3$, $\mathrm{Im}(\zeta) > 0$. Substituting this solution into (4.5) and using the boundary condition we have $U_1 = 0$. Consequently, $u_1 = 0$ everywhere and (4.5) gives also $u_2 = u_3 = 0$. $U_2 = U_3 = 0$. □

Remark 21. Similarly to the proof of the previous result, we can prove that the following boundary conditions satisfy the SL condition :

1. $u_2 = g$ (take $r = -1$).
2. $u_3 = g$ (take $r = 0$).

Remark 22. Since Γ_0 and Γ_1 are disjoints and thanks to the previous statements, the boundary value problem

$$\begin{cases} \gamma_{\alpha\beta}(u) = 0 \text{ on } \Omega, \\ u_2 = 0 \text{ on } \Gamma_0, \\ u_3 = \tilde{u} \text{ on } \Gamma_1. \end{cases} \tag{4.6}$$

is "well posed" in the Agmon, Douglis and Nirenberg sense. Recalling Theorem 3.5 and Remark 14, together with standard regularity theory for elliptic systems, it follows that u is of class C^∞ on $\Omega \cup \Gamma_0$ for any $\tilde{u}$ (either smooth or not). Consequently, up to a kernel of finite dimension composed of smooth functions belonging to $\mathcal{C}^\infty(\overline{\Omega})^3$ (and eventually up to a compatibility condition (to belong to the range of the operator which is a closed subspace of finite codimension), the space $\{v, \ \gamma_{\alpha\beta}(v) = 0 \text{ on } \Omega, \ v_n = 0 \text{ on } \Gamma_0\}$ is isomorphic with the space $\mathcal{C}^\infty(\Gamma_1)$. The previous statements can be rephrased as follows :

up to a finite dimensional space composed of smooth functions, the space $\{v,\ \gamma_{\alpha\beta}(v) = 0 \text{ on } \Omega,\ v_n = 0 \text{ on } \Gamma_0\}$ is isomorphic to the space of traces on Γ_1 :

$$\{\tilde{v} \in C^\infty(\Gamma_1)\}, \tag{4.7}$$

the isomorphism is obtained by solving (4.5).

In the sequel, we shall consider indifferently the functions v (defined up to an additive element of the kernel) or their traces $\tilde{v}$ on Γ_1.

4.2 The system of *membrane tensions*

Consider the *membrane tensions* system $\mathcal{T}$ of three equations with the three unknowns (T^{11}, T^{22}, T^{12}) :

$$\begin{cases} -T^{11}_{|1} - T^{21}_{|2} = f^1 \\ -T^{22}_{|2} - T^{21}_{|1} = f^2 \\ -b_{11}T^{11} - 2b_{12}T^{12} - b_{22}T^{22} = f^3. \end{cases} \tag{4.8}$$

It is apparent that the three unknowns play analagous roles. Concerning the equations, it is clear that the first and the second are similar but different from the third. Therefore, we consider $(1,1,0)$ as indices of equations and $(0,0,0)$ as indices of unknowns. The principal system $\mathcal{T}_P$ is obtained by replacing the covaraint differentiation $_{|\alpha}$ by the usual differentiation ∂_α (i.e. replacing $\Gamma^\lambda_{\alpha\beta}$ by zero). Proceeding as in the proof of Lemma 4.1, we obtain the following result.

Lemma 4.4. *Under the ellipticity assumption of the surface (2.16), the system $\mathcal{T}$ is elliptic of total order two.*

Remark 23. It is worthwhile to study the Cauchy problem for the membrane tension system (4.8). This is done exactly as in Section 4.1.2 for the rigidity system. We eliminate one of the unknowns, T^{11} for instance and (4.8) reduces to an elliptic system of two first order equations in T^{12} and T^{22}. The Cauchy conditions are $T^{12} = T^{22} = 0$ on a part of the boundary. According to our special frame, this amounts to $T^{\alpha\beta}n_\beta = 0$. This Cauchy problem enjoys uniqueness but not existence and stability in usual spaces.

Remark 24. The system of membrane tensions $\mathcal{T}$ (4.8) and the system of rigidity γ (4.1) are adjoint to each other. This is easily checked by covariant integration by parts on S. Indeed, neglecting boundary terms (we are only interested in the equations) and using (2.1) together with the symmetry of the $T^{\alpha\beta}$), we have :

$$\begin{aligned} \int_S T^{\alpha\beta}\gamma_{\alpha\beta}(u)\ \mathrm{d}s &= \int_S T^{\alpha\beta}\Big(\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}u_3\Big)\ \mathrm{d}s \\ &= \int_S T^{\alpha\beta}\Big(u_{\alpha|\beta} - b_{\alpha\beta}u_3\Big)\ \mathrm{d}s \\ &= -\int_S \Big(T^{\alpha\beta}_{|\beta}u_\alpha + T^{\alpha\beta}b_{\alpha\beta}u_3\Big)\ \mathrm{d}s \\ &= \int_S \mathcal{T}(T)u\ \mathrm{d}s \end{aligned}$$

4.3 The *membrane* system

We denote by *membrane* system the system of three equations with three unknowns $u = (u_1, u_2, u_3)$ obtained from (4.8) when the tensions are written in terms of u, i.e.

$$\begin{cases} -T^{11}_{|1}(u) - T^{21}_{|2}(u) = f^1 \\ -T^{22}_{|2}(u) - T^{21}_{|1}(u) = f^2 \\ -b_{11}T^{11}(u) - 2b_{12}T^{12}(u) - b_{22}T^{22}(u) = f^3, \end{cases} \tag{4.9}$$

with

$$T^{\alpha\beta}(u) = A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u), \tag{4.10}$$

and

$$T^{\alpha\beta}_{|k}(u) = \partial_k T^{\alpha\beta}(u) + \Gamma^{\beta}_{kn}T^{\alpha n}(u) + \Gamma^{\alpha}_{km}T^{\beta m}(u). \tag{4.11}$$

In order to prove the ellipticity of the *membrane* system, we replace it by another, equivalent one. Indeed, we shall take as unknowns u_1, u_2, u_3 and the supplementary auxiliary unknowns T^{11}, T^{22}, T^{12}. Inverting the matrix $A^{\alpha\beta\lambda\mu}$ in (4.10) and recalling the definition of γ, we obtain the following equivalent system :

$$\begin{cases} -T^{11}_{|1} - T^{21}_{|2} = f^1 \\ -T^{22}_{|2} - T^{21}_{|1} = f^2 \\ -b_{11}T^{11} - 2b_{12}T^{12} - b_{22}T^{22} = f^3, \end{cases} \tag{4.12}$$

$$\begin{cases} u_{1|1} - b_{11}u_3 - C_{11\alpha\beta}T^{\alpha\beta} = 0 \\ u_{2|2} - b_{22}u_3 - C_{22\alpha\beta}T^{\alpha\beta} = 0 \\ \frac{1}{2}(u_{1|2} + u_{2|1}) - b_{12}u_3 - C_{12\alpha\beta}T^{\alpha\beta} = 0, \end{cases} \tag{4.13}$$

where $C_{\alpha\beta\lambda\mu}$ are the *compliances* (inverse matrix of $A^{\alpha\beta\lambda\mu}$). The system (4.12) and (4.13) is a system of six equations with the six unknowns $(T^{11}, T^{22}, T^{12}, u_1, u_2, u_3)$ (written in this order). We recognize the *membrane tension* system in (4.12) and the *rigidity* system in (4.13). Consider $(1,1,0,0,0,0)$ as indices of equations and $(0,0,0,1,1,0)$ as indices of unknowns. Then replacing the differentiation ∂_α by $-i\xi_\alpha$ and taking the determinant of the obtained system, we have a determinant of the form

$$\begin{vmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{vmatrix} = 0 = \mid D_{11} \mid \mid D_{22} \mid,$$

where the $D_{\alpha\beta}$ are 3×3 matrices. Moreover, D_{11} and D_{22} are precisely those of the *membrane tension* system and the *rigidity* system respectively and ellipticity follows. The same result is obviously obtained without using the auxiliary unknowns $T^{\alpha\beta}$, in fact, we have,

Lemma 4.5. *Under the ellipticity assumption of the surface (2.16), the membrane system with indices (of unknowns and of equations)* $(1,1,0)$ $(1,1,0)$ *is elliptic of total order four.*

Let us now state boundary value problems which will be considered later on. It is to be noticed that only two boundary conditions are considered on Γ_0.

Proposition 4.6. *The boundary value problem*

$$\left\{\begin{array}{l} -\partial_1 T^{11}(u) - \partial_2 T^{21}(u) = f^1 \\ -\partial_2 T^{22}(u) - \partial_1 T^{21}(u) = f^2 \\ -b_{11}T^{11}(u) - 2b_{12}T^{12}(u) - b_{22}T^{22}(u) = f^3 \\ u_1 = u_2 = 0, \text{ on } \Gamma_0 \\ T^{\alpha\beta}(u)n_\alpha = 0 \text{ on } \Gamma_1,\ \beta \in \{1,2\}. \end{array}\right. \tag{4.14}$$

with unknown u *satisfies the SL condition on* Γ_0 *but it does not on* Γ_1.

Remark 25. The partial differential boundary value problem (4.14) is formally associated with the variational problem (2.17) when $\varepsilon = 0$.

Démonstration. Let us fix $x_0 \in \Gamma$. According to the definition of the SL condition, we consider the homogeneous system with constant coefficients in which we only kept the principal terms, i.e. taking $\Gamma^\lambda_{\alpha\beta} = 0$ but $b_{\alpha\beta} \neq 0$ and $f^i = 0$.

After a change of coordinates with local mappings, still denoted by (x_1, x_2), we only have to consider solutions, which are exponentially decreasing in the direction inwards the domain (x_2), of the corresponding boundary value problem obtained by formal tangential Fourier transform. Denoting by $\tilde{u}(\xi_1, x_2)$ such a solution, by periodicity, we can restrict the domain to the strip $B = (0, 2\pi/|\xi_1|) \times (0, +\infty)$ and we can consider the function

$$v(x_1, x_2) = e^{i\xi_1 x_1}\tilde{u}(\xi_1, x_2), \tag{4.15}$$

which is periodic in the tangential direction x_1, decreasing as $x_2 \to +\infty$ and satisfies the homogeneous boundary condition associated with the principal part of (4.14). Recall that v satisfies the equation

$$\left\{\begin{array}{l} -\partial_1 T^{11}(v) - \partial_2 T^{21}(v) = 0 \\ -\partial_2 T^{22}(v) - \partial_1 T^{21}(v) = 0 \\ -b_{11}T^{11}(v) - 2b_{12}T^{12}(v) - b_{22}T^{22}(v) = 0. \end{array}\right. \tag{4.16}$$

We multiply each line of (4.16) by the conjugate $\overline{v}_i$ and we integrate by parts on the periodicity layer B. We see that on the infinite boundary the boundary integral is vanishing thanks to the decreasing condition as $x_2 \to +\infty$. The boundary integral also vanishes on the lateral boundary (which is parallel to x_2) of the strip thanks to the periodicity of v. Recalling the definition of T^{ij}, we obtain

$$\int_B A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(v)\gamma_{\alpha\beta}(\overline{v})\ \mathrm{d}x_1\ \mathrm{d}x_2 = 0, \tag{4.17}$$

where obviously all the $\Gamma^\alpha_{\beta\gamma} = 0$. Consequently, recalling the positivity property (2.10) of A, this yields that

$$\int_B \Sigma_{\alpha\beta}|\gamma_{\alpha\beta}(\overline{v})|^2\ \mathrm{d}x_1\ \mathrm{d}x_2 = 0, \tag{4.18}$$

and then

$$\gamma_{\alpha\beta}(v) = 0 \text{ on } B. \tag{4.19}$$

We have now to distinguish two cases.

If $x_0 \in \Gamma_0$, then reasonning as in Lemma 4.2 (or merely as in Lemma 4.3), we deduce that $v_1 = v_2 = v_3 = 0$, which means that the SL condition is satisfied on Γ_0.

Let now $x_0 \in \Gamma_1$ and

$$\gamma_{\alpha\beta}(v) = 0 \text{ on } B. \tag{4.20}$$

Remembering the definition (4.15) of v, this yields that $\tilde{u}$ is a solution of the following system of ODE of order 2 :

$$\left\{ \begin{array}{l} i\xi_1 \tilde{u}_1 - b_{11}\tilde{u}_3 = 0 \\ \partial_2 \tilde{u}_2 - b_{22}\tilde{u}_3 = 0 \\ \frac{1}{2}(\partial_2 \tilde{u}_1 + i\xi_1 \tilde{u}_2) - b_{12}\tilde{u}_3 = 0. \end{array} \right. \tag{4.21}$$

Thanks to the fact that $b_{11} \neq 0$ and $b_{22} \neq 0$ this can be rewritten as :

$$\left\{ \begin{array}{l} \tilde{u}_1 = -i\frac{b_{11}}{\xi_1}\tilde{u}_3 \\ \tilde{u}_3 = \frac{1}{b_{22}}\partial_2 \tilde{u}_2 \\ b_{11}\partial_2^2 \tilde{u}_2 - 2ib_{12}\xi_1 \partial_2 \tilde{u}_2 - b_{22}\xi_1^2 \tilde{u}_2 = 0. \end{array} \right.$$

Recalling the ellipticity condition (2.16), we obtain after an easy computation that there exists a complex solution $\tilde{u}$, given by $\tilde{u} = we^{\lambda_- x_2}$, where $w \neq 0$ and λ_- is the root with negative real part of

$$b_{11}\lambda^2 - 2ib_{12}\xi_1\lambda - b_{22}\xi_1^2 = 0.$$

This means that there exists non zero v which is exponentially decreasing in the direction inwards the domain

$$v(\xi_1, x_2) = we^{i\xi_1 x_1} e^{\lambda_- x_2},$$

with $\mathrm{Re}(\lambda_-) < 0$ such that

$$\gamma_{\alpha\beta}(v) = 0 \text{ on } B,$$

and hence

$$T^{\alpha\beta}(v)n_\alpha = 0 \text{ on } \Gamma_1.$$

Therefore, the SL condition is not satisfied on Γ_1. □

4.4 The *Koiter shell* system

The boundary value problem associated with the variational problem (2.17) with $\varepsilon > 0$ is classical and well-posed (see for instance [4], [17]). It is elliptic of total order 8, and the boundary conditions satisfy the SL condition. The system of equations is obtained by integration by parts, which yields :

$$\left\{ \begin{array}{l} -T^{\alpha\gamma}_{|\alpha}(u) + \varepsilon^2 b^\gamma_\beta M^{\alpha\beta}_{|\alpha}(u) + \varepsilon^2 \left(b^\gamma_\alpha M^{\alpha\beta}(u) \right)_{|\beta} = f^\gamma \\ -b_{\alpha\beta}T^{\alpha\beta}(u) - \varepsilon^2 M^{\alpha\beta}(u)_{|\alpha\beta} + \varepsilon^2 b^\gamma_\alpha b_{\gamma\beta} M^{\alpha\beta}(u) = f^3, \end{array} \right. \tag{4.22}$$

where the flection moments $M^{\alpha\beta}$ were defined in (2.14), (2.15) The boundary conditions on Γ_0 (supposed clamped) are :

$$u_1 = u_2 = u_3 = \partial_n u_3 = 0 \text{ on } \Gamma_0 \tag{4.23}$$

while the *natural* boundary conditions on Γ_1 are in number of four, are not relevant (they are boundary terms obtained by integration by parts). We have :

Proposition 4.7. *The boundary value problem associated with the variational problem (2.17) when $\varepsilon > 0$ considered as a system of three equations with the unknowns u is elliptic of total order 8 with indices $(1, 1, 2)$ for the unknowns and the equations.*

5 A sensitive singular perturbation problem arising in the Koiter linear shell theory

Very few is known concerning elliptic problems with boundary conditions not satisfying the SL condition and there is no general theory concerning them to our knowledge. Linear shell theory is one physical theory where they are naturally involved.

5.1 Definition of the problem

Let us first recall the variational problem (2.17) we are interested in :

$$\left\{ \begin{array}{l} \text{Find } u^\varepsilon \in V \text{ such that, } \forall v \in V \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v \rangle, \end{array} \right. \tag{5.1}$$

where $f \in V'$ is given, the brackets denote the duality between V' and V. More precisely, we consider the limit boundary partial differential system associated with (5.1) when $\varepsilon = 0$. This is the membrane system, which according to proposition 4.6, is elliptic, satisfies the SL on Γ_0 but does not on Γ_1.

5.2 Sensitive character

Let us now recall the definition of sensitive problem. For a more complete description, see [7] and [15]. Let us comment a little on proposition 4.6.

The SL condition is not satisfied on a free boundary when $\varepsilon = 0$ for the variational problem (5.1). Specifically, the membrane problem is of total order four for elliptic surfaces. The number of boundary conditions should be two. On a fixed boundary Γ_0 they are :

$$u_1 = u_2 = 0. \tag{5.2}$$

Note that the trace of u_3 does not make sense in the membane framework. The previous boundary conditions satisfy the SL condition. Oppositely, on the free boundary Γ_1 the conditions are :

$$T^{\alpha\beta}(u) n_\beta = 0. \tag{5.3}$$

Let us admit that (4.14) has (in some sense) a solution u. Replacing it in the three equations (4.14) and in the boundary conditions on Γ_1 of (4.14), one obtains that the corresponding $T^{\alpha\beta}(u)$ satisfy the elliptic membrane tensions system with Cauchy conditions on the part of Γ_1 of the boundary. As this last problem has in general no solution in usual spaces, it follows that the membrane problem (4.14) cannot (in general) have solution in usual spaces. We shall see that existence of the solution (as well as the convergence for $\varepsilon \to 0$) only holds in very abstract spaces (out of the distribution space).

On the other hand, the boundary condition (5.2) constitutes the Cauchy condition for the rigidity system $\gamma_{\alpha\beta}(u) = 0$. According to the uniqueness theorem for elliptic Cauchy problem (see proof of Lemma 4.2) an elliptic shell is inhibited (or geometrically rigid) provided that it is fixed (or clamped) on a part (or the whole) of the boundary. When the boundary is everywhere free, the shell is not inhibited. Coming back to the inhibited elliptic shells, we see that when the whole boundary is fixed, the membrane problem is classical (the boundary condition satisfies the SL condition). But, when a part of the boundary Γ_0 is fixed whereas another one Γ_1 is not, the boundary conditions satisfy the SL condition on Γ_0 but not on Γ_1. This problem is out of the classical theory of elliptic boundary value problems and is called sensitive for reason which will be self evident later.

Let us consider formally the variationnal formulation of the membrane problem (4.14) (i.e. with $\varepsilon = 0$) :

$$\left\{ \begin{array}{l} \text{Find } u \in V_a \text{ such that, } \forall v \in V_a \\ a(u,v) = \langle f, v \rangle, \end{array} \right. \tag{5.4}$$

where V_a is the completion of the "Koiter space" V with the norm $\|v\|_a = a(v,v)^{1/2}$.

The fact that $\|v\|_a$ is a norm on V follows from lemma 4.2.

At the present state, it should be noticed that the previous completion process is somewhat abstract and the elements of V_a are not necessarly distributions. Indeed, as the SL condition is not satisfied on Γ_1, we may construct corresponding solutions with $u \neq 0$ and $\gamma_{\alpha\beta}(u) = 0$ which are rapidly oscillating along Γ_1 and exponentially decreasing inwards Ω. This is only concerned with the higher order terms. When taking into account lower order terms (which are "small" for rapidly oscillating solutions), we see that we may have "large u" with "small $\gamma_{\alpha\beta}(u)$" (i.e. small $\Sigma_{\alpha,\beta}\|\gamma_{\alpha\beta}(u)\|_{L^2}$) and then small membrane energy. Accordingly, the dual V_a' where f must be taken for (5.4) to make sense is "very small".

The above property originates the term "sensitive". The problem is unstable. Very small and smooth variations of f (even in $\mathcal{D}(\Omega)$) induce modifications of the solution which are large and singular (out of the distribution space).

5.3 Abstract convergence results as $\varepsilon \to 0$

In this section we recall some abstract convergence results (in the norm of the specified spaces), see [3] and [7] for more details.

Recalling the problem we are studying, we know that the shell is geometrically rigid :

$$v \in V \text{ and } \gamma_{\alpha\beta}(v) = 0 \Longrightarrow v = 0. \tag{5.5}$$

Let A and B be the continuous operators from V into V' associated with the forms a and b by :

$$\langle Au, v\rangle = a(u, v) \text{ and } \langle Bu, v\rangle = b(u, v) \ \forall u, v \in V, \tag{5.6}$$

so that equation (5.1) becomes :

$$Au^\varepsilon + \varepsilon^2 Bu^\varepsilon = f. \tag{5.7}$$

Lemma 5.1. *The operator A is injective and its range, $\mathcal{R}(A)$, is dense in V'.*

The proof is not difficult, see [7] if necessary.

It then appears that the operator A is a one-to-one mapping of V onto $\mathcal{R}(A)$, which is a dense subset of V'. Let us define a new norm by

$$\|v\|_{V_A} = \|Av\|_{V'}. \tag{5.8}$$

Obviously V is not complete for the previous norm. But A defines an isomorphism between V (with the norm V_A) and $\mathcal{R}(A)$ (with the norm V'). Automatically, A has an extension by continuity which is an isomorphism between the completions of both spaces. Denoting by $\overline{A}$ the extended operator and by V_A the completion of V with the norm (5.8), $\overline{A}$ is an isomorphism between V_A and V' (which is the completion of $\mathcal{R}(A)$ with the norm of V'). Equation (5.7) may be written as well :

$$\overline{A}u^\varepsilon + \varepsilon^2 Bu^\varepsilon = f. \tag{5.9}$$

Remark 26. In order to pass to the limit as $\varepsilon \to 0$, the classical way consists in obtaining an a priori energy estimate of u^ε by taking the duality product of (5.9) with u^ε. But such a way needs a hypothesis of boundedness of the functional f with respect to the limit form a and this does not work for any $f \in V'$. In the general case, following an idea of Caillerie [3], see also [7], which consists in proving that the term $\varepsilon^2 Bu^\varepsilon$ tends to zero in V', one can pass this latter term to the right-hand side, and show that it tends to f in V'. Then using the fact that $\overline{A}$ is an isomorphism, it is possible to prove the existence of a limit of u^ε in V_A. Specifically we have the following result.

Theorem 5.2. *There exists a unique element u^0 in V_A such that*

$$\overline{A}u^0 = f. \tag{5.10}$$

Moreover the following strong convergence holds in V_A :

$$u^\varepsilon \to u^0 \text{ as } \varepsilon \to 0, \tag{5.11}$$

where $u^\varepsilon \in V$ is the solution of (5.9).

The proof, which follows the trends outlined above, may be seen in [7].

Remark 27. It should be emphasised that theorem 5.2 holds true without special hypothesis on f (besides the obvious one $f \in V'$). The limit $u^0 \in V_A$ is the solution of the abstract problem (5.4), which is not a variational one. The classical variational theory of the limit needs a supplementary hypothesis on f : there exists $C > 0$ such that

$$\|\langle f, v\rangle\| \leq C a(v,v)^{1/2}, \quad \forall v \in V, \tag{5.12}$$

which is very restrictive in shell theory.

For the sake of completness, let us give the elements of the classical limit theory under the assumption (5.12).

We first note that in such a case, $a(v,v)^{1/2}$ defines a norm on V. Let V_a be the completion of V with respect to that norm (which should not be confused with V_A). We then note that (5.12) shows that f may be extended by continuity to an element of V_a'. We shall denote this extension by f again. Obviously, the variational problem

$$\begin{cases} \text{Find } u^0 \in V_a \text{ such that, } \forall v \in V_a \\ a(u^0, v) = \langle f, v\rangle, \end{cases} \tag{5.13}$$

is well posed and has a unique solution. We then have the classical convergence result (see [10] e.g. or even [18])

Theorem 5.3. *Under the assumption (5.12), we have*

$$u^\varepsilon \to u^0 \textit{ strongly in } V_a \textit{ as } \varepsilon \to 0, \tag{5.14}$$

where u^ε and u^0 are the solutions of (5.1) and (5.10) respectively.

Let us now briefly recall the non-inhibited case when (5.5) does not hold. In such a situation, there is a convergence result towards a limit with vanishing membrane energy. More precisely, we define the kernel G of a :

$$G = \{v \in V;\ \gamma_{\alpha\beta}(v) = 0\} = \{v \in V;\ a(v,v) = 0\}. \tag{5.15}$$

It is to be noticed that G is a Hilbert space with the norm of V. But as $a(v,v) = 0$ in G, we see that the norm of V in G is equivalent to $b(v,v)^{1/2}$. As a consequence, the problem

$$\begin{cases} \text{Find } v^0 \in G \text{ such that, } \forall w \in G \\ b(v^0, w) = \langle f, w\rangle, \end{cases} \tag{5.16}$$

is well posed and has a unique solution. Moreover, since the "limit form" a in (5.1) vanishes on G, it implies some kind of weakness in G. The solution will be very large and we should define a new scaling in order to have a finite limit, $v^\varepsilon = \varepsilon^2 u^\varepsilon$, (5.1) becomes

$$\begin{cases} \text{Find } v^\varepsilon \in V \text{ such that, } \forall w \in V \\ \varepsilon^{-2} a(v^\varepsilon, w) + b(v^\varepsilon, w) = \langle f, w\rangle, \end{cases} \tag{5.17}$$

we then have, see [16] e.g. for the proof

Theorem 5.4. *Under the assumption $G \neq \emptyset$,*

$$v^\varepsilon \to v^0 \textit{ strongly in } V, \tag{5.18}$$

where v^ε and v^0 are the solutions of (5.17) and (5.16) respectively.

6 Heuristic asymptotics in the previous problem

The aim of this section is the construction, in a heuristic way, of an approximate description of the solutions u^ε of the linear Koiter model for small values of ε. Indeed, coming back to the Koiter problem for $\varepsilon > 0$, in the sensitive case, the problem is not really to describe the limit problem (which in general has no solution in the distribution space; in particular the space V_A (see (5.8)) where there is always a limit, is not a distribution space), but rather to give a good description of the solution u^ε for very small values of ε. This we shall try to do. We shall see that heuristic considerations allow to construct a simplified model accounting for the main features of the problem.

To do so we shall use the heuristic procedure of [7]. In this latter article, we addressed a model problem including a variational structure, somewhat analogous to the problem studied here, but simpler, as concerning an equation instead of a system. It is shown that the limit problem contains in particular an elliptic Cauchy problem. This problem was handled in both a rigorous (very abstract) framework and using a heuristic procedure for exhibiting the structure of the solutions with very small ε. The main difference is that in the present work, we deal with systems instead of single equations.

We shall see that heuristic considerations involving minimization of energy allow us to reduce the problem to another on the boundary Γ_1. In that context, it is seen that the "pathological" operator A is represented by a smoothing operator S (i.e. sending any distribution to a $\mathcal{C}^\infty$ function), whereas the "classical" operator B is represented by a "classical" elliptic operator Q. Denoting by $s(x,\xi)$ and $q(x,\xi)$ the corresponding symbols (here x is the arc on Γ_1), s is likely exponentially decreasing for $\xi \to \infty$, whereas q is algebraically growing. The action of $S + \varepsilon^2 Q$ on test functions is given by :

$$(S + \varepsilon^2 Q)\theta(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{i\xi x}[s(x,\xi) + \varepsilon^2 q(x,\xi)]\tilde{\theta}(\xi)\ \mathrm{d}\xi. \quad (6.1)$$

It is then apparent that, when ε is small, operator S is significant only for bounded values of ξ, whereas $\varepsilon^2 Q$ describes the behavior for $\xi \to \infty$. If $|\xi| << log(1/\varepsilon)$, then the symbol of the operator $S + \varepsilon^2 Q$ is equal to $(1+o(1))s(x,\xi)$ and for $|\xi| >> log(1/\varepsilon)$, it is $(1+o(1))\varepsilon^2 q(x,\xi)$. The balance of S and $\varepsilon^2 Q$ is obtained for values of ξ such that :

$$|\xi| \sim log(1/\varepsilon). \quad (6.2)$$

This is the window of frequencies allowing a good description of the simultaneous influence of S and $\varepsilon^2 Q$, which is precisely our aim. Moreover, it is easily seen that the range of frequencies (6.2) is responsible for most of the contribution to the integral (6.1). This property is of great interest for the construction of the heuristic approximation. More precisely, the heuristics incorporate approximations for large $|\xi|$. This amounts to saying that only the most singular parts of the solutions are retained, or equivalently, that the approximate solutions are defined up to more regular terms. This is for instance the kind of approximation which is used in the construction of a parametrix. We also note that, as (6.2) involves "moderately large" values of $|\xi|$, the "general quality" of the approximation is not very good, as it is only accurate for very very small values of ε.

It should be noticed that numerical computations [2] carried out with very reliable software (including an adapted mesh procedure) for the Koiter problem with very small values of ε agree with the overall trends of our heuristic procedure. It appears that most of the deformation consists in very large deformations along Γ_1 exponentially decreasing inwards Ω (then in good agreement with the "local lack of uniqueness" implied by the non-satisfied SL condition). As ε decreases, the amplitude increases, whereas the wave length decreases very slowly, verifying fairly well (6.2). The paper [2] also contains numerical comparisons with the case when the shell is fixed all along its boundary, which is classical (as the SL condition is satisfied all along the boundary). The differences are drastic for small values of ε.

6.1 Introduction to the heuristic asymptotic

A first remark in the context described above is that sensitive problems may be considered as "intermediate" between "inhibited" and "non-inhibited". Indeed, "inhibited" means that $v \in V$ and $\gamma_{\alpha\beta}(v) = 0$ implies $v = 0$, whereas "non-inhibited" means that there are non vanishing elements v of V such that $\gamma_{\alpha\beta}(v) = 0$. Strictly speaking, sensitive problems enter in the class "inhibited", but there are non vanishing elements v of V with "very small" $\gamma_{\alpha\beta}(v)$.

In order to minimize the energy

$$a(v,v) + \varepsilon^2 b(v,v) - 2\langle f, v\rangle, \tag{6.3}$$

it is clear that we may proceed as in non-inhibited problems. The solution with small ε "avoids" the (larger) membrane energy a, so that roughly speaking, solutions for small ε should have $\gamma_{\alpha\beta}(v)$ vanishing or at least very small with respect to v.

Obviously, it is impossible to impose the four boundary conditions (4.23) on Γ_0 with the "exact" system $\gamma_{\alpha\beta}(v) = 0$ as they imply $v = 0$.

Nevertheless, we shall see in Section 6.2.1 that it is possible to construct functions satisfying the two boundary conditions $u_n = u_t = 0$ on Γ_0 with the "non exact" system $\gamma_{\alpha\beta}(v) = 0$ in the sense that $\gamma_{\alpha\beta}(v)$ will be "very small" (i.e. $\Sigma_{\alpha,\beta}\|\gamma_{\alpha\beta}(v)\|_{L^2}$ will be very small). This will imply a "membrane boundary layer" in the vicinity of Γ_0 involving the bilinear form a. To this end, we shall first construct a set of functions v with only one vanishing component on Γ_0. Choosing (for instance) the normal component, we define :

$$G^0 = \{v,\ \gamma_{\alpha\beta}(v) = 0 \text{ on } \Omega,\ v_2 = 0 \text{ on } \Gamma_0\}, \tag{6.4}$$

the regularity is not precised as we shall later take the completion, we may consider C^∞ functions for instance. It is to be noticed that v is a triplet of functions.

Recalling Remark 22, we know that up to a finite dimensional space composed of smooth functions, the space G^0 is isomorphic to the space of traces on Γ_1 :

$$\{w \in C^\infty(\Gamma_1)\} \tag{6.5}$$

the isomorphism is obtained by solving the problem :

$$\begin{cases} \gamma_{\alpha\beta}(\tilde{w}) = 0 \text{ on } \Omega, \\ \tilde{w}_2 = 0 \text{ on } \Gamma_0, \\ \tilde{w}_3 = w \text{ on } \Gamma_1. \end{cases} \tag{6.6}$$

In the sequel, when we will consider a function $\tilde{w} \in G^0$, we will consider a function of the equivalence class for the quotient operation described in Remark 22. Moreover, we shall consider indifferently the functions $\tilde{w}$ obtained after a quotient operation on $\overline{\Omega}$ (for the finite dimensional space) or their traces w on Γ_1.

Moreover, the conditions $u_3 = \partial_n u_3 = 0$ on Γ_0 of (4.23) will be satisfied with the help of a "flection sublayer" involving the bilinear form b ; its effect is not relevant (see Section 6.2.2).

According to the previous considerations, we shall consider the minimization problem on G^0 instead of on V. This modified problem obviously involves the a-energy and the $\varepsilon^2 b$-energy. A natural space for handling it should be the completion G of G^0 with the corresponding norm.

The fact that we may "neglect" the functions in the finite dimension space of smooth functions follows from the fact that we are interested in the singular part.

6.2 The boundary layer on Γ_0

Let $\tilde{w}$ be in G^0 (see (6.4)) and let $\varepsilon > 0$ be fixed. The aim of this section is to build a modified function $\tilde{w}^a$ of $\tilde{w}$ in a narrow boundary layer of Γ_0 in order to satisfy the supplementary boundary conditions $\tilde{w}_t = \tilde{w}_3 = \partial_n \tilde{w}_3 = 0$ on Γ_0.

The present problem is analogous to the "model problem" of [7] in the case of a singular perturbation, i.e. [7] Section 7.1.2. Indeed, the membrane problem is of total order 4 allowing 2 boundary conditions ($\tilde{w}_t = \tilde{w}_n = 0$) on Γ_0, whereas the complete Koiter shell problem is of order 8, allowing 4 boundary conditions (we shall add $\tilde{w}_3 = \partial_n \tilde{w}_3 = 0$) on Γ_0. It appears that the two first conditions ($\tilde{w}_t = \tilde{w}_n = 0$) may be obtained from elements of G^0 by modifying them on account of a "membrane layer" which relies on the membrane system, of thickness of order $\frac{1}{\log(1/\varepsilon)}$ on Γ_0, whereas an irrelevant boundary layer will be considered in Section 6.2.2.

6.2.1 The membrane boundary layer on Γ_0

In this subsection, we proceed to modify the element $\tilde{w}$ of G^0 in order to satisfy both conditions $u_1 = u_2 = 0$ on Γ_0.

Let $\tilde{\Gamma}_0$ be a neighborhood of Γ_0 in $\mathbb{R}^2$ disjoint with Γ_1 and sufficiently narrow to be described by the curvilinear coordinates $y_1 =$ arc of Γ_0 and $y_2 =$ distance along the normal to Γ_0. Let $(\psi_j(y_1))_{j\in J}$ be a partition of the unity associated with Γ_0 and let $\eta \in C^\infty(\mathbb{R}_+;\mathbb{R}_+)$ be a cut-off function equal to 1 for small values of y_2.

The mappings θ_j defined by $\theta_j(y_1,y_2) = \psi_j(y_1)\eta(y_2)$, where y_2 is the (inwards) normal coordinate along Γ_0, define a partition of unity in $\tilde{\Gamma}_0$; in particular, for a given $\tilde{w} \in G^0$, we have :

$$\forall (y_1,y_2) \in \tilde{\Gamma}_0,\ \tilde{w}(y_1,y_2) = \Sigma_{j\in J}\theta_j(y_1,y_2)\tilde{w}(y_1,y_2). \tag{6.7}$$

Let us now fix j in J and y_2 such that $(y_1,y_2) \in \tilde{\Gamma}_0$, the function $\theta_j(\cdot,y_2)\tilde{w}(\cdot,y_2)$ has a compact support, we denote by $\tilde{w}^j(\cdot,y_2)$ its extension by zero to $\mathbb{R}$ and by $\mathcal{F}(\tilde{w}^j)$ the tangential Fourier transform, $y_1 \to \xi_1$, of $\tilde{w}^j$.

Let us first exhibit the local structure of the Fourier transform of $\tilde{w}^j$ close to Γ_0. Denoting by θ_j the multiplication operator by θ_j, recalling that the commutator of the operator γ associated with $\gamma_{\alpha\beta}$ and θ_j, denoted by $[\gamma,\theta_j]$, is a differential operator of lower order, taking the γ operator in the new coordinates (y_1,y_2) (which, according to our approximation close to Γ_0, has the same principal part) and using that $\tilde{w} \in G^0$, we see that :

$$\gamma_{\alpha\beta}(\tilde{w}^j) + U_{\alpha\beta}(y,D)\tilde{w}^j = 0 \text{ on } \mathbb{R}\times(0,t), \tag{6.8}$$

for some $t > 0$, $U_{\alpha\beta}$ being differential operator of order less than the order of $\gamma_{\alpha\beta}$.

Now, according to the general trends of our boundary layer approximation, we can neglect the terms of lower order in (6.8) and we can proceed as in the construction of a parametrix (freezing coefficients, dropping lower order terms, solving such simpler equation via tangent Fourier transform and gluing together the solutions for different j), so that (6.8) becomes

$$\gamma_{\alpha\beta}(\tilde{w}^j) = 0 \text{ on } \mathbb{R}\times(0,t). \tag{6.9}$$

The previous system can be rewritten as

$$\begin{cases} \frac{\partial}{\partial y_1}\tilde{w}_1^j - b_{11}\tilde{w}_3^j = 0, \\ \frac{\partial}{\partial y_2}\tilde{w}_2^j - b_{22}\tilde{w}_3^j = 0, \\ \frac{1}{2}\Big(\frac{\partial}{\partial y_2}\tilde{w}_1^j + \frac{\partial}{\partial y_1}\tilde{w}_2^j\Big) - b_{12}\tilde{w}_3^j = 0, \end{cases} \tag{6.10}$$

and taking the tangential Fourier transform denoted by $\mathcal{F}(\tilde{w}^j)(\xi_1,y_2)$ this yields

$$\Big(\hat{\gamma}_0 + \tilde{\gamma}_1\frac{\mathrm{d}}{\mathrm{d}y_2}\Big)\mathcal{F}(\tilde{w}^j) = 0, \tag{6.11}$$

with

$$\hat{\gamma}_0 = \begin{pmatrix} -i\xi_1 & 0 & -b_{11} \\ 0 & 0 & -b_{22} \\ 0 & -i\xi_1 & -2b_{12} \end{pmatrix} \text{ and } \tilde{\gamma}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The general solution of the system (6.11) is :

$$\mathcal{F}(\tilde{w}^j)(\xi_1,y_2) = A\tilde{w}_+e^{\lambda_+(\xi_1)y_2} + B\tilde{w}_-e^{\lambda_-(\xi_1)y_2}, \tag{6.12}$$

with $\tilde{w}_+ = \begin{pmatrix} i\frac{\lambda_+(\xi_1)}{\xi_1}\frac{b_{11}}{b_{22}} \\ 1 \\ \frac{\lambda_+(\xi_1)}{b_{22}} \end{pmatrix}$, $\tilde{w}_- = \begin{pmatrix} i\frac{\lambda_-(\xi_1)}{\xi_1}\frac{b_{11}}{b_{22}} \\ 1 \\ \frac{\lambda_-(\xi_1)}{b_{22}} \end{pmatrix}$, $\lambda_+(\xi_1) = -i\xi_1\frac{b_{12}}{b_{11}} + \frac{|\xi_1|}{b_{11}}\sqrt{b_{11}b_{22} - b_{12}^2}$ and $\lambda_-(\xi_1) = -i\xi_1\frac{b_{12}}{b_{11}} - \frac{|\xi_1|}{b_{11}}\sqrt{b_{11}b_{22} - b_{12}^2}$.

Since $\tilde{w} \in G_0$, it follows that $\mathcal{F}(\tilde{w}_2^j)(\xi_1, 0) = 0$ hence $A = -B$. Consequently, we deduce that

$$\mathcal{F}(\tilde{w}^j)(\xi_1, y_2) = \frac{b_{11}b_{22}}{2|\xi_1|\sqrt{b_{11}b_{22} - b_{12}^2}}\mathcal{F}(\tilde{w}_3^j)(\xi_1, 0)\Big(\tilde{w}_+ e^{\lambda_+(\xi_1)y_2} - \tilde{w}_- e^{\lambda_-(\xi_1)y_2}\Big). \tag{6.13}$$

This expression exhibits the structure of (the Fourier transform of) $\tilde{w}$ in a narrow neigbourhood of Γ_0. It was expressed in terms of (the Fourier transform of) the trace of its third component on Γ_0, but this choice is arbitrary.

We now proceed to the modification of $\tilde{w}^j$ in $\tilde{w}^{ja}$ in a narrow boundary layer of Γ_0 in order to satisfy (always within our approximation) the equation coming from (4.22) for $\varepsilon = 0$ (this is the membrane boundary layer associated with the membrane system of Section 4.3). Using considerations similar to those leading to (6.8), this amounts to

$$\Big(\tilde{\gamma}^* \tilde{A}_1 \tilde{\gamma}\Big)\tilde{w}^{ja} + U(y, D)\tilde{w}^{ja} = 0 \text{ on } \mathbb{R} \times (0, t), \tag{6.14}$$

where U is a differential operator of lower order than four, $\tilde{\gamma}^*$ denotes the operator :

$$\tilde{\gamma}^* = \begin{pmatrix} \partial_1 & 0 & -b_{11} \\ 0 & \partial_2 & -b_{22} \\ \partial_2 & \partial_1 & -2b_{12} \end{pmatrix},$$

and

$$\tilde{A}_1 = \begin{pmatrix} A^{1111} & A^{1122} & A^{1112} \\ A^{2211} & A^{2222} & A^{2212} \\ A^{1211} & A^{1222} & A^{1212} \end{pmatrix}.$$

Therefore dropping as before terms of lower order, we have :

$$\Big(\tilde{\gamma}^* \tilde{A}_1 \tilde{\gamma}\Big)\tilde{w}^{ja} = 0 \text{ on } \mathbb{R} \times (0, t), \tag{6.15}$$

which can be rewritten as

$$\Big((\tilde{\gamma}_0^* - \tilde{\gamma}_1^*\partial_2)\tilde{A}_1(\tilde{\gamma}_0 + \tilde{\gamma}_1\partial_2)\Big)\tilde{w}^{ja} = 0 \text{ on } \mathbb{R} \times (0, t), \tag{6.16}$$

with $\tilde{\gamma}^* = \overline{\tilde{\gamma}}^T$ and

$$\tilde{\gamma}_0 = \begin{pmatrix} \partial_1 & 0 & -b_{11} \\ 0 & 0 & -b_{22} \\ 0 & \partial_1 & -2b_{12} \end{pmatrix}.$$

Hence taking the tangential Fourier transform, we look for solutions of the system :

$$\Big((\overline{\hat{\gamma}_0}^T - \tilde{\gamma}_1^T\frac{\mathrm{d}}{\mathrm{d}y_2})\tilde{A}_1(\hat{\gamma}_0 + \tilde{\gamma}_1\frac{\mathrm{d}}{\mathrm{d}y_2})\Big)\mathcal{F}\Big(\tilde{w}^{ja}\Big)(\xi_1, y_2) = 0, \tag{6.17}$$

with

$$\hat{\gamma}_0 = \begin{pmatrix} -i\xi_1 & 0 & -b_{11} \\ 0 & 0 & -b_{22} \\ 0 & -i\xi_1 & -2b_{12} \end{pmatrix}.$$

At this moment, it is worthwhile to compare (6.17) and (6.11). We see that the given function $\tilde{w}^j$ (rather its Fourier transform) solves the "right half" of (6.17), i.e. the expression on the right of $\tilde{A}_1$ in (6.17). Obviously, the "left half" accounts for the "adjoint part", coming with integration by parts from the bilinear form a (see (2.5)). Our aim in constructing the modified $\tilde{w}^{ja}$ is to satisfy the conditions $\tilde{w}_1^{ja} = \tilde{w}_2^{ja} = 0$ on $y_2 = 0$, whereas for "large y_2" (in the sense of "out of the layer") the modified $\tilde{w}^{ja}$ coincides (up to small terms) with the given $\tilde{w}^j$. We now proceed to write down the general solution of (6.17) on account of its special structure.

For $\lambda \in \{\lambda_-(\xi_1), \lambda_+(\xi_2)\}$, let us consider the function k defined by :

$$k(\xi_1, y_2) = (y_2 w + v)e^{\lambda y_2}, \tag{6.18}$$

where $w \in \{\tilde{w}_-, \tilde{w}_+\}$ is a solution of

$$\Big(\hat{\gamma}_0 + \lambda\tilde{\gamma}_1\Big)w = 0,$$

and v is unknown. We then search for solutions of (6.17) under the form (6.18) i.e. :

$$\Big((\overline{\hat{\gamma}_0}^T - \tilde{\gamma}_1^T \frac{\mathrm{d}}{\mathrm{d}y_2})\tilde{A}_1(\hat{\gamma}_0 + \tilde{\gamma}_1 \frac{\mathrm{d}}{\mathrm{d}y_2})\Big)k(\xi_1, y_2) = 0, \tag{6.19}$$

We check that

$$\Big(\hat{\gamma}_0 + \tilde{\gamma}_1 \frac{\mathrm{d}}{\mathrm{d}y_2}\Big)\Big(y_2 w + e^{\lambda y_2} v\Big) = \Big((\hat{\gamma}_0 + \lambda\tilde{\gamma}_1)v + \tilde{\gamma}_1 w\Big)e^{\lambda y_2}.$$

So that (6.19) becomes

$$\begin{aligned} \Big((\overline{\hat{\gamma}_0}^T - \tilde{\gamma}_1^T \frac{\mathrm{d}}{\mathrm{d}y_2})\tilde{A}_1(\hat{\gamma}_0 + \tilde{\gamma}_1 \frac{\mathrm{d}}{\mathrm{d}y_2})\Big)\Big(y_2 w + v\Big)e^{\lambda y_2} &= \\ (\overline{\hat{\gamma}_0}^T - \tilde{\gamma}_1^T \frac{\mathrm{d}}{\mathrm{d}y_2})\tilde{A}_1\Big((\hat{\gamma}_0 + \lambda\tilde{\gamma}_1)v + \tilde{\gamma}_1 w\Big)e^{\lambda y_2} &= 0. \end{aligned}$$

This amounts to saying that $\tilde{A}_1\Big((\hat{\gamma}_0 + \lambda\tilde{\gamma}_1)v + \tilde{\gamma}_1 w\Big)$ is an eigenvector of $\overline{\hat{\gamma}_0}^T - \lambda\tilde{\gamma}_1^T$ associated with the zero eigenvalue. Since dim Ker $\Big(\overline{\hat{\gamma}_0}^T - \lambda\tilde{\gamma}_1^T\Big) = 1$, denoting by u_0 a non vanishing vector of Ker $\Big(\overline{\hat{\gamma}_0}^T - \lambda\tilde{\gamma}_1^T\Big)$, then v should satisfy

$$(\hat{\gamma}_0 + \lambda\tilde{\gamma}_1)v + \tilde{\gamma}_1 w = \tilde{A}_1^{-1}(\tau u_0), \text{ for some } \tau \in \mathbb{C}. \tag{6.20}$$

According to the Fredholm alternative, a necessary and sufficient condition for the existence of such a v is that

$$\tilde{A}_1^{-1}(\tau u_0) - \tilde{\gamma}_1 w \ \in (\text{Vect } u_0)^{\perp}.$$

Since $\tilde{A}_1$ is positive definite, we deduce that $(\tilde{A}_1^{-1}u_0, u_0) > 0$ hence $\tau = \frac{(\tilde{\gamma}_1 u, u_0)}{(\tilde{A}_1^{-1}u_0, u_0)}$ satisfies

$$(\tau \tilde{A}_1^{-1}u_0 - \tilde{\gamma}_1 w, u_0) = 0.$$

It follows that the vector $v \in \mathbb{C}^3$ exists and is unique (up to an additive and arbitrary eigenvector, which is irrelevant in the sequel). Consequently, k defined as above satisfies (6.19).

Repeating this argument twice (first for $\lambda_+(\xi_1)$, and then for $\lambda_-(\xi_1)$), and denoting by v_+ and v_- the corresponding vectors v, we see that

$$\begin{aligned}\mathcal{F}\Big(\tilde{w}^{ja}\Big)(\xi_1, y_2) \quad &= \quad C_1 \tilde{w}_e^{-\lambda_+(\xi_1)y_2} + C_2 \tilde{w}_- e^{-\lambda_-(\xi_1)y_2} + C_3\Big(y_2\tilde{w}_+ + v_+\Big)e^{\lambda_+(\xi_1)y_2} \\ &\quad + C_4\Big(y_2\tilde{w}_- + v_-\Big)e^{\lambda_-(\xi_1)y_2},\end{aligned} \tag{6.21}$$

with arbitrary C_1, C_2, C_3, C_4 is the general solution of (6.17).

We are now determining C_1, C_2, C_3, C_4 in order to satisfy the boundary conditions $\tilde{w}_1^{ja} = \partial_2 \tilde{w}_1^{ja} = 0$ at $y_2 = 0$ and the "matching condition" with $\tilde{w}^j$, i.e. in the context of boundary layer theory (for large $|\xi_1|$), $\tilde{w}^{ja}$ should become $\tilde{w}^j$ out of the layer.

Let us now explain the process of matching the layer : out of the layer, we want $\tilde{w}^{ja}$ to match with the given function $\tilde{w}^j$. Since $|\xi_1| >> 1$, then $|\xi_1|y_2 >> 1$ and $\frac{\sqrt{b_{11}b_{22}-b_{12}^2}}{b_{11}}|\xi_1|y_2 >> 1$ which means that $y_2 >> \frac{b_{11}}{\sqrt{b_{11}b_{22}-b_{12}^2}}\frac{1}{|\xi_1|}$ (but we still impose that y_2 is small in order to be in a narrow layer of Γ_0 where (6.13) holds) ; this is perfectly consistent, as we will only use the functions for large $|\xi_1|$, hence the terms with coefficients C_2 and C_4 are "boundary layer terms" going to zero out of the layer (i.e. for $|y_2| >> \mathcal{O}\Big(\frac{1}{|\xi_1|}\Big)$).

The matching with (6.13) out of the layer then gives

$$C_3 = 0 \text{ and } C_1 = \frac{b_{11}b_{22}}{2|\xi_1|\sqrt{b_{11}b_{22} - b_{12}^2}}\mathcal{F}(\tilde{w}_3^j)(\xi_1, 0). \tag{6.22}$$

The two other constants C_2 and C_4 are determined by

$$\mathcal{F}(\tilde{w}^{ja})_1(\xi_1, 0) = 0 \text{ and } \mathcal{F}(\tilde{w}^{ja})_2(\xi_1, 0) = 0,$$

which yields the existence of two constants α and β such that

$$C_2 = \alpha C_1 \text{ and } C_4 = \beta C_1.$$

So that the modified solution is of the form :

$$\begin{aligned}\mathcal{F}(\tilde{w}^{ja})(\xi_1, y_2) \quad &= \quad \frac{b_{11}b_{22}}{2|\xi_1|\sqrt{b_{11}b_{22} - b_{12}^2}}\Big(\tilde{w}_+ e^{\lambda_+(\xi_1)y_2} \\ &\quad + ((\alpha + \beta y_2)\tilde{w}_- + \beta v_-)e^{\lambda_-(\xi_1)y_2}\Big)\mathcal{F}(\tilde{w}_3^j)(\xi_1, 0).\end{aligned} \tag{6.23}$$

The modification of the function $\tilde{w}_j$ then consists in adding to it the inverse Fourier transform of

$$\frac{b_{11}b_{22}}{2|\xi_1|\sqrt{b_{11}b_{22} - b_{12}^2}}\Big((\alpha + 1 + \beta y_2)\tilde{w}_- + \beta v_-\Big)e^{\lambda_-(\xi_1)y_2}\mathcal{F}(\tilde{w}_3^j)(\xi_1, 0). \tag{6.24}$$

We shall study in the sequel the behavior of such an expression. The role of the constants α and β is not relevant, and we may assume, for instance that $\alpha = -1$ and $\beta = 1$ (this amounts to change $\tilde{w}_-$ and $\tilde{v}_-$). As the result, the modification of the function $\tilde{w}_j$ consists in adding to it the inverse Fourier transform of

$$\frac{b_{11}b_{22}}{2|\xi_1|\sqrt{b_{11}b_{22}-b_{12}^2}}\Big(y_2\tilde{w}_- + v_-\Big)e^{\lambda_-(\xi_1)y_2}\mathcal{F}(\tilde{w}_3^j)(\xi_1, 0). \tag{6.25}$$

More precisely, on account of considerations at the beginning of Section 6 (see in particular (6.1) and (6.2)), the modification should only be effective for large $|\xi_1|$, accounting for "singular parts" of the solution. Moreover, in order to have $\tilde{w}^a \in V$, we shall also impose $\tilde{w}_1^{ja} = \partial_2\tilde{w}_1^{ja} = 0$ on Γ_0 (the other two conditions $\tilde{w}_3^a = \partial_n\tilde{w}_3^a = 0$ on Γ_0 will be adressed in Section 6.2.2). To this end, we multiply the added term by a cutoff function avoiding low frequencies (It should be remembered that this is one of the typical devices in the construction of a parametrix). More precisely, on account of (6.2), we shall only keep frequencies of order more or equal than $[log(1/\varepsilon)]^{1/2}$, which preserve the useful region (6.2) and are large (then consistent with the fact that the modification is a layer). Moreover, in order to the modified function satisfy the boundary conditions, we must also take into account the low frequencies of the Fourier transform which we multiply by a smooth vector $\rho(y_2)$ such that $\rho_1(0) = \rho_2(0) = 0$ and $\rho(y_2) = 0$ for $y_2 > C$ for a certain C. The division into high and low frequencies is defined by a smooth function $H(z)$ equal to 1 for $|z| > 1$ and vanishing for $|z| < 1/2$, with $z = \frac{\xi}{[log(1/\varepsilon)]^{1/2}}$. Finally, we define the function

$$\begin{aligned} h(\varepsilon, \xi, y_2) = (1 - H(\frac{\xi_1}{[log(1/\varepsilon)]^{1/2}}))\rho(y_2) + \\ \frac{b_{11}b_{22}}{2|\xi_1|\sqrt{b_{11}b_{22}-b_{12}^2}}\Big(y_2\tilde{w}_- + v_-\Big)e^{\lambda_-(\xi_1)y_2}H(\frac{\xi}{[log(1/\varepsilon)]^{1/2}}), \end{aligned} \tag{6.26}$$

which obviously has its first and second components vanishing for $y_2 = 0$. Now we can modify the function $\tilde{w}_j$ by

$$\delta\tilde{w}_j \equiv \tilde{w}_j^a - \tilde{w}_j, \tag{6.27}$$

where $\delta\tilde{w}_j$ is defined by its Fourier transform :

$$\mathcal{F}\Big(\delta\tilde{w}_j\Big) = \mathcal{F}(\tilde{w}_3^j)(\xi_1, 0)h(\varepsilon, \xi, y_2). \tag{6.28}$$

Remark 28. The constant C in the definition of $\rho(y_2)$ is chosen sufficiently small for this function to vanish out of the layer of Ω close to Γ_0 where the curvilinear coordinates y_1, y_2 operate. Rigorously speaking, the rest of the expression should also be multiplied by a cut-off function vanishing for $y_2 > C$, but this is practically not necessary, as this part is exponentially small for large $|\xi_1|$.

Hence summing over j and defining on Γ_0 the family (with parameter y_2) of pseudo-differential smoothing operators $\delta\sigma(\varepsilon, D_1, y_2)$ with symbol :

$$\delta\sigma(\varepsilon,\xi_1,y_2) = \frac{|b_{11}|b_{22}}{2|\xi_1|\sqrt{b_{11}b_{22}-b_{12}^2}}\Big(y_2\tilde{w}_- + v_-\Big)e^{\lambda_-(\xi_1)y_2}, \tag{6.29}$$

we see that the modification of the function $\tilde{w}$:

$$\delta\tilde{w} = \tilde{w}^a - \tilde{w} \tag{6.30}$$

is precisely the action of $\delta\sigma(\varepsilon, D_1, y_2)$ on $\tilde{w}_3^j(y_1,0)$.

Once $\tilde{w}^a$ is constructed, it is worthwhile computing its a-energy. This we proceed to do. More generally, we shall compute the form a for two functions $\tilde{v}^a$ and $\tilde{w}^a$.

Let us now compute the leading terms of the a-energy of the modified function $\tilde{w}^a$.

Let $\tilde{v}$ and $\tilde{w}$ be two elements in G^0 and $\tilde{v}^a, \tilde{w}^a$ the corresponding elements modified in the boundary layer. As the given $\tilde{v}$ and $\tilde{w}$ satisfy $\gamma_{\alpha\beta}(\tilde{v}) = \gamma_{\alpha\beta}(\tilde{w}) = 0$, the a-form is only concerned with the modification terms $\delta\tilde{v}$ and $\delta\tilde{w}$. Then, within our approximation, we have :

$$a(\tilde{v}^a,\tilde{w}^a) = \int_{\Gamma_0} A^{\alpha\beta\lambda\mu} dy_1 \int_0^{+\infty} \gamma_{\alpha\beta}(\delta\tilde{v})\overline{\gamma_{\lambda\mu}(\delta\tilde{w})}\ \mathrm{d}y_2. \tag{6.31}$$

where the integral in $\mathrm{d}y_2$ is only effective in the narrow layer. Using the partition of the unity θ_j and denoting as before by $\delta w_j(\cdot, y_2)$ the extension with value 0 to $\mathbb{R}$ of $\theta_j(\cdot,y_2)\delta w(\cdot,y_2)$, we have

$$a(\tilde{v}^a,\tilde{w}^a) = \Sigma_{j,k}\int_{\Gamma_0} A^{\alpha\beta\lambda\mu} dy_1 \int_0^{+\infty} \gamma_{\alpha\beta}(\delta\tilde{v}_j)\overline{\gamma_{\lambda\mu}(\delta\tilde{w}_k)}\ \mathrm{d}y_2. \tag{6.32}$$

Consequently, using the tangential Fourier transform $y_1 \to \xi_1$ and the Parceval-Plancherel theorem, dropping lower order terms (within our approximation, we only consider expressions with large $|\xi_1|$ which amounts to take $H = 1$ in (6.26)), we deduce that

$$\begin{aligned}
&a(\tilde{v}^a,\tilde{w}^a) = \\
&\Sigma_{j,k}\int_{-\infty}^{+\infty}\tilde{A}_1\ \mathrm{d}\xi_1\int_0^{+\infty}\Big(\hat{\gamma}_0+\tilde{\gamma}_1\frac{\mathrm{d}}{\mathrm{d}y_2}\Big)\delta\sigma(\varepsilon,\xi,y_2)\mathcal{F}(\tilde{v}_3^j)(\xi_1,0)\Big)\times \\
&\overline{\Big(\hat{\gamma}_0+\tilde{\gamma}_1\frac{\mathrm{d}}{\mathrm{d}y_2}\Big)\delta\sigma(\varepsilon,\xi,y_2)\mathcal{F}(\tilde{w}_3^k)(\xi_1,0)\Big)}\ \mathrm{d}y_2 = \\
&\Sigma_{j,k}\int_{-\infty}^{+\infty}\tilde{A}_1\ \mathrm{d}\xi_1\int_0^{+\infty}\frac{b_{11}b_{22}}{2|\xi_1|\sqrt{b_{11}b_{22}-b_{12}^2}}\Big((\hat{\gamma}_0+\lambda_-\tilde{\gamma}_1)v_- + \tilde{\gamma}_1\tilde{w}_-\Big)e^{\lambda_- y_2}\mathcal{F}(\tilde{v}_3^j)(\xi_1,0)\Big)\times \\
&\overline{\frac{b_{11}b_{22}}{2|\xi_1|\sqrt{b_{11}b_{22}-b_{12}^2}}\Big((\hat{\gamma}_0+\lambda_-\tilde{\gamma}_1)v_- + \tilde{\gamma}_1\tilde{w}_-\Big)e^{\lambda_- y_2}\mathcal{F}(\tilde{w}_3^k)(\xi_1,0)\Big)}\ \mathrm{d}y_2
\end{aligned}$$

Hence, on account of the definitions of $\hat{\gamma}_0$, $\tilde{\gamma}_1$, λ_- and $\tilde{w}_-$ integrating in y_2, we know that

$$a(\tilde{v}^a,\tilde{w}^a) = \Sigma_{j,k}\int_{-\infty}^{+\infty}\theta|\xi_1|\mathcal{F}(\tilde{v}^j)_{3|y_2=0}\overline{\mathcal{F}(\tilde{w}^k)_{3|y_2=0}}h^2(\varepsilon,\xi,y_2)\ \mathrm{d}\xi_1, \tag{6.33}$$

with $\theta = \theta(A^{\alpha\beta\lambda\mu}, (v_-)_1(0), b_{\alpha\beta}, \mu_-)$, where $\mu_- = \frac{\lambda_(\xi_1)}{|\xi_1|}$ is independent of ξ_1.

This expression (6.33) only depends on the trace $(\tilde{v}^j)_{3|y_2=0}(y_1)$ and $(\tilde{w}^k)_{3|y_2=0}(y_1)$, which are functions defined on Γ_0.

Remark 29. The important fact in (6.33) is the presence of $|\xi_1|$. This comes from $\int_0^{+\infty} e^{-\lambda_- y_2}\,\mathrm{d}y_2$ and analogous, on account that this integral is equal to $\frac{C}{|\xi_1|}$.

We now simplify this last expression using a sesquilinear form involving pseudo-differential operators.

Then, defining the elliptic pseudo-differential operator $P(y_1, \frac{\partial}{\partial y_1})$ of order 1/2 with principal symbol

$$(\theta|\xi_1|)^{1/2} h(\varepsilon, \xi, y_2), \tag{6.34}$$

and summing over j and k, we obtain

$$a(\tilde{v}^a, \tilde{w}^a) = \int_{\Gamma_0} P(\frac{\partial}{\partial s})(\tilde{v}_3)_{|\Gamma_0} \overline{P(\frac{\partial}{\partial s})(\tilde{w}_3)_{|\Gamma_0}}\,\mathrm{d}s. \tag{6.35}$$

Remark 30. As we only considered the principal terms for large $|\xi_1|$, we may define as well $P(\xi_1)$ by the symbol

$$P(\xi_1) = \theta(1 + |\xi_1|^2)^{1/4}. \tag{6.36}$$

The corresponding pseudo-differential operator is elliptic of order 1/2.

Remark 31. We shall use the definition (6.36), which is more pleasant than (6.34), as such a P defines an isomorphism from $H^s(\Gamma_0)$ onto $H^{s+1/2}(\Gamma_0)$, $s \in \mathbb{R}$.

6.2.2 The flection sublayer on Γ_0

The structure of the flection sublayer, see the beginning of Section 6, accounting for the two new boundary conditions $\tilde{w}_3 = \partial_n \tilde{w}_3 = 0$ follows from classical issues in singular perturbation theory, as in [7] section 7.1.2, [22] and [9]. It is mainly concerned with a drastic change of the normal component $\tilde{w}_3$ (whereas the conditions on $\tilde{w}_1$ and $\tilde{w}_2$ are satisfied). The specific structure is analogous to the layer in [19].

The thickness is of order $\delta = \varepsilon^{1/2}$. This may be easily seen by taking into account only higher order terms in the membrane and the flection systems ; eliminating $\tilde{w}_1$ and $\tilde{w}_2$, we obtain an equation for $\tilde{w}_3$. The membrane terms are of order 4 and the flection terms are of order 8. In the layer, the derivatives of order n have an order of magnitude $\mathcal{O}(\frac{\tilde{w}_3}{\delta^n})$. As both membrane and flection terms are of the same order of magnitude in the layer, we thus have

$$\mathcal{O}(\frac{\tilde{w}_3}{\delta^4}) = \varepsilon^2 \mathcal{O}(\frac{\tilde{w}_3}{\delta^8}),$$

which furnishes $\delta = \mathcal{O}(\varepsilon^{1/2})$.

It is easily seen (as in [7] section 7.1.2) that the presence of this flection sublayer plays a negligible role in the asymptotic behavior.

Indeed, proceeding as in the previous membrane layer, we see that the expression analogous to (6.35) has the form :

$$\varepsilon^2 a_0(\tilde{v}^a, \tilde{w}^a) = \varepsilon^2 \int_{\Gamma_0} P_0(\frac{\partial}{\partial s})(\tilde{v})_{|\Gamma_0} \overline{P_0(\frac{\partial}{\partial s})(\tilde{w})_{|\Gamma_0}} \, \mathrm{d}s, \tag{6.37}$$

where P_0 is an operator of order 0. Going on to next Section 6.4, the action of sublayer amounts to change $\mathcal{A}$ to $\mathcal{A} + \varepsilon^2 \mathcal{C}$ where $\mathcal{C}$ is a smoothing operator. Equivalently, we may change $\mathcal{B}$ to $\mathcal{B} + \mathcal{C}$ which is again a 3-order operator (as $\mathcal{C}$ is smoothing). The asymptotic behavior does not change. Equivalently, in (6.1), the effect of the sublayer is to change s to $s + \varepsilon^2 s_0$ where s_0 is a smoothing symbol, or q to $q + s_0$ which is again the symbol of an operator of order $2m > 0$.

For that reasons, the influence of the sublayer will no more be mentioned.

6.3 Formulation of the problem in the heuristic asymptotics

Presently, our aim is to formulate problem (5.1) on the space of the u^a with $u \in G^0$. The forms $b(u, v)$ and $\langle f, v \rangle$ should be written in the framework of our formal asymptotics, for $\tilde{u}^a$ and $\tilde{v}^a$ obtained from u and v defined on Γ_1 by solving (6.6) and modifying $\tilde{u}$ and $\tilde{v}$ with the Γ_0-layer.

The computation of the b-energy form is exactly analogous to that of [7] Sec. 5.3. It follows the ideas of the previous section in a much simpler situation. As only the third component is involved in the higher order terms of the form b (see (2.15) and (2.2)), we have

$$b(\tilde{u}^a, \tilde{v}^a) \approx \int_\Omega B^{\alpha\beta\lambda\mu} \partial_{\alpha\beta} \tilde{u}_3^a \partial_{\lambda\mu} \tilde{v}_3^a \, \mathrm{d}\xi \, \mathrm{d}x. \tag{6.38}$$

Moreover, from (6.4)–(6.6) and according to our approximations analogous to the construction of a parametrix, $\tilde{u}$, $\tilde{v}$ are only significant in a narrow layer adjacent to Γ_1. The local structure is analogous to (6.12) where obviously the decreasing solution inwards the domain should be chosen. This gives the obvious local asymptotics

$$\hat{\tilde{v}}_3(\xi, y) = \hat{v}_3(\xi_1) e^{\lambda_-(\xi_1) y_2}, \tag{6.39}$$

where $\lambda_-(\xi_1)$ is proportional to $|\xi_1|$. After substitution (6.39) in (6.38) a computation analogous to that of Section 6.2.1 (but much easier) gives (using a partition of unity) :

$$b(\tilde{u}^a, \tilde{v}^a) = \Sigma_{j,k} \int_{-\infty}^{+\infty} \zeta_{jk}(y_1) |\xi_1|^3 \tilde{u}_3^j(\xi_1) \tilde{v}_3^k(\xi_1) \, \mathrm{d}\xi_1$$

where $\zeta_{jk}(y_1)$ are smooth positive functions on Γ_1 depending on the coefficients. The function $|\xi_1|^3$ comes obviously from the integrals in the normal direction of products of second order derivatives of functions of the form $e^{\lambda_-(\xi_1) y_2}$, with $\lambda_-(\xi_1)$ proportional to $|\xi_1|$.

Then, defining the pseudo-differential operator $Q(\frac{\partial}{\partial y_1})$ of order 3/2 with principal symbol

$$\sqrt{\zeta(y_1)|\xi_1|^3}, \tag{6.40}$$

we have within our approximation :

$$\int_\Omega B^{\alpha\beta\lambda\mu}\partial_{\alpha\beta}u_3\partial_{\lambda\mu}v_3 \ \mathrm{d}x = \int_{\Gamma_1} Q(\frac{\partial}{\partial y_1})u \ Q(\frac{\partial}{\partial y_1})v \ \mathrm{d}y_1. \tag{6.41}$$

We observe that the operator Q is only concerned with the trace on Γ_1 and y_1 which denotes its arc.

The formal asymptotic problem becomes :

$$\left\{ \begin{array}{l} \text{Find } \tilde{u}^\varepsilon \in G \text{ such that } \forall \tilde{v} \in G \\ \int_{\Gamma_0} P(\frac{\partial \tilde{u}^\varepsilon}{\partial n})\overline{P(\frac{\partial \tilde{v}}{\partial n})} \ \mathrm{d}s + \varepsilon^2 \int_{\Gamma_1} Q(\tilde{u}^\varepsilon) \ \overline{Q(\tilde{v})} \ \mathrm{d}s = \langle f, w\rangle, \end{array} \right. \tag{6.42}$$

where G is the completion of G^0 for the norm

$$\|\tilde{v}\|_G^2 = \int_{\Gamma_0} \Big|P(\frac{\partial v}{\partial n})\Big|^2 \ \mathrm{d}s + \int_{\Gamma_1} \Big|Q(v_3)\Big|^2 \ \mathrm{d}s$$

Remark 32. For $\varepsilon > 0$, (6.42) is a classical Lax-Milgram problem. Continuity and coerciveness follow from the ellipticity of the operators P and Q.

6.4 The formal asymptotics and its sensitive behaviour

In the sequel, we shall denote

$$\alpha(\tilde{v}^\varepsilon, \tilde{w}) = \int_{\Gamma_0} P(\frac{\partial \tilde{v}^\varepsilon}{\partial n})\overline{P(\frac{\partial \tilde{w}}{\partial n})} \ \mathrm{d}s \tag{6.43}$$

$$\beta(\tilde{v}^\varepsilon, \tilde{w}) = \int_{\Gamma_1} Q(\tilde{v}^\varepsilon) \ \overline{Q(\tilde{w})} \ \mathrm{d}s. \tag{6.44}$$

We observe that the problem (6.42) is again in the same abstract framework as the initial problem (2.17). Nevertheless, the context is different, as the non-local character of the new problem is apparent from the structure of the space G. Let us define the operators

$$\mathcal{A} \in \mathcal{L}(G, G'), \quad \mathcal{B} \in \mathcal{L}(G, G') \tag{6.45}$$

by

$$\alpha(v, w) = \langle \mathcal{A}v, w\rangle \quad \beta(v, w) = \langle \mathcal{B}v, w\rangle. \tag{6.46}$$

Let $G_{\mathcal{A}}$ be the completion of G with the norm

$$\|v\|_{\mathcal{A}} = \|\mathcal{A}v\|_{G'}. \tag{6.47}$$

Denoting again by $\mathcal{A}$ its extension to $\mathcal{L}(G_{\mathcal{A}}, G')$, which is an isomorphism, we may rewrite (6.42) in the form :

$$\Big(\mathcal{A} + \varepsilon\mathcal{B}\Big)\tilde{v}^\varepsilon = F, \tag{6.48}$$

where $F \in G'$ is defined by

$$\langle F, \tilde{w} \rangle = \int_\Omega f \tilde{w} \,\mathrm{d}x, \;\; \forall \tilde{w} \in V. \tag{6.49}$$

It follows that

$$\tilde{v}^\varepsilon \to \tilde{v}^0 \text{ strongly in } G_{\mathcal{A}}, \tag{6.50}$$

where

$$\mathcal{A}\tilde{v}^0 = F. \tag{6.51}$$

Reduction to a problem on Γ_1

In order to exhibit more clearly the unusual character of the problem, we shall now write (6.42) in another, equivalent form involving only the traces on Γ_1 . Coming back to (6.6), let us define $\mathcal{R}_0$ as follows. For a given $w \in C^\infty(\Gamma_1)$ we solve (6.6) and we obtain

$$\tilde{w}_3 = \mathcal{R}_0 w. \tag{6.52}$$

Using the regularity properties of the solution of (6.6), it follows that $\mathcal{R}_0 w$ is in $C^\infty(\Gamma_0)$. Moreover, we may take in (6.6) a w in any $H^s(\Gamma_1)$, $s \in \mathbb{R}$ and the corresponding solution is of class C^∞ on Γ_0 and its neighbourhood, so that $\mathcal{R}_0$ has an extension which is continuous from $H^s(\Gamma_1)$ to $C^\infty(\Gamma_0)$. We shall denote by $\mathcal{R}_0$ such an extension, so that

$$\mathcal{R}_0 \in \mathcal{L}(H^s(\Gamma_1), H^r(\Gamma_0)), \forall s, r \in \mathbb{R}. \tag{6.53}$$

Then, (6.42) may be written as a problem for the traces on Γ_1 :

$$\left\{ \begin{array}{l} \text{Find } v^\varepsilon \in H^{3/2}(\Gamma_1) \text{ such that } \forall w \in H^{3/2}(\Gamma_1) \\ \int_{\Gamma_0} P(\frac{\partial}{\partial s})\mathcal{R}_0 v^\varepsilon \overline{P(\frac{\partial}{\partial s})\mathcal{R}_0 w} \,\mathrm{d}s + \varepsilon^2 \int_{\Gamma_1} Q(\frac{\partial}{\partial s}) v^\varepsilon \,\overline{Q(\frac{\partial}{\partial s}) w} \,\mathrm{d}s = \int_\Omega F \tilde{w} \,\mathrm{d}x, \end{array} \right. \tag{6.54}$$

where the configuration space is obviously $H^{3/2}(\Gamma_1)$. The left hand side with $\varepsilon > 0$ is continuous and coercive.

Remark 33. Coerciveness follows from the ellipticity of Q, as it is of order 3/2. Strictly speaking, this only ensures coerciveness on the leading order terms, which may "forget" a finite-dimensional kernel. But this is controlled by $\mathcal{R}_0$, as it is a surjective operator. Indeed, $\mathcal{R}_0 v = 0$ implies $\gamma_{\alpha\beta}(\tilde{v}) = 0$ with $\tilde{v}_3 = \tilde{v}_2 = 0$ on Γ_0, which implies $\tilde{v} = 0$ (and then $v = 0$) using the uniqueness of the Cauchy problem for the rigidity system.

Here $F \in H^{-3/2}(\Gamma_1)$ is defined for $f \in V'$ by

$$\langle F, w \rangle_{H^{-3/2}(\Gamma_1), H^{3/2}(\Gamma_1)} = \langle f, \tilde{w} \rangle. \tag{6.55}$$

We note that, for instance, when the "loading" f is defined by a "force" F on Γ_1, this function is the F in (6.54). Obviously, (6.54) may be written :

$$\Big(\mathcal{R}_0^* P^*(\frac{\partial}{\partial s}) P(\frac{\partial}{\partial s}) \mathcal{R}_0 + \varepsilon^2 Q^*(\frac{\partial}{\partial s}) Q(\frac{\partial}{\partial s}) \Big) \tilde{v}^\varepsilon = F. \tag{6.56}$$

From (6.53) we see that $\mathcal{R}_0^*$ is also a smoothing operator, i. e. :

$$\mathcal{R}_0^* \in \mathcal{L}(H^{-r}(\Gamma_1), H^{-s}(\Gamma_0)), \forall s, r \in \mathbb{R}. \tag{6.57}$$

Now we define the new operators (but we use the same notations)

$$\begin{aligned}\mathcal{A} &= \mathcal{R}_0^* P^* P \mathcal{R}_0 \ \in \mathcal{L}(H^s(\Gamma_1), H^r(\Gamma_0)), \forall s, r \in \mathbb{R}, &(6.58)\\ \mathcal{B} &= Q^* Q \ \in \mathcal{L}(H^{3/2}(\Gamma_1), H^{-3/2}(\Gamma_1)). &(6.59)\end{aligned}$$

Obviously, $\mathcal{B}$ is an elliptic pseudo-differential operator of order 3, whereas $\mathcal{A}$ is a smoothing (non local) operator. Then (6.56) becomes

$$\Big(\mathcal{A} + \varepsilon^2 \mathcal{B}\Big) v^\varepsilon = F \text{ in } H^{-3/2}(\Gamma_1). \tag{6.60}$$

Once more, the problem (6.54) is in the general framework of (2.17), so that we can define the space $\mathcal{V} = H^{3/2}(\Gamma_1)$ and its completion $\mathcal{V}_A$ with the norm

$$\|v\|_{\mathcal{A}} = \|\mathcal{A}v\|_{H^{-3/2}(\Gamma_1)}. \tag{6.61}$$

Denoting similarly by $\mathcal{A}$ the continuous extension of $\mathcal{A}$, which is an isomorphism between $\mathcal{V}_{\mathcal{A}}$ and $\mathcal{V}'$, we obtain

$$u^\varepsilon \to u^0 \text{ strongly in } \mathcal{V}_{\mathcal{A}}, \tag{6.62}$$

where $u^0 \in \mathcal{V}_{\mathcal{A}}$ satisfies

$$\mathcal{A}u^0 = F. \tag{6.63}$$

Obviously, this equation is uniquely solvable in $\mathcal{V}_{\mathcal{A}}$ for $F \in \mathcal{V}' = H^{-3/2}(\Gamma_1)$. But, the unusual character of this equation appears now clearly :

Proposition 6.1. *Let $F \in H^{-3/2}(\Gamma_1)$ and $F \notin C^\infty(\Gamma_1)$, then the problem (6.63) has no u^0 solution in $\mathcal{D}'(\Gamma_1)$.*

Démonstration. If $u^0 \in \mathcal{D}'(\Gamma_0)$ was a solution of (6.63), as Γ_1 is compact, u^0 should be in some H^s, then recalling (6.58), we should have $\mathcal{A}u^0 \in C^\infty(\Gamma_0)$, which is not possible. Moreover, (6.60) is clearly of the form (6.1). □

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