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# Stabilization of Neutral Systems with Saturating Control Inputs

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## Abstract

This paper focuses on the stabilization problem of neutral systems in the presence of time-varying delays and control saturation. Based on a descriptor approach and the use of a modified sector relation, global and local stabilization conditions are derived using Lyapunov-Krasovskii functionals. These conditions, formulated directly as linear matrix inequalities (LMIs), allow to relate the control law to be computed to a set of admissible initial conditions, for which the asymptotic and exponential [stabilities of the closed-loop system are ensured](#). An extension of these conditions to the particular case of retarded systems is also provided. From the theoretical conditions, optimization problems with LMI constraints are therefore proposed to compute stabilizing state feedback gains with the aim of ensuring stability for a given set of admissible initial conditions or the global stability of the closed-loop system. A numerical example illustrates the application of the proposed results.

**Keywords:** time delay, saturation, stabilization, stability domains, robustness.

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# 1 Introduction

In the last years great attention has been paid to stability and control of time-delay systems [11] [9] [15]. This is due to the fact that the behavior of many physical systems (mechanical, chemical processes, telecommunication, etc.) can be modeled by functional differential equations. Delays can appear in the state, input or output variables (retarded systems), as well as in the state derivative (neutral systems). Furthermore, it is well known that the presence of the delays in control systems can lead to bad time-domain performances or even to the instability of the closed-loop system. Hence, we can find in the literature a great amount of techniques and methodologies dealing with the stability and stabilization of time-delay systems (retarded and also neutral), and associated problems such as performance, robustness and filtering.

The difficulty in controlling time-delay systems becomes even greater if the control signal is bounded. Unfortunately, this is a practical constraint, which comes from the impossibility of actuators to drive signals with unlimited amplitude or energy to the controlled plants. For retarded systems, some works addressing the stability analysis and stabilization in the presence of saturating control signals can be found in the literature. In [14] and [13] globally stabilizing control laws are proposed. In [2] and [21], conditions for stability or stabilization are proposed with state feedback and sampled-data state feedback. However, in these papers, the set of admissible initial conditions, for which the asymptotic stability is ensured (i.e. the domain of attraction) in the presence of control saturation, is not mentioned or explicitly defined. Based on invariance properties, in [3] the control was computed to avoid the (input and state) saturations. In [18], [1] and [5], methods for computing stabilizing state feedback control laws aiming at enlarging well defined estimates of the domain of attraction of the closed-loop system have been proposed. These methods are based on the use of polytopic differential inclusions for describing the behavior of the closed-loop system with saturating inputs. In [20] and [19], the synthesis of stabilizing static anti-windup loops is addressed for the case of retarded systems presenting fixed delays. On the other hand, considering neutral systems, we can cite only [17]. In that paper, using a polytopic approach for modeling saturation effects, a method for computing stabilizing state feedback controls with the aim of maximizing the set of admissible initial conditions is proposed. It should be pointed out that the results in [17] are derived in the delay independent context and the obtained conditions are in the form of nonlinear matrix inequalities. Furthermore, due to the use of a polytopic approach, only local stability can be ensured.

As in [17], this paper is concerned with the asymptotic as well as the exponential stabilization problem of neutral systems in the presence of control saturation<sup>1</sup>. Based on a Lyapunov-Krasovskii functional and on the application of a modified sector condition [19], global and local stabilization conditions are derived in a delay dependent context. Differently from [17], these conditions allow to consider the case of time-varying delays in a delay dependent context and they are formulated directly as linear matrix inequalities (LMIs). In addition, the extension of these conditions to the particular case of retarded systems with delays is also

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<sup>1</sup>A preliminary version of the present work has been presented in [7]

presented. Optimization problems are then formulated with the aim of computing stabilizing state feedback control laws. These optimization problems allow to search the maximal delay bound for which a global stabilizing control law can be found. On the other hand, when only local stabilization is possible (e.g. when the open-loop system is unstable), the optimization objective consists in finding a control law that maximizes an estimate of the domain of attraction or that ensures the stability for a given set of admissible initial states.

The paper is organized as follows. In section 2, the problem to be treated is formally stated. The results concerning the asymptotic stabilization are presented in section 3. Exponential stabilization results are provided in section 4. Optimization problems to compute stabilizing gains are proposed and discussed in section 5. Finally, in section 6, numerical examples illustrate the application of the results.

**Notations.** Throughout the paper  $\mathfrak{R}^n$  denotes the  $n$  dimensional Euclidean space.  $A_i$  denotes the  $i$ th row of matrix  $A$ . For two symmetric matrices,  $A$  and  $B$ ,  $A > B$  means that  $A - B$  is positive definite.  $A'$  denotes the transpose of  $A$ .  $I$  denotes an identity matrix of appropriate order.  $\lambda_{max}(P)$  and  $\lambda_{min}(P)$  denote respectively the maximal and the minimal eigenvalues of matrix  $P$ .  $\mathcal{C}_h = \mathcal{C}([-h, 0], \mathfrak{R}^n)$  is the Banach space of continuous vector functions mapping the interval  $[-h, 0]$  into  $\mathfrak{R}^n$  with the norm  $\|\phi\|_c = \sup_{-h \leq t \leq 0} \|\phi(t)\|$ .  $\|\cdot\|$  refers to either the Euclidean vector norm or the induced matrix 2-norm.  $\mathcal{C}_h^v$  is the set defined by  $\mathcal{C}_h^v = \{\phi \in \mathcal{C}_h ; \|\phi\|_c < v, v > 0\}$ .

## 2 Problem statement

Consider the following neutral type linear system:

$$\begin{aligned} \dot{x}(t) - F\dot{x}(t - \tau(t)) &= Ax(t) + A_d x(t - \tau(t)) + Bu(t) \\ x(t_0 + \theta) &= \phi(\theta), \forall \theta \in [-h, 0], t_0 \in \mathfrak{R}_+, \phi(\theta) \in \mathcal{C}_h^v, \end{aligned} \quad (1)$$

where  $x(t) \in \mathfrak{R}^n$  and  $u(t) \in \mathfrak{R}^m$  are respectively the state and the input vectors,  $\tau(t)$  corresponds to a time-varying delay that satisfies

$$0 \leq \tau(t) \leq h, \quad \dot{\tau}(t) \leq d < 1$$

The initial function  $\phi(\theta)$  is supposed to be continuously differentiable. Matrices  $A$ ,  $A_d$ ,  $B$  and  $F$  are real constant matrices of appropriate dimensions. To apply the Lyapunov stability Theorem (see p.337 in [9]) we assume that  $\|F\| < 1$ .

We suppose that the input vector  $u$  is subject to amplitude limitations defined as follows:

$$|u_i| \leq u_{0i}, \quad u_{0i} > 0, \quad i = 1, \dots, m \quad (2)$$

Consider now a state feedback control law  $u(t) = Kx(t)$ . Due to the control bounds defined in (2), the effective control signal to be applied to the system is given by

$$u(t) = \text{sat}(Kx(t))$$

where  $u_i(t) = \text{sat}(K_i x(t)) = \text{sign}(K_i x(t)) \min\{u_{0i}, K_i x(t)\}$ . Hence, the closed-loop system reads

$$\dot{x}(t) - F\dot{x}(t - \tau(t)) = Ax(t) + A_d x(t - \tau(t)) + B \text{sat}(Kx(t)) \quad (3)$$

System (3) will be said *globally* asymptotically stable if for any differentiable initial condition  $\phi(\theta) \in \mathcal{C}_h$ , the trajectories of the system converge asymptotically to the origin [14], [13]. Similarly to the case of delay-free ( $\tau(t) = 0$ ), the determination of a global stabilizing controller is possible only when some stability assumptions are verified by the open-loop system ( $u(t) = 0$ ) [10]. When this hypothesis is not verified, it is only possible to achieve local stabilization. In this case, given a stabilizing matrix  $K$ , we associate a *basin of attraction* to the equilibrium point  $x_e(t) \equiv 0$  of system (5). The basin of attraction corresponds to all initial conditions  $\phi(\theta) \in \mathcal{C}_h$  such that the corresponding trajectories of system (5) converge asymptotically to the origin. Since the determination of the exact basin of attraction is practically impossible, a problem of interest is to ensure the asymptotic stability for a set of admissible initial conditions  $\phi(\theta)$  [18], [1],[4]. Of course, this set is included in the basin of attraction. Hence, from the considerations above, in this paper we are interested in studying the stabilization problems stated as follows.

1. Given  $h$  and  $d$ , find  $K$  and a set of admissible initial conditions, as large as possible, for which the asymptotic (or exponential) stability of the closed-loop system is ensured.
2. Given  $h$ ,  $d$  and a set of admissible initial conditions, find  $K$  such that the asymptotic (or exponential) stability is ensured for all initial conditions of the admissible set.
3. Maximize the bound on the delay  $h$ , for which the asymptotic (or exponential) stability of the closed-loop system can be ensured for some set of admissible initial conditions and a given  $d$ .

Of course, when it is possible, the objective will be the global stabilization of the closed-loop system. Otherwise, the set of admissible initial conditions will be defined from bounds on  $\|\phi(\theta)\|_c$  and  $\|\dot{\phi}(\theta)\|_c$ . In the sequel, theoretical conditions that allow to address the stabilization problems above are proposed. Based on these conditions, optimization problems will be formulated in the section 5.

## 3 Asymptotic Stabilization

### 3.1 Preliminaries

Define the following function

$$\psi(Kx(t)) = Kx(t) - \text{sat}(Kx(t)) \quad (4)$$

Note that,  $\psi(Kx(t))$  corresponds to a decentralized deadzone nonlinearity. Considering the function  $\psi(Kx(t))$ , the closed-loop system can be re-written as

$$\dot{x}(t) - F\dot{x}(t - \tau(t)) = (A + BK)x(t) + A_d x(t - \tau(t)) - B\psi(Kx(t)) \quad (5)$$

Considering a matrix  $G \in \mathfrak{R}^{m \times n}$  and defining the following polyhedral set

$$\mathcal{S} \triangleq \{x \in \mathfrak{R}^n ; |(K_i - G_i)x| \leq u_{0i}, i = 1, \dots, m\} \quad (6)$$

the following Lemma, concerning the nonlinearity  $\psi(Kx(t))$  can be stated.

**Lemma 1** [19] *Consider the function  $\psi(Kx)$  defined in (4). If  $x \in \mathcal{S}$  then the relation*

$$\psi(Kx)'T[\psi(Kx) - Gx] \leq 0 \quad (7)$$

*is verified for any matrix  $T \in \mathfrak{R}^{m \times m}$  diagonal and positive definite.*

The result in Lemma 1 can be seen as a generalized sector condition. As will be seen in the sequel, differently from the classical sector condition (used for instance in [20]), this condition will allow to obtain stability conditions directly in an LMI form.

Another instrumental result, needed in the sequel to devise the stabilization conditions, is given by the following Lemma.

**Lemma 2** *Consider two scalars  $a < b$  and a positive definite matrix  $R \in \mathfrak{R}^{n \times n}$ . For any continuous function  $\omega : [a, b] \rightarrow \mathfrak{R}^n$  and any strictly positive continuous function  $f : [a, b] \rightarrow \mathfrak{R}$ , the following inequality holds:*

$$\int_a^b \omega'(s)f(s)R\omega(s)ds \geq \left( \int_a^b \omega(s)ds \right)' \left( \int_a^b (f(s))^{-1}ds \right)^{-1} R \left( \int_a^b \omega(s)ds \right) \quad (8)$$

**Proof:** Consider any  $\epsilon \in [0, 1)$ . By virtue of the Schur complement, we can write that:

$$\begin{bmatrix} f(s)\omega'(s)R\omega(s) & \omega'(s) \\ \omega(s) & (\epsilon f(s)R)^{-1} \end{bmatrix} \geq 0$$

Then the proof consists in integrating the previous inequality, applying the Schur complement and taking  $\epsilon \rightarrow 1$ .  $\square$

**Remark 1** *From (8), it is simple to see that if  $f(s) = 1$ , the classical Jensen's inequality is obtained (see [8] p. 322).*

### 3.2 Neutral Systems

**Theorem 1** *If there exist symmetric positive definite matrices  $Q_1, L, J, X$ , matrices  $Q_2, Q_3, Y, W$  and a diagonal matrix  $S$  of appropriate dimensions satisfying the LMIs (9) and (10),*

$$\begin{bmatrix} \tilde{\mathcal{K}} & \begin{bmatrix} J/h \\ A_d Q_1 \end{bmatrix} & \begin{bmatrix} 0 \\ FL \end{bmatrix} & \begin{bmatrix} Y' \\ -BS \end{bmatrix} & h \begin{bmatrix} Q'_2 \\ Q'_3 \end{bmatrix} & \begin{bmatrix} Q'_2 \\ Q'_3 \end{bmatrix} \\ \star & (d-1)X - J/h & 0 & 0 & 0 & 0 \\ \star & \star & (d-1)L & 0 & 0 & 0 \\ \star & \star & \star & -2S & 0 & 0 \\ \star & \star & \star & \star & -2hQ_1 + hJ & 0 \\ \star & \star & \star & \star & \star & -L \end{bmatrix} < 0 \quad (9)$$

$$\text{with } \tilde{\mathcal{K}} = \begin{bmatrix} Q_2 + Q'_2 + X - J/h & Q_1 A' + W' B' - Q'_2 + Q_3 \\ \star & -Q_3 - Q'_3 \end{bmatrix}, \quad (10)$$

$$\begin{bmatrix} Q_1 & (W - Y)'_j \\ \star & u_{0j}^2 \end{bmatrix} \geq 0, \quad j = 1, \dots, m$$

then, for  $K = WQ_1^{-1}$  and all initial conditions satisfying

$$\delta = (\lambda_{\max}(Q_1^{-1}) + h\lambda_{\max}(Q_1^{-1}XQ_1^{-1}))\|\phi(\theta)\|_c^2 + \left(\frac{h^2}{2}\lambda_{\max}(Q_1^{-1}JQ_1^{-1}) + h\lambda_{\max}(L^{-1})\right)\|\dot{\phi}(\theta)\|_c^2 \leq 1, \quad (11)$$

the corresponding trajectories of system (5) converge asymptotically to the origin.

**Proof:** Consider that  $x(t) \in \mathcal{S}$  and the following Lyapunov-Krasovskii functional, proposed in [6] for dealing with [time-varying](#) delays:

$$V(t) = x'(t)P_1x(t) + \int_{t-\tau(t)}^t x'(s)Mx(s)ds + \int_{t-\tau(t)}^t \dot{x}'(s)U\dot{x}(s)ds + \int_{-h}^0 \int_{t+\theta}^t \dot{x}'(s)R\dot{x}(s)dsd\theta$$

with  $R, M, U > 0$ . Noting that  $(A + BK)x(t) + A_dx(t - \tau(t)) - B\psi(Kx(t)) + F\dot{x}(t - \tau(t)) - \dot{x}(t) = 0$ , it follows that the derivative of the functional is given by:

$$\begin{aligned} \dot{V}(t) &= x'(t)P_1\dot{x}(t) + \dot{x}'(t)P_1x(t) + x'(t)Mx(t) - (1 - \dot{\tau}(t))x'(t - \tau(t))Mx(t - \tau(t)) \\ &\quad + \dot{x}'(t)U\dot{x}(t) - (1 - \dot{\tau}(t))\dot{x}'(t - \tau(t))U\dot{x}(t - \tau(t)) + h\dot{x}'(t)R\dot{x}(t) - \int_{-h}^0 \dot{x}'(t + \theta)R\dot{x}(t + \theta)d\theta \\ &\quad + 2x'(t)P_2'((A + BK)x(t) + A_dx(t - \tau(t)) - B\psi(Kx(t)) + F\dot{x}(t - \tau(t)) - \dot{x}(t)) \\ &\quad + 2\dot{x}'(t)P_3'((A + BK)x(t) + A_dx(t - \tau(t)) - B\psi(Kx(t)) + F\dot{x}(t - \tau(t)) - \dot{x}(t)) \end{aligned}$$

Introducing the vectors  $\bar{x}'(t) = \begin{bmatrix} x'(t) & \dot{x}'(t) \end{bmatrix}$  and  $\xi'(t) = \begin{bmatrix} \bar{x}'(t) & \bar{x}'(t - \tau(t)) & \psi'(Kx(t)) \end{bmatrix}$  and the

matrix  $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$ , we will follow the descriptor approach [6]. In this case, the derivative of the functional is expressed as:

$$\begin{aligned} \dot{V}(t) &= \bar{x}'(t)\mathcal{L}\bar{x}(t) + 2\bar{x}'(t)P' \begin{bmatrix} 0 & A'_d \end{bmatrix}' x(t - \tau(t)) - 2\bar{x}'(t)P' \begin{bmatrix} 0 & B' \end{bmatrix}' \psi(Kx(t)) \\ &\quad + 2\bar{x}'(t)P' \begin{bmatrix} 0 & F' \end{bmatrix}' \dot{x}(t - \tau(t)) + x'(t)Mx(t) - x'(t - \tau(t))(1 - \dot{\tau}(t))Mx(t - \tau(t)) \\ &\quad + \dot{x}'(t)U\dot{x}(t) - (1 - \dot{\tau}(t))\dot{x}'(t - \tau(t))U\dot{x}(t - \tau(t)) + h\dot{x}'(t)R\dot{x}(t) - \int_{t-h}^t \dot{x}'(s)R\dot{x}(s)ds \end{aligned} \quad (12)$$

with  $\mathcal{L} = \begin{bmatrix} 0 & I \\ (A+BK) & -I \end{bmatrix}' P + P' \begin{bmatrix} 0 & I \\ (A+BK) & -I \end{bmatrix}$ .

Provided that  $x(t) \in \mathcal{S}$ , from Lemma 1, it follows that:

$$\dot{V}(t) \leq \dot{V}(t) - 2\psi(Kx)'T[\psi(Kx) - Gx] \quad (13)$$

where  $T$  is a diagonal positive definite matrix.

Applying now the Jensen's inequality to the last term of (12), the following inequality holds:

$$-\int_{t-h}^t \dot{x}'(s)R\dot{x}(s)ds \leq -(x(t) - x(t - \tau(t)))' \frac{R}{h} (x(t) - x(t - \tau(t))) \quad (14)$$

Combining (13) and (14), it follows that  $\dot{V}(t) \leq \xi'(t)\Gamma\xi(t)$  with

$$\Gamma = \begin{bmatrix} \mathcal{L} + \Phi & \begin{bmatrix} R/h \\ 0 \end{bmatrix} + P' \begin{bmatrix} 0 \\ A_d \end{bmatrix} & P' \begin{bmatrix} 0 \\ F \end{bmatrix} & \begin{bmatrix} G'T \\ 0 \end{bmatrix} & -P' \begin{bmatrix} 0 \\ B \end{bmatrix} \\ \star & (d-1)M - R/h & 0 & 0 & 0 \\ \star & \star & (d-1)U & 0 & 0 \\ \star & \star & \star & \star & -2T \end{bmatrix}$$

where  $\Phi = \begin{bmatrix} M - R/h & 0 \\ 0 & hR + U \end{bmatrix}$ .

Suppose now that  $\Gamma < 0$ . Applying the Schur's complement to the terms  $\begin{bmatrix} 0 & 0 \\ 0 & hR \end{bmatrix}$  and  $\begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix}$ , it follows that  $\Gamma < 0$  is equivalent to:

$$\begin{bmatrix} \tilde{\mathcal{L}} & \begin{bmatrix} R/h \\ 0 \end{bmatrix} + P' \begin{bmatrix} 0 \\ A_d \end{bmatrix} & P' \begin{bmatrix} 0 \\ F \end{bmatrix} & \begin{bmatrix} G'T \\ 0 \end{bmatrix} & -P' \begin{bmatrix} 0 \\ B \end{bmatrix} & h \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \star & (d-1)M - R/h & 0 & 0 & 0 & 0 & 0 \\ \star & \star & (d-1)U & 0 & 0 & 0 & 0 \\ \star & \star & \star & -2T & 0 & 0 & 0 \\ \star & \star & \star & \star & -hR^{-1} & 0 & 0 \\ \star & \star & \star & \star & \star & -U^{-1} & 0 \end{bmatrix} < 0 \quad (15)$$

where  $\tilde{\mathcal{L}} = \mathcal{L} + \begin{bmatrix} M - R/h & 0 \\ 0 & 0 \end{bmatrix}$ . Note now that if the previous matrix inequality is satisfied, one has  $\tilde{\mathcal{L}} < 0$ , which implies that  $-P'_3 - P_3$  is negative definite. Hence, since  $P_1 > 0$ , it follows that matrix  $P$  is invertible. Denote now the matrix  $P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}$  and define a block diagonal matrix



$\Xi = \text{diag}\{Q, Q_1, U^{-1}, T^{-1}, I, I\}$ . By pre and post-multiplying (15) by  $\Xi'_1$  and  $\Xi_1$  respectively, one obtains:

$$\begin{bmatrix} Q'\tilde{\mathcal{L}}Q & \begin{bmatrix} Q_1RQ_1/h \\ A_dQ_1 \end{bmatrix} & \begin{bmatrix} 0 \\ FU^{-1} \end{bmatrix} & \begin{bmatrix} Q_1G' \\ -BT^{-1} \end{bmatrix} & h \begin{bmatrix} Q'_2 \\ Q'_3 \end{bmatrix} & \begin{bmatrix} Q'_2 \\ Q'_3 \end{bmatrix} \\ \star & Q_1((d-1)M - R/h)Q_1 & 0 & 0 & 0 & 0 \\ \star & \star & (d-1)U^{-1} & 0 & 0 & 0 \\ \star & \star & \star & -2T^{-1} & 0 & 0 \\ \star & \star & \star & \star & -hR^{-1} & 0 \\ \star & \star & \star & \star & \star & -U^{-1} \end{bmatrix} < 0 \quad (16)$$

Consider now the change of variables:  $X = Q_1MQ_1$ ,  $J = Q_1RQ_1$ ,  $L = U^{-1}$ ,  $S = T^{-1}$ ,  $Y = GQ_1$  and  $KQ_1 = W$ . Noting that  $Q'\tilde{\mathcal{L}}Q = \tilde{\mathcal{K}}$  and that the development of  $(Q_1 - R^{-1})R(Q_1 - R^{-1}) \geq 0$  implies  $R^{-1} \geq 2Q_1 - J$ , it follows that the LMI condition (9) is obtained. Thus, we conclude that (9) implies that  $\Gamma < 0$ , which implies that  $\dot{V}(t) < 0$ , provided that  $x(t) \in \mathcal{S}$ ,  $t > 0$ .

From the definition of  $V(t)$ , it follows that:

$$\begin{aligned} V(0) &\leq x'(0)P_1x(0) + \int_{-h}^0 x'(s)Mx(s)ds + \int_{-h}^0 \dot{x}'(s)U\dot{x}(s)ds + \int_{-h}^0 \int_{\theta}^0 \dot{x}'(s)R\dot{x}(s)dsd\theta \\ &\leq (\lambda_{\max}(Q_1^{-1}) + h\lambda_{\max}(Q_1^{-1}XQ_1^{-1}))\|\phi(\theta)\|_c^2 + (\frac{h^2}{2}\lambda_{\max}(Q_1^{-1}JQ_1^{-1}) + h\lambda_{\max}(L^{-1}))\|\dot{\phi}(\theta)\|_c^2 = \delta \end{aligned}$$

If  $\dot{V}(t) < 0$ ,  $\forall t \geq 0$ , then we can conclude that

$$x(t)'P_1x(t) \leq V(t) \leq V(0) \leq \delta, \quad \forall t \geq 0 \quad (17)$$

Consider the ellipsoidal set  $\mathcal{E} = \{x \in \mathfrak{R}^n ; x'P_1x \leq 1\}$ , where  $P_1 = Q_1^{-1}$ . It is easy to see [18] that (10) implies that  $\mathcal{E} \subset \mathcal{S}$ , with  $\mathcal{S}$  as defined in (6). Suppose now that the initial condition  $\phi(\theta)$  satisfies (11), i.e.  $\delta \leq 1$ , and conditions (9)-(10) hold. From (17), it follows that the state trajectory is confined in the ellipsoid  $\mathcal{E}$ ,  $\forall t \geq 0$ , which ensures that  $x(t) \in \mathcal{S}$ ,  $\forall t \geq 0$ . Then,  $\dot{V}(t) < 0$ ,  $\forall t \geq 0$  is effectively satisfied for all initial conditions verifying (11), which concludes the proof.  $\square$

Theorem 1 considers the local (or regional) stabilization, in the sense that the computed gain  $K$  ensures asymptotic stability just for the initial conditions satisfying (11). As pointed out in Section 2, provided the open-loop system is asymptotically stable, it can be possible to compute globally stabilizing gains. The next result, which can be seen as a particularization of Theorem 1, allows to address this problem.

**Corollary 1** *If there exist positive definite matrices  $Q_1, L, J, X$ , matrices  $Q_2, Q_3, W$  and a diagonal matrix  $S$  of appropriate dimensions satisfying (9) with  $Y = W$ , then, for  $K = WQ_1^{-1}$  the origin of system (5) is globally asymptotically stable.*

**Proof:** The proof mimics the one of Theorem 1. In this case it follows that  $G = WP_1 = WQ_1^{-1} = K$ . Hence (7) is verified for all  $x \in \mathfrak{R}^n$  and the global asymptotic stability follows.  $\square$

### 3.3 Retarded Systems

We focus now on the stabilization of the following retarded system:

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)) + B \text{sat}(Kx(t)) \quad (18)$$

This system can be seen as a particular case of system (3) when  $F = 0$ . The following Theorem gives a condition to stabilize system (18).

**Theorem 2** *If there exist positive definite matrices  $Q_1, X, J$ , matrices  $Q_2, Q_3, Y, W$  and a diagonal matrix  $S$  of appropriate dimensions satisfying the LMIs (10) and (19)*

$$\begin{bmatrix} \tilde{K} & \begin{bmatrix} J/h \\ A_d Q_1 \end{bmatrix} & \begin{bmatrix} Y' \\ -BS \end{bmatrix} & h \begin{bmatrix} Q'_2 \\ Q'_3 \end{bmatrix} \\ \star & (d-1)X - J/h & 0 & 0 \\ \star & \star & -2S & 0 \\ \star & \star & \star & -2hQ_1 + hJ \end{bmatrix} < 0 \quad (19)$$

then, for  $K = WQ_1^{-1}$  and all initial conditions satisfying

$$\delta_r = (\lambda_{\max}(Q_1^{-1}) + h\lambda_{\max}(Q_1^{-1}XQ_1^{-1}))\|\phi(\theta)\|_c^2 + \frac{h^2}{2}\lambda_{\max}(Q_1^{-1}JQ_1^{-1})\|\dot{\phi}(\theta)\|_c^2 \leq 1$$

the corresponding trajectories of system (18) converge asymptotically to the origin.

**Proof:** Considering the following Lyapunov-Krasovskii functional

$$V(t) = x'(t)P_1x(t) + \int_{t-\tau(t)}^t x'(s)Mx(s)ds + \int_{-h}^0 \int_{t+\theta}^t \dot{x}'(s)R\dot{x}(s)dsd\theta$$

with  $P_1, M, R > 0$ , it suffices to follow the same steps of the proof of Theorem 1 considering  $U = 0$ .  $\square$

Concerning the global stabilization, the following result follows in this case.

**Corollary 2** *If there exist positive definite matrices  $Q_1, X, J$ , matrices  $Q_2, Q_3, W$  and a diagonal matrix  $S$  of appropriate dimensions satisfying the LMI (19) with  $Y = W$ , then the state feedback gain  $K = WQ_1^{-1}$  ensures that the origin of system (18) is globally asymptotically stable.*

**Remark 2** *The result presented in Theorem 2 can be easily adapted to consider the case of delays that can vary arbitrarily fast (i.e.  $\dot{\tau}(t)$  is not bounded by  $d$ ). This can be done by setting the matrix  $X$  equal to zero in (19), which corresponds to set  $M = 0$  in the Lyapunov-Krasovskii functional.*

## 4 Exponential Stabilization

Exponential stability properties could be an interesting way to characterize the convergence rate of the system. As usual [12, 22, 16], given some rate  $\alpha > 0$ , a system (5) is said to be  $\alpha$ -stable, or “exponentially

stable with the rate  $\alpha$ ", if there exists a scalar  $\beta \geq 1$  such that its solution  $x(t; t_0, \phi(\theta))$ , with any initial continuously differentiable function  $\phi(\theta)$ , satisfies:

$$\|x(t, t_0; \phi(\theta))\| \leq \beta [\|\phi(\theta)\|_c + \|\dot{\phi}(\theta)\|_c] e^{-\alpha(t-t_0)}. \quad (20)$$

**Theorem 3** *If, for a positive number  $\alpha$ , there exist positive definite matrices  $Q_1, X, L, J$ , matrices  $Q_2, Q_3, Y, W$  and a diagonal matrix  $S$  of appropriate dimensions satisfying the LMIs (10) and (21),*

$$\begin{bmatrix} \tilde{\mathcal{K}}_\alpha & \begin{bmatrix} Je^{-2\alpha h}/\eta_1 \\ A_d Q_1 \end{bmatrix} & \begin{bmatrix} 0 \\ FL \end{bmatrix} & \begin{bmatrix} Y' \\ -BS \end{bmatrix} & h \begin{bmatrix} Q'_2 \\ Q'_3 \end{bmatrix} & \begin{bmatrix} Q'_2 \\ Q'_3 \end{bmatrix} \\ \star & ((d-1)X - J/\eta_1)e^{-2\alpha h} & 0 & 0 & 0 & 0 \\ \star & \star & (d-1)Le^{-2\alpha h} & 0 & 0 & 0 \\ \star & \star & \star & -2S & 0 & 0 \\ \star & \star & \star & \star & -2hQ_1 + hJ & 0 \\ \star & \star & \star & \star & \star & -L \end{bmatrix} < 0 \quad (21)$$

with  $\tilde{\mathcal{K}}_\alpha = \begin{bmatrix} Q_2 + Q'_2 + X + 2\alpha Q_1 - Je^{-2\alpha h}/\eta_1 & Q_1 A' + W' B' - Q'_2 + Q_3 \\ \star & -Q_3 - Q'_3 \end{bmatrix}$

then, for  $K = WQ_1^{-1}$  and all initial condition satisfying

$$\delta_e = (\lambda_{\max}(Q_1^{-1}) + \eta_1 \lambda_{\max}(Q_1^{-1} X Q_1^{-1})) \|\phi(\theta)\|_c^2 + (\eta_1 \lambda_{\max}(L^{-1}) + \eta_2 \lambda_{\max}(Q_1^{-1} J Q_1^{-1})) \|\dot{\phi}(\theta)\|_c^2 \leq 1 \quad (22)$$

with

$$\eta_1 = \frac{1 - e^{-2\alpha h}}{2\alpha}, \quad \eta_2 = \frac{e^{-2\alpha h} - 1 + 2\alpha h}{4\alpha^2}, \quad (23)$$

the corresponding trajectories of system (5) converge exponentially to the origin, with a decay rate  $\alpha$ .

**Proof:** Consider the following Lyapunov-Krasovskii functional (LKF)

$$\begin{aligned} V_\alpha(t) = & x'(t) P_1 x(t) + \int_{t-\tau(t)}^t x'(s) e^{2\alpha(s-t)} M x(s) ds + \int_{t-\tau(t)}^t \dot{x}'(s) e^{2\alpha(s-t)} U \dot{x}(s) ds \\ & + \int_{-h}^0 \int_{t+\theta}^t \dot{x}'(s) e^{2\alpha(s-t)} R \dot{x}(s) ds d\theta \end{aligned}$$

with  $R, M, U > 0$ . Following the proof of Theorem 1, the differentiation of the LKF along the trajectories of system (1) leads to:

$$\begin{aligned} \dot{V}_\alpha(t) = & \bar{x}'(t) \mathcal{L} \bar{x}(t) + 2\bar{x}'(t) P' \begin{bmatrix} 0 & A'_d \end{bmatrix}' x(t - \tau(t)) - 2\bar{x}'(t) P' \begin{bmatrix} 0 & B' \end{bmatrix}' \psi(Kx(t)) \\ & + 2\bar{x}'(t) P' \begin{bmatrix} 0 & F' \end{bmatrix}' \dot{x}(t - \tau(t)) + x'(t) M x(t) + \dot{x}'(t) U \dot{x}(t) \\ & - (1 - \dot{\tau}(t)) x'(t - \tau(t)) e^{-2\alpha\tau(t)} M x(t - \tau(t)) - (1 - \dot{\tau}(t)) \dot{x}'(t - \tau(t)) e^{-2\alpha\tau(t)} U \dot{x}(t - \tau(t)) \\ & + h \dot{x}'(t) R \dot{x}(t) - \int_{t-h}^t \dot{x}'(s) e^{2\alpha(s-t)} R \dot{x}(s) ds + 2\alpha x'(t) P_1 x(t) - 2\alpha V_\alpha(t) \end{aligned}$$

Applying Lemma 2 to the integral term of the previous expression, the following inequality is obtained:

$$- \int_{t-h}^t \dot{x}'(s) e^{2\alpha(s-t)} R \dot{x}(s) ds \leq - \frac{2\alpha}{e^{2\alpha h} - 1} (x(t) - x(t - \tau(t)))' R (x(t) - x(t - \tau(t)))$$

Noting that  $\frac{2\alpha}{e^{2\alpha h}-1} = e^{-2\alpha h}/\eta_1$  and that for all delay  $\tau \in [0, h]$ ,  $e^{-2\alpha\tau(t)} \geq e^{-2\alpha h}$ , then the following inequality holds:

$$\begin{aligned} \dot{V}_\alpha(t) + 2\alpha V_\alpha(t) &\leq \bar{x}'(t)\mathcal{L}\bar{x}(t) + 2\bar{x}'(t)P' \begin{bmatrix} 0 & A'_d \end{bmatrix}' x(t - \tau(t)) - 2\bar{x}'(t)P' \begin{bmatrix} 0 & B' \end{bmatrix}' \psi(Kx(t)) \\ &\quad + 2\bar{x}'(t)P' \begin{bmatrix} 0 & F' \end{bmatrix}' \dot{x}(t - \tau(t)) + x'(t)Mx(t) + \dot{x}'(t)U\dot{x}(t) \\ &\quad - (1 - \dot{\tau}(t))x'(t - \tau(t))e^{-2\alpha h}Mx(t - \tau(t)) - (1 - \dot{\tau}(t))\dot{x}'(t - \tau(t))e^{-2\alpha h}U\dot{x}(t - \tau(t)) \\ &\quad + h\dot{x}'(t)R\dot{x}(t) - e^{-2\alpha h}/\eta_1(x(t) - x(t - \tau(t)))'R(x(t) - x(t - \tau(t))) + 2\alpha x(t)'P_1x(t) \end{aligned}$$

The end of the proof strictly follows the line of Theorem 1. Thus if LMI (21) is satisfied, it follows that  $\dot{V}_\alpha(t) + 2\alpha V_\alpha(t) < 0$  for all  $x(t) \in \mathcal{S}$  and consequently, by integration, that  $V_\alpha(t)$  exponentially decreases with the decay rate  $2\alpha$ . This implies that the condition (20), for the exponential stability of the solution of system (1), holds [16].

On the other hand, one has

$$\begin{aligned} V_\alpha(0) &\leq \left( \lambda_{\max}(P_1) + \left( \int_{-h}^0 e^{2\alpha s} ds \right) \lambda_{\max}(M) \right) \|\phi(\theta)\|_c^2 \\ &\quad + \left( \left( \int_{-h}^0 e^{2\alpha s} ds \right) \lambda_{\max}(U) + \left( \int_{-h}^0 \int_{\theta}^0 e^{2\alpha s} ds d\theta \right) \lambda_{\max}(R) \right) \|\dot{\phi}(\theta)\|_c^2, \\ &\leq (\lambda_{\max}(P_1) + \eta_1 \lambda_{\max}(M)) \|\phi(\theta)\|_c^2 + (\eta_1 \lambda_{\max}(U) + \eta_2 \lambda_{\max}(R)) \|\dot{\phi}(\theta)\|_c^2, \end{aligned} \quad (24)$$

From the definition of  $Q_1$ ,  $X$ ,  $J$  and  $L$  as in Theorem 1, it follows that  $\lambda_{\max}(P_1) = \lambda_{\max}(Q_1^{-1})$ ,  $\lambda_{\max}(M) = \lambda_{\max}(Q_1^{-1}XQ_1^{-1})$ ,  $\lambda_{\max}(R) = \lambda_{\max}(Q_1^{-1}JQ_1^{-1})$  and  $\lambda_{\max}(U) = \lambda_{\max}(L^{-1})$ . Hence, if  $\phi(\theta)$  verifies (22), we can conclude that

$$x'(t)P_1x(t) \leq V_\alpha(t) \leq e^{-2\alpha t}V_\alpha(0) \leq \delta_e \leq 1, \quad \forall t \geq 0,$$

which implies that  $x(t) \in \mathcal{E}$ ,  $\forall t \geq 0$ . Then, since (10) implies that  $\mathcal{E} \subset \mathcal{S}$ , as in Theorem 1, we can effectively conclude that (21) implies  $\dot{V}_\alpha(t) < -2\alpha V_\alpha(t) < 0$ ,  $\forall t \geq 0$ , for all initial conditions verifying (22).  $\square$

**Remark 3** Since the exponential function is convex, from (23) it follows that  $\eta_1 \geq h$  and  $\eta_2 \geq \frac{h^2}{2}$ . This ensures that the set of initial conditions for the asymptotic case is greater than the one for the exponential case. Moreover, when  $\alpha \rightarrow 0$ ,  $\eta_1 \rightarrow h$  and  $\eta_2 \rightarrow \frac{h^2}{2}$ , which ensures the continuity of the set with respect to  $\alpha$ . Thus the set of admissible initial conditions of Theorem 1 is recovered when  $\alpha \rightarrow 0$ .

**Remark 4** Since the LMIs in Theorems 1, 2 and 3, as well as in Corollaries 1 and 2, are affine in the system matrices  $A$ ,  $A_d$ ,  $B$  and  $F$ , the extension of the conditions to consider uncertain systems described by polytopic uncertainties is straightforward. Note that if these matrices can be computed as a convex combination of the vertices of a polytope of matrices, given by  $(A^i, A_d^i, B^i, F^i)$ ,  $i = 1, \dots, N$ , then, by convexity, it suffices to verify the LMIs at each vertex of the polytope simultaneously. Furthermore, for each vertex, different matrices  $Q_3^i$  and  $Q_2^i$  can be considered.

## 5 Optimization Problems

In this section we show how the theoretical conditions can be casted into LMI-based optimization problems to determine a suitable stabilization gain  $K$ . In particular, three criteria are considered: the maximization of the delay bound  $h$  for which global stability can be ensured; the maximization of the set of admissible initial conditions, which indirectly corresponds to determine  $K$  in order to maximize the region of attraction of the closed-loop system; and maximization of the delay bound  $h$  or a quadratic performance criteria, while ensuring the stability for a given set of admissible initial conditions.

### 5.1 Maximization of the delay for which global stability is ensured

In the case where the system can be globally asymptotically stabilized in the absence of the delays, an interesting problem consists in finding the maximal bound  $h^*$  on the time-varying delay  $\tau(t)$ , for which system (5) can be globally stabilized, considering a given bound  $d$  on  $\dot{\tau}(t)$ . This can be accomplished by solving the following optimization problem:

$$\begin{aligned} & \max h \\ & \text{subject to} \\ & (9) \text{ with } Y = W \end{aligned} \tag{25}$$

Note that, due to the product between  $h$  and the variables  $Q_2, Q_3$  and  $J$ , the solution of this problem can be obtained by iteratively increasing  $h$  and testing the feasibility of (9), which is an LMI for a fixed  $h$ .

### 5.2 Maximization of the set of admissible initial conditions

Consider given  $h$  and  $d$ . In order to ensure the stability of system (5) by using the Theorem 1, the admissible initial conditions must verify condition (11). Assume that  $\|\phi(\theta)\|_c^2 = \delta_1$  and  $\|\dot{\phi}(\theta)\|_c^2 = \delta_2$ . Note that the smaller the maximal eigenvalues of  $Q_1^{-1}, Q_1^{-1}JQ_1^{-1}, Q_1^{-1}XQ_1^{-1}$ , and  $L^{-1}$ , the larger  $\delta_1$  and  $\delta_2$  for which (11) is verified. Hence, the problem of finding  $K$  leading to the maximization of the region of stability of the closed-loop system can be achieved by minimizing these maximal eigenvalues. With this aim, consider the following auxiliary LMIs:

$$\begin{bmatrix} \lambda_{Q_1} I & I \\ \star & Q_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \lambda_X I & I \\ \star & 2Q_1 - X \end{bmatrix} \geq 0, \quad \begin{bmatrix} \lambda_J I & I \\ \star & 2Q_1 - J \end{bmatrix} \geq 0, \quad \begin{bmatrix} \lambda_L I & I \\ \star & L \end{bmatrix} \geq 0, \tag{26}$$

From the fact that  $R^{-1} \geq 2Q_1 - J$  and  $M^{-1} \geq 2Q_1 - X$  it follows that these LMIs are respectively equivalent to  $\lambda_{Q_1} \geq \lambda_{\max}(Q_1^{-1})$ ,  $\lambda_X \geq \lambda_{\max}(Q_1^{-1}XQ_1^{-1})$ ,  $\lambda_J \geq \lambda_{\max}(Q_1^{-1}JQ_1^{-1})$  and  $\lambda_L \geq \lambda_{\max}(L^{-1})$ .

Hence, the following optimization problem can be considered:

$$\begin{aligned} & \min \beta_1 \lambda_{Q_1} + \beta_2 \lambda_X + \beta_3 \lambda_J + \beta_4 \lambda_L \\ & \text{subject to} \\ & (9), (10) \text{ and } (26) \end{aligned} \tag{27}$$

where  $\beta_1, \beta_2, \beta_3$  and  $\beta_4$  are weights that should be tuned in order to satisfy some trade-off between  $\delta_1$  and  $\delta_2$ . The choice of these weighting parameters are performed in an *ad hoc* way. In general, the minimization of one of the eigenvalues is more critical to obtain larger values of  $\delta_1$  and/or  $\delta_2$ . In this case, the weight associated to the appropriated eigenvalue should be increased.

### 5.3 Maximizations for a given set of admissible initial conditions

Consider now  $\delta_1 > 0$  and  $\delta_2 > 0$ . The idea is then to compute  $K$  in order to guarantee the stability for all initial conditions satisfying  $\|\phi(\theta)\|_c^2 \leq \delta_1$  and  $\|\dot{\phi}(\theta)\|_c^2 \leq \delta_2$ . This case can be addressed considering the auxiliary LMIs (26) and the following additional constraint:

$$(\lambda_{Q_1} + h\lambda_X)\delta_1 + (0.5h^2\lambda_J + h\lambda_L)\delta_2 - 1 \leq 0 \quad (28)$$

Note that if  $\|\phi(\theta)\|_c^2 \leq \delta_1$  and  $\|\dot{\phi}(\theta)\|_c^2 \leq \delta_2$ , (28) implies that (11) is verified. In this case, for instance, the following optimization criteria can be considered.

#### 5.3.1 Maximization of the bound $h$ for which is possible to find a stabilizing gain

In this case, a problem analogous to (25) can be formulated as follows:

$$\begin{aligned} & \max h \\ & \text{subject to} \\ & (9), (10), (26) \text{ and } (28) \end{aligned} \quad (29)$$

#### 5.3.2 Minimization of an upper bound to a given cost function (guaranteed cost problem)

A natural performance measure is given by the following quadratic criterion on plant states:

$$\mathcal{J} = \int_0^\infty x'(t)C'Cx(t)dt \quad \text{where } C'C \geq 0, \quad C'C \in \mathfrak{R}^{n \times n}.$$

If we are now able to show that:

$$\dot{V}(t) + \frac{1}{\gamma}\bar{x}' \begin{bmatrix} C' \\ 0 \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \bar{x} < 0, \quad (30)$$

it follows that  $\mathcal{J} < \gamma V(0) < \gamma, \forall \phi(\theta)$  satisfying (11).

Note that (30) is satisfied if the following matrix inequality is verified:

$$\begin{bmatrix} \tilde{\mathcal{K}} & \begin{bmatrix} J/h \\ A_d Q_1 \end{bmatrix} & \begin{bmatrix} 0 \\ FL \end{bmatrix} & \begin{bmatrix} Y' \\ -BS \end{bmatrix} & h \begin{bmatrix} Q'_2 \\ Q'_3 \end{bmatrix} & \begin{bmatrix} Q'_2 \\ Q'_3 \end{bmatrix} & \begin{bmatrix} Q'_1 C' \\ 0 \end{bmatrix} \\ * & (d-1)X - J/h & 0 & 0 & 0 & 0 & 0 \\ * & * & (d-1)L & 0 & 0 & 0 & 0 \\ * & * & * & -2S & 0 & 0 & 0 \\ * & * & * & * & -2hQ_1 + hJ & 0 & 0 \\ * & * & * & * & * & -L & 0 \\ * & * & * & * & * & 0 & -\gamma I \end{bmatrix} < 0 \quad (31)$$

Hence, the following optimization problem can be formulated in order to minimize the bound  $\gamma$  (guaranteed cost) on the performance quadratic criterion:

$$\begin{aligned} & \min \gamma \\ & \text{subject to} \\ & (31), (10), (26) \text{ and } (28) \end{aligned} \tag{32}$$

**Remark 5** *The optimization problems above can be straightforwardly adapted to the problem of retarded systems and to the exponential stabilization. It suffices to consider the conditions stated in Theorems 2 and 3. In particular, for the case of exponential stabilization, another problem of interest is the maximization of the decay rate  $\alpha$ , for which it is possible to ensure the stability for a given set of admissible initial states or the global stability.*

**Remark 6** *It should be noticed that the derived results apply also to the analysis problem. In this case, the results and the optimization problems can be straightforwardly adapted to consider a given gain  $K$ .*

## 6 Numerical Examples

**Example 1** *Consider system (1) with*

$$A = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad u_0 = 15;$$

*Considering the optimization problem (27) with  $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$ , the gain  $K = [-0.1325 \ 0.0153]$  is obtained for  $h = 1$  and  $d = 0.1$ . This gain ensures the asymptotic stability for any  $\phi(\theta)$  satisfying:*

$$11.4\|\phi(\theta)\|_c^2 + 8.57\|\dot{\phi}(\theta)\|_c^2 \leq 10^5 \tag{33}$$

*It is worth to notice that in our previous work [7], where a direct descriptor approach was adopted, considering the same conditions, the asymptotic stability was ensured for  $\phi(\theta)$  satisfying  $51\|\phi(\theta)\|_c^2 + 9.34\|\dot{\phi}(\theta)\|_c^2 \leq 10^4$ . This shows that the [set of admissible initial conditions obtained from the application of Theorem 1](#) are significantly less conservative.*

*Note that the set of admissible  $\phi(\theta)$  given by (33) denotes a trade-off between the amplitude and the derivative of the initial conditions. Hence, for instance, if we consider that  $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c$  the stability is ensured for  $\phi(\theta)$  such that  $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c < 70.74$ . On the other hand, if we consider that the initial states are constant over the interval  $[-h, 0]$ , that is  $\|\dot{\phi}(\theta)\|_c = 0$ , it follows that all  $\phi(\theta)$  such that  $\|\phi(\theta)\|_c \leq 93.65$  are admissible.*

*Consider now  $d = 0.1$  and the optimization problem (27) with  $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$ . In [Table 1](#), considering initial conditions such that  $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c \leq \bar{\delta}$ , the maximal value of  $\bar{\delta}$  and the respective*

$h$	$\bar{\delta}$	K
1	70.74	[-0.1325 0.0153]
2	56.17	[-0.1201 - 0.0421]
3	18.17	[-0.1681 - 0.0137]
3.53	$62.8 \times 10^{-3}$	[-1.2062 11.1614]

Table 1:  $h \times$  region of stability

$\alpha$	0.0	0.2	0.4	0.6	0.8	1.0	1.2
$\bar{\delta}$	70.74	44.38	35.50	26.38	21.33	18.13	9.58

Table 2:  $\alpha \times$  region of stability

gain, obtained from the solution of (27), are shown for different values of  $h$ . As expected, the set of admissible initial conditions reduces as the upperbound on the delay increases. The value of  $h = 3.53$  corresponds to the maximum upperbound on the delay for which the LMIs are feasible. The same behavior appears when  $h$  is fixed and the parameter  $d$  varies. For instance, for  $h = 1$ , the set of initial conditions reduces as  $d$  increases and it is almost empty for  $d = 0.947$ .

Concerning the performance analysis with Theorem 3 together with the optimization problem (27), one can see that the set of admissible initial conditions reduces as the exponential decay rate  $\alpha$  increases. This fact is illustrated in Table 2, where the maximal value of  $\bar{\delta}$  obtained considering  $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c \leq \bar{\delta}$ ,  $h = 1$  and  $d = 0.1$  is shown for different values of the decay rate  $\alpha$ . For  $h = 1$  and  $d = 0.1$ , the maximum exponential decay rate, for which the LMIs are feasible, is  $\alpha = 1.85$ .

**Example 2** Consider a retarded system given by (18), with the matrices  $A$ ,  $A_d$  and  $B$  defined in the Example 1,  $u_0 = 15$ ,  $h = 1$  and  $d = 0.1$ . It is possible to ensure the asymptotic stability of initial conditions satisfying  $\|\phi(\theta)\|_c = \|\dot{\phi}(\theta)\|_c < 83.55$  with the gain  $K = [-0.1950 \ 0.0649]$ . Note that the bound on the admissible conditions is larger than the ones obtained in [4] (79.43) and in [7] (79.54). This indicates that the proposed method is less conservative than the previous approaches.

## 7 Concluding Remarks

The synthesis of stabilizing gains for linear neutral systems in the presence of saturating inputs and time-varying delays has been addressed. First, conditions that allow the computation of a state feedback matrix associated to a set of initial conditions, for which the asymptotic closed-loop stability can be ensured, have been derived. Considering the case of open-loop asymptotically stable systems, this condition can be slightly modified to address the problem of computing globally stabilizing gains. It has also been shown that the



conditions can be particularized to consider retarded systems. Following the same approach, exponential stabilization conditions have been derived.

Based on the theoretical conditions, convex optimization problems (with LMI constraints) have been proposed in order to compute the stabilizing gains aiming at: maximizing the delay for which global stability can be ensured; maximizing the set of admissible initial conditions, which indirectly corresponds to determine  $K$  in order to maximize the region of attraction of the closed-loop system; or maximizing the delay or a quadratic performance criterion, while ensuring the stability for a given set of admissible initial conditions.

The extension of the results to uncertain polytopic systems is straightforward. Another interesting possible extension regards the problem of static anti-windup design.

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