On the isotopic meshing of an algebraic implicit surface

Daouda Niang Diatta, Bernard Mourrain, Olivier Ruatta

On the Isotopic Meshing of an Algebraic Implicit Surface

Daouda Niang Diatta

XLIM University of Limoges;CNRS;INRIA Méditerranée

Bernard Mourrain

INRIA Méditerranée, 06902 Sophia Antipolis, France

Olivier Ruatta

XLIM University of Limoges;CNRS

Abstract

We present a new and complete algorithm for computing the topology of an algebraic surface S given by a squarefree polynomial in $\mathbb{Q}[X, Y, Z]$. Our algorithm involves only subresultant computations and entirely relies on rational manipulation, which makes it direct to implement. We extend the work in [15], on the topology of *non-reduced* algebraic space curves, and apply it to the polar curve or apparent contour of the surface S. We exploit simple algebraic criterion to certify the pseudo-genericity and genericity position of the surface. This gives us rational parametrizations of the components of the polar curve, which are used to lift the topology of the projection of the polar curve. We deduce the connection of the two-dimensional components above the cell defined by the projection of the polar curve. A complexity analysis of the algorithm is provided leading to a bound in $\widetilde{\mathcal{O}}_B(d^{15}\tau)$ for the complexity of the computation of the topology of an implicit algebraic surface defined by integer coefficients polynomial of degree d and coefficients size τ. Examples illustrate the implementation in Mathemagix of this first complete code for certified topology of algebraic surfaces.

1. Introduction

The problem of computing a triangulation of a real (semi)-algebraic variety S is an old but fundamental problem in real algebraic geometry. It has been studied in the literature

Email addresses: `daouda-niang.diatta@xlim.fr` (Daouda Niang Diatta), `mourrain@sophia.inria.fr` (Bernard Mourrain), `olivier.ruatta@xlim.fr` (Olivier Ruatta).

[23], [24], mainly from a theoretical point of view. From a more computational point of view, in works such as [11], [13], [9], the triangulation problem is tackled effectively via Cylindrical Algebraic Decomposition. It consists in decomposing a semi-algebraic set S into cells, defined by sign conditions on polynomial sequences. Such polynomial sequences are obtained by (sub)-resultant computations, corresponding to successive projections from $\mathbb{R}^{k+1}$ to $\mathbb{R}^k$. The degree of the polynomials in these sequences is bounded by $\mathcal{O}(d^{2^{n-1}})$ and their number by $\mathcal{O}((m\,d)^{3^{n-1}})$, where m is the number of polynomials defining the semi-algebraic set S, d is a bound on the degree of these polynomials and n the number of different variables appearing in these polynomials [9]. This Cylindrical Algebraic Decomposition does not directly yield a triangulation, nor any global topological information on the set S, because the representation lacks information about the adjacency of the cells. Additional work is required to obtain a triangulation of S, using for instance, Thom encoding of algebraic numbers or numerical approximations (see e.g., [13], [11], [28]). But this requires the evaluation of signs of many polynomials at many real algebraic numbers. It also explains why practical efficient implementations of these algorithms are not available.

For semi-algebraic sets in small dimension ($n \leqslant 3$), the problem has been investigated in more details. A wide literature exists on the computation of the topology of curves in $\mathbb{R}^2$. See e.g. [20], [16], [25], [6], [7], [14], [32], [4] and applications in Computer Aided Geometric Design [33], [8], [27].

In $\mathbb{R}^3$, the problem of computing the topology of space curves has been less investigated. In [22], the case of intersections curves of parametric surfaces is considered, based on the analysis of planar curves in the parameter domains. In [5], Alcázar and Sendra give a symbolic-numeric algorithm for *reduced* space curves using subresultant and gcd computations of approximated polynomials. In [29], Owen, Rockwood and Alyn give a numerical algorithm for *reduced* space curve using subdivision method. In [17], Elkaoui gives a certified symbolic-numeric algorithm for space curve defined as the intersection of the vanishing sets of n trivariate polynomials, which requires the computation of generators of the radical of the ideal, that involves Gröbner basis computation.

The special case of surfaces in $\mathbb{R}^3$ has also received a lot of attention (see e.g. [19], [18], [1], and references in [12]), but these works deal only with *smooth* surfaces. See also [26], where Cylindrical Algebraic Decomposition approach has been further investigated to analyze the topology of critical sections of an implicit surface, by exploiting the properties of delineability.

The contribution of this paper is a new and complete algorithm for computing the topology of an algebraic surface S given by a squarefree polynomial in $\mathbb{Q}[X, Y, Z]$. Our algorithm involves only subresultant computations and entirely relies on simple rational operations, which makes it direct to implement. In particular, compared to [3], we avoid to compute the topolgy of plane sections of the surface at critical values.

The approach extend the work in [15], which provides a certified algorithm for the topology of *non-reduced* algebraic space curves. It is essential to be able to treat non reduced spaces, since we apply it to the polar curve or apparent contour of the surface S. We exploit simple algebraic criterion to certify the pseudo-genericity and genericity position of the surface. This gives us rational parametrizations of the components of the polar curve, used to lift the topology computed after projection, without any supplementary effort. The topology of the polar curve is then used to deduce the connection of the

two-dimensional components above the cell defined, in the plane, by the projection of the polar curve.

Furthermore, this algorithm allows a complete complexity analysis. An upper bound on the bit complexity of the computation of the topology of implicit space curves and surfaces is given. Our algorithm realizes this task with complexity in $\widetilde{\mathcal{O}}_B(d^{15}\tau)$.

At last, we describe the results of the implementation of this algorithm in MATHEMAGIX. To our knowledge, it is the first complete code for certified topology of algebraic surfaces. Examples of experimentations for surfaces with isolated or one-dimensional singularities are given.

2. Topology of a plane algebraic curve

To be able to describe the topology of space curves, we need to do so with planar curves. In this section, we recall definitions and describe an algorithm allowing to compute with certainty the topology of plane algebraic curves.

2.1. Description of the problem

Let $f \in \mathbb{Q}[X, Y]$ be a square free polynomial and $\mathcal{C}(f) := \{(\alpha, \beta) \in \mathbb{R}^2, f(\alpha, \beta) = 0\}$ be the real algebraic curve associated to f. We want to compute the topology of $\mathcal{C}(f)$. For curves in generic position, computing its critical fibers and one regular fiber between two critical ones is sufficient to obtain the topology using a sweeping algorithm (see [20]). But for a good computational behaviour, it is essential to certify the genericity of the position of the curve.

We propose an effective test allowing to certify the computation and connection, in a deterministic way. This is an important tool in order to address the case of space curves and surfaces.

Now, let us introduce the definitions of generic position, critical, singular and regular points.

2.2. Genericity conditions for plane algebraic curves

Definition 2.1. Let $f \in \mathbb{Q}[X, Y]$ be a square free polynomial and $\mathcal{C}(f) = \{(\alpha, \beta) \in \mathbb{R}^2 : f(\alpha, \beta) = 0\}$ be the curve defined by f. A point $(\alpha, \beta) \in \mathcal{C}(f)$ is called:

- a x-critical point if $\partial_Y f(\alpha, \beta) = 0$,
- a singular point if $\partial_X f(\alpha, \beta) = \partial_Y f(\alpha, \beta) = 0$,
- a regular point if $\partial_X f(\alpha, \beta) \neq 0$ or $\partial_Y f(\alpha, \beta) \neq 0$.

Definition 2.2. Let $f \in \mathbb{Q}[X, Y]$ be a square free polynomial and $\mathcal{C}(f) = \{(\alpha, \beta) \in \mathbb{R}^2 : f(\alpha, \beta) = 0\}$ be the curve defined by f. Let $\mathcal{N}_x(\alpha) := \#\{\beta \in \mathbb{R}$, such that (α, β) is a x-critical point of $\mathcal{C}(f)\}$. $\mathcal{C}(f)$ is in generic position for the x-direction, if:

(1) $\forall \alpha \in \mathbb{C}, \mathcal{N}_x(\alpha) \leqslant 1$,
(2) There is no asymptotic direction of $\mathcal{C}(f)$ parallel to the y-axis.

This notion of genericity also appears in [20] or [16]. In [20], the algorithm succeed if genericity conditions are satisfied. The authors give a numerical test that do **not guarantee to reject the curve** if it is not in generic position. So for some input curves the computed topology might not be exact.

A change of coordinates such that $\mathrm{lcoef}_Y(f) \in \mathbb{Q}^*$ is sufficient to place $\mathcal{C}(f)$ in a position such that any asymptotic direction is not parallel to the y-axis. It remains to find an efficient way to verify the first condition. Using the next propositions, we give an algorithm to do so. We refer to [20], for proofs.

Proposition 2.3. Let $f \in \mathbb{Q}[X,Y]$ be a square free polynomial with $\mathrm{lcoef}_Y(f) \in \mathbb{Q}^*$, $\mathrm{Res}_Y(f, \partial_Y f)$ be the resultant with respect to Y of the polynomials f, $\partial_Y f$ and $\{\alpha_1, \ldots, \alpha_l\}$ be the set of the roots of $\mathrm{Res}_Y(f, \partial_Y f)$ in $\mathbb{C}$. Then $\mathcal{C}(f)$ is in generic position if and only if $\forall i \in \{1, \ldots, l\}, \gcd(f(\alpha_i, Y), \partial_Y f(\alpha_i, Y))$ has at most one root.

Let $f \in \mathbb{Q}[X,Y]$ be a square free polynomial with $\mathrm{lcoef}_Y(f) \in \mathbb{Q}^*$ and $d := \deg_Y(f)$. We denote by $\mathrm{Sr}_i(X,Y)$ the ith subresultant polynomial of f and $\partial_Y f$ and $\mathrm{sr}_{i,j}(X)$ the coefficient of Y^j in $\mathrm{Sr}_i(X,Y)$ (see appendix for definitions). We define inductively the following polynomials:

$$\Phi_0(X) = \frac{\mathrm{sr}_{0,0}(X)}{\gcd(\mathrm{sr}_{0,0}(X), \mathrm{sr}'_{0,0}(X))};$$

$$\forall i \in \{1, \ldots, d-1\}, \Phi_i(X) = \gcd(\Phi_{i-1}(X), \mathrm{sr}_{i,i}(X)) \,\text{and}\, \Gamma_i(X) = \frac{\Phi_{i-1}(X)}{\Phi_i(X)}.$$

Proposition 2.4.

(1) $\Phi_0(X) = \prod_{i=1}^{d-1} \Gamma_i(X)$ and $\forall i,\, j \in \{1, \ldots, d-1\}, i \neq j \Longrightarrow \gcd(\Gamma_i(X), \Gamma_j(X)) = 1$;

(2) Let $k \in \{1, \ldots, d-1\}$, $\alpha \in \mathbb{C}$. $\Gamma_k(\alpha) = 0 \Longleftrightarrow \gcd(f(\alpha, Y), \partial_Y f(\alpha, Y)) = \mathrm{Sr}_k(\alpha, Y)$;

(3) $\{(\alpha, \beta) \in \mathbb{R}^2 : f(\alpha, \beta) = \partial_Y f(\alpha, \beta) = 0\} = \bigcup_{k=1}^{d-1} \{(\alpha, \beta) \in \mathbb{R}^2 : \Gamma_k(\alpha) = \mathrm{Sr}_k(\alpha, \beta) = 0\}$.

In the following theorem, we give an effective and efficient algebraic test to certify the genericity of the position of a curve with respect to a given direction.

Theorem 2.5. Let $f \in \mathbb{Q}[X,Y]$ be a square free polynomial with $\mathrm{lcoef}_Y(f) \in \mathbb{Q}^*$ and $d := \deg_Y(f)$. Then $\mathcal{C}(f)$ is in generic position for the projection on the x axis if and only if $\forall k \in \{1, \ldots, d-1\}$, $\forall i \in \{0, \ldots, k-1\}$, $k(k-i)\,\mathrm{sr}_{k,i}(X)\,\mathrm{sr}_{k,k}(X) - (i+1)\,\mathrm{sr}_{k,k-1}(X)\,\mathrm{sr}_{k,i+1}(X) = 0 \bmod \Gamma_k(X)$.

Proof. Assume that $\mathcal{C}(f)$ is in generic position and let $\alpha \in \mathbb{C}$ be a root of $\Gamma_k(X)$. According to Proposition 7.3 (2) $\gcd(f(\alpha, Y), \partial_Y f(\alpha, Y)) = \mathrm{Sr}_k(\alpha, Y) = \sum_{j=0}^{k} \mathrm{sr}_{k,j}(\alpha) Y^j$. According to Proposition 2.3, $\mathrm{Sr}_k(\alpha, Y)$ has only one root $\beta(\alpha) = -\frac{\mathrm{sr}_{k,k-1}(\alpha)}{k\,\mathrm{sr}_{k,k}(\alpha)}$, so $\mathrm{Sr}_k(\alpha, Y) = \mathrm{sr}_{k,k}(\alpha)(Y-\beta)^k$. Binomial Newton formula gives $\mathrm{Sr}_k(\alpha, Y) = \mathrm{sr}_{k,k}(\alpha)(Y-\beta)^k = \mathrm{sr}_{k,k}(\alpha) \sum_{i=0}^{k} \binom{k}{i} (-\beta)^{k-i} Y^i$. So by identification $\forall k \in \{1, \ldots, d-1\}, \forall i \in \{0, \ldots, k-1\}$ and $\forall \alpha \in \mathbb{C}$ such that $\Gamma_k(\alpha) = 0$,

$$k(k-i)\,\mathrm{sr}_{k,i}(\alpha)\,\mathrm{sr}_{k,k}(\alpha) - (i+1)\,\mathrm{sr}_{k,k-1}(\alpha)\,\mathrm{sr}_{k,i+1}(\alpha) = 0.$$

It is to say that $\forall k \in \{1, \ldots, d-1\}, \forall i \in \{0, \ldots, k-1\}$, $k(k-i)\,\mathrm{sr}_{k,i}(X)\,\mathrm{sr}_{k,k}(X) - (i+1)\,\mathrm{sr}_{k,k-1}(X)\,\mathrm{sr}_{k,i+1}(X) = 0 \bmod \Gamma_k(X)$.

Conversely, let α be a root of $\Gamma_k(X)$ such that

$$k(k-i)\,\mathrm{sr}_{k,i}(\alpha)\,\mathrm{sr}_{k,k}(\alpha) - (i+1)\,\mathrm{sr}_{k,k-1}(\alpha)\,\mathrm{sr}_{k,i+1}(\alpha) = 0.$$

With the same argument used in the first part of this proof we obtain

$$gcd\,(f(\alpha,Y), \partial_Y f(\alpha,Y)) = Sr_k(\alpha,Y) = \sum_{j=0}^{k} sr_{k,j}(\alpha)Y^j = sr_{k,k}(\alpha)(Y-\beta)^k,$$

with $\beta := -\frac{\mathrm{sr}_{k,k-1}(\alpha)}{k\,\mathrm{sr}_{k,k}(\alpha)}$.
Then we conclude that $\gcd(f(\alpha,Y), \partial_Y f(\alpha,Y))$ has only one distinct root and, according to Proposition 2.3, $\mathcal{C}(f)$ is in generic position. □

Remark 2.6. Theorem 2.5 shows that it is possible to check with certainty if a plane algebraic curve is in generic position or not. If not, we can put it in generic position by a basis change. In fact, it is well known that there is only a finit number of bad changes of coordinates of the form $X := X + \lambda Y$, $Y := Y$, such that if $\mathcal{C}(f)$ is not in generic position then the transformed curve remains in a non-generic position.

Let us remind the connection algorithm, before talking about space curve.

2.3. Connection algorithm

Let $c := (\alpha, \beta)$ be a x-critical point of the curve and Crit the x-critical fiber containing c. Let $\mathrm{Up} := \{(\alpha, b) \in \mathrm{Crit} : b > \beta\}$ and $\mathrm{Down} := \{(\alpha, b) \in \mathrm{Crit} : b < \beta\}$. So $\mathrm{Crit} := \{\mathrm{Up}, c, \mathrm{Down}\}$.
Let r be the smalest regular value such that $r > \alpha$ and $\mathrm{Reg} := \{(r, y) \in \mathbb{R}^2 : f(r, y) = 0\}$ the regular fiber define on r. Up, Down and Reg are ordered sets. Here below we remind Grandine's sweeping algorithm which connect a x-critical fiber to a regular one.

Algorithm 2.1: `Plane Curve Connection`

Input: $\mathrm{Crit} := \{\mathrm{Up}, c, \mathrm{Down}\}$ and Reg.
Output: The set of segments linking Crit to Reg.
For i from 1 to #Up, link Reg[i] to Up[i];
For i from 1 to #Down, link Reg[$\#\,\mathrm{Reg} - i + 1$] to Down[$\#\,\mathrm{Down} - i + 1$];
For i in $\#\,\mathrm{Up} + 1$ to $\#\mathrm{Reg} - (\#\,\mathrm{Down})$, link Reg[$i$] to c;

Remark 2.7. Let p be a point of Reg and q be the point of the x-critical fiber Crit connected to p. The map $\varphi_c : \mathrm{Reg} \longrightarrow \{-1, 0, 1\}$ defined by:
- $\varphi_c(p) = -1$ if $q \in \mathrm{Up}$
- $\varphi_c(p) = 0$ if $q = c$
- $\varphi_c(p) = 1$ if $q \in \mathrm{Down}$

will be very helpfull for the description of the connection algorithm in section 4.

3. Topology of an implicit space algebraic curve

This section is devoted to the description of an algorithm allowing to compute the topology of algebraic space curves with certainty.

3.1. *Description of the problem*

Let $P_1, P_2 \in \mathbb{Q}[X, Y, Z]$ and $\mathcal{C}_{\mathbb{R}} := \{(x, y, z) \in \mathbb{R}^3 : P_1(x, y, z) = P_2(x, y, z) = 0\}$ be the intersection of the surfaces defined by $P_1 = 0$ and $P_2 = 0$. We assume that $\gcd(P_1, P_2) = 1$ so that $\mathcal{C}_{\mathbb{R}}$ is a space curve. The ideal (P_1, P_2) is not necessary radical.

Definition 3.1 (Reduced space curve)**.** The space curve $\mathcal{C}_{\mathbb{R}}$ is **reduced** if the ideal genereated by P_1 and P_2 is radical, else it is **non-reduced**.

Our goal is to analyze the geometry of $\mathcal{C}_{\mathbb{R}}$ in the following sense: We want to compute a piecewise linear structure of $\mathbb{R}^3$ isotopic to our original space curve. The algorithm computes the topology of the space curve by lifting the topology of one of its projection on a plane. To make the lifting possible using only one projection, a new definition of generic position for space curves and an algebraic characterization of it are given. We will also need to distinguish between to kind of singularities of the projected curve, namely the "apparent singularities" and the "real singularities". A **certified** algorithm is given to **distinguish** these two kinds of **singularities**. For the lifting phase, using the new notion of curve in **pseudo-generic position**, we give an algorithm that computes rational parametrizations of the space curve. The use of these rationals parametrizations allows us to lift the topology of the projected curve without any supplementary computation.

3.2. *Genericity conditions for space curves*

Let $\Pi_z : (x, y, z) \in \mathbb{R}^3 \mapsto (x, y) \in \mathbb{R}^2$. Let $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}}) \subset \mathbb{R}^2$ be the curve obtained by projection of $\mathcal{C}_{\mathbb{R}}$. We assume that $\deg_Z(P_1) = \deg(P_1)$ and $\deg_Z(P_2) = \deg(P_2)$ (by a basis change, these conditions are always satisfied). Let $h(X, Y)$ be **the squarefree** part of $\mathrm{Res}_Z(P_1, P_2) \in \mathbb{Q}[X, Y]$ and $\mathcal{C}_{\mathbb{C}} := \left\{(x, y, z) \in \mathbb{C}^3 | P_1(x, y, z) = P_2(x, y, z) = 0\right\}$.

Definition 3.2 (Pseudo-generic position)**.** The curve $\mathcal{C}_{\mathbb{R}}$ is in pseudo-generic position with respect to the (x, y)-plane if and only if almost every point of $\Pi_z(\mathcal{C}_{\mathbb{C}})$ has only one geometric inverse-image, i.e. generically, if $(\alpha, \beta) \in \Pi_z(\mathcal{C}_{\mathbb{C}})$, then $\Pi_z^{-1}(\alpha, \beta)$ consists in one point possibly multiple.

Let m be the minimum of $\deg_Z(P_1)$ and $\deg_Z(P_2)$. The following theorems give us an effective way to test if a curve is in pseudo-generic position or not.

Theorem 3.3. Let $(\mathrm{Sr}_j(X, Y, Z))_{j \in \{0,\dots,m\}}$ be the subresultant sequence and $(\mathrm{sr}_j(X, Y))_{j \in \{0,\dots,m\}}$ be the principal subresultant coefficient sequence. Let $(\Delta_i(X, Y))_{i \in \{1,\dots,m\}}$ be the sequence of $\mathbb{Q}[X, Y]$ defined by the following relations:

- $\Delta_0(X, Y) = 1; \Theta_0(X, Y) = h(X, Y)$;
- For $i \in \{1, ..., m\}$,
  $\Theta_i(X, Y) = \gcd(\Theta_{i-1}(X, Y), \mathrm{sr}_i(X, Y)), \Delta_i(X, Y) = \frac{\Theta_{i-1}(X,Y)}{\Theta_i(X,Y)}$.

For $i \in \{1, \dots, m\}$, let $\mathcal{C}(\Delta_i) := \left\{(x, y) \in \mathbb{R}^2 | \Delta_i(x, y) = 0\right\}$ and $\mathcal{C}(h) := \{(x, y) \in \mathbb{R}^2 | h(x, y) = 0\}$ then

(1) $h(X, Y) = \prod_{i=1}^{m} \Delta_i(X, Y)$,

(2) $\mathcal{C}(h) = \bigcup_{i=1}^{m} \mathcal{C}(\Delta_i)$,

(3) $\mathcal{C}_{\mathbb{R}}$ is in pseudo-generic position with respect to the (x,y)-plane if and only if $\forall i \in \{1,\dots,m\}, \forall (x,y) \in \mathbb{C}^2$ such that $\mathrm{sr}_{i,i}(x,y) \neq 0$ and $\Delta_i(x,y) = 0$ we have $\mathrm{Sr}_i(x,y,Z) = \mathrm{sr}_{i,i}(x,y)\left(Z + \frac{\mathrm{sr}_{i,i-1}(x,y)}{\mathrm{isr}_{i,i}(x,y)}\right)^i$.

Proof. (1) By definition, $\forall i \in \{1,\dots,m\}, \Delta_i(X,Y) = \frac{\Theta_{i-1}(X,Y)}{\Theta_i(X,Y)}$. So by a trivial induction

$$\prod_{i=1}^{m} \Delta_i(X,Y) = \frac{\Theta_0(X,Y)}{\Theta_m(X,Y)}.$$

$\deg_Z(P_1) = \deg(P_1)$ and $\deg_Z(P_2) = \deg(P_2)$ imply $\mathrm{sr}_m(X,Y) \in \mathbb{Q}^*$ (Remark 7.2). So $\Theta_m(X,Y) = \gcd(\Theta_{m-1}(X,Y), \mathrm{sr}_m(X,Y)) = 1$, then $\prod_{i=1}^{m} \Delta_i(X,Y) = \Theta_0(X,Y) = h(X,Y)$.

(2) Knowing that $h(X,Y) = \prod_{i=1}^{m} \Delta_i(X,Y)$, it is clear that $\mathcal{C}(h) = \bigcup_{i=1}^{m} \mathcal{C}(\Delta_i)$.

(3) Assume that $\mathcal{C}_{\mathbb{R}}$ is in pseudo-generic position with respect to the (x,y)-plane. Let $i \in \{1,\dots,m\}$ and $(\alpha,\beta) \in \mathbb{C}^2$ such that $\mathrm{sr}_i(\alpha,\beta) \neq 0$ and $\Delta_i(\alpha,\beta) = 0$. Then $\Delta_i(X,Y) = \frac{\Theta_{i-1}(X,Y)}{\Theta_i(X,Y)} \implies \Theta_{i-1}(\alpha,\beta) = 0$. Knowing that $\Theta_{i-1}(X,Y) = \gcd(\Theta_{i-2}(X,Y), \mathrm{sr}_{i-1}(X,Y))$, so it exists $d_1, d_2 \in \mathbb{Q}[X,Y]$ such that $\Theta_{i-2}(X,Y) = d_1(X,Y)\Theta_{i-1}(X,Y)$ and$\mathrm{sr}_{i-1}(X,Y) = d_2(X,Y)\Theta_{i-1}(X,Y)$. In this way,$\Theta_{i-1}(\alpha,\beta)$ implies $\Theta_{i-2}(\alpha,\beta) = 0$ and $\mathrm{sr}_{i-1}(\alpha,\beta) = 0$. By the same arguments, $\Theta_{i-2}(\alpha,\beta) = 0$ implies $\Theta_{i-3}(\alpha,\beta) = 0$ and $\mathrm{sr}_{i-2}(\alpha,\beta) = 0$. By repeating the same argument, we show $\mathrm{sr}_{i-1}(\alpha,\beta) = \ldots = \mathrm{sr}_0(\alpha,\beta) = 0$. Because $\mathrm{sr}_i(\alpha,\beta) \neq 0$, the fundamental theorem of subresultant gives:
$\gcd((P_1(\alpha,\beta,Z), P_2(\alpha,\beta,Z)) = \mathrm{Sr}_i(\alpha,\beta,Z) = \sum_{j=0}^{i} \mathrm{sr}_{i,i-j}(\alpha,\beta)Z^{i-j}$.
Knowing that $\mathcal{C}_{\mathbb{R}}$ is in pseudo-generic position with respect to the (x,y)-plane and $\Delta_i(\alpha,\beta) = 0$ then the polynomial $\mathrm{Sr}_i(\alpha,\beta,Z)$ has only one distinct root which can be written $-\frac{\mathrm{sr}_{i,i-1}(\alpha,\beta)}{i\,\mathrm{sr}_{i,i-1}(\alpha,\beta)}$ depending on the relation between coefficients and roots of a polynomial. So $\mathrm{Sr}_i(\alpha,\beta,Z) = \sum_{j=0}^{m} \mathrm{sr}_{,i,i-j}(\alpha,\beta)Z^{i-j} = \mathrm{sr}_{i,i}(\alpha,\beta)\left(Z + \frac{\mathrm{sr}_{i,i-1}(\alpha,\beta)}{i\,\mathrm{sr}_{i,i-1}(\alpha,\beta)}\right)^i$.
Conversely, assume that $\forall i \in \{1,\dots,m\}$, $\forall (x,y) \in \mathbb{C}^2$ such that $\mathrm{sr}_i(x,y) \neq 0$ and $\Delta_i(x,y) = 0$, we have $\mathrm{Sr}_i(x,y,Z) = \sum_{j=0}^{m} \mathrm{sr}_{,i,i-j}(x,y)Z^{i-j} = \mathrm{sr}_{i,i}(x,y)\left(Z + \frac{\mathrm{sr}_{i,i-1}(x,y)}{i\,\mathrm{sr}_{i,i}(x,y)}\right)^i$.
Let (α,β) be a point such that $\Delta_i(\alpha,\beta) = 0$ and $\mathrm{sr}_i(\alpha,\beta) \neq 0$. Now if we define $\gamma := -\frac{\mathrm{sr}_{i,i-1}(\alpha,\beta)}{i\,\mathrm{sr}_{i,i}(\alpha,\beta)}$, then we obtain that $\mathrm{Sr}_i(\alpha,\beta,\gamma) = 0$, and (α,β,γ) is the only point of $\mathcal{C}_{\mathbb{C}}$ with (α,β) as projection. Furthermore there are only finitely many points such that $\Delta_i(x,y) = 0$ and $\mathrm{sr}_i(x,y) = 0$. So $\mathcal{C}_{\mathbb{R}}$ is in pseudo-generic position with respect to the (x,y)-plane.
□

The following proposition is a corollary of the third point of the previous theorem. If $\mathcal{C}_{\mathbb{R}}$ is in pseudo-generic position with respect to the (x,y)-plane, it gives a rational parametrization for each regular points of a connected component of a given multiplicity of $\mathcal{C}_{\mathbb{R}}$.

Proposition 3.4. Assume that $\mathcal{C}_{\mathbb{R}}$ is in pseudo-generic with respect to the (x, y)-plane and let $(\alpha, \beta, \gamma) \in \mathcal{C}_{\mathbb{R}}$ such that $\mathrm{sr}_i(\alpha, \beta) \neq 0$ and $\Delta_i(\alpha, \beta) = 0$. Then,

$$\gamma := -\frac{\mathrm{sr}_{i,i-1}(\alpha, \beta)}{i\,\mathrm{sr}_{i,i}(\alpha, \beta).} \tag{1}$$

Remark 3.5. By construction, the parametrization given in Proposition 3.4 is valid when $\mathrm{sr}_{i,i}(\alpha, \beta) \neq 0$. In pseudo-generic postion, if $\mathrm{sr}_{i,i}(\alpha, \beta) = 0$ then either $\Delta_j(\alpha, \beta) = 0$ for some $j > i$ or (α, β) is a x-critical point of $\mathcal{C}(\Delta_i)$ (see section 3.3).

The following theorem gives an algebraic certificate for the pseudo-genericity of the position of a space curve with respect to a given plane.

Theorem 3.6. Let $(\mathrm{Sr}_j(X, Y, Z))_{j\in\{0,\ldots,m\}}$ be the subresultant sequence associated to $P_1(X, Y, Z)$ and $P_2(X, Y, Z)$ and $(\Delta_i(X, Y))_{i\in\{0,\ldots,m\}}$ be the sequence of $\mathbb{Q}[X, Y]$ as defined in Theorem 3.3. The curve $\mathcal{C}_{\mathbb{R}}$ is in pseudo-generic position with respect to the (x, y)-plane if and only if $\forall i \in \{1, \ldots, m-1\}$, $\forall j \in \{1, \ldots, i-1\}$,
$i(i-j)\,\mathrm{sr}_{i,j}(X, Y)\,\mathrm{sr}_{i,i}(X, Y) - (j+1)\,\mathrm{sr}_{i,i-1}(X, Y)\,\mathrm{sr}_{i,j+1}(X, Y) = 0 \bmod \Delta_i(X, Y)$.

Proof. Assume $\mathcal{C}_{\mathbb{R}}$ be in pseudo-generic position. Let $i \in \{1, \ldots, m-1\}$, $j \in \{0, \ldots, i-1\}$, $(\alpha, \beta) \in \mathbb{R}^2$ such that $\Delta_i(\alpha, \beta) = 0$. If $\mathrm{sr}_{i,i}(\alpha, \beta) = 0$, then, by Proposition 7.3, $\mathrm{sr}_{i,i-1}(\alpha, \beta) = 0$, so $i(j+1)\,\mathrm{sr}_{i,j+1}(\alpha, \beta)\,\mathrm{sr}_{i,i}(\alpha, \beta) - (i-j)\,\mathrm{sr}_{i,i-1}(\alpha, \beta)\,\mathrm{sr}_{i,j}(\alpha, \beta) = 0$. If $\mathrm{sr}_{i,i}(\alpha, \beta) \neq 0$, then according to the third point of the Theorem 3.3, $\mathrm{Sr}_i(\alpha, \beta, Z) = \mathrm{sr}_{i,i}(\alpha, \beta)\,(Z-\gamma)^i$ where $\gamma := -\frac{\mathrm{sr}_{i,i-1}(\alpha,\beta)}{i\,\mathrm{sr}_{i,i-1}(\alpha,\beta)}$, then $\mathrm{Sr}_i(\alpha, \beta, Z) = \sum_{j=0}^{i} \mathrm{sr}_{,i,i-j}(\alpha, \beta) Z^{i-j} = \mathrm{sr}_{i,i}(\alpha, \beta)\,(Z-\gamma)^i$. Using the binomial Newton formula we obtain:
$\mathrm{Sr}_i(\alpha, \beta, Z) = \mathrm{sr}_{i,i}(\alpha, \beta) \sum_{j=0}^{i} \binom{i}{j} (-\gamma)^{i-j} Z^j$.
So by identification, it comes that $\forall i \in \{1, \ldots, m-1\}, \forall j \in \{0, \ldots, i-1\}$, $\forall(\alpha, \beta)$, st .$\Delta_i(\alpha, \beta) = 0$,

$$i(i-j)\,\mathrm{sr}_{i,j}(\alpha, \beta)\,\mathrm{sr}_{i,i}(\alpha, \beta) - (i+j)\,\mathrm{sr}_{i,i-1}(\alpha, \beta)\,\mathrm{sr}_{i,j+1}(\alpha, \beta) = 0.$$

The reciprocal uses the same arguments. □

The following algorithm tests the pseudo-genericity of position of the curve :

Algorithm 3.1: `PseudoGenerTest`

Input: $P_1, P_2 \in \mathbb{Q}[X, Y, Z]$ two squarefree polynomials such that $\gcd(P_1, P_2) = 1$

Output: true if the curve defined by $P_1 = 0, P_2 = 0$ is in generic position and false otherwise.

Step 1: *Making P_1 and P_2 monic with respect to z by a change of coordinates.*

If $\deg_Z(P_2) \neq \deg(P_2)$ *or* $\deg_Z(P_1) \neq \deg(P_1)$ *do*

$(X, Y, Z) \leftarrow (X + \lambda Z, Y + \mu Z, Z)$ in P_1 and P_2, with $\lambda, \mu \in \mathbb{Q}^*$.

Step 2: *Computing the Δ_k polynomials.*

Using a subresultant algorithm, compute $\mathrm{Sr}_m(X, Y, Z), \ldots, \mathrm{Sr}_0(X, Y, Z)$ the subresultants sequence associated to P_1 and P_2 and denote

$\mathrm{Sr}_i(X, Y, Z) = \sum_{j=0}^{i} \mathrm{sr}_{i,j}(X, Y) Z^j$.

$h(X, Y) := \mathrm{squarefree}(\mathrm{sr}_{0,0}(X, Y))$,

$\Theta_1(X, Y) = \gcd\Big(h(X, Y), \mathrm{sr}_1(X, Y)), \Delta_1(X, Y) = \frac{h(X,Y)}{\Theta_1(X,Y)}$,

for i from 1 to m do

$\Theta_i(X, Y) = \gcd(\Theta_{i-1}(X, Y), \mathrm{sr}_i(X, Y))$

$\Delta_i(X, Y) = \frac{\Theta_{i-1}(X,Y)}{\Theta_i(X,Y)}$

end do

Step 3: *The Test.*

for i from 1 to m do

if $\Delta_i(X, Y) \neq 0$ *then*

for j from 0 to i do

$(i * (i - j) * \mathrm{sr}_{i,j}(X, Y) * \mathrm{sr}_{i,i}(X, Y) - (j + 1) * \mathrm{sr}_{i,i-1}(X, Y) * \mathrm{sr}_{i,j+1}(X, Y)) \bmod \Delta_i(X, Y)$.

If the result is zero then continue else break and return false;

end do

end if

end do;

return true;

Remark 3.7. Theorem 3.6 shows that it is possible to check with certainty if a space algebraic curve is in pseudo-generic position or not. If it is not, we can put it in pseudo-generic position by a change of coordinates.

Let us introduce the definitions of generic position, critical, singular, regular points, apparent singularity and real singularity for a space algebraic curve.

Definition 3.8. Let $(g_1, \ldots, g_s)$ be the radical ideal of the ideal (P_1, P_2). Let $M(X, Y, Z)$ be the s×3 Jacobian matrix with $(\partial_X g_i, \partial_Y g_i, \partial_Z g_i)$ as its i th row.

(1) A point p$\in \mathcal{C}_{\mathbb{R}}$ is regular (or smooth) if the rank of $M(p)$ is 2.

(2) A point p$\in \mathcal{C}_{\mathbb{R}}$ which is not regular is called singular.

(3) A point $p = (\alpha, \beta, \gamma) \in \mathcal{C}_{\mathbb{R}}$ is x-critical (or critical for the projection on the x-axis) if the curve $\mathcal{C}_{\mathbb{R}}$ is tangent at this point to a plane parallel to the (y,z)-plane. The corresponding α is called a x-critical value.

Definition 3.9 (Apparent singularity, Real singularity)**.** Let $\mathcal{D} = \prod_z(\mathcal{C}_{\mathbb{R}})$ and let P be a singular point of $\mathcal{D}$. We define:

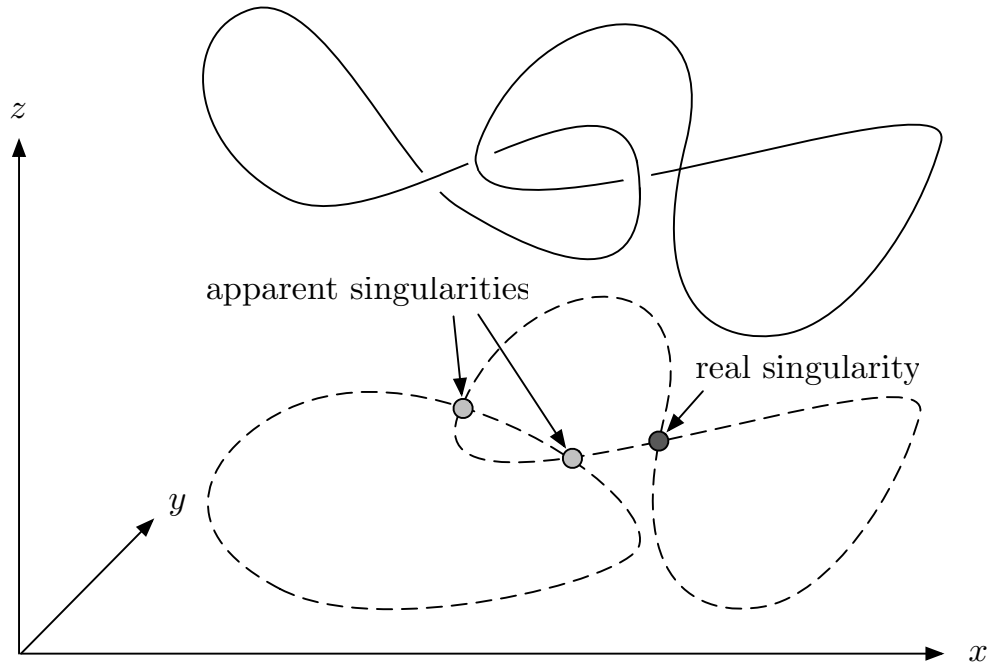

Fig. 1. Apparent and real singularities.

(1) The point P is called an **apparent singularity** if the fiber$\prod_z^{-1}(P) \cap \mathcal{C}_{\mathbb{R}}$ above P contains strictly more than one point.
(2) The point P is called a **real singularity** if the fiber $\prod_z^{-1}(P) \cap \mathcal{C}_{\mathbb{R}}$ above P contains exactly one point. In this case, the point of $\prod_z^{-1}(P) \cap \mathcal{C}_{\mathbb{R}}$ is a singularity of $\mathcal{C}_{\mathbb{R}}$.

A geometric illustration of those definitions can be found in figure 1.

Definition 3.10 (Node)**.** We call a node an ordinary double point (both arcs have different tangential directions).

Definition 3.11 (Generic position)**.** The curve $\mathcal{C}_{\mathbb{R}}$ is in generic position with respect to the (x, y)-plane if
(1) $\mathcal{C}_{\mathbb{R}}$ is in pseudo-generic position with respect to the (x, y)-plane,
(2) $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$ is in generic position (as a plane algebraic curve) with respect to the x-direction,
(3) any apparent singularity of $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$ is a node.

This notion of genericity also appears in a slightly more restrictive form in [5] and [17].

The aim of the next section is to give an algorithm to certify the third point of the previous definition of generic position. We also give an effective way to distinguish the real singularities from the apparent ones.

3.3. Distinguish real singularities and apparent singularities

In this section, we suppose that $\mathcal{C}_{\mathbb{R}}$ is in **pseudo-generic position** and $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$ is in **generic position as a plane** algebraic curve.
Let $(\Gamma_j(X))_{j\in\{1,\dots,n\}}$ be the sequence of Γ polynomials associated to the plane curve $\mathcal{D}$ and $(\beta_j(X))_{j\in\{1,\dots,n\}}$ be the sequence of associated rational parametrization $(\beta_j(X) := -\frac{\mathrm{sr}_{j,j-1}(X)}{j\,\mathrm{sr}_{j,j}(X)})$.

Let $(\mathrm{Sr}_j(X,Y,Z))_{j\in\{0,\dots,m\}}$ be the subresultant sequence associated to $P_1, P_2 \in \mathbb{Q}[X,Y,Z]$. For any $(k,i) \in \{1,\dots,m\} \times \{0,\dots,k-1\}$ let $R_{k,i}(X,Y)$ be defined by $R_{k,i}(X,Y) = k(k-i)\,\mathrm{sr}_{k,i}(X,Y)\,\mathrm{sr}_{k,k}(X,Y) - (i+1)\,\mathrm{sr}_{k,k-1}(X,Y)\,\mathrm{sr}_{k,i+1}(X,Y)$.

The following lemma allows us to characterize fibers containing only one point.

Lemma 3.12. Let $(a,b) \in \mathbb{R}^2$ such that $\mathrm{sr}_{k,k}(a,b) \neq 0$, so $\mathrm{Sr}_k(a,b,Z) = \sum_{i=0}^{k} \mathrm{sr}_{k,i}(a,b)Z^i \in \mathbb{R}[Z]$ has one and only one root if and only if $\forall i \in \{0,\dots,k-1\}$ $R_{k,i}(a,b) = 0$.

By construction, for any $j \in \{1,\dots,n\}$, $\gcd(\Gamma_j(X), \mathrm{sr}_{j,j}(X)) = 1$. So modulo $\Gamma_j(X)$, $\beta_j(X)$ can be written as a polynomial $\tilde{\beta}_j$. We will use the polynomial expression $\tilde{\beta}_j$ of $\beta_j(X)$ in the following constructions.

For any $j \in \{1,\dots,n\}$ let us define the sequence $(u_{k,j}(X))_{k\in\{1,\dots,j\}}$ and $(v_{k,j}(X))_{k\in\{2,\dots,j\}}$ by
- $u_{1,j}(X) := \gcd(\Gamma_j(X), \mathrm{sr}_{1,1}(X, \tilde{\beta}_j(X)))$,
- $u_{k,j}(X) := \gcd(\mathrm{sr}_{k,k}(X, \tilde{\beta}_j(X)), u_{k-1,j}(X))$
- $v_{k,j}(X) := \mathrm{quo}(u_{k-1,j}(X), u_{k,j}(X))$.

For $k \in \{2,\dots,j\}$, $i \in \{0, k-1\}$, we define $(w_{k,i,j}(X))$ by
- $w_{k,0,j}(X) := v_{k,j}(X)$,
- $w_{k,i+1,j}(X) := \gcd(R_{k,i}(X, \tilde{\beta}_j(X)), w_{k,i,j}(X))$.

More intuitively, for some j, the polynomials $v_{k,j}$ are exactly those with roots α such that the gcd of the projected plane curve and its derivative, localized at α, has degree j, and the gcd of the two surfaces, localized at $(\alpha, \beta_j(\alpha))$, has degree k.

Remark 3.13. By construction, for any $j \in \{1,\dots,n\}$, $\gcd(\Gamma_j(X), \mathrm{sr}_{j,j}(X)) = 1$

Theorem 3.14. For any $j \in \{1,\dots,n\}$, let $(\Gamma_{j,k}(X))_{k\in\{1,\dots,j\}}$ and $(\chi_{j,k}(X))$ be the sequence defined by the relations: $\Gamma_{j,1}(X) = \mathrm{quo}(\Gamma_j(X), u_{1,j}(X))$ and $\Gamma_{j,k}(X) := w_{k,k,j}(X)$. $\chi_{j,k}(X) := \mathrm{quo}(w_{k,0,j}(X), \Gamma_{j,k}(X))$.

(1) For any root α of $\Gamma_{j,k}(X)$, the x-critical fiber $(\alpha, \beta_j(\alpha))$ contains only the point $(\alpha, \beta_j(\alpha), \gamma_j(\alpha))$ with $\gamma_j(\alpha) := -\frac{\mathrm{sr}_{k,k-1}(\alpha,\beta_j(\alpha))}{k\,\mathrm{sr}_{k,k}(\alpha,\beta_j(\alpha))}$ so $(\alpha, \beta_j(\alpha))$ is a real singularity.
(2) For any root α of $\chi_{j,k}(X)$, $(\alpha, \beta_j(\alpha))$ is an apparent singularity.

Proof. (1) Let α be a root of $\Gamma_{j,k}(X) := w_{k,k,j}(X) = \gcd(R_{k,k-1}(X, \tilde{\beta}_j(X)), w_{k,k-1,j}(X))$. Then $w_{k,k-1,j}(\alpha) = R_{k,k-1}(\alpha, \beta_j(\alpha)) = 0$.
$w_{k,k-1,j}(X) := \gcd(R_{k,k-2}(X, \tilde{\beta}_j(X)), w_{k,k-2,j}(X))$, so $w_{k,k-2,j}(\alpha) = R_{k,k-2}(\alpha, \beta_j(\alpha)) = 0$. By induction, using the same argument, it comes that for

i from 0 to $(k-1)$, $w_{k,i,j}(\alpha) = R_{k,i}(\alpha, \beta_j(\alpha)) = 0$. $w_{k,0,j}(X) := v_{k,j}(X)$, so $v_{k,j}(\alpha) = 0$. Knowing that $v_{k,j}(X) := \mathrm{quo}(u_{k-1,j}(X), u_{k,j}(X))$; $u_{k,j}$ and $u_{k-1,j}$ are square free, then $u_{k-1,j}(\alpha) = 0$ and $u_{k,j}(\alpha) \neq 0$. Knowing that
$u_{k,j}(X) = \gcd(\mathrm{sr}_{k,k}(X, \tilde{\beta}_j(X)), u_{k-1,j}(X))$, then $\mathrm{sr}_{k,k}(\alpha, \beta_j(\alpha)) \neq 0$.
$u_{k-1,j}(X) = \gcd(\mathrm{sr}_{k-1,k-1}(X, \tilde{\beta}_j(X)), u_{k-2,j}(X))$ and $u_{k-1,j}(\alpha) = 0$,
so $\mathrm{sr}_{k-1,k-1}(\alpha, \beta_j(\alpha)) = u_{k-2,j}(\alpha) = 0$. By induction, using the same argument, it comes that for i from 0 to $k-1$, $\mathrm{sr}_{i,i}(\alpha, \beta_j(\alpha)) = 0$. For i from 0 to $k-1$ $\mathrm{sr}_{i,i}(\alpha, \beta_j(\alpha)) = 0$ and $\mathrm{sr}_{k,k}(\alpha, \beta_j(\alpha)) \neq 0$, so by the fundamental theorem of subresultants, $\gcd(P_1(\alpha, \beta_j(\alpha), Z), P_2(\alpha, \beta_j(\alpha), Z)) = \sum_{i=0}^{k} \mathrm{sr}_{k,i}(\alpha, \beta_j(\alpha)) Z^i$.
Knowing that $\gcd(P_1(\alpha, \beta_j(\alpha), Z), P_2(\alpha, \beta_j(\alpha), Z)) = \mathrm{Sr}_k(\alpha, \beta_j(\alpha), Z) = \sum_{i=0}^{k} \mathrm{sr}_{k,i}(\alpha, \beta_j(\alpha)) Z^i$ and for i from 0 to $(k-1)$, $R_{k,i}(\alpha, \beta_j(\alpha)) = 0$ then by the previous lemma the polynomial $\gcd(P_1(\alpha, \beta_j(\alpha), Z), P_2(\alpha, \beta_j(\alpha), Z)$ have only one root $\gamma_j(\alpha) := -\frac{\mathrm{sr}_{k,k-1}(\alpha, \beta_j(\alpha))}{k\, \mathrm{sr}_{k,k}(\alpha, \beta_j(\alpha))}$.

(2) Let α be a root of the polynomial $\chi_{j,k}(X) := \mathrm{quo}(w_{k,0,j}(X), \Gamma_{j,k}(X))$. Then $w_{k,0,j}(\alpha) = 0$ and $\Gamma_{j,k}(\alpha) = w_{k,k,j}(\alpha) \neq 0$ because $w_{k,0,j}(X)$ and $\Gamma_{j,k}(X)$ are square free. For i from 0 to $k-1$, knowing that $w_{k,i+1,j}(X) := \gcd(R_{k,i}(X, \tilde{\beta}_j(X)), w_{k,i,j}(X))$, $w_{k,0,j}(\alpha) = 0$ and $w_{k,k,j}(\alpha) \neq 0$, then there exists $i \in \{0, \ldots, k-1\}$ such that $R_{k,i}(\alpha, \beta_j(\alpha)) \neq 0$. So by Lemma 1, the polynomial $\mathrm{Sr}_k(\alpha, \beta_j(\alpha), Z) = \sum_{i=0}^{k} \mathrm{sr}_{k,i}(\alpha, \beta_j(\alpha)) Z^i$ has at least two distinct roots. It is clear that $\gcd(P_1(\alpha, \beta_j(\alpha), Z), P_2(\alpha, \beta_j(\alpha), Z)) = \mathrm{Sr}_k(\alpha, \beta_j(\alpha), Z) = \sum_{i=0}^{k} \mathrm{sr}_{k,i}(\alpha, \beta_j(\alpha)) Z^i$. $\gcd(P_1(\alpha, \beta_j(\alpha), Z), P_2(\alpha, \beta_j(\alpha), Z)) = \mathrm{Sr}_k(\alpha, \beta_j(\alpha), Z)$ and $\mathrm{Sr}_k(\alpha, \beta_j(\alpha), Z)$ has at least two distinct roots imply that $(\alpha, \beta_j(\alpha))$ is an apparent singularity.
□

Proposition 3.15. For $(j,k) \in \{1, \ldots, n\} \times \{2, \ldots j\}$, let α be a root of $\chi_{j,k}(X)$. The apparent singularity $(\alpha, \beta_j(\alpha))$ is a node if and only if

$$(\partial^2_{XY} h(\alpha, \beta_j(\alpha)))^2 - \partial^2_{X^2} h(\alpha, \beta_j(\alpha)) \partial^2_{Y^2} h(\alpha, \beta_j(\alpha)) \neq 0.$$

Proof. Let us write $h(X+\alpha, Y+\beta_j(\alpha)) = F_1 + F_2 + \ldots + F_n$, where F_i is a form in $\mathbb{Q}[\alpha][X,Y]$ of degree i. The apparent singularity $(\alpha, \beta_j(\alpha))$ is a double point if and only if $F_2 \neq 0$. It is well known that $F_2 = 1/2(\partial^2_{Y^2} h(\alpha, \beta_j(\alpha)) Y^2 + 2\partial^2_{XY} h(\alpha, \beta_j(\alpha)) XY + \partial^2_{X^2} h(\alpha, \beta_j(\alpha)) X^2)$ and its factorization gives the tangent lines of $\mathcal{C}(h(X,Y) = 0)$ at $(\alpha, \beta_j(\alpha))$. So the apparent singularity $(\alpha, \beta_j(\alpha))$ is a node if and only the discriminant in Y of the polynomial $F_2(1,Y)$ is different to 0. □

Theorem 3.16. $\mathcal{C}_{\mathbb{R}}$ is in generic position if and only if for any $(j,k) \in \{1, \ldots, n\} \times \{2, \ldots, j\}$ the polynomials $\partial^2_{XY} h(X, \beta_j(X)))^2 - \partial^2_{X^2} h(X, \beta_j(X)) \partial^2_{Y^2} h(X, \beta_j(X))$ and $\chi_{j,k}(X)$ are coprime.

Proof. $\mathcal{C}_{\mathbb{R}}$ is in generic position if and only if any apparent singularity is a node. So the result comes clearly from the previous proposition. □

3.4. Lifting and connection phase

In this section, we suppose that $\mathcal{C}_{\mathbb{R}}$ is in **generic position** that means that $\mathcal{C}_{\mathbb{R}}$ is in **pseudo-generic position**, $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$ is in **generic position as a plane** algebraic curve and any **apparent singularity** of $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$ is a **node**. To compute the topology

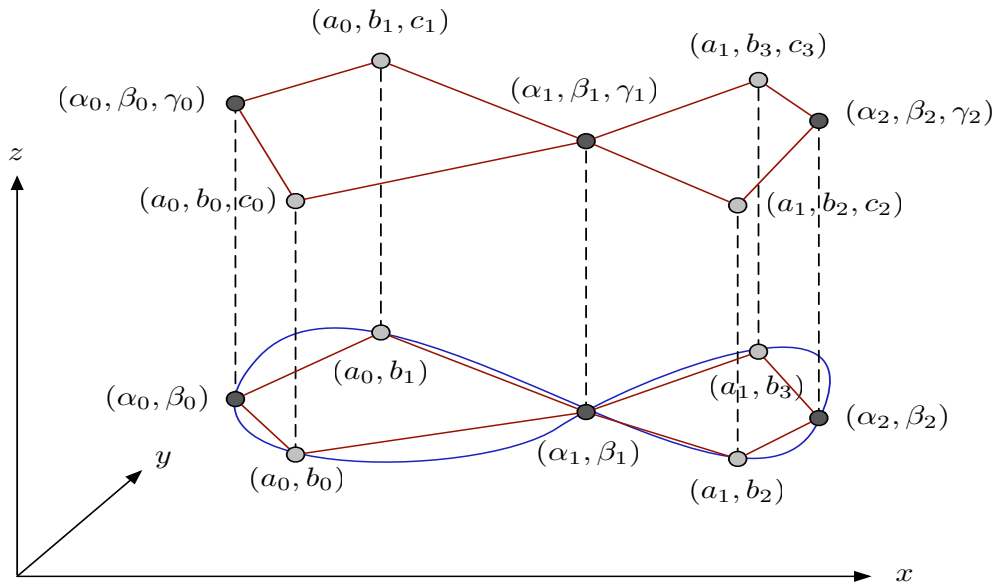

Fig. 2. Connection between real singularities and regular points.

of $\mathcal{C}_{\mathbb{R}}$, we first compute the topology of its projection on the (x, y)-plane and in second we lift the computed topology.

As mentioned in section 2, to compute the topology of a plane algebraic curve in generic position, we need to compute its critical fibers and one regular fiber between two critical ones. So to obtain the topology of $\mathcal{C}_{\mathbb{R}}$, we just need to lift the critical and regular fibers of $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$.

Here after we explain how this lifting can be done without any supplementary computation for the regular fibers and the real critical fibers. And for the special case of the apparent singular fibers, we present a new approach for the lifting and the connections.

3.4.1. Lifting of the regular points of $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$

The lifting of the regular fibers of $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$ is done by using the rational parametrizations given in Proposition 3.4.

3.4.2. Lifting of the real singularities of $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$

The lifting of the real singularities of $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$ is done by using the rational parametrizations given by 1. of Theorem 3.14.

3.4.3. Connection between real singularities and regular points

For a space curve in pseudo-generic position, the connections between real singularities and regular points are exactly those obtained on the projected curve using Grandine's sweeping algorithm [20] (see figure 3.4.3).

3.4.4. Lifting of the apparent singularities

Lifting of the topology around an apparent singularity is a little more complex. Above an apparent singularity of $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}})$, we first have to compute the z-coordinates and

secondly to decide which of the two branches passes over the other (see figure 3). We solve these problems by analyzing the situation at an apparent singularity.

According to Theorem 3.3 (2), $\mathcal{D} = \Pi_z(\mathcal{C}_{\mathbb{R}}) = \bigcup_{i=1}^{m} \mathcal{C}(\Delta_i)$, so in generic position, an apparent singularity is a cross point of a branch of $\mathcal{C}(\Delta_i)$ and a branch of $\mathcal{C}(\Delta_j)$ with $i, j \in \{1, \ldots, m\}$. So we have the following proposition.

Proposition 3.17. In generic position, if (α, β) is an apparent singularity of $\mathcal{D}$ such that $\Delta_i(\alpha, \beta) = \Delta_j(\alpha, \beta) = 0$, then the degree of the polynomial $\gcd(P_1(\alpha, \beta, Z), P_2(\alpha, \beta, Z) \in \mathbb{R}[Z]$ will be $(i + j)$.

Proof. Let (α, β) be an apparent singularity of $\mathcal{D}$ such that $\Delta_i(\alpha, \beta) = \Delta_j(\alpha, \beta) = 0$. It comes that $\deg_Z(\gcd(P_1(\alpha, \beta, Z), P_2(\alpha, \beta, Z)) \geqslant (i + j)$. Assume that $\deg_Z(\gcd(P_1(\alpha, \beta, Z), P_2(\alpha, \beta, Z)) > (i + j)$. So it exist $k \in \{1, \ldots, m\} \backslash \{i, j\}$ such that $\Delta_k(\alpha, \beta) = 0$ then (α, β) is a cross point of a branch of $\mathcal{C}(\Delta_i)$, a branch of $\mathcal{C}(\Delta_j)$ and a branch of $\mathcal{C}(\Delta_k)$ and is not a node. This is not possible because $\mathcal{C}_{\mathbb{R}}$ is in generic position and any apparent singularty is a node, so $\deg_Z(\gcd(P_1(\alpha, \beta, Z), P_2(\alpha, \beta, Z)) = (i+j)$. $\square$

Let (α, β) be an apparent singularity of $\mathcal{D}$ such that $\Delta_i(\alpha, \beta) = \Delta_j(\alpha, \beta) = 0$ and let γ_1, γ_2 be the corresponding z-coordinates. So by Proposition 3.17 and Proposition 7.3 $\mathrm{sr}_{0,0}(\alpha, \beta) = \ldots = \mathrm{sr}_{i,i}(\alpha, \beta) = \ldots = \mathrm{sr}_{j,j}(\alpha, \beta) = \ldots. = \mathrm{sr}_{i+j-1,i+j-1}(\alpha, \beta) = 0$. By Proposition 3.4, for any $(a, b, c) \in \mathcal{C}_{\mathbb{R}}$ such that $\Delta_i(a, b) = 0$ and $\mathrm{sr}_{i,i}(a, b) \neq 0$, we have $c = -\frac{\mathrm{sr}_{i,i-1}(a,b)}{i\,\mathrm{sr}_{i,i}(a,b)}$. So the function $(x, y) \longmapsto Z_i(x, y) := -\frac{\mathrm{sr}_{i,i-1}(x,y)}{i\,\mathrm{sr}_{i,i}(x,y)}$ gives the z-coordinate of any $(a, b, c) \in \mathcal{C}_{\mathbb{R}}$ such that $\Delta_i(a, b) = 0$ and $\mathrm{sr}_{i,i}(a, b) \neq 0$. $\Delta_i(\alpha, \beta) = 0$ but $\mathrm{sr}_{i,i}(\alpha, \beta) = 0$, so the function Z_i is not defined on (α, β). However, the function Z_i is continuously extensible on (α, β). Let u_1 be the slope of the tangent line of $\mathcal{C}(\Delta_i)$ at (α, β) and $t \in \mathbb{R}^*$. Let $\gamma_i(t) := Z_i(\alpha, \beta + tu_1) = -\frac{\mathrm{sr}_{i,i-1}(\alpha,\beta+tu_1)}{i\,\mathrm{sr}_{i,i}(\alpha,\beta+tu_1)}$. Knowing that the algebraic curve $\mathcal{C}_{\mathbb{R}}$ hasn't any discontinuity, it comes $\lim_{t\to 0^+} \gamma_i(t) = \lim_{t\to 0^-} \gamma_i(t) = \gamma_1$. By the same arguments, if we denote u_2 the slope of the tangent line of $\mathcal{C}(\Delta_j)$ at (α, β) and $\gamma_j(t) := Z_j(\alpha, \beta + tu_2) = -\frac{\mathrm{sr}_{j,j-1}(\alpha,\beta+tu_2)}{j\,\mathrm{sr}_{j,j}(\alpha,\beta+tu_2)}$, then $\lim_{t\to 0^+} \gamma_j(t) = \lim_{t\to 0^-} \gamma_j(t) = \gamma_2$. The values u_1, u_2, γ_1 and γ_2 are computed using Taylor formulas and certified numerical approximations.
Now it remains to decide which of the two branches pass over the other. This problem is equivalent to the problem of deciding the connection around an apparent singularity. Let (a, b_1, c_1) and (a, b_2, c_2) the regular points that we have to connect to $(\alpha, \beta, \gamma_1)$ and $(\alpha, \beta, \gamma_2)$. The question is which of the points (a, b_1, c_1) and (a, b_2, c_2) will be connected to $(\alpha, \beta, \gamma_1)$ and the other to $(\alpha, \beta, \gamma_2)$ (see figure 3)? In [5] Alcázar and Sendra give a solution using a second projection of the space curve but it costs a computation of a Sturm Habicht sequence of P_1 and P_2. Our solution does not use any supplementary computation. It comes from the fact that γ_1 is associated to u_1 and γ_2 to u_2. Knowing that u_1 is the slope of the tangent line of $\mathcal{C}(\Delta_i)$ at (α, β) and u_2 the slope of the tangent line of $\mathcal{C}(\Delta_j)$ at (α, β), so $(\alpha, \beta, \gamma_1)$ will be connected to (a, b_1, c_1) if (a, b_1) is on the branch associated to u_1. If (a, b_1) is not on the branch associated to u_1, then (a, b_1) is on the branch associated to u_2, so $(\alpha, \beta, \gamma_2)$ will be connected to (a, b_1, c_1) (see figure 4).

Remark 3.18. For a curve in generic position any apparent singularity is a node, so the slopes at an apparent singularity are always distinct that is to say $u_1 \neq u_2$.

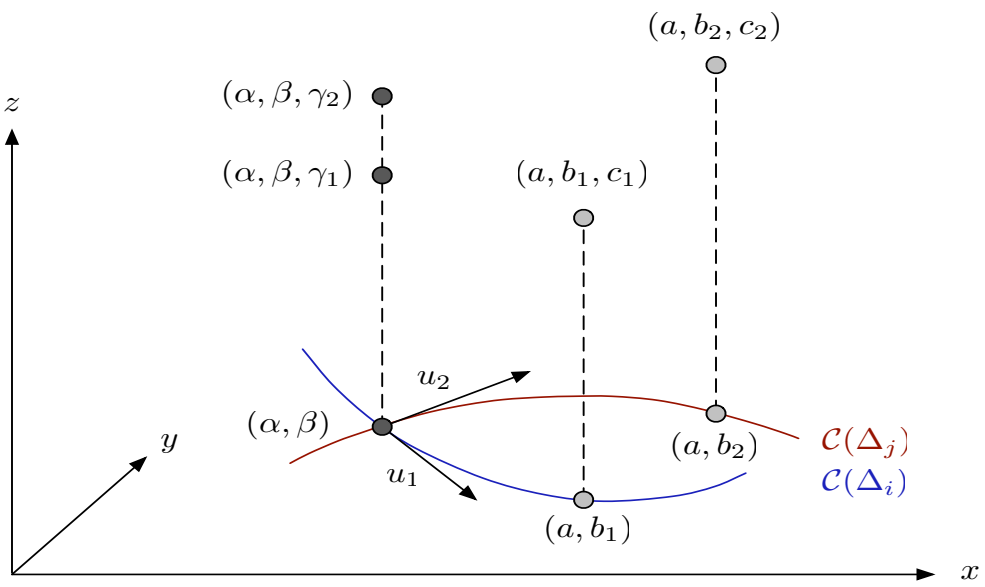

Fig. 3. Lifting of an apparent singularity.

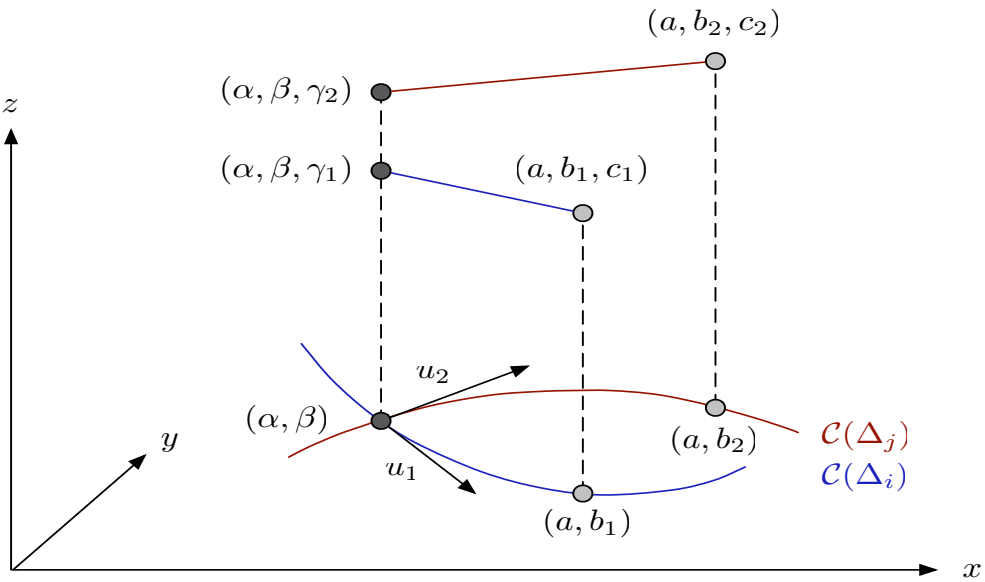

Fig. 4. Connection above an apparent singularity.

4. Isotopic meshing of an algebraic implicit surface

In this section, we describe an algorithm producing a piecewize linear structure isotopic to an algebraic surface. This algorithm strongly relies on the computation of the topology of a polar curve of the surface which is an implicit non-reduced space curve.

4.1. Description of the problem

Let $P \in \mathbb{Q}[X, Y, Z]$ be a square free polynomial and $\mathcal{S} := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : P(\alpha, \beta, \gamma) = 0\}$ be the real algebraic surface defined by P. Our goal is to compute a "correct" meshing of the surface $\mathcal{S}$.
Meshing is the process of computing, for a given surface, a representation consisting of pieces of simple surfaces patches. "Correct" means that the result should be topologically correct and geometrically close. It is not sufficient to require that a surface $\mathcal{S}$ and its mesh $\mathcal{S}'$ are homeomorphic. A torus and a knotted torus are homeomorphic when viewed as surfaces in isolation, but one would certainly not accept one as a topologically correct representation of the other. The following definition combines the strongest notion of having the correct topology with the requirement of geometric closeness.

Definition 4.1. An isotopy between two surfaces $\mathcal{S}, \mathcal{S}' \subset \mathbb{R}^3$ is a continuous mapping $\gamma : \mathcal{S} \times [0, 1] \to \mathbb{R}^3$ which, for any fixed $t \in [0, 1]$, $\gamma(., t)$ is a homeomorphism from $\mathcal{S}$ onto its image, and wich continuously deforms $\mathcal{S}$ into $\mathcal{S}'$: $\gamma(\mathcal{S}, 1) = \mathcal{S}'$.

We give a meshing algorithm for algebraic implicit surfaces that is based on sweeping a vertical plane over the surface. To guide the sweep, we use the topology of the polar variety of the surface. In contrast to previous methods our algorithm makes no smoothness or regularity assumptions about the input surface. The algorithm works for surfaces with self-intersections, fold lines, or other singularities.
In [2] the authors give an algorithm that needs the cuts of the surface on the singularies of its polar variety to be able to reconstruct the topology. This operation is quite difficult to certify because it requires the computation of the topology of plane algebraic curve of equation $P(\alpha, Y, Z) = 0$ where α is an algebraic real number.
With our new connection algorithm, we do not need to cut the surface on the x-critical points of its polar variety to be able to construct its topology. The connection is completly guided by the topology of the polar variety and the structure of its x-critical fibers, avoiding at the same time the difficulty of the computation of the cuts of the surface on the singularities of its polar variety and its cost.
Let us give a rough overview, concentrating geometric ideas before discussing the primitive geometric operations that are necessary for the algorithm.

4.2. Geometric ideas

Our goal is to find uniform regions in the (x, y)-plane where the surface can be regarded as a family of a constant number of function graphs of the form $z = h(x, y)$. We therefore analyze the surface outside the singular points and outside the points that have vertical tangents (the apparent contour). These points form the polar variety $\mathcal{P}$ of the surface $\mathcal{S}$.

Definition 4.2 (Polar Variety)**.** Let $P \in \mathbb{Q}[X, Y, Z]$ be a square free polynomial and $\mathcal{S} := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : P(\alpha, \beta, \gamma) = 0\}$ be the real algebraic surface defined by P. We call polar variety of $\mathcal{S}$ the following set:

$$\mathcal{P} := \left\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : P(\alpha, \beta, \gamma) = \partial_z P(\alpha, \beta, \gamma) = 0\right\}$$

When we cut away our surface at its polar variety, we obtain (x, y)-monotone surface patches that can be parameterized in x and y. After the computation of the polar variety of the surface, we subdivide the surface into vertical slabs by planes perpendicular to the x-axis. In contrast to previous approaches, these points do not include any x-critical point and any apparent singularity of the polar variety. Finally we mesh the resulting patches of the surfaces by computing a set of points, open segments and open triangles, wich are not self-intersecting, defining a simplicial complex isotopic to the original surface.

We summarize the geometric primitives that we need to provide in oder to make the algorithm work:

(1) We must be able to compute the topology of the polar variety of our surface.
(2) We must be able to compute the topology of the sections of our surface in slab points
(3) We must be able to connect two consecutives sections by exploiting the topology of the polar variety of the surface.
(4) We must be able to triangulate surface patches, avoiding self-intersection of segments and triangles.

Because the polynomial $P(X, Y, Z)$ is supposed to be square free, the dimension of its polar variety $\mathcal{P} := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : P(\alpha, \beta, \gamma) = \partial_z P(\alpha, \beta, \gamma) = 0\}$ is at most equal to one. So the computation of the topology of $\mathcal{P}$ will be done using the algorithm described in section 3. The computation of the topology of the plane sections of the surface will be done using the algorithm described in section 2. In the following section we describe the genericity conditions required and the connection algorithm of two consecutives section of the surface.

4.3. Genericity conditions, arcs ordering and connection algorithm

4.3.1. Genericity conditions

Definition 4.3. Let $P \in \mathbb{Q}[X, Y, Z]$ be a square free polynomial and $\mathcal{S} := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : P(\alpha, \beta, \gamma) = 0\}$ be the real algebraic surface defined by P. We say that $\mathcal{S}$ is in generic position if and only if its polar variety is in generic position (as an implicit space curve).

This condition excludes, for instance, a surface which consists of two equal spheres vertically above each other. The two silhouettes (equators) would coincide in the projection. It also excludes for example vertical cylinder (for wich the polar variety would be two dimentional).

The genericity of the position of a given surface will be certified during the step of the computation of the topology of its polar variety. If the genericity conditions are not fulfilled by the surface, we perform a change of coordinates and a restart of the algorithm.

We assume hereafter that our surface $\mathcal{S}$ is in generic position. Let us outline briefly the algorithm for algebraic surfaces before going into the details.

The first step consists in computing the topology of the polar variety. We apply the algorithm section 3 with $P_1 := P, P_2 := \partial_z P$, which computes a polygonal approximation of the polar variety which is isotopic to it. In this way, the algorithm computes x-critical values corresponding to x-critical points of the polar variety and singular points of its projection on the (x, y)-plane, adds intermediate "regular" x-values between them and computes the points of the polar variety corresponding to the regular x-values.
Next, we cut the surface by planes perpendicular to the x-axis on the "regular" x-values

of the polar variety, using the algorithm described in section 2. Note that the intersection of the polar variety with the vertical planes become critical points for the plane curves problem (see next proposition).
Then, we compute all the fibers of the surface containing the x-critical points of the polar variety.
Finally, we connect all the plane cuts of the surface using the topology of the polar variety and the structure of the computed fibers.

Before talking about connection algorithm let us give two simples but important properties of surfaces in generic position.

Let $(c_i := (\alpha_i, \beta_i, \gamma_i))_{i\in[\![1,n]\!]}$ be the ordered sequence, according to the first coordinate, of the x-critical points of the projection of $\mathcal{P}$ on the (x, y)-plane. Let $(r_i)_{i\in[\![1,n+1]\!]}$ be a sequence of rational values such that: $r_1 < \alpha_1 < r_2 < \alpha_2 < \ldots < \alpha_n < r_{n+1}$.
For $i \in [\![1, n+1]\!]$, let $\mathcal{R}_{r_i} := \{(r_i, y, z) \in \mathbb{R}^3 : P(r_i, y, z) = \partial_z P(r_i, y, z) = 0\}$ be the section of the polar variety by the plane of equation $x = r_i$ and $n_i := \#\mathcal{R}_{r_i}$. Let us denote $(p_1^{r_i}, \ldots, p_{n_i}^{r_i})$ the grading sequence, with respect to their y-coordinate, of the elements of $\mathcal{R}_{r_i}$.
For any $i \in [\![1, n+1]\!]$, let $\mathcal{C}_{r_i} := \{(r_i, y, z) \in \mathbb{R}^2 : P(r_i, y, z) = 0\}$ be the section of the surface $\mathcal{S}$ by the plane of equation $x = r_i$.

Proposition 4.4. For $i \in [\![1, n+1]\!]$, the y-critical points of the (y, z)-plane curve $\mathcal{C}_{r_i}$ are exactely the intersection points, $(p_1^{r_i}, \ldots, p_{n_i}^{r_i})$, of the polar variety with the vertical plane of equation $x = r_i$.

Proof. let $i \in [\![1, n+1]\!]$, by definition, the y-critical points of the (y, z)-plane curve $\mathcal{C}_{r_i} := \{(r_i, y, z) \in \mathbb{R}^2 : P(r_i, y, z) = 0\}$ are the solutions of $P(r_i, y, z) = \partial_z P(r_i, y, z) = 0$. The solutions of this system are exactely the intersection points, $(p_1^{r_i}, \ldots, p_{n_i}^{r_i})$, of the polar variety with the vertical plane of equation $x = r_i$. □

From the previous proposition it comes:

Proposition 4.5. For any $i \in [\![1, n+1]\!]$, the plane curve $\mathcal{C}_{r_i}$ is in generic position.

For any $i \in [\![1, n+1]\!]$, the topology of the plane curve $\mathcal{C}_{r_i}$ will be described by giving the arcs linking its y-critical points $(p_1^{r_i}, \ldots, p_{n_i}^{r_i})$. For the connection algorithm, it will be necessary to be able to order the computed arcs. That's the aim of the next sub-section.

4.3.2. Arcs ordering

Definition 4.6. Let $i \in [1, n+1]$ and $\mathcal{C}_{r_i} := \{(r_i, y, z) \in \mathbb{R}^2 : P(r_i, y, z) = 0\}$. We mean by *arc* of the plane curve $\mathcal{C}_{r_i}$, a connected smooth open subset of the curve which closure contains two distinct points of the sequence $(p_1^{r_i}, \ldots, p_{n_i}^{r_i})$ of its y-critical points. It is represented hereafter by a segment.

Definition 4.7. Let q and g be two distinct points of the sequence $(p_1^{r_i}, \ldots, p_{n_i}^{r_i})$ and $\mathcal{A}$ be the *class of the arcs* of $\mathcal{C}_{r_i}$ linking q and g. We will call support of $\mathcal{A}$ the bipoint (q,g).

The arcs of $\mathcal{C}_{r_i}$ linking two given points q and g of the sequence $(p_1^{r_i}, \ldots, p_{n_i}^{r_i})$ are naturally ordered. The order relation is the following one:

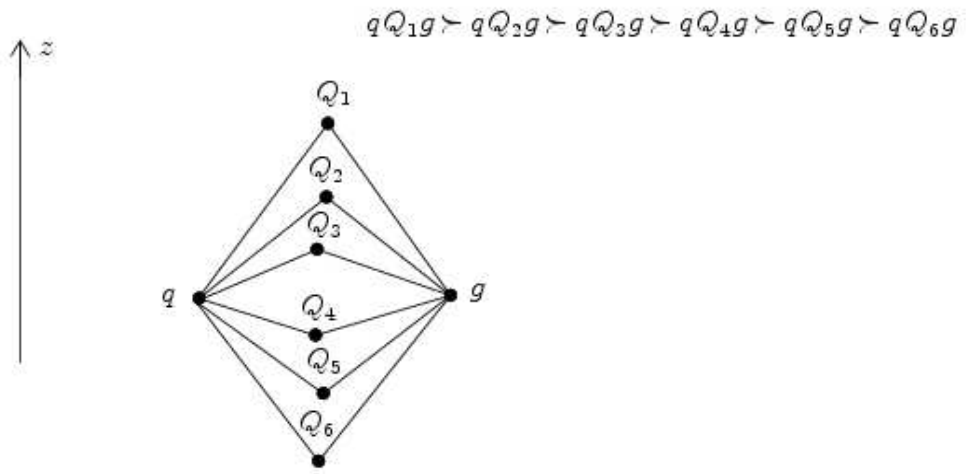

Fig. 5. An example of a class of arcs

Definition 4.8. We define the *arc ordering* as follows: Let qQ_1g and qQ_2g two different arcs linking two distinct points, q and g of the sequence $(p_1^{r_i}, \ldots, p_{n_i}^{r_i})$. The coordinates of the two points Q_1 and Q_2 verify: $x_{Q_1} = x_{Q_2}$, $y_{Q_1} = y_{Q_2}$ and $z_{Q_1} \neq z_{Q_2}$. So we define the following order relation: $qQ_1g \succ qQ_2g$ if and only if $z_{Q_1} > z_{Q_2}$ (see figure5)

Remark 4.9. We will use the previous order relation to arrange a given class of arcs with the same support.

Hereafter we extend the previous order relation to the case of arcs with support verifying a particular relation.

Definition 4.10. Let $q_1Q_1g_1$ and $q_2Q_2g_2$ be two arcs of $\mathcal{C}_{r_i}$ such that $[y_{q_1}, y_{g_1}] \subseteq [y_{q_2}, y_{g_2}]$ and $x_{Q_1} = x_{Q_2}$, $y_{Q_1} = y_{Q_2} \in [y_{q_1}, y_{g_1}]$ and $z_{Q_1} \neq z_{Q_2}$. So we define the following order relation: $q_1Q_1g_1 \succ q_2Q_2g_2$ if and only if $z_{Q_1} > z_{Q_2}$ (see figure 6)

The next step of the computation of an isotopic meshing of the surface $\mathcal{S}$ consists in connecting consecutively the computed regular sections $(\mathcal{C}_{r_i})_{i \in [\![1,n+1]\!]}$ of the surface. Hereafter, we will describe the algorithm to connect $\mathcal{C}_{r_1}$ to $\mathcal{C}_{r_2}$. For the other connections, we will apply recursively the same algorithm.

4.3.3. Connection algorithm of two consecutive sections, $\mathcal{C}_{r_1}$ and $\mathcal{C}_{r_2}$

We remind that $\mathcal{R}_{r_1} := \{(r_1, y, z) \in \mathbb{R}^3 : P(r_1, y, z) = \partial_z P(r_1, y, z) = 0\}$ (resp. $\mathcal{R}_{r_2} := \{(r_2, y, z) \in \mathbb{R}^3 : P(r_2, y, z) = \partial_z P(r_2, y, z) = 0\}$) is the section of the polar variety by the plane of equation $x = r_1$ (resp. $x = r_2$) and $n_1 := \#\mathcal{R}_{r_1}$ (resp. $n_2 := \#\mathcal{R}_{r_2}$), $(p_1^{r_1}, \ldots, p_{n_1}^{r_1})$ (resp. $(p_1^{r_2}, \ldots, p_{n_2}^{r_2})$) is the grading sequence, with respect to their y-coordinate, of the elements of $\mathcal{R}_{r_1}$ (resp.$\mathcal{R}_{r_2}$).
We denote $c := (\alpha, \beta, \gamma)$ the only x-critical point of the projection of the polar variety $\mathcal{P}$

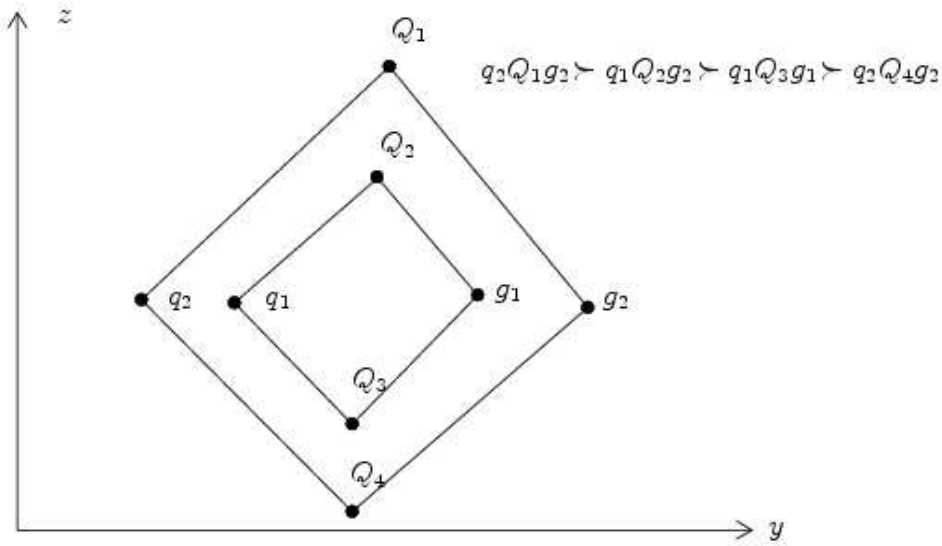

Fig. 6.

on the (x, y)-plane such that $r_1 < \alpha < r_2$.
Let UpPoints $:= \{(\alpha, \beta, z) \in \mathbb{R}^3 : P(\alpha, \beta, z) = 0 \,\text{and}\, z > \gamma\}$ be the grading sequence, with respect to their z-coordinate, of the points of the x-critical fiber of $\mathcal{S}$ located on top of the critical point $c := (\alpha, \beta, \gamma)$ and DownPoints $:= \{(\alpha, \beta, z) \in \mathbb{R}^3 : P(\alpha, \beta, z) = 0 \,\text{and}\, z < \gamma\}$ be the one of the points located under the x-critical point $c := (\alpha, \beta, \gamma)$. The x-critical fiber containing c is completely described by the give of the set $\{\text{UpPoints}, c, \text{DownPoints}\}$.

We also remind that the topology of the section $\mathcal{C}_{r_1}$ (resp. $\mathcal{C}_{r_2}$) is completly described by the arcs linking together the points of the sequence $(p_1^{r_1}, \ldots, p_{n_1}^{r_1})$ (resp. $(p_1^{r_2}, \ldots, p_{n_2}^{r_2})$). Let us denote $(\mathcal{A}_1, \ldots, \mathcal{A}_{m_1})$ (resp. $(\mathcal{B}_1, \ldots, \mathcal{B}_{m_2})$ the list of the classes of arcs of $\mathcal{C}_{r_1}$ (resp. $\mathcal{C}_{r_2}$).
The aim of the connection algorithm is to link the sequences $(\mathcal{A}_1, \ldots, \mathcal{A}_{m_1})$ to the sequences $(\mathcal{B}_1, \ldots, \mathcal{B}_{m_2})$ using the structure of the fiber {UpPoints, c, DownPoints} and the connections between $(p_1^{r_1}, \ldots, p_{n_1}^{r_1})$ and $(p_1^{r_2}, \ldots, p_{n_2}^{r_2})$ given .
During the connection algorithm, we will only need the following subroutines to connect:
(1) an arc to an arc,
(2) an arc to a point,
(3) an arc to an arc by passing at a given point.
Hereafter we describe three small algorithms to do that.

Algorithm 1: ConnectArcToArc

The aim of this algorithm is to connect two given arcs. Let $p_i^{r_1} Q_1 p_j^{r_1}$ *and* $p_k^{r_2} Q_2 p_l^{r_2}$ be two given arcs. We describe in figure 7 how the connections are done on an example.

Algorithm 2: ConnectArcToPoint

The aim of this algorithm is to connect a given arc to a given point. Let $p_i^{r_1} Q_1 p_j^{r_1}$ be a given arc and R be a given point. We describe in figure 8 how the connections are done on an example.

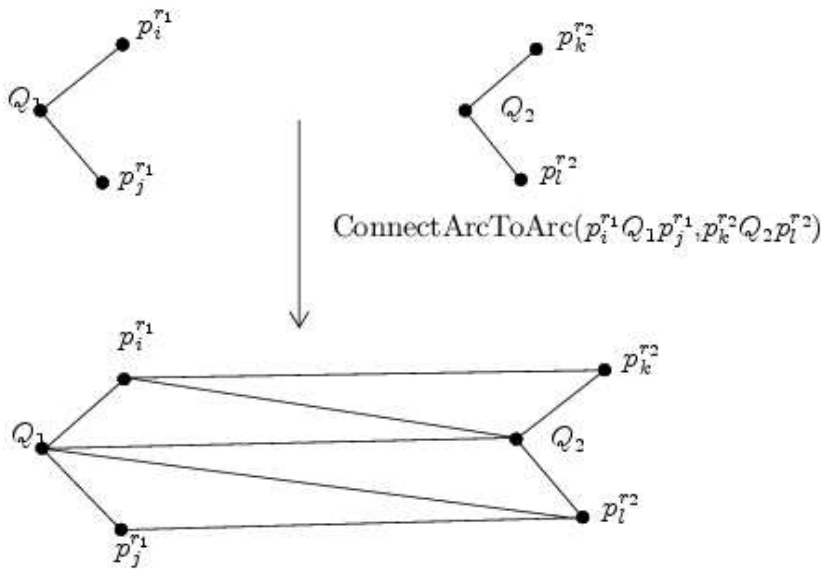

Fig. 7. Connecting an arc to an arc.

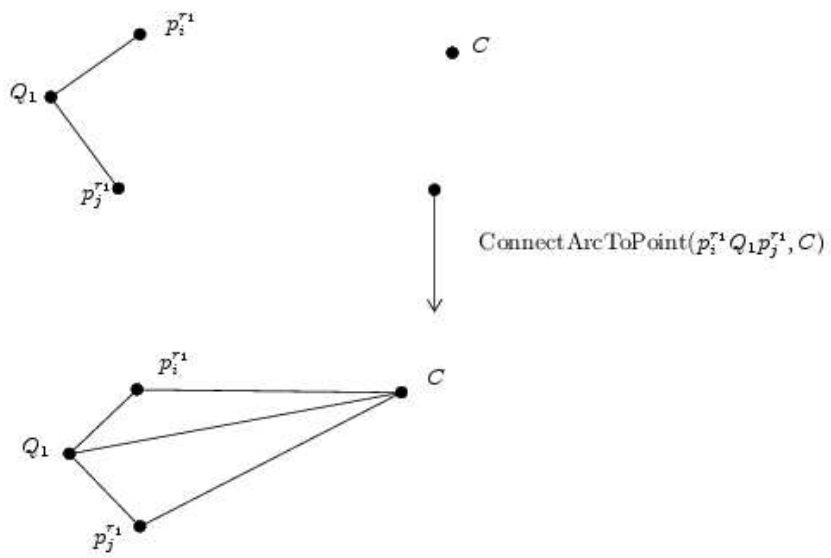

Fig. 8. Connecting an arc to a point.

Algorithm 3: ConnectArcToPointToArc

The aim of this algorithm is to connect two given arcs to given a point. We show in figure 9 how the connections are done.

Now we are going to describe the complete connection algorithm of two consecutive

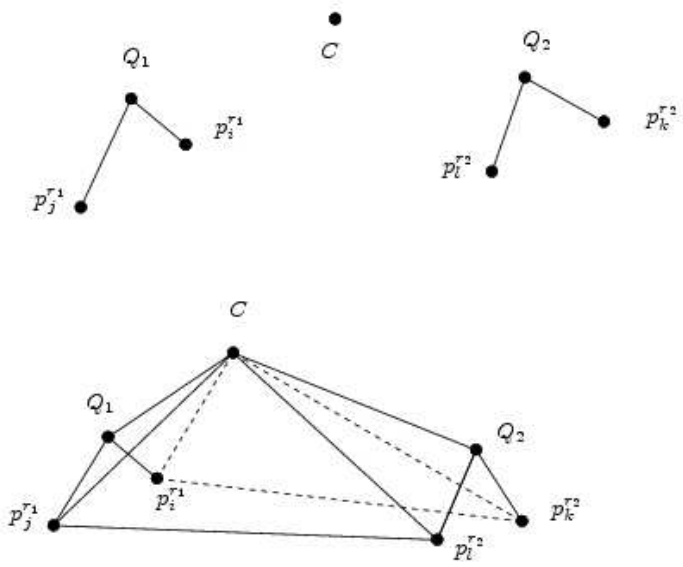

Fig. 9. Connecting two arcs to a point.

sections, $\mathcal{C}_{r_1}$ and $\mathcal{C}_{r_2}$, of our implicit surface. Let us remind that for any $j \in [\![1, m_1]\!]$, $\mathcal{A}_j$ is the ordered collection of arcs linking two given points of the sequence $(p_1^{r_1}, \ldots, p_{n_1}^{r_1})$.

For $j \in [\![1, m_1]\!]$, let (μ_1, μ_2) be the support of the arcs in $\mathcal{A}_j$ and let (θ_1, θ_2) be the points of $(p_1^{r_2}, \ldots, p_{n_2}^{r_2})$ connected to (μ_1, μ_2) via the polar variety of $\mathcal{S}$. The connection algorithm will be guided by the value of the integer $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2))$ where φ_c is the function defined in Remark 2.7. The integer $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2))$ may only takes three value, $\{-1, 0, 1\}$, corresponding to three distinct geometrics configurations.

(1) $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = -1$:
So $\varphi_c(\Pi_z(\mu_1)) = -1$ and $\varphi_c(\Pi_z(\mu_2)) = 1$ or $\varphi_c(\Pi_z(\mu_1)) = 1$ and $\varphi_c(\Pi_z(\mu_2)) = -1$. Whithout any loss of generality we can consider $\varphi_c(\Pi_z(\mu_1)) = -1$ and $\varphi_c(\Pi_z(\mu_2)) = 1$. By the Remark 2.7, this case corresponds to the geometric configuration described in figure 10.
In the connection algorithm, we will at first collect in $\mathcal{K}$ (resp. $\mathcal{L}$) the arcs of $\mathcal{C}_{r_1}$ (resp. $\mathcal{C}_{r_2}$) verifiying this constraint. Then, using the order relation given in Definition 4.10, we reorder the arcs in $\mathcal{K}$ and in $\mathcal{L}$. The connection of the arcs will be guided by the situation on the x-critical fiber $\{\text{UpPoints}, c, \text{DownPoints}\}$. We connect the first $\#$ UpPoints's arcs of $\mathcal{K}$ to the first $\#$ UpPoints arcs of $\mathcal{L}$, we connect the last $\#$ DownPoints arcs of $\mathcal{K}$ to the last $\#$ DownPoints arcs of $\mathcal{L}$, then we connect the reminded non connected arcs in $\mathcal{K}$ and in $\mathcal{L}$ to the x-critical point c.

(2) $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = 1$:
So $\varphi_c(\Pi_z(\mu_1)) = -1$ and $\varphi_c(\Pi_z(\mu_2)) = -1$ or $\varphi_c(\Pi_z(\mu_1)) = 1$ and $\varphi_c(\Pi_z(\mu_2)) = 1$. Whithout any loss of generality we can consider $\varphi_c(\Pi_z(\mu_1)) = 1$ and $\varphi_c(\Pi_z(\mu_2)) = 1$. By Remark 2.7, this corresponds to the geometric configuration described in figure 11.
When $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = 1$, the number of arcs of support (μ_1, μ_2) is equal

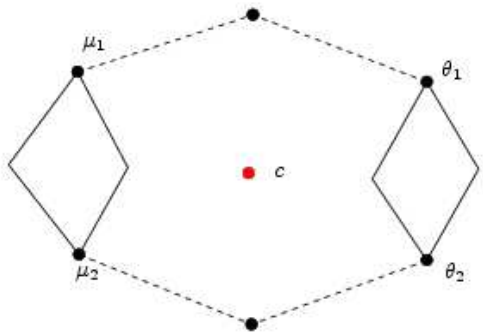

Fig. 10. $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = -1$

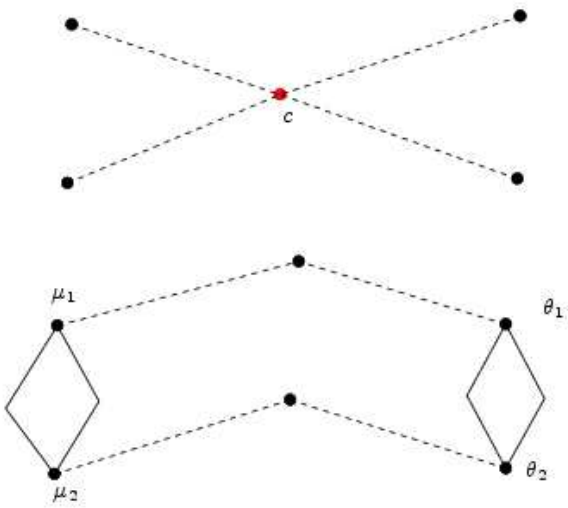

Fig. 11. $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = 1$

to the number of arcs of support (θ_1, θ_2) because if it wasn't this will mean that μ_1 or μ_2 is connected to c.

So we will just have to connect them one to one by respecting their ordering.

(3) $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = 0$:

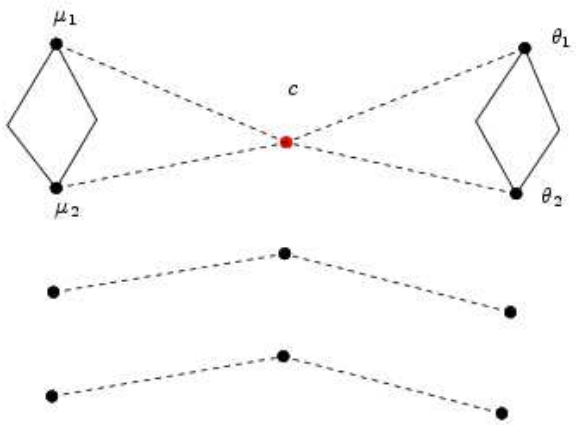

Fig. 12. $\varphi_c(\Pi_z(\mu_1)) = 0$ and $\varphi_c(\Pi_z(\mu_2)) = 0$

So $\varphi_c(\Pi_z(\mu_1)) = -1$ and $\varphi_c(\Pi_z(\mu_2)) = 0$ or $\varphi_c(\Pi_z(\mu_1)) = 1$ and $\varphi_c(\Pi_z(\mu_2)) = 0$ or $\varphi_c(\Pi_z(\mu_1)) = 0$ and $\varphi_c(\Pi_z(\mu_2)) = 1$ or $\varphi_c(\Pi_z(\mu_1)) = 0$ and $\varphi_c(\Pi_z(\mu_2)) = -1$ or $\varphi_c(\Pi_z(\mu_1)) = 0$ and $\varphi_c(\Pi_z(\mu_2)) = 0$. Whithout any lost of generality we can consider the cases $\varphi_c(\Pi_z(\mu_1)) = 1$, $\varphi_c(\Pi_z(\mu_2)) = 0$ and $\varphi_c(\Pi_z(\mu_1)) = 0$ and $\varphi_c(\Pi_z(\mu_2)) = 0$. By Remark 2.7, these cases correspond to the geometric configurations described in figure 12 and 13:

When $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = 0$, we will just have to connect all the arcs of support (μ_1, μ_2) to the x-critical point c.

The proof of the correctness of the algorithm is a direct adaptation of the proof given in [2]. For the sake of completness we give the main ideas and results of the proof and we refer to [2] for more details.

4.3.4. Why we get the topology

The general idea of the algorithm is to detect where *some topological changes* in the surface $\mathcal{S}$ happen. We recall why in-between the events that we have computed in the previous sub-sections, the topology is locally trivial. This result is used to describe explicitly the isotopy between the mesh and the surface.

The fundamental notion is the Whitney stratification. It is a decomposition of the variety into smooth parts that fit together "regularly". Here are some definitions:

Definition 4.11. A stratification of a (semi-algebraic) variety $A \subset \mathbb{R}^n$ is a locally finite partition of A into smooth submanifolds called strata.

Definition 4.12. Let (X, Y) be two strata and $p \in \overline{X} \cap Y \subset \mathbb{R}^n$. X is Whitney-regular at p along Y if for any sequences $x_n \in X$, $y_n \in Y$ converging to p, $l = \lim_{n\to+\infty} \overline{x_n y_n} \subset \mathcal{T} = \lim_{n\to+\infty} T_{x_n} X$, where $T_x X$ is the tangent space of X at the point x.

Algorithm 4.1: `Surface Connection`

h

Input:

- $(\mathcal{A}_1, \ldots, \mathcal{A}_{m_1})$ and $(\mathcal{B}_1, \ldots, \mathcal{B}_{m_2})$ the classes of arcs describing the topology of $\mathcal{C}_{r_1}$ and $\mathcal{C}_{r_2}$
- the x-critical fiber $\{\text{UpPoints}, c, \text{DownPoints}\}$,
- the function φ_c, defined in the Remark 2.7, associated to the projection of $\mathcal{P}$.

Output:

$\mathcal{E} := (\mathcal{B}_1, \ldots, \mathcal{B}_{m_2})$; $j := 1$; $\mathcal{K} := \{\}$; $\mathcal{L} := \{\}$;

While $j < m_1$ do

- $\Omega := \mathcal{A}_j$; (μ_1, μ_2) :=the support of the arcs in $\mathcal{A}_j$,
- *In this step we collect the arcs that will be connected by passing probably over the critical point c.*

  if $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = -1$ then do {
  $\mathcal{K} := \mathcal{K} \bigcup \Omega$;
  (θ_1, θ_2) :=the points of $(p_1^{r_2}, \ldots, p_{n_2}^{r_2})$ connected to (μ_1, μ_2),
  Λ :=the element of $\mathcal{E} := (\mathcal{B}_1, \ldots, \mathcal{B}_{m_2})$ of support (θ_1, θ_2),
  $\mathcal{L} := \mathcal{L} \bigcup \Lambda$; $\mathcal{E} := \mathcal{E} \backslash \Lambda$,
  }
- *In this step, we collect and connect the arcs living in the same cylinder*

  if $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = 1$ then do {
  $(\theta_1, \theta_2) :=$ the points of $(p_1^{r_2}, \ldots, p_{n_2}^{r_2})$ connected to (μ_1, μ_2),
  $\Lambda :=$ the element of $\mathcal{E} := (\mathcal{B}_1, \ldots, \mathcal{B}_{m_2})$ of support (θ_1, θ_2),
  For k from 1 to $\#\Omega$ do { ConnectArcToArc$(\Omega[k], \Lambda[k])$; }
  $\mathcal{E} := \mathcal{E} \backslash \Lambda$,
  }
- *In this step, we collect and connect to c the arcs which support points are linked to c via the polar variety curve.*

  if $\varphi_c(\Pi_z(\mu_1)) * \varphi_c(\Pi_z(\mu_2)) = 0$ then do
  For k from 1 to $\#\Omega$ do { ConnectArcToPoint$(\Omega[k], c)$; }

end while.

if $\#\mathcal{E} \neq 0$ then for i in 1 to $\#\mathcal{E}$ do {
  $\Lambda := \mathcal{E}[i]$;
  For k from 1 to $\#\Lambda$ do { ConnectArcToPoint$(\Lambda[k], c)$; }
}

Reorder the arcs in $\mathcal{K}$ and in $\mathcal{L}$ using the order relation given in Definition 4.10, then do
  Σ_1 :=the ordered sequence of the points in UpPoints;
  Σ_2 :=the ordered sequence of the points in DownPoints;
  For k from 1 to $\#\Sigma_1$ do { ConnectArcToPointToArc$(\mathcal{K}[k], \Sigma_1[k], \mathcal{L}[k])$; };
  For k from 1 to $\#\Sigma_2$ do {
  ConnectArcToPointToArc$(\mathcal{K}[(\#\mathcal{K}) - k], \Sigma_2[\#(\Sigma_2) - k], \mathcal{L}[\#(\mathcal{L}) - k])$; };
  For k from $\#\Sigma_1 + 1$ to $\#\mathcal{K} - \#\Sigma_2$ do { ConnectArcToPoint$(\mathcal{K}[k], c)$; };
  For k from $\#\Sigma_1 + 1$ to $\#\mathcal{L} - \#\Sigma_2$ do { ConnectArcToPoint$(\mathcal{L}[k], c)$; };

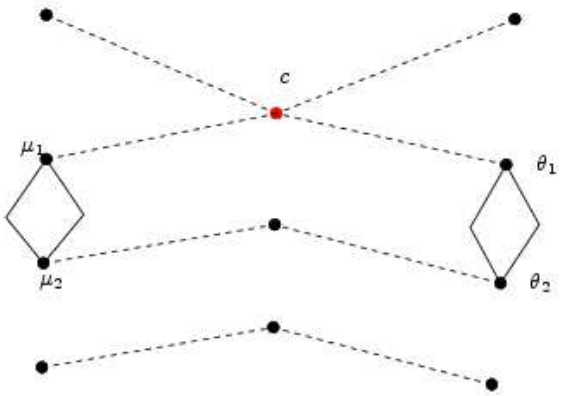

Fig. 13. $\varphi_c(\Pi_z(\mu_1)) = 0$ and $\varphi_c(\Pi_z(\mu_2)) = 1$

A Whitney stratification of a variety S is a stratification of S so that all pairs of strata are Whitney-regular.

We recall that any semi-algebraic variety $A \subset \mathbb{R}^n$ admits a Whitney stratification [21].

Definition 4.13. For Z and W two stratified sets, a differential map $f : Z \to W$ is a stratified submersion at a point p of Z if the differential map at p of f, $Df : T_p(Z_\sigma) \to T_{f(p)}(W_\tau)$ is surjective. Where Z_σ and W_τ are the strata of Z and W containing p and $f(p)$.

Definition 4.14. If Z and W are two stratified sets, a continuous map $f : Z \to W$ is proper if the inverse image of any compact set of W is a compact of Z.

The main used theorem is Thom's lemma [21].

Theorem 4.15 (Thom's first isotopy lemma)**.** Let Z be a Whitney stratified subset of $\mathbb{R}^m$ and $\pi : Z \to \mathbb{R}^n$ be a proper stratified submersion. Then there is a stratum preserving homeomorphism

$$h : Z \to (\pi^{-1}(0) \cap Z) \times \mathbb{R}^n$$

which is smooth on each stratum and such that π factorizes via the projection to the

second component $\mathbb{R}^n$.

$$\begin{tikzcd} Z \arrow[r, "\pi"] \arrow[d, "h"'] & \mathbb{R}^n \\ (\pi^{-1}(0) \cap Z) \times \mathbb{R}^n \arrow[ur, "\nu"'] & \end{tikzcd}$$

This means that Z is homeomorphic to the cylinder with base $\pi^{-1}(0) \cap Z$. In our case, we will apply the theorem with $Z = S_B$, $m = 3$, $n = 1$ and π the projection on the x-axis which is automatically proper as we work in a ball B which is compact.

Remind that $P \in \mathbb{Q}[X, Y, Z]$ is a square free polynomial, $\mathcal{S} := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : P(\alpha, \beta, \gamma) = 0\}$ is the real algebraic surface defined by P and
$\mathcal{P} := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : P(\alpha, \beta, \gamma) = \partial_z P(\alpha, \beta, \gamma) = 0\}$ is the polar variety of $\mathcal{S}$.
We suppose that $\mathcal{S}$ is in generic position with respect to the (x, y)-plane. So we have the following theorem:

Theorem 4.16. Let
- S^0 be the inverse image of the set of the singular points of $\Pi_z(\mathcal{P})$, each point is considered as a stratum,
- S^1 the set of the connected components of $\mathcal{P} - S^0$, (each connected component is a stratum),
- S^2 the set of the connected components of $\mathcal{S} - \mathcal{P}$ (each connected component is a stratum).
- S^3 the set of connected components of $\mathbb{R}^3 - \mathcal{S}$ (each connected component is a stratum).

Then (S^0, S^1, S^2, S^3) is a Whitney stratification of $\mathbb{R}^3$ compatible with $\mathcal{S}$.

By Theorem 4.16 and using Thom's lemma (Theorem 4.15), we deduce that in between two consecutive critical sections, the topology of the sections is constant. We have computed the topology of regular sections, in between two successive critical ones. So now, in order to prove the isotopy of the surface and the mesh, we have two things to verify:

a) From a topological point of view, we define the good connections between the sections.

b) The mesh is isotopic to the surface.

It is clear that the triangulation we compute does not create holes, because it refines the topological complex of $\mathcal{P}$ and we do not create intersection of open triangles because the arcs are connected by respecting their ordering. For the construction of the isotopy between the surface and its mesh we, refer to [2].

5. Complexity analysis

This section is devoted to the complexity analysis of our approach. Two main points are considered:

(1) the intrinsec complexity of the approach, i.e. the number of points computed in the algorithm and their bit size in the worst case which is a measure of the size of the output independently of the algorithm to compute those points;

(2) the binary complexity of our algorithm.

We consider that the input polynomial $P(X,Y,Z)$ defining the surface $\mathcal{S}$ lies in $\mathbb{Z}[X,Y,Z]$. We show that the complexity of our algorithm is $\tilde{\mathcal{O}}_B(d^{15}\tau)$. To our knowledge, this is the first time that a bound on the binary complexity of the computation of the topology of implicit surfaces is given.

5.1. Notations and basic results

Let $a \in \mathbb{Z} \setminus \{0\}$, we denote $\mathcal{L}(a) = \lceil \log_2 |a| \rceil$. The notation $\mathcal{O}_B$ means the binary complexity and $\tilde{\mathcal{O}}_B$ means the binary complexity where logarithmic factors are ignored. We denote $M(\tau)$ binary cost of the multiplication of two interger of size τ and $M(d,\tau)$ the binary complexity of the multiplication of two polynomials of degree d with size of coefficients bounded by τ. Using fast Fourier transform we have $M(\tau) \in \mathcal{O}_B(\tau \log_2^a(\tau))$ and $M(d,\tau) \in \mathcal{O}_B(d\tau \log_2^a(d\tau))$ for a constant a.

If A is a polynomial of with integer coefficients (with one or several variables), we denote $\mathcal{L}(A)$ the maximal size of its coefficients. The following result can be found in [34]:

Proposition 5.1. Let A and $B \in \mathbb{Z}[X]$ with degree at most d and $\tau = \max(\mathcal{L}(A), \mathcal{L}(B))$. There is an algorithm computing the Sturm-Habicht sequence of $A(X)$ and $B(X)$ with complexity lying in $\mathcal{O}_B(d^2 M(d\tau))$.

The roots isolation can be given using an algorithm computing root approximations with a choosen precision that do not depend on root isolation. Using an algorithm describe by Pan in [30], computing approximation of roots with relative precision ε in $\tilde{\mathcal{O}}_B(d^3\tau + d\varepsilon)$, with $\varepsilon \in \tilde{\mathcal{O}}(d\tau)$ which is the separation bound for a polynomial of degree with integer coefficients of size τ, isolation is achieved according to the following proposition:

Proposition 5.2. Let $A \in \mathbb{Z}[X]$ with degree d and with $\mathcal{L}(A) = \tau$ and $\tau \in \mathcal{O}(d)$. The cost of root isolation is bounded by $\tilde{\mathcal{O}}_B(d^3\tau)$ and the endpoints bit-size of the isolating intervals is $\tilde{\mathcal{O}}(d\tau)$.

Now let F and G be two polynomials lying in $\mathbb{Z}[Y_1,\ldots,Y_k][X]$ with $\deg_X(F) = p \geqslant q = \deg_Y(G)$, $\deg_{Y_i}(F)$ and $\deg_{Y_i}(G) \leqslant d_i$. We denote $d = \prod_{i=1}^{k} d_i$ and we assume that the coefficients of both F and G are bounded by τ. The four next results can be easily deduced from [31]:

Proposition 5.3. There is an algorithm computing $\mathrm{Sr}_{i,X}(F,G)$ with cost in $\tilde{\mathcal{O}}_B(q(p+q)^{k+2}d\tau)$.

Proposition 5.4. We have $\mathcal{L}(Sr_{X,i}(F,G)) = \mathcal{O}(\max\{p,q\}\tau)$.

Corollary 5.5. If F and G are polynomials in $\mathbb{Z}[X,Y]$ of degree d and e respectively and with coefficients bounded by τ, then $Sr_Y(F,G)$ can be computed in $\tilde{\mathcal{O}}_B(ed(d+e)^3\tau)$, and so, if $d = e+1$, this can be done in $\tilde{\mathcal{O}}_B(d^5\tau)$ and the size of $\mathcal{L}(Sr_{Y,i}(F,G)) = \mathcal{O}(d\tau)$.

Corollary 5.6. If F and G are polynomials in $\mathbb{Z}[X,Y,Z]$ of degree d and e respectively and with coefficients bounded by τ, then $Sr_Z(F,G)$ can be computed in $\tilde{\mathcal{O}}_B(ed(d+e)^4\tau)$, and so, if $d = e+1$, this can be done in $\tilde{\mathcal{O}}_B(d^6\tau)$ and the size of $\mathcal{L}(Sr_{Z,i}(F,G)) = \mathcal{O}(d\tau)$.

5.2. Bound on the size of the output

The following theorem gives an asymptotic bound on the number of points to be computed in our approach. Even if our approach improve several pratical aspects of the algorithm proposed in [2], the worst case gives the same asymptotic bound.

Theorem 5.7. If the degree of the implicit equation defining the surface is d the number of points needed is at most $\mathcal{O}(d^7)$.

Proof. We recall that our approach consists in a sweeping along a line, so we compute slices and make conection between slices. Denote $P(X,Y,Z)$ the implicit equation of $\mathcal{S}$. The first resultant $\mathrm{Res}_Z(P(X,Y,Z),\partial_Z P(X,Y,Z)) = \Delta(X,Y)$ as at most degree d^2 and we denote $h(X,Y)$ its squarefree part which has the same degree in the worst case. Now, we have to compute the topology of the planar curve defined by $h(X,Y)$ and to do so, we have to compute its x-critical values. Those values are given by the roots of $\Theta(X) = \mathrm{Res}_y(\Delta(X,Y),\partial_Y\Delta(X,Y))$ and it degree is bounded by d^4. We compute the topology of the curve defined by $\Delta(X,Y)$ by computing the critical fibers and one regular fiber in each intervals defined by those critical values. The fibers contain at most d^2 points, so we have computed $\mathcal{O}(d^6)$ points. Now, we have to compute the topology of the regular slice corresponding to regular fibers of the planar curve. The number of points needed to do that is proportional the number of points on the fibers of the points already computed on the planar curve and there is at most d such point on the fibers. This leads us to the given bound. □

5.3. Complexity of our algorithm

The two more complex types of computations done in our algorithm are the elemination steps (computation of Sturm-Habicht sequences) and roots isolation. We will see that the complexity is dominated by the root isolation of the last computed resultant. Remark that, from the computational point of view, the worst case is reach when the computed resultants during the algorithm ($\Delta(X,Y)$ and $\Theta(X)$) are squarefree. We will need to bound the complexity of the computation together with the size of the computed objects. Let $P \in \mathbb{Z}[X,Y,Z]$ denotes the implicit equation of the surface supposed to be of degree d and with $\mathcal{L}(P) = \tau$.

Theorem 5.8. The computation of the topology of a polar curve of the surface defined by P can be given in $\tilde{\mathcal{O}}_B(d^{15}\tau)$.

Proof. The first step is the computation of the Sturm-Habicht sequence associated to $P(X,Y,Z)$ and $\partial_Z P(X,Y,Z)$ with respect to the variable Z. This is done with complexity in $\tilde{\mathcal{O}}_B(d^6\tau)$ and the resultant $\Delta(X,Y)$ has degree $\mathcal{O}(d^2)$ and $\mathcal{L}(\Delta(X,Y)) \in \mathcal{O}(d\tau)$ (using Corollary 5.6). We take the squarefree part of $\Delta(X,Y)$ but since we still have $\mathcal{O}(d\tau) \in \mathcal{O}(d^2)$ because $\tau \in \mathcal{O}(d)$, it does not change the complexity of the algorithm (we have already compute the Sturm-Habicht sequence and we can use it to compute the squarefree part of Δ). The next step is the computation of the Sturm-Habicht sequence associated to $\Delta(X,Y)$ and $\partial_Y\Delta(X,Y)$ with respect to the variable Y. This step is done in $\tilde{\mathcal{O}}(d^{11}\tau)$ and the resultant computed has coefficients of size at most $\mathcal{O}(d^3\tau)$ and degree $\mathcal{O}(d^4)$ (using Corollary 5.5). Finally, we have to isolate the roots of the last resultant and this is done using at most $\tilde{\mathcal{O}}_B(d^{15}\tau)$ and the bit-size of the end-points of the isolating interval is $\tilde{\mathcal{O}}_B(d^7\tau)$ (using Proposition 5.2). □

Remark 5.9. In general, the knowledge of a factorization of the last resultant $\Theta(X) := \mathrm{Res}_Y(\Delta(X,Y), \partial_Y \Delta(X,Y))$ is a great improvement for practical computation together with the knowledge of the Sturm-Habicht sequence.

For the lifting phases, it is not difficult to see that we use parametrization using subresultant sequences, so that computed values are bounded by evaluation of coefficients of subresultants at rational points of size bounded by $\mathcal{O}(d^7\tau)$ and that finally, all other algebraic computation costs are dominated by the one of root isolation of $\Theta(X)$. The connection algorithm use only basic comparisons and very few algebraic computation using objects already computed. This shows that the cost of the computation of the surface topology is dominated by the computation of the topology of one of it polar variety. We finally have the following theorem:

Theorem 5.10. Let $\mathcal{S}$ an implicit surface defined by a degree d polynomial with integer coefficients of size at most τ. The topology of $\mathcal{S}$ can be computed with complexity in $\tilde{\mathcal{O}}_B(d^{15}\tau)$.

6. Implementation and experiments

A complete implementation of our algorithm has been written using the Computer Algebra System MATHEMAGIX [1]. Results are visualized using the algebraic geometric modeler Axel [2], which allows the manipulation of geometric objects with algebraic representation such as implicit or parametric curves or surfaces.
Since existing methods have no publicly available implementations, the following table only reports our experiments, performed on an Intel(R) Core machine clocked at 2GHz with 1GB RAM.

Surface	$P(X,Y,Z)$	Time (s)
1	$x^4 - y^4 - z^2$	0.33
2	$x^5 - y^2 - z^2$	0.36
3	$x^4 + y^2 - z^2$	0.31
4	$-xz^2 - z^3 + y^2$	0.40
5	$x^5 + y^2 - z^2$	0.32
6	$z(x^2+y^2+z^2-1)(x^2+4x+y^2+z^2+3)$	0.82
7	$-x^2z - 2xyz + 2yz^2 + z^3 + x^2 +$ $2xy + 2y^2 + 2yz + 2z^2 - 1$	0.90

[1] www.mathemagix.org
[2] axel.inria.fr

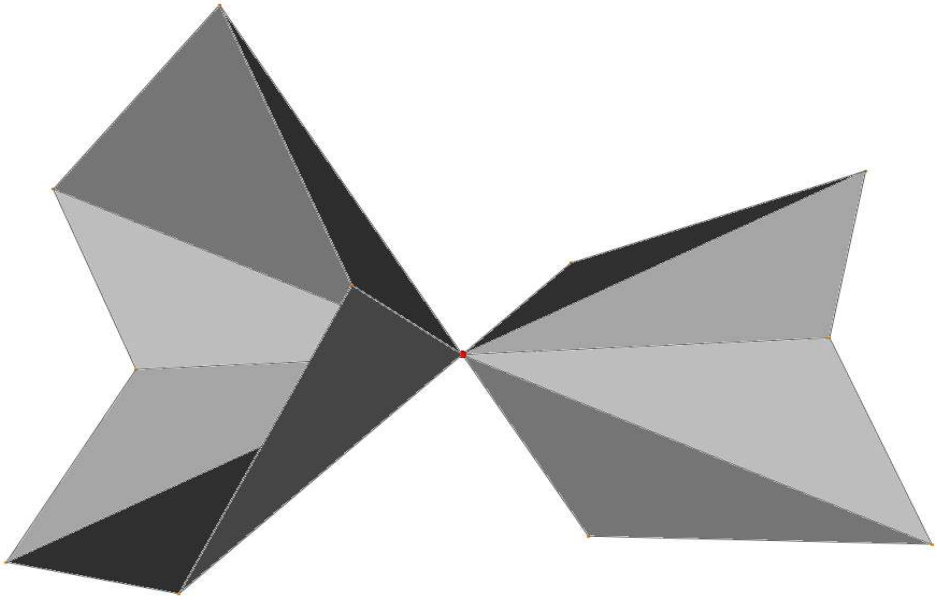

Fig. 14. Surface 3

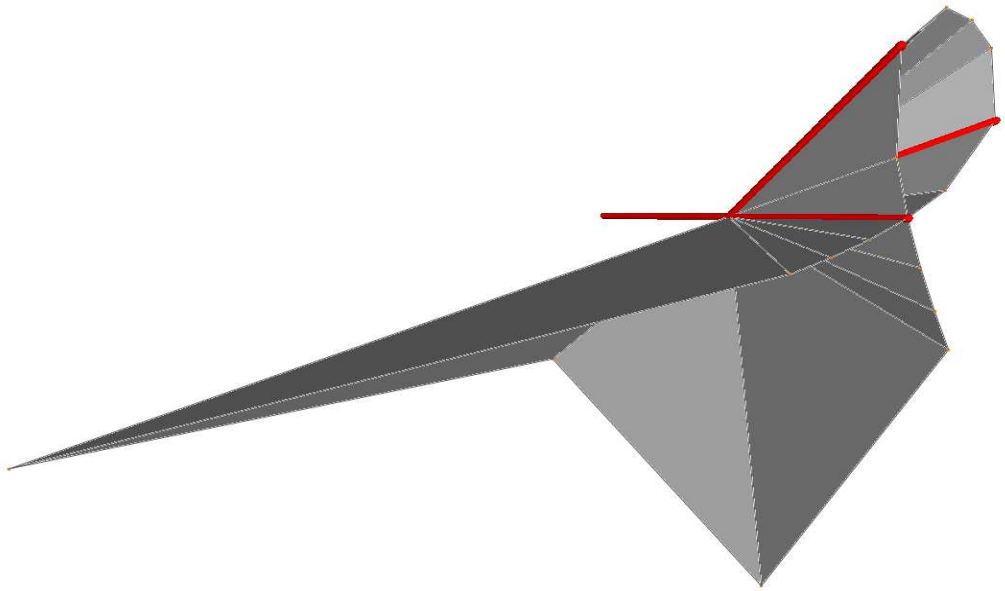

Fig. 15. Surface 4

References

[1] Lionel Alberti, Georges Comte, and Bernard Mourrain. Meshing implicit algebraic surfaces: the smooth case. In L.L. Schumaker M. Maehlen, K. Morken, editor,

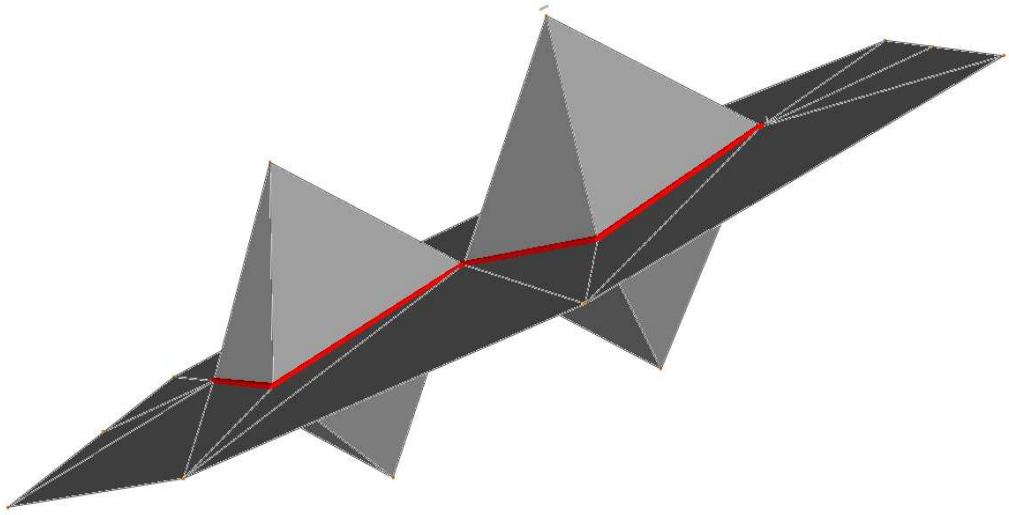

Fig. 16. Surface 6

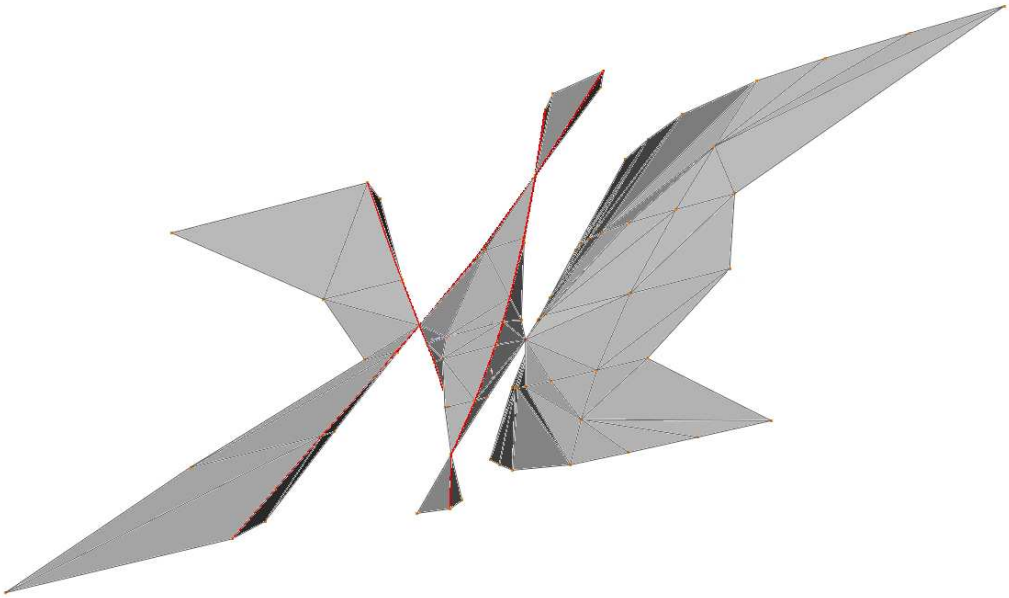

Fig. 17. Surface 7

Mathematical Methods for Curves and Surfaces: Tromso'04, pages 11–26. Nashboro, 2005.

[2] Lionel Alberti, Bernard Mourrain, and Jean P. Tecourt. Isotopic triangulation of a real algebraic surface. In *Preprint*. 2009.

[3] Lionel Alberti, Bernard Mourrain, and Jean-Pierre Técourt. Isotopic triangulation of a real algebraic surface. *J. of Symbolic Computation*, 2008.

[4] Lionel Alberti, Bernard Mourrain, and Julien Wintz. Topology and arrangement computation of semi-algebraic planar c urves. *Computer Aided Geometric Design*, 25:631–651, 2008.

[5] Juan Gerardo Alcázar and J. Rafael Sendra. Computation of the topology of real algebraic space curves. *J. Symbolic Comput.*, 39(6):719–744, 2005.

[6] Stefan Arnborg and Huichun Feng. Algebraic decomposition of regular curves. *J. Symbolic Comput.*, 5(1-2):131–140, 1988.

[7] Dennis S. Arnon and Scott McCallum. A polynomial-time algorithm for the topological type of a real algebraic curve. *J. Symbolic Comput.*, 5(1-2):213–236, 1988.

[8] Chandrajit. Bajaj, Christoph M. Hoffmann, Eric. Lynch, and John. E. H. Hopcroft. Tracing surface intersections. *Comput. Aided Geom. Design*, 5(4):285–307, 1988.

[9] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in Real Algebraic Geometry.* Springer-Verlag, Berlin, 2003. ISBN 3-540-00973-6.

[10] Riccardo Benedetti and Jean-Jacques Risler. *Real algebraic and semi-algebraic sets.* Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1990.

[11] J. Bochnak, M. Coste, and M.-F. Roy. *Géométrie Algébrique Réelle.* Springer-Verlag, Heidelberg, 1987.

[12] Jean-Daniel Boissonnat, David Cohen-Steiner, Bernard Mourrain, Günter Rote, and Gert Vegter. *Meshing of Surfaces*, pages 459–478. Mathematics and Visualisation. Springer-Verlag, 2006.

[13] Michel Coste. An introduction to semi-algebraic geometry. RAAG network school, 2002.

[14] Michel Coste and Marie-Françoise Roy. Thom's lemma, the coding of real algebraic numbers and the computation of the topology of semi-algebraic sets. *J. Symbolic Comput.*, 5(1-2):121–129, 1988.

[15] Daouda Niang Diatta, Olivier Ruatta, and Bernard Mourrain. On the computation of the topology of a non-reduced space curve. In *ISSAC 2008*, pages 47–54. ACM, 2008.

[16] Arno Eigenwillig, Michael Kerber, and Nicola Wolpert. Fast and exact geometric analysis of real algebraic plane curves. In *ISSAC 2007*, pages 151–158. ACM, New York, 2007.

[17] M'hammed El Kahoui. Topology of real algebraic space curves. *J. Symbolic Comput.*, 43(4):235–258, 2008.

[18] Elisabetta Fortuna, Patricia M. Gianni, and Domenico Luminati. Algorithmical determination of the topology of a real algebraic surface. *J. Symb. Comput.*, 38(6):1551–1567, 2004.

[19] Elisabetta Fortuna, Patricia M. Gianni, Paola Parenti, and Carlo Traverso. Algorithms to compute the topology of orientable real algebraic surfaces. *J. Symb. Comput.*, 36(3-4):343–364, 2003.

[20] Laureano Gonzalez-Vega and Ioana Necula. Efficient topology determination of implicitly defined algebraic plane curves. *Comput. Aided Geom. Design*, 19(9):719–743, 2002.

[21] Mark Goresky and Richard MacPherson. *Stratified Morse Theory.* Springer-Verlag, 1988.

[22] Thomas A. Grandine and Frederick W. Klein, IV. A new approach to the surface intersection problem. *Comput. Aided Geom. Design*, 14(2):111–134, 1997.
[23] Robert M. Hardt. Triangulation of subanalytic sets and proper light subanalytic maps. *Invent. Math.*, 38(3):207–217, 1976/77.
[24] H. Hironaka. Triangulations of algebraic sets. In *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974)*, pages 165–185. Amer. Math. Soc., Providence, R.I., 1975.
[25] Hoon Hong. An efficient method for analyzing the topology of plane real algebraic curves. *Math. Comput. Simulation*, 42(4-6):571–582, 1996. Symbolic computation, new trends and developments (Lille, 1993).
[26] J. R. Sendra J. G. Alcazar, J. Schicho. A delineability-based method for computing critical sets of algebraic surfaces. *J.Symb.Comp.*, 42(6):678–691, 2007.
[27] John Keyser, Shankar Krishnan, and Dinesh Manocha. Efficient and accurate B-rep generation of low degree sculptured solids using exact arithmetic. I. Representations. *Comput. Aided Geom. Design*, 16(9):841–859, 1999.
[28] Scott McCallum and G.E. Collins. Local box adjacency algorithms for cylindrical algebraic decompositions. *J. Symbolic Comput.*, 33(3):321–342, 2002.
[29] John C. Owen and Alyn P. Rockwood. Intersection of general implicit surfaces. In *Geometric modeling*, pages 335–345. SIAM, Philadelphia, PA, 1987.
[30] Victor Y. Pan. Optimal and nearly optimal algorithms for approximating polynomial zeros. *Comput. Math. Appl.*, 31(12):97–138, 1996.
[31] Daniel Reischert. Asymptotically fast computation of subresultants. In *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation (Kihei, HI)*, pages 233–240 (electronic), New York, 1997. ACM.
[32] Takis Sakkalis. The topological configuration of a real algebraic curve. *Bull. Austral. Math. Soc.*, 43(1):37–50, 1991.
[33] Wolfgang Schreiner, Christian Mittermaier, and Franz Winkler. Plotting algebraic space curves by cluster computing. In *Computer mathematics (Chiang Mai, 2000)*, volume 8 of *Lecture Notes Ser. Comput.*, pages 49–58. World Sci. Publ., River Edge, NJ, 2000.
[34] Joachim von zur Gathen and Jürgen Gerhard. *Modern computer algebra*. Cambridge University Press, Cambridge, second edition, 2003.

7. Appendix : Subresultants

In this section, we describe the main algebraic ingredient of our algorithm based on subresultant.

Let $P_1, P_2 \in \mathbb{Q}[X, Y, Z]$ and $\mathcal{C}_{\mathbb{R}} := \{(x, y, z) \in \mathbb{R}^3 | P_1(x, y, z) = P_2(x, y, z) = 0\}$ be the intersection of the vanishing sets of P_1 and P_2. Our curve analysis needs to compute a plane projection of $\mathcal{C}_{\mathbb{R}}$. Subresultant sequences are a suitable tool to do it. For the reader's convenience, we recall their definition and relevant properties. For all the results of this section, we refer to [10], for proofs.

Let $\mathbb{A}$ be a integral domain. Let $P = \sum_{i=0}^{p} a_i X^i$ and $Q = \sum_{i=0}^{q} b_i X^i$ be two polynomials with coefficients in $\mathbb{A}$. We shall always assume $a_p \neq 0$, $b_q \neq 0$ and $p \geqslant q$.
Let $\mathbb{A}[X]_r$ be the set of polynomials in $\mathbb{A}[X]$ of degree not exceeding r, with the basis (as an $\mathbb{A}$-module) $1, X, \ldots, X^r$. If $r < 0$, we set $\mathbb{A}_r = \{0\}$ by convention. We will identify an element $S = s_0 + \ldots + s_r X^r$ of $\mathbb{A}[X]_r$ with the row vector $(s_0, \ldots, s_r)$.

Let k be an integer such that $0 \leqslant k \leqslant q$, and let Ψ_k: $\mathbb{A}[X]_{q-k-q} \times \mathbb{A}[X]_{p-k-1} \to \mathbb{A}[X]_{q+p-k-1}$ be the $\mathbb{A}$-linear map defined by $\Psi_k(U,V) = P\,U + Q\,V$, with $M_k(P,Q)$ the $(p+q-k) \times (p+q-k)$ matrix of Ψ_k. As we write vectors as row vectors, we have:

$$M_k(P,Q) = \begin{pmatrix} a_0 & & & b_0 & & \\ \vdots & \ddots & & \vdots & \ddots & \\ a_p & & a_0 & b_q & & b_0 \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_p & & & b_q \end{pmatrix}$$

That is $M_0(P,Q)$ is the classical Sylvester matrix associated to P, Q. To be coherent with the degree of polynomials, we will attach index $i-1$ to the ith column of $M_k(P,Q)$, so the indices of the columns go from 0 to $p+q-k-1$.

Definition 7.1. For $j \leqslant p+q-k-1$ and $0 \leqslant k \leqslant q$, let $\mathrm{sr}_{k,j}$ be the determinant of the submatrix of $M_k(P,Q)$ formed by the last $p+q-2k-1$ columns, the column of index j and all the $(p+q-2k)$ rows. The polynomial $\mathrm{Sr}_k(P,Q) = \mathrm{sr}_{k,0} + \ldots + \mathrm{sr}_{k,k}\,X^k$ is the kth sub-gcd of P and Q, and its leading term $\mathrm{sr}_{k,k}$ (also denoted sr_k) is the kth subresultant of P and Q. So, it follows that $\mathrm{Sr}_0(P,Q) = \mathrm{sr}_0$ is the usual resultant of P and Q.

Remark 7.2. (1) For $k < j \leqslant p+q-k-1$, we have $\mathrm{sr}_{k,j} = 0$, because it is the determinant of a matrix with two equal columns.
(2) If $q < p$, we have $\mathrm{Sr}_q = (b_q)^{p-q-1}Q$ and $\mathrm{sr}_q = (b_q)^{p-q}$.

The following proposition justify the name of sub-gcd given to the polynomial Sr_k.

Proposition 7.3. Let d be the degree of the gcd of P an Q (d is defined because $\mathbb{A}$ is an integral domain, so we may compute the gcd over the quotient field of $\mathbb{A}$). Let k be an integer such that $k \leqslant d$.
(1) The following assertions are equivalent:
  (a) $k < d$;
  (b) $\mathrm{Sr}_k = 0$;
  (c) $\mathrm{sr}_k = 0$.
(2) $\mathrm{sr}_d \neq 0$ and Sr_d is the gcd of P and Q.

Theorem 7.4. Fundamental property of subresultants
The first polynomial Sr_k associated to P and Q with $\mathrm{sr}_k \neq 0$ is the greatest common divisor of P and Q.

Theorem 7.5. Specialization property of subresultants
Let $P_1, P_2 \in \mathbb{A}[Y,Z]$ and $(\mathrm{Sr}_i(Y,Z))_i$ be their subresultant sequence with respect to Z. Then for any $\alpha \in \mathbb{A}$ with: $\deg_Z(P(Y,Z)) = \deg_Z(P(\alpha,Z))$; $\deg_Z(Q(Y,Z)) = \deg_Z(Q(\alpha,Z))$, $(\mathrm{Sr}_i(\alpha,Z))_i$ is the subresultant sequence of the polynomials $P(\alpha,Z)$ and $Q(\alpha,Z)$.