# Mathematical Analysis for some Hyperbolic-Parabolic Coupled Problems

Julien Jimenez, Laurent Levi

# Mathematical Analysis for some Hyperbolic-Parabolic Coupled Problems

Julien Jimenez, Laurent Lévi

September 22, 2009

Université de Pau et des Pays de l'Adour,
Laboratoire de Mathématiques Appliquées - UMR 5142 CNRS
BP 1155 - 64013 - PAU Cedex - FRANCE
(julien.jimenez@univ-pau.fr, laurent.levi@univ-pau.fr)

**Abstract.** We deal with the mathematical analysis of the coupling problem in a bounded domain of $\mathbb{R}^n$, $n \geq 1$, between a purely quasilinear first-order hyperbolic equation set on a subdomain and a parabolic one, set on its complementary. We start by providing the definition of a weak solution through an entropy inequality on the whole domain. The uniqueness property relies on a pointwise inequality along the interface between the two subdomains and on the method of doubling variables. The existence proof is based on a vanishing viscosity method.

## 1 Introduction

### 1.1 Presentation

Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ with a Lipschitz boundary $\Gamma = \partial\Omega$, $n \geq 1$, such that $\overline{\Omega} = \overline{\Omega}_h \cup \overline{\Omega}_p$. We suppose that $\Omega_h$ (the hyperbolic zone) and $\Omega_p$ (the parabolic one) are two disjoint bounded domains with Lipschitz boundaries $\Gamma_l \equiv \partial\Omega_l$, $l \in \{h, p\}$. We denote the interface by $\Gamma_{hp} = \Gamma_h \cap \Gamma_p$ and assume that for $l$ in $\{h, p\}$, the set $(\overline{\Gamma}_{hp} \cap (\overline{\Gamma_l \setminus \Gamma_{hp}}))$ has a zero $\mathcal{H}^{n-1}$-measure. Let $T$ be a finite positive real number: we are interested in the uniqueness and existence of a measurable and bounded function $u$ on $Q \equiv (0, T) \times \Omega$ satisfying (at least in a distributional sense)

$$\begin{cases} \partial_t u + \mathrm{div}_x(b(x)\boldsymbol{f}(u)) + g(t, x, u) &= \mathrm{div}_x(\mathbb{I}_{\Omega_p}(x)\nabla\phi(u)) \quad \text{in } Q, \\ u &= 0 \quad \text{on } \Sigma \equiv (0, T) \times \Gamma, \\ u(0, .) &= u_0 \quad \text{on } \Omega, \end{cases} \tag{1}$$

for discontinuous fluxes and reaction terms given by:

$$b(x)\boldsymbol{f}(u) = b_h(x)\boldsymbol{f}_h(u)\mathbb{I}_{\Omega_h} + b_p(x)\boldsymbol{f}_p(u)\mathbb{I}_{\Omega_p},$$

$$g(t, x, u) = g_p(t, x, u)\mathbb{I}_{\Omega_p}(x) + g_h(t, x, u)\mathbb{I}_{\Omega_h}(x).$$

Here for each set $A \subset \Omega$, $\mathbb{I}_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ else.} \end{cases}$

With these notations, the equation in (1) is reduced to the hyperbolic first-order equation

$$\partial_t u + \mathrm{div}_x(b_h(x)\boldsymbol{f}_h(u)) + g_h(t, x, u) = 0 \text{ in } Q_h \equiv (0, T) \times \Omega_h,$$

and to the parabolic one

$$\partial_t u + \mathrm{div}_x(b_p(x)\boldsymbol{f}_p(u)) + g_p(t,x,u) = \Delta\phi(u) \text{ in } Q_p \equiv (0,T)\times\Omega_p.$$

Furthermore it implicitly contains the (*formal*) transmission condition along the interface:

$$\big(b_p\boldsymbol{f}_p(u) - b_h\boldsymbol{f}_h(u)\big)\cdot\nu_h = \nabla\phi(u)\cdot\nu_h \text{ on } \Sigma_{hp}. \tag{2}$$

As mentioned in [6], this type of problem arises from several physical applications that are modelled by a global advection-diffusion-reaction in the whole $\Omega$. However, in these problems, the diffusive term may be relevant only in a subregion $\Omega_p$ (which clearly depends on the problem in hands) while it can be neglected in the rest of the domain $\Omega$, without affecting the solution in a sensible way.

Fluid dynamics is among the field that benefit largely from a coupling approach of the type considered here. For example, we may consider viscous-compressible flows around a rigid profile (*e.g.* an aerofoil). Physical evidence suggests that viscosity effects are negligible apart from a small region close to the rigid body. So that the mathematical modelling of the problem may lead to use equations with different character (precisely Euler, Navier-Stokes equations) in separate regions, by dropping viscous terms when they are very small.

Another example is provided by a heat transfer problem such as a forced incompressible flow over a heated plate. In such a case, the thermal diffusivity is much more important in the boundary layer than elsewhere (here the reduced equation of conservation of energy can be assumed to describe the flows field). The velocity field can be evaluated independently from the temperature while the latter is the solution to an advection-diffusion equation in which the transport field is given precisely by the (known) velocity. Away from the boundary layer, the diffusive term may be neglected.

We complete this introduction with a last example, within the framework of infiltration processes through a stratified subsoil viewed as an heterogeneous porous medium with different geological characteristics in each layer, and such that, depending of the physical properties of the rocks, the diffusivity effects may be neglected with respect to the transport ones. This approach has mainly motivated the previous studies in [1],[2], and [7].

## 1.2 Notations and main assumptions on data

Throughout this paper, we give by a subscript $h$ when referring to the hyperbolic zone and a subscript $p$ for the parabolic one. Then, for $l \in \{h,p\}$,

- the coefficients $b_l$ are elements of $W^{2,\infty}(\Omega_l)$ and the vector flux function $\boldsymbol{f}_l = (f_{l,1},\ldots,f_{l,n})$ belongs to $W^{2,\infty}(\mathbb{R})^n$. For $i=1,\ldots,n$, the positive real $M_{f_i}$ denotes the Lipschitz constant of $f_i$ and set $M_{\boldsymbol{f}} = \max_{i=1,\ldots,n} M_{f_i}$,
- the source term $g_l$ is in $W^{1,\infty}(]0,T[\times\Omega_l\times\mathbb{R})$, such that

  $$\exists\ M_{g_l}\in\mathbb{R}^+,\ \text{a.e. on } ]0,T[\times\Omega_l\times\mathbb{R},\ |\partial_u g_l| \le M_{g_l}.$$

  We set $M_g = M_{g_h} + M_{g_p}$.
- the diffusion term $\phi$ is an increasing function of $W^{1,\infty}(\mathbb{R})$. By normalization, we suppose that $\phi(0)=0$. In addition, we assume that $\phi^{-1}$ exists on $\mathrm{Im}(\phi)$. That means that the second order operator set on the parabolic area is *weakly degenerated*. This is in particular fulfilled when $\{x\in\mathbb{R}, \phi'(x)=0\}$ has a zero Lebesgue measure.
- in order to deal with bounded solutions, the initial datum $u_0$ belongs to $L^\infty(\Omega)$ and takes values in $[m,M]$ where $m$ and $M$ are two fixed real numbers and we introduce a nondecreasing smooth function $M_1$ of the time variable and a nonincreasing smooth function $M_2$ of the time variable such that

  $$\begin{cases} M_1(0)\ge M,\\ \forall t\in(0,T)\\ M_1'(t) + \nabla b(.)\cdot\boldsymbol{f}(M_1(t)) + g(t,.,M_1(t)) \ge 0 \text{ a.e. on } \Omega_L\cup\Omega_R, \end{cases}$$

and

$$\begin{cases} M_2(0) \le m, \\ \forall t \in (0,T) \\ M_2'(t) + \nabla b(.) \cdot \boldsymbol{f}(M_2(t)) + g(t,.,M_2(t)) \le 0 \text{ a.e. on } \Omega_L \cup \Omega_R. \end{cases}$$

Observe that one can propose

$$M_1 : t \in [0,T] \longrightarrow M_1(t) = \operatorname{ess\,sup}_{\Omega} u_0^+ e^{N_1 t} + \frac{N_2}{N_1}(e^{N_1 t} - 1),$$

and

$$M_2 : t \in [0,T] \longrightarrow M_2(t) = \operatorname{ess\,inf}_{\Omega} (-u_0^-) e^{N_1 t} - \frac{N_2}{N_1}(e^{N_1 t} - 1),$$

with

$$N_1 = \max(\|\nabla b_h\|_{L^\infty(\Omega_h)^n}, \|\nabla b_p\|_{L^\infty(\Omega_p)^n}) \sum_{i=1}^{n} M_{f_i} + M_g \ ,$$

and $N_2 = \sum_{l=h,p} \max_{[0,T]\times\overline{\Omega}} |g(t,x,0) + \nabla b_l(x) \cdot \boldsymbol{f}(0)|$.

- Since $\Gamma_p \setminus \Gamma_{hp}$ has a non zero $\mathcal{H}^{n-1}$-measure we may consider the Hilbert space

  $$V = \{v \in H^1(\Omega_p), v = 0 \text{ a.e. on } \Gamma_p \setminus \Gamma_{hp}\},$$

  endowed with the norm $\|v\|_V = \|\nabla v\|_{L^2(\Omega_p)^n}$, which is equivalent to the classical $H^1(\Omega_p)$-norm. Then we denote $\langle\langle .,. \rangle\rangle$ the duality pairing between $V$ and $V'$.

- The notation $|.|_n$ stands for the Euclidian norm on $\mathbb{R}^n$.

- For any real numbers $a, b$, we set $I(a,b)$ is the closed interval bounded by $a$ and $b$.

- The function $sgn_\eta$ stands for the Lipschitzian approximation of the function $sgn$ given for any positive $\eta$ and any nonnegative real number $x$ by $sgn_\eta(x) = \min(\frac{x}{\eta}, 1)$ and $sgn_\eta(-x) = -sgn_\eta(x)$.

  Lastly, for any real numbers $\tau, k$, we introduce the *classical* Otto flux [9, 10] which is now a standard and useful tool to transcript - through an integral inequality - the boundary condition along the hyperbolic frontier (see (5) below)

  $$\mathcal{F}_h(\tau,k) = \frac{1}{2}\{sgn(\tau)(\boldsymbol{f}_h(\tau) - \boldsymbol{f}_h(0)) \\ -sgn(k)(\boldsymbol{f}_h(k) - \boldsymbol{f}_h(0)) + sgn(\tau - k)(\boldsymbol{f}_h(\tau) - \boldsymbol{f}_h(k))\}.$$

This paper is organized as follows. In Section 2 we give the definition of a weak entropy solution to (1). We show in Section 3 the uniqueness of such a solution while Section 4 is mainly devoted to the proof of the existence property, by the vanishing viscosity method.

# 2 A notion of weak entropy solution

We want to take into account not only the coexistence of known hyperbolic and parabolic areas in the studied field but also the unknown ones in $\Omega_p$ coming from the degeneracy of the second order operator set in $\Omega_p$. With this view we define a weak solution to (1) through a global entropy inequality on the whole domain $Q$ in the same spirit as in [7].

To this end we introduce the next mollified entropy pairs:

$$I_\eta(a,b) = \int_b^a sgn_\eta(\phi(\tau) - \phi(b)) d\tau$$

and

$$\boldsymbol{F}_{l,\eta}(a,b) = \int_{\phi(b)}^{\phi(a)} \boldsymbol{f}_l \circ \phi^{-1}(\tau) sgn'_\eta(\tau - \phi(b)) d\tau.$$

Moreover we set $\boldsymbol{F}_\eta = \boldsymbol{F}_{h,\eta}\mathbb{I}_{\Omega_h} + \boldsymbol{F}_{p,\eta}\mathbb{I}_{\Omega_p}$. Then it will be said that

**Definition 1.** *A measurable function $u$ on $Q$ is a weak entropy solution to* (1) *if*
*(i)* $u \in L^\infty(Q)$, $\phi(u) \in L^2(0,T;V)$,
*(ii)* $\forall \varphi \in \mathcal{D}(Q)$ *with* $\varphi \geq 0$, $\forall k \in \mathbb{R}$,

$$\begin{aligned}
&\int_Q I_\eta(u,k)\partial_t\varphi dxdt - \int_Q sgn_\eta(\phi(u)-\phi(k))\nabla\phi(u)\cdot\nabla\varphi dxdt \\
&+\int_Q b(x)\{sgn_\eta(\phi(u)-\phi(k))\boldsymbol{f}(u) - \boldsymbol{F}_\eta(u,k)\}\cdot\nabla\varphi dxdt \\
&-\int_Q \{sgn_\eta(\phi(u)-\phi(k))g(t,x,u) + \nabla b(x)\cdot\boldsymbol{F}_\eta(u,k)\}\varphi dxdt \\
&+\int_{\Sigma_{hp}} (b_h(\overline{\sigma})\boldsymbol{F}_{h,\eta}(u,k) - b_p(\overline{\sigma})\boldsymbol{F}_{p,\eta}(u,k))\varphi\cdot\boldsymbol{\nu}_h dtd\mathcal{H}^{n-1}_{\overline{\sigma}} \geq 0
\end{aligned} \tag{3}$$

*(iii) $u$ satisfies the initial condition in the $L^1$-sense i.e.*

$$\operatorname*{ess\,lim}_{t\to 0^+}\int_\Omega |u(t,x)-u_0(x)|dx = 0. \tag{4}$$

*(iv) For any $\zeta$ in $L^1(\Sigma_h\backslash\Sigma_{hp})$, $\zeta \geq 0$, $\forall k \in \mathbb{R}$,*

$$\operatorname*{ess\,lim}_{s\to 0^-}\int_{\Sigma_{hp}} b(\overline{\sigma})\mathcal{F}_h(u(\sigma+s\boldsymbol{\nu}_h),k)\cdot\boldsymbol{\nu}_h dtd\mathcal{H}^{n-1}_{\overline{\sigma}} \geq 0, \tag{5}$$

*where $\sigma = (t,\overline{\sigma}) \in \Sigma$.*

Entropy inequality (3) is expressed with a mollification of the classical Kruzhkov entropy pairs. This will be used in Lemma 2 to transcript an entropy jump condition across the interface $\Sigma_{hp}$ for a weak entropy solution. This regularization was not needed in [7], where the characteristics along $\Sigma_{hp}$ were known so that this jump condition was automatically fulfilled. This is the main feature and contribution of this work in comparison with the previous ones [1, 2, 7]

**Remark 1.** *When we take the limit with respect to $\eta$ in* (3)*, since for $l = h, p$,*

$$\lim_{\eta\to 0^+}\boldsymbol{F}_{l,\eta}(u,k) = sgn(\phi(u)-\phi(k))\boldsymbol{f}_l(k) = sgn(u-k)\boldsymbol{f}_l(k),$$

*$\phi$ being nondecreasing, we obtain an entropy inequality written with the standard Kruzhkov entropy pairs*

$$\begin{aligned}
&\int_Q |u-k|\partial_t\varphi dxdt - \int_Q \nabla|\phi(u)-\phi(k)|\cdot\nabla\varphi dxdt \\
&+\int_Q b(x)\boldsymbol{\Phi}(u,k)\cdot\nabla\varphi dxdt \\
&-\int_Q sgn(u-k)(g(t,x,u)+\nabla b(x)\cdot\boldsymbol{f}(k))\varphi dxdt \\
&+\int_{\Sigma_{hp}} sgn(\phi(u)-\phi(k))(b_h\boldsymbol{f}_h(k) - b_p f_p(k))\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} \geq 0
\end{aligned} \tag{6}$$

*where $\boldsymbol{\Phi}(u,k) = sgn(u-k)(\boldsymbol{f}(u)-\boldsymbol{f}(k))$ is the Kruzhkov flux.*

**Remark 2.** *If $u$ is a weak entropy solution then $u$ is a solution to* (1) *in the sense of distributions, that is to say, for all $\varphi \in \mathcal{D}(Q)$,*

$$\int_Q (u\partial_t\varphi + (b(x)\boldsymbol{f}(u) - \mathbb{I}_{\Omega_p}\nabla\phi(u))\cdot\nabla\varphi - g(t,x,u)\varphi)dxdt = 0, \tag{7}$$

*so that $u$ fulfills*

$$\partial_t u + \operatorname{div}_x(b_h(x)\boldsymbol{f}_h(u)) + g_h(t,x,u) = 0 \ \textit{in}\ \mathcal{D}'(Q_h),$$
$$\partial_t u + \operatorname{div}_x(b_p(x)\boldsymbol{f}_p(u)) + g_p(t,x,u) = \Delta\phi(u) \ \textit{in}\ \mathcal{D}'(Q_p).$$

*and the transmission condition* (2) *in a formal sense at this stage.*

## 3 The uniqueness property

Our aim is to establish a time-Lipschitzian dependence in $L^1(\Omega)$ of a weak entropy solution $u$ to (1) with respect to its corresponding initial datum. The idea of the proof is to derive from (3) two local formulations, one in $Q_h$ and one in $Q_p$ and an entropy jump condition along the interface $\Sigma_{hp}$. To transcript the latter, we need to define, in a certain way, a trace for $u$ coming from the hyperbolic zone. Observe that a trace for $u$ coming from the parabolic zone is given by the trace of $\phi(u)$ that belongs to $L^2(0,T;H^{1/2}(\Gamma_p))$. So we assume now that the flux function satisfies a non-degeneracy condition i.e. for almost all $x \in \Omega_h$, for all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, the function

$$\begin{gathered}\lambda \longmapsto \xi \cdot b_h(x)\boldsymbol{f}_h(\lambda) \text{ is not linear on any nondegenerated interval} \\ \text{included in } [M_2(T), M_1(T)].\end{gathered} \tag{8}$$

Assumption (8) will allow us to define *strong traces* as explained in Section 2 (see Lemma 1). Indeed, following the works of E. Yu. Panov [12] or A. Vasseur [15], it comes:

**Lemma 1.** *Let $u$ be a function of $L^\infty(Q)$ satisfying (3). Under (8) there exists a function $u^\tau$ of $L^\infty(\Sigma_h)$ such that, for every compact $K$ of $\Sigma_h$ and every regular Lipschitz deformation $\Psi$ of $\Omega_h$,*

$$\operatorname{ess}\lim_{s\to 0^+} \int_K |u(\Psi(s,\sigma)) - u^\tau(\sigma)| dt d\mathcal{H}^{n-1} = 0. \tag{9}$$

Let us note that a regular Lipschitz deformation can be defined if $\Omega_h$ is, for example, a star-shaped domain (see [4] for more details).

In this framework, the boundary condition (5) on the outer frontier of the hyperbolic zone is written for almost all $t \in (0,T)$, $\mathcal{H}^{n-1}$-a.e. on $\Gamma_h \setminus \Gamma_{hp}$, $\forall k \in \mathbb{R}$,

$$b_h \mathcal{F}_h(u^\tau, k) \cdot \boldsymbol{\nu}_h \geq 0, \tag{10}$$

which is equivalent to the well-known pointwise Bardos, LeRoux and Nédélec formulation given in [3].

To begin with, let us highlight some local informations included in inequality (3). Indeed, it first contains an entropy formulation on the hyperbolic domain since,

**Proposition 1.** *Let $u$ be a weak entropy solution to (1). Then for any real number $k$ and any $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^n)$ with $\varphi \geq 0$,*

$$\begin{aligned} &\int_{Q_h} |u-k|\partial_t\varphi dxdt + \int_{Q_h} b_h(x)\boldsymbol{\Phi}_h(u,k)\cdot\nabla\varphi dxdt - \int_{Q_h} G_h(u,k)\varphi dxdt \\ \geq\ & \int_{\Sigma_{hp}} b_h(\overline{\sigma})\boldsymbol{\Phi}_h(u^\tau,k)\cdot\boldsymbol{\nu}_h\varphi dt d\mathcal{H}^{n-1} + \int_{\Sigma_h\setminus\Sigma_{hp}} b_h(\overline{\sigma})\boldsymbol{\Phi}_h(0,k)\cdot\boldsymbol{\nu}_h\varphi dt d\mathcal{H}^{n-1} \\ & - \int_{\Sigma_h\setminus\Sigma_{hp}} b_h(\overline{\sigma})\boldsymbol{\Phi}_h(u^\tau,0)\cdot\boldsymbol{\nu}_h\varphi dt d\mathcal{H}^{n-1} \end{aligned} \tag{11}$$

*where*

$$G_h(u,k) = sgn(u-k)(g_h(t,x,u) + \nabla b_h(x)\cdot\boldsymbol{f}_h(k))$$

*and*

$$\boldsymbol{\Phi}_h(a,b) = sgn(u-k)(\boldsymbol{f}_h(a) - \boldsymbol{f}_h(b))$$

*Proof.* Thanks to Remark 1, we have for any $\varphi \in \mathcal{D}(Q_h)$ with $\varphi \geq 0$,

$$\int_{Q_h} (|u-k|\partial_t\varphi + b_h(x)\boldsymbol{\Phi}_h(u,k)\cdot\nabla\varphi - G_h(u,k)\varphi) dxdt \geq 0. \tag{12}$$

Let $(\omega_\varepsilon)_{\varepsilon>0}$ be a sequence of functions such that for every $\varepsilon$, $\omega_\varepsilon \in \mathcal{C}^\infty(\overline{\Omega}_h)$ and

$$\begin{cases} 0 \le \omega_\varepsilon \le 1 \text{ on } \Omega_h, \\ \omega_\varepsilon(x) = 1 \text{ if } x \in \Gamma_h, \\ \omega_\varepsilon(x) = 0 \text{ if } d(x,\Gamma_h) > \varepsilon, \\ (\varepsilon\nabla\omega_\varepsilon)_\varepsilon \text{ is bounded on } \Omega_h. \end{cases}$$

We choose in (12) the test function $\varphi(1-\omega_\varepsilon)$ and we take the limit with respect to $\varepsilon$. From the non-degeneracy condition (8) and Lemma 1, it comes

$$\lim_{\varepsilon\to 0^+} \int_{Q_h} b_h(x)\mathbf{\Phi}_h(u,k)\cdot\nabla\omega_\varepsilon\varphi dxdt = \int_{\Sigma_h} b_h(\overline{\sigma})\mathbf{\Phi}_h(u^\tau,k)\cdot\boldsymbol{\nu}_h dtd\mathcal{H}^{n-1}.$$

Thus

$$\begin{aligned} &\int_{Q_h} (|u-k|\partial_t\varphi + b_h(x)\mathbf{\Phi}_h(u,k)\cdot\nabla\varphi - G_h(u,k)\varphi)dxdt \\ \ge &\int_{\Sigma_h} b_h(\overline{\sigma})\mathbf{\Phi}(u^\tau,k)\cdot\boldsymbol{\nu}_h dtd\mathcal{H}^{n-1}. \end{aligned}$$

We split the frontier of $Q_h$ into $\Sigma_h$ and $\Sigma_h \setminus \Sigma_{hp}$. The boundary condition (10) provides

$$\begin{aligned} \int_{\Sigma_h\setminus\Sigma_{hp}} b_h(\overline{\sigma})\mathbf{\Phi}_h(u^\tau,k)\cdot\boldsymbol{\nu}_h dtd\mathcal{H}^{n-1} \quad &\ge \int_{\Sigma_h\setminus\Sigma_{hp}} b_h(\overline{\sigma})\mathbf{\Phi}_h(0,k)\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} \\ &- \int_{\Sigma_h\setminus\Sigma_{hp}} b_h(\overline{\sigma})\mathbf{\Phi}_h(u^\tau,k)\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} \end{aligned}$$

The conclusion follows. □

Entropy inequality (11) is not sufficient to prove the uniqueness on the hyperbolic area: as stated forward in ,in (19), it only provides an $L^1(Q_h)$-error between two solutions with respect to their interface values and initial data. That is why we are interested now in the behavior of a weak entropy solution $u$ to (1) on the parabolic zone. We prove that it fulfills a local variational equality that involves entering data from the hyperbolic domain:

**Proposition 2.** *Let $u$ be a weak entropy solution to* (1)*. Then $\partial_t u$ belongs to $L^2(0,T;V')$. Moreover, for any $\varphi \in L^2(0,T;V)$,*

$$\begin{aligned} &\int_0^T \langle\langle \partial_t u, \varphi\rangle\rangle dt + \int_{Q_p} (\nabla\phi(u) - b_p\boldsymbol{f}_p(u))\cdot\nabla\varphi dxdt \\ &+ \int_{Q_p} g_p(t,x,u)\varphi dxdt - \int_{\Sigma_{hp}} b_h\boldsymbol{f}_h(u^\tau)\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} = 0. \end{aligned} \tag{13}$$

*Proof.* We sketch this proof, the reasoning being the same as in [7]. By a density argument (7) is still true for any function $\varphi$ of $\mathcal{D}(0,T;H_0^1(\Omega))$. Let $\xi \in \mathcal{D}(0,T;V)$ and $\hat{\xi}$ an extension of $\xi$ to $\mathcal{D}(0,T;H_0^1(\Omega))$ . We choose in (7) the test function $\varphi = \hat{\xi}\lambda_\varepsilon$ with, for any positive constant $\varepsilon$, $\lambda_\varepsilon \in \mathcal{C}^\infty(\overline{\Omega}_p)$ such that

$$\begin{cases} \lambda_\varepsilon(x) = 1 \text{ if } x \in \overline{\Omega}_p, \\ \lambda_\varepsilon(x) = 0 \text{ if } x \in \Omega_h, d(x,\Gamma_{hp}) \ge \varepsilon, \\ \|\varepsilon\nabla\lambda_\varepsilon\|_\infty \text{ is bounded.} \end{cases}$$

Then we take the limit with respect to $\varepsilon$ in (7). Thanks to (8) and (9) we assert that

$$\lim_{\varepsilon\to 0^+} \int_{Q_h} b_h(x)\hat{\xi}\boldsymbol{f}_h(u)\cdot\nabla\lambda_\varepsilon dxdt = \int_{\Sigma_{hp}} b_h\boldsymbol{f}_h(u^\tau)\cdot\boldsymbol{\nu}_h\xi dtd\mathcal{H}^{n-1},$$

and that provides (13). □

Now we give a consequence of Proposition 2 where we may recognize a certain form of the relation (2) and that will be helpful to express the transmission condition along $\Sigma_{hp}$. To this purpose we introduce a sequence of $\mathcal{D}(\overline{\Omega})$, denoted by $(\beta_\varepsilon)_{\varepsilon>0}$, such that

$$\left\{\begin{array}{l}\forall\varepsilon>0,\ 0\le\beta_\varepsilon\le 1,\ \beta_\varepsilon=1 \text{ if } x\in\Gamma_{hp},\\ \forall\varepsilon>0,\ \beta_\varepsilon=0 \text{ if } x\in\Omega,\ d(x,\Gamma_{hp})\ge\varepsilon,\\ \|\varepsilon\nabla\beta_\varepsilon\|_\infty \text{ is bounded},\\ \forall x\in\Omega\setminus\Gamma_{hp},\ \lim\limits_{\varepsilon\to0^+}\beta_\varepsilon(x)=0.\end{array}\right. \tag{14}$$

Then we have

**Lemma 2.** *Let $u$ in $L^\infty(Q)$ satisfying (13). Then for any $\psi$ of $H_0^1(Q)$,*

$$\lim_{\varepsilon\to0^+}\int_{Q_p} sgn_\eta(\phi(u)-\phi(k))\psi\nabla\phi(u)\cdot\nabla\beta_\varepsilon dxdt =$$
$$\int_{\Sigma_{hp}}(b_h\boldsymbol{f}_h(u^\tau)-b_p\boldsymbol{f}_p(u^\phi))\cdot\boldsymbol{\nu}_h sgn_\eta(\phi(u)-\phi(k))\psi dtd\mathcal{H}^{n-1},$$

*where $u^\phi=\phi^{-1}(\phi(u)_{|\Sigma_{hp}})$.*

*Proof.* From Proposition 2 it comes for any $\varphi$ of $L^2(0,T;V)$:

$$\int_0^T\langle\langle\partial_t u,\varphi\beta_\varepsilon\rangle\rangle dt+\int_{Q_p}(\nabla\phi(u)-b_p\boldsymbol{f}_p(u))\cdot\nabla\varphi\beta_\varepsilon dxdt+\int_{Q_p}g_p(t,x,u)\varphi\beta_\varepsilon dxdt$$
$$+\int_{Q_p}(\nabla\phi(u)-b_p\boldsymbol{f}_p(u))\cdot\nabla\beta_\varepsilon\varphi dxdt-\int_{\Sigma_{hp}}b_h\boldsymbol{f}_h(u^\tau)\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1}=0.$$

We choose the test function $\varphi=sgn_\eta(\phi(u)-\phi(k))\psi_{|Q_p}$, where $\psi\in\mathcal{D}(Q)$, $\psi\ge0$. We use the F.Mignot time-integration by parts formula (see [5], p.31) to state:

$$\int_0^T\langle\langle\partial_t u,\varphi\beta_\varepsilon\rangle\rangle dt=-\int_{Q_p}I_\eta(u,k)\beta_\varepsilon\partial_t\psi dxdt.$$

Therefore,

$$\lim_{\varepsilon\to0^+}\int_0^T\langle\langle\partial_t u,\varphi\beta_\varepsilon\rangle\rangle dt=0.$$

Since $\boldsymbol{f}_p(u)=\boldsymbol{f}_p\circ\phi^{-1}(\phi(u))$ and $\boldsymbol{f}_p\circ\phi^{-1}$ is continuous, we claim that:

$$\lim_{\varepsilon\to0^+}\int_{Q_p}b_p\boldsymbol{f}_p(u)\cdot\nabla\beta_\varepsilon\varphi dxdt=-\int_{\Sigma_{hp}}b_p\boldsymbol{f}(u^\phi)sgn_\eta(\phi(u)-\phi(k))\cdot\boldsymbol{\nu}_h\psi dxdt$$

Thus

$$\lim_{\varepsilon\to0^+}\int_{Q_p}\nabla\phi(u)\cdot\nabla\beta_\varepsilon sgn_\eta(\phi(u)-\phi(k))\psi dxdt$$
$$=\int_{\Sigma_{hp}}(b_h\boldsymbol{f}_h(u^\tau)-b_p\boldsymbol{f}_p(u^\phi))\cdot\boldsymbol{\nu}_h sgn_\eta(\phi(u)-\phi(k))\psi dtd\mathcal{H}^{n-1}.$$

That completes the proof. □

Lemma 2 allows us to write a transmission condition along $\Sigma_{hp}$ that is in fact included in the global formulation (3) on the whole $Q$. This is a key point of the uniqueness proof. Observe that this interface relation is written as a pointwise inequality on $\Sigma_{hp}$ that requires the knowledge of strong traces coming from the hyperbolic area and from the parabolic one for a weak entropy solution to (1). Indeed the next entropy jump condition holds:

**Lemma 3.** *Let $u$ be a weak entropy solution to* (1)*. Then a.e. in* $(0,T)$*,* $\mathcal{H}^{n-1}$*-a.e. on* $\Gamma_{hp}$*, for any* $k \in I(u^\tau, u^\phi)$*,*

$$sgn(u^\tau - u^\phi) b_h(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(k)) \cdot \boldsymbol{\nu}_h \geq 0. \tag{15}$$

*Proof.* For any real number $\varepsilon > 0$, we consider in (3) the test function $\varphi_\varepsilon = \psi\beta_\varepsilon$ with $\psi \in H_0^1(Q)$, $\psi \geq 0$ (that is possible by a density argument). Thus, for any real number $k$,

$$\begin{aligned}
&\int_Q I_\eta(u,k)\beta_\varepsilon \partial_t \psi dxdt - \int_Q sgn_\eta(\phi(u) - \phi(k))\beta_\varepsilon \nabla\phi(u) \cdot \nabla\psi dxdt \\
&- \int_Q sgn_\eta(\phi(u) - \phi(k))\psi \nabla\phi(u) \cdot \nabla\beta_\varepsilon dxdt \\
&+ \int_Q b(x)\beta_\varepsilon \{sgn_\eta(\phi(u) - \phi(k))\boldsymbol{f}(u) - \boldsymbol{F}_\eta(u,k)\} \cdot \nabla\psi dxdt \\
&+ \int_Q b(x)\psi \{sgn_\eta(\phi(u) - \phi(k))\boldsymbol{f}(u) - \boldsymbol{F}_\eta(u,k)\} \cdot \nabla\beta_\varepsilon dxdt \\
&- \int_Q \{sgn_\eta(\phi(u) - \phi(k))g(t,x,u) + \nabla b(x) \cdot \boldsymbol{F}_\eta(u,k)\}\psi\beta_\varepsilon dxdt \\
&+ \int_{\Sigma_{hp}} (b_h \boldsymbol{F}_{h,\eta}(u,k) - b_p \boldsymbol{F}_{p,\eta}(u,k)) \cdot \boldsymbol{\nu}_h \psi dt d\mathcal{H}^{n-1} \geq 0.
\end{aligned}$$

All the integrals involving the function $\beta_\varepsilon$ itself goes to zero with $\varepsilon$. To take the limit with respect to $\varepsilon$ in the second line we use Lemma 2 and for the forth one we split the integration field into $Q_p$ and $Q_h$ so that on the parabolic zone, we have by introducing the trace $u^\phi$,

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \int_{Q_p} b_p(x)\psi\{sgn_\eta(\phi(u) - \phi(k))\boldsymbol{f}_p(u) - \boldsymbol{F}_{p,\eta}(u,k)\} \cdot \nabla\beta_\varepsilon dxdt \\
&= - \int_{\Sigma_{hp}} b_p \{sgn_\eta(\phi(u) - \phi(k))\boldsymbol{f}_p(u^\phi) - \boldsymbol{F}_{p,\eta}(u,k)\} \cdot \boldsymbol{\nu}_h \psi dt d\mathcal{H}^{n-1}
\end{aligned}$$

and on the hyperbolic one, in view of (9), we ensure that

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \int_{Q_h} b_h \psi\{sgn_\eta(\phi(u) - \phi(k))\boldsymbol{f}_h(u) - \boldsymbol{F}_{h,\eta}(u,k)\} \cdot \nabla\beta_\varepsilon dxdt \\
&= \int_{\Sigma_{hp}} b_h \{sgn_\eta(\phi(u^\tau) - \phi(k))\boldsymbol{f}_h(u^\tau) - \boldsymbol{F}_{h,\eta}(u^\tau,k)\} \cdot \boldsymbol{\nu}_h \psi dt d\mathcal{H}^{n-1}.
\end{aligned}$$

Eventually it follows, for any $\psi \in H_0^1(Q)$,

$$\begin{aligned}
&- \int_{\Sigma_{hp}} (b_h \boldsymbol{f}_h(u^\tau) - b_p \boldsymbol{f}_p(u^\phi)) \cdot \boldsymbol{\nu}_h sgn_\eta(\phi(u) - \phi(k))\psi dt d\mathcal{H}^{n-1}. \\
&- \int_{\Sigma_{hp}} b_p \{sgn_\eta(\phi(u) - \phi(k))\boldsymbol{f}_p(u^\phi) - \boldsymbol{F}_{p,\eta}(u,k)\} \cdot \boldsymbol{\nu}_h \psi dt d\mathcal{H}^{n-1} \\
&+ \int_{\Sigma_{hp}} b_h \{sgn_\eta(\phi(u^\tau) - \phi(k))\boldsymbol{f}_h(u^\tau) - \boldsymbol{F}_{h,\eta}(u^\tau,k)\} \cdot \boldsymbol{\nu}_h \psi dt d\mathcal{H}^{n-1} \\
&+ \int_{\Sigma_{hp}} (b_h \boldsymbol{F}_{h,\eta}(u,k) - b_p \boldsymbol{F}_{p,\eta}(u,k)) \cdot \boldsymbol{\nu}_h \psi dt d\mathcal{H}^{n-1} \geq 0.
\end{aligned}$$

Consequently, for any positive $\eta$ and any real number $k$,

$$\begin{aligned}
&- (b_h \boldsymbol{f}_h(u^\tau) - b_p \boldsymbol{f}_p(u^\phi)) \cdot \boldsymbol{\nu}_h sgn_\eta(\phi(u) - \phi(k)) \\
&- b_p \{sgn_\eta(\phi(u) - \phi(k))\boldsymbol{f}_p(u^\phi) - \boldsymbol{F}_{p,\eta}(u,k)\} \cdot \boldsymbol{\nu}_h \\
&+ b_h \{sgn_\eta(\phi(u^\tau) - \phi(k))\boldsymbol{f}_h(u^\tau) - \boldsymbol{F}_{h,\eta}(u^\tau,k)\} \cdot \boldsymbol{\nu}_h \\
&+ (b_h \boldsymbol{F}_{h,\eta}(u,k) - b_p \boldsymbol{F}_{p,\eta}(u,k)) \cdot \boldsymbol{\nu}_h \geq 0
\end{aligned}$$

a.e. on $(0,T)$, $\mathcal{H}^{n-1}$ a.e. on $\Gamma_{hp}$. By taking the limit with respect to $\eta$ in the above inequality we obtain

$$(sgn(u^\tau - k) - sgn(u^\phi - k))b_h(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(k)) \cdot \boldsymbol{\nu}_h \geq 0,$$

that is (15) when $k$ belongs to $I(u^\tau, u^\phi)$. □

Now we are able to state the uniqueness property that is a time-Lipschitzian dependence in $L^1(\Omega)$ of a weak entropy solution to (1) with respect to its initial condition.

**Theorem 1.** *Assume that there exists a constant $C > 0$ and a real number $\theta \in [\frac{1}{2}, 1]$ such that, for all $(a,b) \in [\phi(M_2(T)), \phi(M_1(T))]^2$,*

$$|(\boldsymbol{f}_p \circ \phi^{-1})(a) - (\boldsymbol{f}_p \circ \phi^{-1})(b)|_n \leq C|a-b|^\theta. \tag{16}$$

*Let $u$ and $v$ be two weak entropy solutions to (1) for initial data $u_0$ and $v_0$. Then for almost all $t \in (0,T)$,*

$$\int_\Omega |u(t,.) - v(t,.)|dx \leq e^{M_g t} \int_\Omega |u(t,.) - v(t,.)|dx.$$

*Proof.* First we introduce $(\rho_j)_{j\in\mathbb{N}^\star}$ a sequence of mollifiers on $\mathbb{R}$ and $\mathcal{W}_j$, the sequence of mollifiers on $\mathbb{R}^{n+1}$ defined by

$$\forall j > 0,\ \forall p = (t,x) \in \mathbb{R}^{n+1},\ \mathcal{W}_j(p) = \rho_j(t) \prod_{i=1}^n \rho_j(x_i).$$

In a first step, we focus on the parabolic area and we use the method of doubling the time variable only. Let $\gamma$ be a nonnegative element of $\mathcal{D}(0,T)$. We consider the mapping $\alpha_j : (t,\tilde{t}) \to \gamma((t+\tilde{t})/2)\rho_j((t-\tilde{t})/2)$. Note that, for $j$ small enough, $\alpha_j$ belongs to $\mathcal{D}((0,T) \times (0,T))$. Then we choose in (13) written for $u$ in variables $(t,x)$, the test function $\varphi(t,x) = sgn_\eta(\phi(u(t,x)) - \phi(v(\tilde{t},x)))\alpha_j(t,\tilde{t})$ and we integrate with respect to $\tilde{t}$ over $[0,T]$. In (13) written for $v$ in variables $(\tilde{t},x)$ the test function $\varphi(\tilde{t},x) = -sgn_\eta(\phi(u(t,x)) - \phi(v(\tilde{t},x)))$ and we integrate with respect to $t$ over $[0,T]$. We add up and it comes (by sake of simplicity, we add a “tilde” superscript to any unknown in the $\tilde{t}$ variable).

$$\begin{aligned}
& \int_0^T \int_0^T \langle\langle \partial_t u - \partial_{\tilde{t}} \tilde{v}, sgn_\eta(\phi(u) - \phi(\tilde{v}))\rangle\rangle \alpha_j dt d\tilde{t} \\
& + \int_0^T \int_{Q_p} sgn'_\eta(\phi(u) - \phi(\tilde{v}))|\nabla(\phi(u) - \phi(\tilde{v}))|_n^2 \alpha_j dx dt d\tilde{t} \\
& - \int_0^T \int_{Q_p} sgn'_\eta(\phi(u) - \phi(\tilde{v}))b_p(\boldsymbol{f}_p(u) - \boldsymbol{f}_p(\tilde{v})) \cdot \nabla(\phi(u) - \phi(\tilde{v}))\alpha_j dx dt d\tilde{t} \\
& + \int_0^T \int_{Q_p} (g_p(t,x,u) - g_p(\tilde{t},x,\tilde{v}))sgn_\eta(\phi(u) - \phi(\tilde{v}))\alpha_j dx dt d\tilde{t} \\
= & \int_0^T \int_{\Sigma_{hp}} sgn_\eta(\phi(u) - \phi(\tilde{v}))b_h \boldsymbol{f}_h(u^\tau) \cdot \boldsymbol{\nu}_h \alpha_j dt d\mathcal{H}^{n-1} d\tilde{t} \\
& - \int_0^T \int_{\Sigma_{hp}} sgn_\eta(\phi(u) - \phi(\tilde{v}))b_h \boldsymbol{f}_h(\tilde{v}^\tau) \cdot \boldsymbol{\nu}_h \alpha_j dt d\mathcal{H}^{n-1} d\tilde{t}
\end{aligned} \tag{17}$$

By using the Young's inequality (with $p=2$) in the third line of (17) we obtain

$$\begin{aligned}
& \int_0^T \int_{Q_p} sgn'_\eta(\phi(u) - \phi(\tilde{v}))|\nabla(\phi(u) - \phi(\tilde{v}))|_n^2 \alpha_j dx dt d\tilde{t} \\
& - \int_0^T \int_{Q_p} sgn'_\eta(\phi(u) - \phi(\tilde{v}))b_p(\boldsymbol{f}_p(u) - \boldsymbol{f}_p(\tilde{v})) \cdot \nabla(\phi(u) - \phi(\tilde{v}))\alpha_j dx dt d\tilde{t} \\
\geq & -\tfrac{1}{2} \int_0^T \int_{Q_p} sgn'_\eta(\phi(u) - \phi(\tilde{v}))b_p^2 |\boldsymbol{f}_p(u) - \boldsymbol{f}_p(\tilde{v})|_n^2 \alpha_j dx dt d\tilde{t} \\
& + \tfrac{1}{2} \int_0^T \int_{Q_p} sgn'_\eta(\phi(u) - \phi(\tilde{v}))|\nabla(\phi(u) - \phi(\tilde{v}))|_n^2 \alpha_j dx dt d\tilde{t}
\end{aligned}$$

So, by virtue of (16), we claim the existence of a constant $C' > 0$ such that

$$\begin{aligned} & \int_0^T \int_{Q_p} sgn'_\eta(\phi(u)-\phi(\tilde{v}))|\nabla(\phi(u)-\phi(\tilde{v}))|^2 \alpha_j dxdtd\tilde{t} \\ & - \int_0^T \int_{Q_p} sgn'_\eta(\phi(u)-\phi(\tilde{v})) b_p(\boldsymbol{f}_p(u)-\boldsymbol{f}_p(\tilde{v}))\cdot\nabla(\phi(u)-\phi(\tilde{v}))\alpha_j dxdtd\tilde{t} \\ \geq \; & -C' \int_0^T \int_{Q_p} b_p^2 |\phi(u)-\phi(\tilde{v})|^{2\theta} sgn'_\eta(\phi(u)-\phi(\tilde{v}))\alpha_j dxdtd\tilde{t}. \end{aligned}$$

Since $\theta \geq \frac{1}{2}$, the term in the right-hand side of the above inequality tends to 0 as $\eta$ goes to $0^+$.
For the evolutionary term of (17) we use again the time integration by parts of F.Mignot Lemma ([5] p.31) to assert that

$$\int_0^T \langle\langle \partial_t u, sgn_\eta(\phi(u)-\phi(\tilde{v}))\rangle\rangle \alpha_j dt = -\int_{Q_p} \left( \int_{\tilde{v}}^u sgn_\eta(\phi(\tau)-\phi(\tilde{v}))d\tau \right) \partial_t \alpha_j dxdt$$

and

$$\int_0^T \langle\langle \partial_{\tilde{t}} \tilde{v}, sgn_\eta(\phi(u)-\phi(\tilde{v}))\rangle\rangle \alpha_j d\tilde{t} = -\int_{Q_p} \left( \int_{\tilde{v}}^u sgn_\eta(\phi(u)-\phi(\tau))d\tau \right) \partial_{\tilde{t}} \alpha_j dxd\tilde{t}$$

Then we take the limit with respect to $\eta$ in (17) and that gives:

$$\begin{aligned} & -\int_0^T \int_{Q_p} |u-\tilde{v}|(\partial_t\alpha_j + \partial_{\tilde{t}}\alpha_j) dxdtd\tilde{t} \\ \leq \; & \int_0^T \int_{Q_p} |g_p(t,x,u) - g_p(t,x,\tilde{v})|\alpha_j dxdtd\tilde{t} \\ & + \int_0^T \int_{\Sigma_{hp}} sgn(\phi(u)-\phi(\tilde{v})) b_h(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(\tilde{v}^\tau))\cdot\boldsymbol{\nu}_h \alpha_j d\mathcal{H}^{n-1} dtd\tilde{t}. \end{aligned}$$

Finally when $j$ goes to $+\infty$ we get

$$\begin{aligned} -\int_{Q_p} |u-v|\gamma'(t)dxdt \leq M_{g_p} \int_{Q_p} |u-v|\gamma(t)dxdt \\ \int_{\Sigma_{hp}} sgn(u^\phi - v^\phi) b_h(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(v^\tau))\cdot\boldsymbol{\nu}_h\gamma(t)dtd\mathcal{H}^{n-1}. \end{aligned} \tag{18}$$

Next we focus on the hyperbolic area. We use here the classical method of doubling (all) the variables due to S.N.Kruzhkov [8]. Briefly speaking, for all $(p,\tilde{p}) \in ((0,T)\times\mathbb{R}^n)^2$, we set

$$\psi_j(t,x,\tilde{t},\tilde{x}) = \psi_j(p,\tilde{p}) = \gamma((t+\tilde{t})/2)\zeta((x+\tilde{x})/2)\mathcal{W}_j(p-\tilde{p})$$

where $\gamma \in \mathcal{D}(0,T)$ with $\gamma \geq 0$, $\zeta \in \mathcal{D}(\mathbb{R}^n)$ with $\zeta \geq 0$. The variables $(\tilde{t},\tilde{x})$ being frozen, we choose in (11) written for $u$ in variables $(t,x)$, $k = \tilde{v} = v(\tilde{t},\tilde{x})$ and $\varphi(t,x) = \psi_j(p,\tilde{p})$. Now, the variable $(t,x)$ being frozen in (11) written for $\tilde{v}$ in variables $(\tilde{t},\tilde{x})$, we choose $k = u = u(t,x)$ and $\varphi(\tilde{t},\tilde{x}) = \psi_j(p,\tilde{p})$. We integrate over $Q_h$ with respect to the corresponding frozen variables and we add the two resulting inequalities. That yields

$$\begin{aligned}
&\int_{Q_h\times Q_h} |u-\tilde v|(\partial_t\psi_j+\partial_{\tilde t}\psi_j)dpd\tilde p\\
&+\int_{Q_h\times Q_h} \mathbf{\Phi}(u,\tilde v)\cdot(b_h\nabla_x\psi_j+\tilde b_h\nabla_{\tilde x}\psi_j)dpd\tilde p\\
&-\int_{Q_h\times Q_h}(G_h(u,\tilde v)+G_h(\tilde v,u))\psi_j dpd\tilde p\\
\geq\ &\int_{Q_h}\int_{\Sigma_h\setminus\Sigma_{hp}} b_h sgn(\tilde v)(\boldsymbol{f}_h(\tilde v)-\boldsymbol{f}_h(0))\cdot\boldsymbol{\nu}_h\psi_j(\sigma,\tilde p)dtd\mathcal{H}^{n-1}_\sigma d\tilde p\\
&+\int_{Q_h}\int_{\Sigma_h\setminus\Sigma_{hp}} \tilde b_h sgn(u)(\boldsymbol{f}_h(u)-\boldsymbol{f}_h(0))\cdot\boldsymbol{\nu}_h\psi_j(p,\tilde\sigma)d\tilde td\mathcal{H}^{n-1}_{\tilde\sigma} dp\\
&-\int_{Q_h}\int_{\Sigma\setminus\Sigma_{hp}} b_h sgn(u^\tau)(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(0))\cdot\boldsymbol{\nu}_h\psi_j(\sigma,\tilde p)dtd\mathcal{H}^{n-1}_\sigma d\tilde p\\
&-\int_{Q_h}\int_{\Sigma\setminus\Sigma_{hp}} \tilde b_h sgn(\tilde v^\tau)(\boldsymbol{f}_h(\tilde v^\tau)-\boldsymbol{f}_h(0))\cdot\boldsymbol{\nu}_h\psi_j(p,\tilde\sigma)d\tilde td\mathcal{H}^{n-1}_{\tilde\sigma} dp\\
&+\int_{Q_h}\int_{\Sigma_{hp}} b_h\mathbf{\Phi}_h(u^\tau,\tilde v)\cdot\boldsymbol{\nu}_h\psi_j(\sigma,\tilde p)dtd\mathcal{H}^{n-1}_\sigma d\tilde p\\
&+\int_{Q_h}\int_{\Sigma_{hp}} \tilde b_h\mathbf{\Phi}_h(\tilde v^\tau,u)\cdot\boldsymbol{\nu}_h\psi_j(\tilde\sigma,p)d\tilde td\mathcal{H}^{n-1}_{\tilde\sigma} dp
\end{aligned}$$

There is no difficulty to take the limit with respect to $j$ in the previous inequality and we obtain

$$\begin{aligned}
-\int_{Q_h}|u-v|\gamma'(t)dxdt\leq-\int_{Q_h} sgn(u-v)(g_h(t,x,u)-g_h(t,x,v))\gamma(t)dxdt\\
-\int_{\Sigma_{hp}} sgn(u^\tau-v^\tau)(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(v^\tau))\cdot\boldsymbol{\nu}_h b_h\gamma(t)d\mathcal{H}^{n-1}dt.
\end{aligned}\tag{19}$$

Observe that as soon as, for a.e. $\bar\sigma$ in $\Sigma_{hp}$, the mapping $\tau\to b_h(\bar\sigma)\boldsymbol{f}_h(\tau)\cdot\boldsymbol{\nu}_h(\bar\sigma)$ is nondecreasing, the second term in the right hand side of (19) falls. As a consequence, it warrants the uniqueness on the hyperbolic zone in first and then on the parabolic one by coming back to (18) and using that $u^\tau=v^\tau$ a.e. on $\Sigma_{hp}$. This framework that has been investigated in earlier works [1, 7]. Here, we add inequalities (18) and (19). Consequently

$$\begin{aligned}
-\int_Q|u-v|\gamma'(t)dxdt\leq M_g\int_Q|u-v|\gamma(t)dxdt\\
+\int_{\Sigma_{hp}} b_h sgn(u^\phi-v^\phi)(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(v^\tau))\cdot\boldsymbol{\nu}_h\gamma(t)d\mathcal{H}^{n-1}dt\\
-\int_{\Sigma_{hp}} b_h sgn(u^\tau-v^\tau)(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(v^\tau))\cdot\boldsymbol{\nu}_h\gamma(t)d\mathcal{H}^{n-1}dt.
\end{aligned}\tag{20}$$

For almost all $t$ in $]0,T[$, $\mathcal{H}^{n-1}$ a.e. on $\Gamma_{hp}$, we set

$$J=(sgn(u^\phi-v^\phi)-sgn(u^\tau-v^\tau))b_h(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(v^\tau))\cdot\boldsymbol{\nu}_h.$$

Our aim is to prove that $J\leq 0$ on $\Sigma_{hp}$. To do so we make a pointwise reasoning and so we have to distinguish several cases.
If $sgn(u^\phi-v^\phi)=sgn(u^\tau-v^\tau)$ or $u^\tau=v^\tau$, then $J=0$.
If $(sgn(u^\phi-v^\phi)=-sgn(u^\tau-v^\tau)$ or $u^\phi=v^\phi)$ and $u^\tau\neq v^\tau$ , then

$$J=-2b_h sgn(u^\tau-v^\tau)(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(v^\tau))\cdot\boldsymbol{\nu}_h.$$

Let us assume that $u^\tau<v^\tau$, the reasoning when $u^\tau>v^\tau$ being similar. In this framework $J=2b_h(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(v^\tau))\cdot\boldsymbol{\nu}_h$. Here we consider three different cases.

i) if $v^\phi\in]u^\tau,v^\tau[$, we choose $k=v^\phi$ in (15) written for $u$ and for $v$ and we add the two resulting inequalities. We have

$b_h(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(v^\tau)) \cdot \boldsymbol{\nu}_h \leq 0$.

ii) if $v^\phi \leq u^\tau$, we have $v^\phi \leq u^\tau < v^\tau$. Then we choose $k = u^\tau$ in (15) written for $v$. That gives $b_h(\boldsymbol{f}_h(v^\tau) - \boldsymbol{f}_h(u^\tau)) \cdot \boldsymbol{\nu}_h \geq 0$.

iii) if $v^\phi \geq v^\tau$, then $u^\tau < v^\tau < u^\phi$. We choose $k = v^\tau$ in (15) written for $u$ and $-b_h(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(v^\tau)) \geq 0$.

Consequently we deduce from (20) that

$$-\int_Q |u-v|\gamma'(t)dxdt \leq M_g \int_Q |u-v|\gamma(t)dxdt.$$

Finally we consider, for almost all $t$ in $(0,T)$, a sequence of test functions approximating the characteristic function $\mathbb{I}_{[0,t]}$. Then we use the initial condition (4) and the Gronwall's Lemma to close the proof of Theorem 1. □

# 4 The existence property

We use the vanishing viscosity method to approximate a weak entropy solution to (1). To do so we introduce a viscous problem related to (1) and prove that it has a unique weak solution that fulfills some *a priori* estimates uniform with respect to the viscous parameter. Then by applying E.Yu.Panov's works [11, 13] providing - under suitable non-degeneracy conditions for the flux $\boldsymbol{f}_h$ and $\boldsymbol{f}_p$ - a precompactness property in $L^1$ for the sequence of viscous solutions - we are able to establish that this sequence converges (in the $L^1$-sense) towards a weak entropy solution to (1).

## 4.1 The viscous problem

Let $\mu$ be a positive real number. We set

$$\lambda_\mu(x) = \mathbb{I}_{\Omega_p}(x) + \mu\mathbb{I}_{\Omega_h}(x),\ \phi_\mu = \phi + \mu\mathbb{I}_{\mathbb{R}},$$

and we consider of viscous problem related to (1) that means that we are interested in the existence and uniqueness of a measurable and bounded function $u_\mu$ satisfying

$$\left\{\begin{array}{rcll} \partial_t u_\mu + \mathrm{div}_x(b(x)\boldsymbol{f}(u_\mu)) + g(t,x,u_\mu) &=& \mathrm{div}_x(\lambda_\mu(x)\nabla\phi_\mu(u_\mu)) & \text{in } Q,\\ u_\mu &=& 0 & \text{on } \Gamma,\\ u_\mu(0,.) &=& u_0 & \text{on } \Omega. \end{array}\right. \tag{21}$$

In order to deal with bounded solutions, we introduce the following assumptions on $\boldsymbol{f}_h$ and $\boldsymbol{f}_p$ that will discussed below, see proof of (24) in Proposition 3: we suppose that $\mathcal{H}^{n-1}$ a.e. on $\Gamma_{hp}$, for almost all $t$ in $(0,T)$,

$$(b_p\boldsymbol{f}_p(M_1(t)) - b_h\boldsymbol{f}_h(M_1(t))) \cdot \boldsymbol{\nu}_h \geq 0, \tag{22}$$

$$(b_p\boldsymbol{f}_p(M_2(t)) - b_h\boldsymbol{f}_h(M_2(t))) \cdot \boldsymbol{\nu}_h \leq 0. \tag{23}$$

We also introduce the functional space

$$W(0,T) = \{v \in L^2(0,T;H^1_0(\Omega)), \partial_t v \in L^2(0,T;H^{-1}(\Omega))\},$$

and denote by $\langle .,. \rangle$ the pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

This way we may state

**Proposition 3.** *Under* (22) *and* (23) *there exists a unique solution* $u_\mu \in W(0,T) \cap L^\infty(Q)$ *to* (21) *such that*

$$M_2(t) \leq u_\mu(t,.) \leq M_1(t) \text{ for all } t \in [0,T],\ \text{a.e. in } \Omega, \tag{24}$$

$$u_\mu(0,.) = u_0 \text{ a.e. in } \Omega. \tag{25}$$

*Moreover, for any* $v \in H^1_0(\Omega)$, *and for almost all* $t \in (0,T)$, $u_\mu$ *satisfies the variational equality*

$$\langle \partial_t u_\mu, v\rangle + \int_\Omega ((\lambda_\mu(x)\nabla\phi_\mu(u_\mu) - b(x)\boldsymbol{f}(u_\mu)) \cdot \nabla v + g(t,x,u_\mu)v)dx = 0. \tag{26}$$

*Proof.* We focus on the proof of (24). In a first step we use a troncation process. For any real numbers $a$, $b$ and $c$ we set $\mathcal{B}(a,b,c) = \max\{a, \min\{b,c\}\}$. Then, for a fixed * $\mu$, we introduce the auxiliary problem

$$\left\{\begin{array}{l} \text{Find } u_\mu \text{ in } W(0,T) \text{ such that a.e. on } (0,T) \text{ and for all } v \in H_0^1(\Omega), \\ \langle \partial_t u_\mu, v\rangle + \displaystyle\int_\Omega ((\lambda_\mu(x)\phi'_\mu(u^\star_\mu)\nabla u_\mu - b(x)\boldsymbol{f}(u^\star_\mu)) \cdot \nabla v + g(t,x,u^\star_\mu)v)dx = 0 \\ u^\star_\mu(0,.) = u_0 \text{ a.e. on } \Omega \end{array}\right. \tag{27}$$

where $u^\star_\mu = \mathcal{B}(M_2(t), u_\mu, M_1(t))$.

Let us prove that (24) - (26) is equivalent to (27). It is clear that if $u_\mu$ satisfies (24) - (26) then $u_\mu$ fulfills (27). Conversely let $u_\mu$ be a solution to (27). To obtain the majorization for $u_\mu$ in (24) we may consider in (27) the test-function $v_\eta = sgn_\eta(u_\mu - M_1(t))^+$ and we integrate over $(0,s)$, for any $s \in (0,T)$. One adds and substracts $\langle \partial_t M_1(t), u_\mu\rangle$ in the evolution term. Furthermore, since $v_\eta$ is supported on $\{u_\mu > M_1(t)\}$ we have (by denoting $Q_s = (0,s)\times\Omega$)

$$\begin{aligned} &\int_{Q_s} (-b(x)\boldsymbol{f}(u^\star_\mu)) \cdot \nabla v_\eta + g(t,x,u^\star_\mu)v_\eta)dxdt \\ =\ &\int_{Q_s} (-b(x)\boldsymbol{f}(M_1(t))) \cdot \nabla v_\eta + g(t,x,M_1(t))v_\eta)dxdt \end{aligned}$$

For the convection term, we use a Green's formula to obtain

$$\begin{aligned} -\int_{Q_s} b(x)\boldsymbol{f}(u^\star_\mu)) \cdot \nabla v_\eta dxdt = &\sum_{i\in\{h,p\}} \int_{Q_{i,s}} \boldsymbol{f}_i(M_1(t)) \cdot \nabla b_i(x) v_\eta dxdt \\ &+ \int_{\Sigma_{hp}} (b_p \boldsymbol{f}_p(M_1(t)) - b_h \boldsymbol{f}_h(M_1(t))) \cdot \boldsymbol{\nu}_h v_\eta dt d\mathcal{H}^{n-1} \end{aligned}$$

Then, in view of (22), we ensure that the interface integral is nonnegative.
Owing to the definition of $v_\eta$ the diffusion term is also nonnegative. Then, when $\eta$ goes to $0^+$ thanks to the Lebesgue dominated convergence Theorem it yields,

$$\begin{aligned} &\int_\Omega (u_\mu(s,x) - M_1(s))^+ dx + \int_{Q_s} M_1'(t) sgn(u_\mu - M_1(t))^+ dxdt \\ &+ \sum_{i\in\{h,p\}} \int_{Q_{i,s}} (\boldsymbol{f}_i(M_1(t)) \cdot \nabla b_i + g_i(t,x,M_1(t))) sgn(u_\mu - M_1(t))^+ dxdt \leq 0. \end{aligned}$$

Due to the definition of $M_1$, everywhere on $Q_{i,s}$,

$$M_1'(t) + (\boldsymbol{f}_i(M_1(t)) \cdot \nabla b_i + g_i(t,x,M_1(t))) \geq 0,$$

and the conclusion follows.

In a similar way, to prove the minorization in (24) we may consider the test function $v_\eta = -sgn_\eta(u_\mu - M_2(t))^-$ in (27) and we use (23) to state that the integral over $\Sigma_{hp}$ that appears when we deal with the convection term is nonnegative.

Thus the existence property for (21) is reduced to an existence result to (27). To do so we use the Schauder-Tychonoff fixed point Theorem as in [7] while the uniqueness of a solution to (24)-(26) is obtained by a Holmgren-type duality method (see [7]).

□

Let us now collect an *a priori* estimate for the sequence $(u_\mu)_{\mu>0}$ proper to study its limit when $\mu$ goes to $0^+$.

**Proposition 4.** *There exists a positive constant $C$ independent of $\mu$ such that*

$$\|(\lambda_\mu)^{1/2}\nabla\widehat{\phi}(u_\mu)\|^2_{L^2(Q)^n} + \|(\mu\lambda_\mu)^{1/2}\nabla u_\mu\|^2_{L^2(Q)^n} \leq C, \tag{28}$$

*where* $\widehat{\phi}(u_\mu) = \displaystyle\int_0^{u_\mu} \sqrt{\phi'(\tau)}d\tau$.

*Proof.* We choose $v = u_\mu$ in (26) and integrate over $]0,T[$. We have:

$$\int_0^T \langle \partial_t u, u\rangle dt = \frac{1}{2}\|u_\mu(T,.)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|u_\mu(T,.)\|^2_{L^2(\Omega)}.$$

Moreover,

$$\begin{aligned}\int_Q b(x)\boldsymbol{f}(u_\mu)\cdot\nabla u_\mu dxdt &= \int_Q b(x)\mathrm{div}\left(\int_0^{u_\mu}\boldsymbol{f}(\tau)d\tau\right)dxdt\\ &= \sum_{i\in\{h,p\}}\int_{Q_i} b_i(x)\mathrm{div}\boldsymbol{F}_i(u_\mu)dxdt.\end{aligned}$$

where $\boldsymbol{F}_i(u_\mu) = \int_0^{u_\mu}\boldsymbol{f}_i(\tau)d\tau$.

So by a Green's formula on each $Q_i$, $i = h, p$, the convection term can be written as

$$\sum_{i\in\{h,p\}}\int_{Q_i}\nabla b_(x)\boldsymbol{F}_i(u_\mu)dxdt + \int_{\Sigma_{hp}}(b_p(\overline{\sigma})\boldsymbol{F}_p(u_\mu) - b_h(\overline{\sigma})\boldsymbol{F}_h(u_\mu))\cdot\boldsymbol{\nu}_h dtd\mathcal{H}^{n-1}.$$

Thanks to (24), we assert that the convection term is bounded independently of $\mu$. The reaction term is also clearly bounded (as a consequence of (24) and for the diffusion term we write:

$$\int_Q \lambda_\mu\nabla\phi_\mu(u_\mu)\cdot\nabla u_\mu dxdt = \int_Q \lambda_\mu\phi'(u_\mu)|\nabla u_\mu|^2_n dxdt + \mu\int_Q \lambda_\mu|\nabla u_\mu|^2_n dxdt$$

where,

$$\int_Q \lambda_\mu\phi'(u_\mu)|\nabla u_\mu|^2_n dxdt = \int_Q(\sqrt{\lambda_\mu}|\nabla\widehat{\phi}(u_\mu)|_n)^2 dxdt.$$

This way,

$$\int_Q \lambda_\mu\nabla\phi_\mu(u_\mu)\cdot\nabla u_\mu dxdt = \|(\lambda_\mu)^{1/2}\nabla\widehat{\phi}(u_\mu)\|^2_{L^2(Q)^n} + \|(\mu\lambda_\mu)^{1/2}\nabla u_\mu\|^2_{L^2(Q)^n}$$

The conclusion follows. □

## 4.2 The viscous limit

Since the estimates (24) and (28) are not sufficient not allow us to pass to the limit with $\mu$ on the parabolic zone (mainly it misses an estimate on the time derivative of $u_{\mu|Q_p}$ in $L^2(0,T;V')$), we use E.Yu.Panov's results in [13]. The author introduces a so-called nonlinear condition for the flux $\boldsymbol{f}_p$ so that the sequence $(u_{\mu|Q_p})_{\mu>0}$ is precompact in $L^1(Q_p)$. Indeed assume that:

**Assumption 1.** *For almost all $x \in \Omega_p$, for all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, the functions $\lambda \mapsto b_p(x)\boldsymbol{f}_p(\lambda)\cdot\xi$ and $\lambda \mapsto \phi(\lambda)\xi^2$ are not linear simultaneously on any non-degenerate intervals.*

Then, as soon as (24) and (28) hold, there exists a subsequence, still denoted $(u_{\mu|Q_p})_{\mu>0}$, that converges strongly in $L^1(Q_p)$. This way, we may state:

**Theorem 2.** *The sequence $(u_\mu)_{\mu>0}$ of solutions to* (21)*, admits a subsequence that converges strongly in $L^1(Q)$ towards a function $u \in L^\infty(Q)$.*
*Moreover $u$ is the weak entropy solution of the problem* (1)*.*

**Remark 3.** *The strong convergence property on $Q_p$ could also be obtained if we replace Assumption 1 by the following one*

$$\phi \text{ is Hölder continuous with an exponent } \tau \in (0,1),$$

*as it is supposed in [7].*

*Proof of theorem 2.* Thanks to Assumption 1 we make sure that we can extract a subsequence of $(u_{\mu|_{Q_p}})_{\mu>0}$ that strongly converges in $L^1(Q_p)$. Besides, from [13] and by virtue of (8) we know that we can extract a subsequence of $(u_{\mu|_{Q_h}})_{\mu>0}$ that strongly converges in $L^1(Q_h)$. Thus there exists a subsequence of $(u_\mu)_{\mu>0}$ that converges strongly in $L^1(Q)$ towards a function $u$ of $L^\infty(Q)$. It remains to show that $u$ is a weak entropy solution to (1). To this purpose we choose in (26) the test function $v_\mu^\eta = sgn_\eta(\phi(u_\mu) - \phi(k))\varphi$, where $k \in \mathbb{R}$, $\varphi \in \mathcal{D}([0,T) \times \Omega)$, $\varphi \geq 0$. We integrate over $[0,T]$ to have

$$\int_0^T \langle \partial_t u_\mu, v_\mu^\eta \rangle dt + \int_Q ((\lambda_\mu \nabla \phi_\mu(u_\mu) - b(x)\boldsymbol{f}(u_\mu)) \cdot \nabla v_\mu^\eta + g(t,x,u_\mu)v_\eta^\mu) dxdt = 0. \quad (29)$$

An integration-by-parts in the evolution term gives

$$\int_0^T \langle \partial_t u_\mu, v_\mu^\eta \rangle dt = -\int_Q I_\eta(u_\mu, k)\partial_t \varphi dxdt - \int_\Omega I_\eta(u_0, k)\varphi(0,.)dx.$$

Due to the definition of $sgn_\eta$ and $\phi_\mu$, we obtain for the diffusion term

$$\int_Q \lambda_\mu \nabla \phi_\mu(u_\mu) \cdot \nabla v_\mu^\eta dxdt \geq \int_Q \lambda_\mu sgn_\eta(\phi(u_\mu) - \phi(k)) \nabla \phi(u_\mu) \cdot \nabla \varphi \, dxdt$$
$$+\mu \int_Q \lambda_\mu sgn_\eta(\phi(u_\mu) - \phi(k)) \nabla u_\mu \cdot \nabla \varphi \, dxdt.$$

Note that the second term of the right-hand side goes to zero with $\mu$ thanks to estimate (28). The convection term is written as

$$-\sum_{i\in\{h,p\}} \int_Q b_i(x)\boldsymbol{f}_i(u_\mu) \cdot \nabla \phi(u_\mu) sgn_\eta'(\phi(u_\mu) - \phi(k))\varphi \, dxdt$$
$$-\sum_{i\in\{h,p\}} \int_Q b_i(x) sgn_\eta(\phi(u_\mu) - \phi(k))\boldsymbol{f}_i(u_\mu) \cdot \nabla \varphi \, dxdt.$$

Since the $\mu$ and $\eta$-limits in second line does not bring difficulties we focus on the first line for $i = h$, the reasoning for $i = p$ being similar. We denote

$$J_{\mu,\eta} = -\int_{Q_h} b_h(x)\boldsymbol{f}_h(u_\mu) \cdot \nabla \phi(u_\mu) sgn_\eta'(\phi(u_\mu) - \phi(k))\varphi \, dxdt.$$

Thus, owing to definition of $\boldsymbol{F}_{h,\eta}$,

$$J_{\mu,\eta} = -\int_{Q_h} b(x) \text{div} \boldsymbol{F}_{h,\eta}(u_\mu, k)\varphi \, dxdt.$$

So, by a Green's formula, we have

$$J_{\mu,\eta} = \int_{Q_h} \boldsymbol{F}_{h,\eta}(u_\mu, k) \cdot (\nabla b_h \varphi_2 + \nabla \varphi b_h) dxdt$$
$$-\int_{\Sigma_{hp}} b_h \boldsymbol{F}_{h,\eta}(u_\mu, k) \cdot \boldsymbol{\nu}_h \varphi \, d\mathcal{H}^{n-1} dt.$$

Now we take the limit with respect to $\mu$. For the interface integral we assert that the sequence $(\boldsymbol{F}_{h,\eta}(u_\mu, k)\varphi)_{\mu>0}$ weakly converges towards $\boldsymbol{F}_{h,\eta}(u,k)\varphi$ in $L^2(\Sigma_{hp})^n$. Indeed, we notice first that since $\phi(u_\mu) \in L^2(0,T;H^1(\Omega))$, for almost all $t \in (0,T)$, $\mathcal{H}^{n-1}$ a.e., $(\phi(u_\mu)_{|\Omega_h})_{|\Gamma_{hp}} = (\phi(u_\mu)_{|\Omega_p})_{|\Gamma_{hp}}$.
Moreover, $\boldsymbol{F}_{h,\eta}(.,k)$ being a Lipschitz function, for $1 \leq q < \infty$,
$(\boldsymbol{F}_{h,\eta}(u_\mu, k))_{\mu>0}$ strongly converges towards $\boldsymbol{F}_{h,\eta}(u,k)$ in $L^q(Q_p)^n$.
Besides, by virtue of (28), the sequence
$(\boldsymbol{F}_{h,\eta}(u_\mu, k))_{\mu>0}$ is uniformly bounded in $L^2(0,T;V)^n \cap L^\infty(Q)^n$.
So $(\boldsymbol{F}_{h,\eta}(u_\mu, k)\varphi)_{\mu>0}$ weakly converges, up to a subsequence, towards

$\boldsymbol{F}_{h,\eta}(u,k)\varphi$ in $L^2(0,T;V)^n$.
Therefore, as the trace operator from $L^2(0,T;V)$ into $L^2(\Sigma_{hp})$ is linear and continuous, $(\boldsymbol{F}_{h,\eta}(u_\mu,k)\varphi)_{\mu>0}$ weakly converges towards
$\boldsymbol{F}_{h,\eta}(u,k)\varphi$ in $L^2(\Sigma_p)^n$ and then in $L^2(\Sigma_{hp})^n$.
Consequently, $\lim_{\mu\to 0^+} J_{\mu,\eta} = J_\eta$ where

$$\begin{aligned} J_\eta = \ & \int_{Q_h} \boldsymbol{F}_{h,\eta}(u,k)\cdot(\nabla b_h\varphi + \nabla\varphi b_h)dxdt \\ & - \int_{\Sigma_{hp}} b_h \boldsymbol{F}_{h,\eta}(u,k)\cdot\boldsymbol{\nu}_h\varphi\, d\mathcal{H}^{n-1}dt. \end{aligned}$$

Then, we can pass to the limit with respect to $\mu$ in each term of (26) to have

$$\begin{aligned} & \int_Q I_\eta(u_\mu,k)\partial_t\varphi dxdt + \int_\Omega I_\eta(u_0,k)\varphi(0,.)\,dx \\ & - \int_Q sgn_\eta(\phi(u)-\phi(k))\nabla\phi(u)\cdot\nabla\varphi\, dxdt \\ & + \int_Q b(x)\{sgn_\eta(\phi(u)-\phi(k))\boldsymbol{f}(u) - \boldsymbol{F}_\eta(u,k)\}\cdot\nabla\varphi\, dxdt \\ & - \int_Q \{sgn_\eta(\phi(u)-\phi(k))g(t,x,u) + \nabla b(x)\cdot\boldsymbol{F}_\eta(u,k)\}\varphi\, dxdt \\ & + \int_{\Sigma_{hp}} (b_h\boldsymbol{F}_{h,\eta}(u,k) - b_p\boldsymbol{F}_{p,\eta}(u,k))\varphi\cdot\boldsymbol{\nu}_h dtd\mathcal{H}^{n-1} \geq 0, \end{aligned} \tag{30}$$

and (3) follows.

Now it remains to prove that $u$ fulfills (4)-(5). We consider in (30) a test function $\varphi$ in $\mathcal{D}([0,T[\times\Omega_h)$ and take the limit with respect to $\eta$. We obtain

$$\begin{aligned} & -\int_{Q_h} (|u-k|\partial_t\varphi + b_h(x)\boldsymbol{\Phi}(u,k)\cdot\nabla\varphi)\,dxdt \\ & + \int_{Q_h} sgn(u-k)(g_h(t,x,u) + \nabla b_h(x)\cdot\boldsymbol{f}_h(k))\varphi\, dxdt \\ & \leq \int_{\Omega_h} |u_0-k|\varphi(0,.)\,dx. \end{aligned}$$

Thus, we refer to F.Otto's work in [10] (see also [9, Chap. 2]) to ensure that

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\Omega_h} |u(t,x)-u_0(x)|dx = 0.$$

In order to show that $u$ fulfills the initial condition on the parabolic zone, we choose a test function $\varphi = \varphi_1\varphi_2$ where $\varphi_1$ belongs to $\mathcal{D}([0,T[)$ and $\varphi_2$ to $\mathcal{D}(\Omega_p)$, $\varphi_1,\varphi_2\geq 0$. We pass to the limit with respect to $\eta$ and we can state that

$$-\int_0^T \left(\int_{\Omega_p} |u-k|\varphi_2 dx + h(t)\right)\varphi_1'(t) \leq \int_{\Omega_p} |u_0-k|\varphi_2\varphi_1(0)dx.$$

where

$$\begin{aligned} & h(t) = \int_{\Omega_p}\int_0^t b_p(x)\boldsymbol{\Phi}_p(u(\tau,x),k)\cdot\nabla\varphi_2 d\tau dx \\ & + \int_{\Omega_p}\int_0^t sgn(u(\tau,x)-k)g_p(\tau,x,u(\tau,x))\varphi_2 - |\phi(u(\tau,x)) - \phi(k)|\Delta\varphi_2 d\tau dx. \end{aligned}$$

So the time-depending function

$$t \mapsto \int_{\Omega_p} |u-k|\varphi_2 dx + h(t)$$

is identified with a nonincreasing and bounded function and consequently has an essential limit when $t$ tends to $0^+$. Since $h$ goes to zero with $t$ it comes, for any $\varphi_2 \in \mathcal{C}_c^\infty(\Omega_p)$ with $\varphi_2 \geq 0$,

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\Omega_p} |u(t,x) - k|\varphi_2 dx \leq \int_{\Omega_p} |u_0 - k|\varphi_2 dx.$$

The above inequality implies that $u$ satisfies (4) (see [9] for more details).

To prove (5), we consider the family of boundary entropy - entropy flux pair (see [9, 10]) defined by, for any $\delta > 0$,

$$H_\delta(\tau, k) = ((\mathrm{dist}(\tau, I[0,k]))^2 + \delta^2)^{1/2} - \delta$$

and

$$\boldsymbol{Q}_{h,\delta}(\tau, k) = \int_k^\tau \partial_1 H_\delta(\lambda, k)\boldsymbol{f}'_h(\lambda)d\lambda.$$

Then we come back to the viscous problem (21) and choose in (26) the test function $v = \partial_1 H_\delta(u_\mu, k)\varphi$ where $\varphi$ belongs to $\mathcal{D}((0,T)\times\overline{\Omega}_h)$, $\varphi \geq 0$ and $\varphi_2 = 0$ on $\Gamma_{hp}$. Let us note that $\partial_1 H_\delta(u_\mu, k)\varphi$ is an element of $L^2(0,T; H_0^1(\Omega_h))$ so that Green's formula does not give rise to integrals over $\Sigma_{hp}$. We integrate over $(0,T)$ and, noticing that $\tau \mapsto H_\delta(\tau, .)$ is a convex function, it provides

$$\begin{aligned} &-\int_{Q_h} (H_\delta(u_\mu, k)\partial_t\varphi + b_h\mathbf{Q}_{h,\delta}(u_\mu, k)\cdot\nabla\varphi - \boldsymbol{G}_{h,\delta}(u_\mu, k)\varphi)\, dxdt \\ &\leq -\mu \int_{Q_h} \partial_1 H_\delta(u_\mu, k)\nabla\varphi\cdot\nabla\phi(u_\mu)dxdt \end{aligned}$$

where

$$\boldsymbol{G}_{h,\delta}(u_\mu, k) = \int_k^{u_\mu} \partial_{11}^2 H_\delta(\tau, k)\boldsymbol{f}_h(\tau)d\tau\cdot\nabla b_h + g_h(t,x,u_\mu)\partial_1 H_\delta(u_\mu, k).$$

By virtue of Theorem 2 and (28) to deal with the right hand side, there is no difficulties to take the limit with respect to $\mu$. That gives

$$-\int_{Q_h} (H_\delta(u,k)\partial_t\varphi + b_h\mathbf{Q}_{h,\delta}(u,k)\nabla\varphi - \boldsymbol{G}_{h,\delta}(u,k)\varphi)\, dxdt \leq 0.$$

From [10] it follows that, for any $\zeta$ in $L^1(\Sigma_h \setminus \Sigma_{hp})$ with $\zeta \geq 0$,

$$\operatorname*{ess\,lim}_{s\to 0^-} \int_{\Sigma_h\setminus\Sigma_{hp}} b_h\boldsymbol{G}_{h,\delta}(u(\sigma + s\boldsymbol{\nu}_h), k)\cdot\boldsymbol{\nu}_h\zeta\, dtd\mathcal{H}^{n-1} \geq 0,$$

that is

$$\int_{\Sigma_h\setminus\Sigma_{hp}} b_h\boldsymbol{G}_{h,\delta}(u^\tau, k)\cdot\boldsymbol{\nu}_h\zeta\, dtd\mathcal{H}^{n-1} \geq 0,$$

in view of (9).

That yields first to boundary condition (5) by observing that $(\boldsymbol{Q}_{h,\delta})_\delta$ uniformly converges towards $\mathcal{F}_h$ as $\delta$ goes to $0^+$ et so to (10). □

# References


[1] G. Aguilar, L. Lévi and M. Madaune-Tort, *Coupling of Multidimensional Parabolic and Hyperbolic Equations*, Journal of Hyperbolic Differential Equations, **3**, No. 1, (2006) 53-80.

[2] G. Aguilar, F. Lisbona and M. Madaune-Tort, *Analysis of a nonlinear parabolic-hyperbolic problem*, Adv. in Math. Sci. and Appl. **9** 597-620 (1999).

[3] C. Bardos, A.Y LeRoux, J.C Nédélec: *First order quasilinear equations with boundary conditions*, Comm. in partial differential equations, **4**, 1017-1034, 1979.

[4] G.-Q. Chen, H. Frid : *Divergence-Measure fields and hyperbolic conservation laws*, Arch. Rational Mech. Anal., **147**, 89-118, 1999.

[5] G. Gagneux, M. Madaune-Tort: *Analyse mathématique de modèles nonlinéaires de l'ingénierie pétrolière,* Mathématiques et Applications, **22**, Springer-Verlag, Berlin, 1996.

[6] F. Gastaldi and A. Quateroni, Coupling of two-dimensional hyperbolic and elliptic equations, *Comput. Methods Appl. Mech. Eng.* **80** 1-3 (1990) 347-354 .

[7] J. Jimenez, L. Lévi: *Entropy formulations for a class of scalar conservation laws with space-discontinuous flux functions in a bounded domain*, J. Eng. Math., **60**, 319-335, 2008.

[8] S.N. Kruzhkov: *First-order quasilinear equations with several independent variables*, Mat. Sb. **81**, 228-255, 1970.

[9] J. Màlex, J. Nečas, M. Rokyta, M. Růžička : *Weak and mesure-valued solutions to evolutionary PDEs* Applied Mathematics and Mathematical Computation, **13**, Chapman & Hall, London, 1996.

[10] F. Otto: *Initial-boundary value problem for a scalar conservation law*, C.R. Acad. Sci. Paris, **322**, série I, pp. 729-734, 1996.

[11] E. Yu. Panov: *Property of strong precompactness for bounded sets of measure valued solutions of a first-order quasilinear equation*, Sbornik: Mathematics, **190**:3, 427-446, 1999.

[12] E. Yu. Panov: *Existence of strong traces for generalized solutions of multidimensional scalar conservation laws*, Journal of Hyperbolic Differential Equations, **2**:4, 885-908, 2005.

[13] E. Yu. Panov: *On the strong precompactness property for entropy solutions of an ultra-parabolic equation with discontinuous flux*, Preprint (available on http://www.math.ntnu.no/conservation/).

[14] S. A. Sazhenkov: *The genuinely nonlinear Graetz-Nusselt ultraparabolic equation*, Siberian Mathematical Journal, **47**:2, 355-375, 2006.

[15] A. Vasseur: *Strong traces for solutions of multidimensional scalar conservation laws*, Arch. Rational Mech. Anal. **160**, 181-193, 2001.