A note on strong convergence to common fixed points of nonexpansive mappings in Hilbert spaces

Jean-Philippe Chancelier

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Résumé

The aim of this paper is to investigate the links between $\mathcal{T}_C$-class algorithms [1], CQ Algorithm [6, 8] and shrinking projection methods [9]. We show that strong convergence of these algorithms are related to coherent $\mathcal{T}_C$-class sequences of mapping. Some examples dealing with nonexpansive finite set of mappings and nonexpansive semigroups are given. They extend some existing theorems in [1, 6, 9, 7].

1 Introduction

Let C be a closed convex subset of a Hilbert space $\mathcal{H}$. A mapping T of C into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all} x, y \in C .$$

We denote by $\mathrm{Fix}(T)$ the set of fixed points of T. That is

$$\mathrm{Fix}(T) \stackrel{\text{def}}{=} \{x \in C \, : \, Tx = x\} . \tag{1}$$

There are many iterative methods for approximation of common fixed points of a family of nonexpansive mapping in a Hilbert space. In Section 2 we recall the CQ Algorithm [6, 8] (Algorithm 2) associated to a sequence of mappings $(T_n)_{n\geq 0}$ of C into itself. The CQ Algorithm when applied to a sequence of mappings of $\mathcal{H}$ into itself is the same as a Haugazeau method [4] studied in [1, Algorithm 3.1] and applied to $\mathcal{T}$-class mappings.

We straighforwardly generalize, in Section 2, the $\mathcal{T}$-class to take into account mappings of C into itself. We denote this new class by the $\mathcal{T}_C$-class. Using this extension, the CQ Algorithm (Algorithm 2) coincides with the Haugazeau method (Algorithm 1) and a strong convergence theorem can be obtained by following results from [1]. Note that the convergence theorem is obtained for $\mathcal{T}_C$-class sequences which are coherent (Definition 3).

In [9] another algorithm called the shrinking projection method is also studied. One of our aims in this article is to prove that, rephrased in the context

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of $\mathcal{T}_C$-class algorithm, the convergence results of this new algorithm (Algorithm 3) is also related to coherent sequences of $\mathcal{T}_C$-class mappings. We give in Theorem 6 a strong convergence result of Algorithm 3 for $\mathcal{T}_C$-class coherent sequence of mappings. Section 4 is devoted to the proof. The strong convergence of Algorithm 3 is also proved in [9, Theorem 3.3] for sequence of nonexpansive mappings satisfying the NST-condition(I) (Definition 9). It is easy to prove that if R is a nonexpansive mapping of C into itself then $T = (R + Id)/2$ belongs to the $\mathcal{T}_C$-class and that a sequence of nonexpansive mappings satisfying the NST-condition(I) is coherent. Thus Theorem 6 extends [9, Theorem 3.3 and Theorem 3.4].

In Section 3 we show that specific sequences of mappings are coherent. Combined with Theorem 6 it can be considered as an extension to some existing theorems in [6, 9, 7].

2 The $\mathcal{T}_C$-class iterative algorithms, CQ algorithm and the shrinking projection method

We first recall here the $\mathcal{T}$-class iterative algorithms as defined by H. Bauschke and P. L. Combettes [1].

For $(x, y) \in \mathcal{H}^2$ and S a subset of $\mathcal{H}$, we define the mapping H_S as follows :

$$H_S(x, y) \stackrel{\text{def}}{=} \{z \in S \quad | \quad \langle z - y, x - y\rangle \leq 0\} \ . \tag{2}$$

We also define the mapping H by $H \stackrel{\text{def}}{=} H_{\mathcal{H}}$. Note that $H_S(x, x) = S$ and for $x \neq y$, $H(x, y)$ is a closed affine half space. For a nonempty closed convex C, we denote by $Q_C(x, y, z)$ the projection, when it exists, of x onto $H_C(x, y) \cap H_C(y, z)$ and Q the projection when $C = \mathcal{H}$, that is $Q \stackrel{\text{def}}{=} Q_{\mathcal{H}}$. As an intersection of two closed affine half spaces and a closed convex, $H_C(x, y) \cap H_C(y, z)$ is a possibly empty closed convex.

It is easy to check, from the definition of H, that y is the projection of x onto $H(x, y)$ and we therefore have $Q(x, x, y) = P_{H(x,y)}x = y$. Where P_C is the metric projection from $\mathcal{H}$ onto C. Moreover, if $y \in C$ then we also have that y is the projection of x onto $H_C(x, y)$ which gives $Q_C(x, x, y) = y$.

The algorithm studied in [1] is the following

Algorithm 1 *Given $x_0 \in C$ and a sequence $(T_n)_{n\geq 0}$ of mappings $T_n : C \to \mathcal{H}$, we consider the sequence $(x_n)_{n\geq 0}$ generated by the following algorithm :*

$$x_{n+1} = Q_C(x_0, x_n, T_n x_n)$$

A very similar algorithm exists under the name of CQ algorithm [6, 8] :

Algorithm 2 *Given $x_0 \in C$, we consider the sequence $(x_n)_{n\geq 0}$ generated by the following algorithm :*

$$\begin{cases} y_n = R_n x_n, \\ C_n \stackrel{def}{=} \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\} \ , \\ D_n \stackrel{def}{=} \{z \in C \mid \langle x_n - z, x_0 - x_n\rangle \geq 0\} \ , \\ x_{n+1} = P_{(C_n \cap D_n)} x_0. \end{cases}$$

The link between the two algorithms is described by the following lemma.

Lemma 1 *The sequence generated by Algorithm 2 coincides with the sequence given by* $x_{n+1} = Q_C(x_0, x_n, T_n x_n)$ *with* $T_n \stackrel{def}{=} (R_n + Id)/2$.

Proof : Following [1], the proof easily follows from the equality

$$4 \langle z - Tx, x - Tx \rangle = \|Rx - z\|^2 - \|x - z\|^2 .$$

□

The convergence of Algorithm 1 and therefore of Algorithm 2 when $C = \mathcal{H}$ is studied in [1]. It relies on two requested properties of the sequence $(T_n)_{n\geq 0}$. First, the sequence $(T_n)_{n\geq 0}$ must belong the $\mathcal{T}$-class which means that for all $n \in \mathbb{N}$ we must have $T_n \in \mathcal{T}$ where $\mathcal{T}$ is defined as follows :

Definition 2 *A mapping* $T : C \mapsto \mathcal{H}$ *belongs to the* $\mathcal{T}_C$*-class if it is an element of the set* $\mathcal{T}_C$ *:*

$$\mathcal{T}_C \stackrel{def}{=} \{T : C \mapsto C \,|\, dom(T) = C \quad and \quad (\forall x \in C)\, Fix(T) \subset H(x, Tx)\} .$$

When $C = \mathcal{H}$, we use the notation $\mathcal{T} = \mathcal{T}_{\mathcal{H}}$. Second, the sequence $(T_n)_{n\geq 0}$ must be coherent as defined below.

Definition 3 *[1] A sequence* $(T_n)_{n\geq 0}$ *such that* $T_n \in \mathcal{T}_C$ *is* coherent *if for every bounded sequence* $\{z_n\}_{n\geq 0} \in C$ *the following holds :*

$$\left\{ \begin{array}{l} \sum_{n\geq 0} \|z_{n+1} - z_n\|^2 < \infty \\ \sum_{n\geq 0} \|z_n - T_n z_n\|^2 < \infty \end{array} \right. \Rightarrow \mathcal{M}(z_n)_{n\geq 0} \subset \bigcap_{n\geq 0} Fix(T_n) \tag{3}$$

where $\mathcal{M}(z_n)_{n\geq 0}$ *is the set of weak cluster points of the sequence* $(z_n)_{n\geq 0}$.

Theorem 4 *[1, Theorem 4.2] Suppose that* $C = \mathcal{H}$ *and the* $\mathcal{T}_C$*-class sequence* $(T_n)_{n\geq 0}$ *is coherent. Then, for an arbitrary orbit of Algorithm 1, exactly one of the following alternatives holds :*

(a) $F \neq \emptyset$ *and* $x_n \to_n P_F x_0$ *;*

(b) $F = \emptyset$ *and* $x_n \to_n +\infty$ *;*

(c) $F = \emptyset$ *and the algorithm terminates,*

where the set F *is defined by* $F \stackrel{def}{=} \bigcap_{n\geq 0} Fix(T_n)$.

Remark 5 *In the previous proof, it is supposed that* $C = \mathcal{H}$. *If* C *is a nonempty closed convex subset of* $\mathcal{H}$, *Theorem 4 (a) remains valid.*

In [9] another iterative algorithm called the *shrinking projection method* is studied. Using our notation it can be rephrased as follows :

Algorithm 3 *Given* $x_0 \in C$ *and* $C_0 \stackrel{def}{=} C$*, we consider the sequence* $(x_n)_{n\geq 0}$ *(when it exists) generated by the following algorithm :*

$$\begin{cases} C_{n+1} \stackrel{def}{=} C_n \cap H(x_n, T_n x_n) \quad \textit{with} \quad T_n \stackrel{def}{=} (R_n + Id)/2\,, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases}$$

The previous algorithm is stopped once $C_n = \emptyset$. One of the results of this paper is the proof that the convergence of Algorithm 3 is governed by the same rules as for the convergence of Algorithm 1.

Theorem 6 *Suppose that the* $\mathcal{T}_C$*-class sequence* $(T_n)_{n\in\mathbb{N}}$ *is coherent and let*

$$F \stackrel{def}{=} \bigcap_{n\in\mathbb{N}} Fix(T_n)\,.$$

Then, if $F \neq \emptyset$ *the sequence* $(x_n)_{n\geq 0}$ *produced by Algorithm 3 and Algorithm 1 converges to* $P_F x_0$*.*

Proof : As pointed out in the introduction the case of Algorithm 1 when $C = \mathcal{H}$ is proved in Theorem 4. The extension to the case of a closed nonempty subset C of $\mathcal{H}$ is straightforward and we will not give an explicit proof. The proof of the case of Algorithm 3 is postponed to Section 4. $\square$

Remark 7 *The first condition for the convergence is the fact that the sequence* $(T_n)_{n\geq 0}$ *must belong to the* $\mathcal{T}_C$*-class. Note that by [1, Proposition 2.3]* $T \in \mathcal{T}$ *iff the mapping* $2T - Id$ *is quasi nonexpansive and* $dom(T) = \mathcal{H}$*. The equivalence remains true for* $\mathcal{T}_C$*-class if* $dom(T) = \mathcal{H}$ *is replaced by* $dom(T) = C$*.*

Thus, if $T_n \stackrel{def}{=} (R_n + Id)/2$*, a necessary and sufficient condition for the sequence* $(T_n)_{n\geq 0}$ *to belong to the* $\mathcal{T}_C$*-class is that the sequence* $(R_n)_{n\geq 0}$ *is a sequence of quasi nonexpansive mappings.*

Remark 8 *Moreover, it is a well known fact [3, Theorem 12.1] that* $2T - Id$ *is nonexpansive iff* T *is firmly nonexpansive. Thus, a sufficient condition for the mapping* T *to belong to the* $\mathcal{T}_C$*-class is that* T *is a* firmly nonexpansive *mapping,* i.e *:*

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle \quad \forall (x, y) \in C^2 \tag{4}$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(T - Id)x - (T - Id)y\|^2 \quad \forall (x, y) \in C^2\,. \tag{5}$$

We recall here the definition of the NST-condition (I) [5]. Let $(T_n)_{n\geq 0}$ and $\mathcal{F}$ be two families of nonexpansive mappings of C into itself such that

$$\emptyset \neq \mathrm{Fix}(\mathcal{F}) \stackrel{\mathrm{def}}{=} \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(T_n)\,,$$

where $\mathrm{Fix}(\mathcal{F})$ is the set of all common fixed points of mappings from the family $\mathcal{F}$.

Definition 9 *The sequence* $(T_n)_{n\geq 0}$ *of mappings is said to satisfy the NST-condition (I) with* $\mathcal{F}$ *if, for each bounded sequence* $(z_n)_{n\geq 0} \subset C$*, we have that* $\lim_{n\mapsto\infty} \|z_n - T_n z_n\| = 0$ *implies that* $\lim_{n\mapsto\infty} \|z_n - T z_n\| = 0$ *for all* $T \in \mathcal{F}$.

Remark 10 *Suppose that* $\mathcal{F}$ *is a family of nonexpansive mappings. It is easy to see that a sequence* $(T_n)_{n\geq 0}$ *of mappings satisfying a NST-condition (I) with* $\mathcal{F}$ *is coherent. Indeed, from a demi-closed principle or using [9, Lemma 3.1] if* $\|x_n - T x_n\| \mapsto 0$ *for all* $T \in \mathcal{T}$ *then* $\mathcal{M}(x_n)_{n\geq 0} \subset Fix(\{T\}_{T\in\mathcal{T}})$.

3 Coherent sequences of mappings

We consider here Algorithms 1 and 3 for a sequence of mappings $(R_n)_{n\geq 0}$ built by N level iterations. Our aim is to give conditions under which the sequence $(R_n)_{n\geq 0}$ or equivalently $(T_n)_{n\geq 0} \stackrel{\text{def}}{=} (R_n + Id)/2$ is coherent[1] and apply Theorem 6 to get convergence results.

Let $N \geq 1$ and $(T_n^{(j)})_{n\geq 0} : C \to \mathcal{H}$ for $1 \leq j \leq N$ be a finite set of sequences of nonexpansive mappings. Given also a family of sequences of real parameters $(\alpha_n^{(j)})_{n\geq 0}$ for $1 \leq j \leq N$, we define new sequences $(\Gamma_n^{(j)})_{n\geq 0} : C \to \mathcal{H}$ by the recursive equations :

$$\Gamma_n^{(j)} x \stackrel{\text{def}}{=} \alpha_n^{(j)} x + (1 - \alpha_n^{(j)}) T_n^{(j)} \Gamma_n^{(j+1)} x \quad \text{and} \quad \Gamma_n^{(N+1)} x \stackrel{\text{def}}{=} x \tag{6}$$

$\mathbf{H}_\alpha$: We will assume that the sequences of real parameters $(\alpha_n^{(j)})_{n\geq 0}$ satisfy the following condition : for $2 \leq j \leq N$ and for all $n \in \mathbb{N}$ we have $\alpha_n^{(j)} \in (a, b)$ with $0 < a < b < 1$ and $\alpha_n^{(1)} \in [0, b)$.

Using the sequence of mappings $R_n \stackrel{\text{def}}{=} \Gamma_n^{(1)}$ in Algorithms 1 and 3 gives N level algorithms. We will consider the following specific examples :

$\mathbf{H}_1$ Each sequence $(T_n^{(j)})_{n\geq 0}$ is constant, *i*.e $T_n^{(j)} = T^{(j)}$ for $1 \leq j \leq N$ and $F \stackrel{\text{def}}{=} \text{Fix}\left(\left\{T^{(j)}, 1 \leq j \leq N\right\}\right)$ is nonempty.

$\mathbf{H}_2$ The $(T_n^{(j)})_{n\geq 0}$ sequences for $1 \leq j \leq N$ are given by $T_n^{(j)} = T^{(j)}(t_n)$, where $\left\{T^{(j)}(t) : t \geq 0\right\}$ is a finite set of given semigrougs and $(t_n)_{n\geq 0}$ is a sequence of real numbers such that $\liminf_n t_n = 0$, $\limsup_n t_n > 0$ and $\lim_n (t_{n+1} - t_n) = 0$. We assume that $F \stackrel{\text{def}}{=} \text{Fix}\left(\left\{T^{(j)}(t), 1 \leq j \leq N, t \geq 0\right\}\right)$ is nonempty.

$\mathbf{H}_3$ The $(T_n^{(j)})_{n\geq 0}$ sequences for $1 \leq j \leq N$ are given by

$$T_n^{(j)} x = \frac{1}{t_n} \int_0^{t_n} T^{(j)}(s) x ds\,, \tag{7}$$

where $\left\{T^{(j)}(t) : t \geq 0\right\}$ is a finite set of given semigrougs and $(t_n)_{n\geq 0}$ is a positive divergent sequence of real numbers. We assume that $F \stackrel{\text{def}}{=} \text{Fix}\left(\left\{T^{(j)}(t), 1 \leq j \leq N, t \geq 0\right\}\right)$ is nonempty.

[1] By [1, Proposition 4.5] if $(T_n)_{n\geq 0} \in \mathcal{T}$ and $T_n' \stackrel{\text{def}}{=} Id + \lambda_n (T_n - Id)$ with $\lambda_n \in [\delta, 1]$ and $\delta \in]0, 1]$. Then $(T_n)_{n\geq 0}$ is coherent iff $(T_n')_{n\geq 0}$ is coherent.

Theorem 11 *Given a finite set of N nonexpansive sequences $(T_n^{(j)})_{n\geq 0}$ satisfying $\mathbf{H}_1$, $\mathbf{H}_2$, or $\mathbf{H}_3$. The sequence $(x_n)_{n\geq 0}$ produced by Algorithm 1 and Algorithm 3 with $R_n \stackrel{def}{=} \Gamma_n^{(1)}$ and $(T_n)_{n\geq 0} \stackrel{def}{=} (R_n + Id)/2$ converges to $P_F x_0$. The mappings $\Gamma_n^{(j)}$ being defined by equation (6) with parameters $\alpha_n^{(j)}$ satisfying $\mathbf{H}_\alpha$.*

Proof : The proof is obtained by showing that the sequence of mappings $(T_n)_{n\geq 0}$ is coherent in each given case and by applying Theorem 6 to conclude. The coherence is proved in the sequel in Proposition 15 for the case $\mathbf{H}_1$, in Proposition 17 for the case $\mathbf{H}_2$ and in Proposition 19 for the case $\mathbf{H}_3$. $\square$

We start here by a set of lemmata which are common to all cases.

Lemma 12 *Let T be a F-quasi nonexpansive mapping and for $\beta \in (0,1)$ the mapping $T_\beta \stackrel{def}{=} \beta Id + (1-\beta)T$. For $p \in F$ and all $x \in H$ we have :*

$$\beta(1-\beta)\|x - Tx\|^2 \leq 2(\|x-p\| - \|T_\beta x - p\|)\|x-p\| \tag{8}$$

Proof : For $p \in F$ and all $x \in H$ we have :

$$\begin{aligned}
\|T_\beta x - p\|^2 &= \|\beta(x-p) + (1-\beta)(Tx-p)\|^2 \\
&= \beta\|x-p\|^2 + (1-\beta)\|Tx-p\|^2 - \beta(1-\beta)\|Tx-x\|^2 \\
&\leq \|x-p\|^2 - \beta(1-\beta)\|Tx-x\|^2 .
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\beta(1-\beta)\|Tx-x\|^2 &\leq (\|x-p\| - \|T_\beta x - p\|)(\|x-p\| + \|T_\beta x - p\|) \\
&\leq 2(\|x-p\| - \|T_\beta x - p\|)\|x-p\| .
\end{aligned}$$

$\square$

Lemma 13 *Let T a F-quasi nonexpansive mapping. For $\beta \in (0,1)$ we define the mapping $T_\beta \stackrel{def}{=} \beta Id + (1-\beta)T$. For $p \in F$, all $x \in H$ and S a F-quasi nonexpansive mapping, we have :*

$$\beta(1-\beta)\|x - Tx\|^2 \leq 2\|x - ST_\beta x\|\|x-p\|. \tag{9}$$

If moreover S is nonexpansive we also have :

$$\|x - Sx\| \leq \|x - ST_\beta x\| + \|Tx - x\|. \tag{10}$$

Proof : For $p \in F$ and all $x \in H$ we have :

$$\begin{aligned} \|x-p\| &\leq \|x - ST_\beta x\| + \|ST_\beta x - p\| \\ &\leq \|x - ST_\beta x\| + \|T_\beta x - p\| . \end{aligned}$$

We thus have $\|x-p\| - \|T_\beta x - p\| \leq \|x - ST_\beta x\|$ which combined with Lemma 12 gives equation (9).

Now if S is nonexpansive,

$$\begin{aligned} \|x - Sx\| &\leq \|x - ST_\beta x\| + \|ST_\beta x - Sx\| \leq \|x - ST_\beta x\| + \|T_\beta x - x\| \\ &\leq \|x - ST_\beta x\| + (1-\beta)\|Tx - x\| \leq \|x - ST_\beta x\| + \|Tx - x\| . \end{aligned}$$

□

Lemma 14 *Suppose that* $F \stackrel{def}{=} \bigcap_{\{n\in\mathbb{N};1\leq j\leq N\}} Fix(T_n^{(j)})$ *is not empty suppose that for a bounded sequence* $(x_n)_{n\geq 0}$ *and a fixed value of* j *we have* $\|x_n - T_n^{(j)}\Gamma_n^{(j+1)} x_n\| \to 0$. *Moreover, suppose that for* $2 \leq j \leq N$ *and all* $n \in \mathbb{N}$ *we have* $\alpha_n^{(j)} \in (a,b)$ *with* $0 < a < b < 1$. *Then for all* $k \geq j$ *we have* $\|x_n - T_n^{(k)} x_n\| \to 0$.

Proof : Note first that the sequences $(T^{(j)})_{1\leq j\leq N}$ and $(\Gamma^{(j)})_{1\leq j\leq N+1}$ are composed of nonexpansive mappings. Indeed the composition of nonexpansive mappings is nonexpansive and for $\beta \in (0,1)$ $\beta Id + (1-\beta)S$ is nonexpansive when S is nonexpansive. The sequences are also F-quasi nonexpansive since it is straightforward that $F \subset \mathrm{Fix}(\Gamma_n^{(j)})$ for all $j \in [1,N]$ and $n \in \mathbb{N}$ and if S is nonexpansive it is also $\mathrm{Fix}(S)$-quasi nonexpansive.

The proof then follows by backward induction on j. Assume that the result is true for $j+1$ then we will prove that it is true for j. Using the definition of $\Gamma_n^{(j+1)}$ and using equation (9) for $p \in F$, $S = T_n^{(j)}$, $T = T_n^{(j+1)}\Gamma_n^{(j+2)}$ and $\beta = \alpha_n^{(j+1)}$ (we thus have $T_\beta = \Gamma_n^{(j+1)}$) we obtain :

$$\alpha_n^{(j+1)}(1-\alpha_n^{(j+1)})\|x_n - T_n^{(j+1)}\Gamma_n^{(j+2)} x_n\|^2 \leq 2\|x_n - T_n^{(j)}\Gamma_n^{(j+1)} x_n\|\|x_n - p\| \tag{11}$$

We thus obtain that $\|x_n - T_n^{(j+1)}\Gamma_n^{(j+2)} x_n\| \to 0$ and by induction hypothesis we obtain $\|x_n - T_n^{(k)} x_n\| \to 0$ for $k \geq j+1$. Now using equation (10) with $S \stackrel{\text{def}}{=} T_n^{(j)}$, $T \stackrel{\text{def}}{=} T_n^{(j+1)}\Gamma_n^{(j+2)}$ and $\beta = \alpha_n^{(j+1)}$ we get :

$$\|x_n - T_n^{(j)} x_n\| \leq \|x_n - T_n^{(j)}\Gamma_n^{(j+1)} x_n\| + \|T_n^{(j+1)}\Gamma_n^{(j+2)} x_n - x_n\| \tag{12}$$

and the result follows for j. □

3.1 The case $\mathbf{H}_1$

Proposition 15 *In the case* $\mathbf{H}_1$, *the sequence* $(R_n)_{n\geq 0}$, *defined by* $R_n \stackrel{def}{=} \Gamma_n^{(1)}$ *with parameters satisfying* $\mathbf{H}_\alpha$, *satisfy the NST-condition(I) with* $\mathcal{F} \stackrel{def}{=} Fix\{T^{(j)}{}_{1\leq j\leq N}\}$ *and the sequence* $T_n = (R_n + Id)/2$ *is a* $\mathcal{T}_C$*-class and coherent sequence.*

Proof : We have $\|x_n - R_n x_n\| = \|x_n - T_n^{(1)}\Gamma_n^{(2)}x_n\|(1-\alpha_n^{(1)})$. Thus, if for each bounded sequence $(x_n)_{n\geq 0}$ $\|x_n - R_n x_n\| \mapsto 0$ we also have $\|x_n - T_n^{(1)}\Gamma_n^{(2)}x_n\| \mapsto 0$ since $(1-\alpha_n^{(1)})$ is bounded from zero. Using Lemma 14 we have $\|x_n - T^{(j)}x_n\| \mapsto 0$ for $1 \leq j \leq N$ which gives use the NST-condition(I) with $\mathcal{F}$. Now we consider the sequence $(T_n)_{n\geq 0}$. The sequence belongs to the $\mathcal{T}_C$-class since $2T_n - Id = R_n$ is nonexpansive and thus quasi nonexpansive. Now if $\|x_n - T_n x_n\| \mapsto 0$ we also have $\|x_n - R_n x_n\| \mapsto 0$ and thus using the NST-condition(I) we have $\|x_n - T^{(j)}x_n\| \mapsto 0$ for $1 \leq j \leq N$. Since the $T^{(j)}$ are nonexpansive they are also demi-closed [2, Lemma 4] and thus we must have $\mathcal{M}(x_n)_{n\geq 0} \subset \mathrm{Fix}(\{T^{(j)}, 1 \leq j \leq N\}) = \mathrm{Fix}(\{T_n\}_{n\in\mathbb{N}})$. The sequence $(T_n)_{n\geq 0}$ is thus in the $\mathcal{T}_C$-class and coherent. □

Remark 16 *For $N = 1$ we recover [9, Theorem 1.1] and [9, Theorem 4.1].*

3.2 The case $\mathbf{H}_2$

Let $\{T(t) : t \geq 0\}$ be a family of mappings from a subset C of $\mathcal{H}$ into itself. We call it a nonexpansive semigroup on C if the following conditions are satisfied :

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous ;
(iv) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \geq 0$.

Proposition 17 *In the case $\mathbf{H}_2$, the sequence $(R_n)_{n\geq 0}$, defined by $R_n \stackrel{def}{=} \Gamma_n^{(1)}$ with parameters satisfying $\mathbf{H}_\alpha$, satisfy the NST-condition(I) with $\mathcal{F} \stackrel{def}{=} Fix\{T^{(j)}(t)_{1\leq j\leq N, t\geq 0}\}$ and the sequence $T_n = (R_n + Id)/2$ is a $\mathcal{T}_C$-class and coherent sequence.*

Proof : As in the proof of Proposition 15 we obtain that for each bounded sequence $(x_n)_{n\geq 0}$ such that $\|x_n - R_n x_n\| \mapsto 0$ we also have $\|x_n - T^{(j)}(t_n)x_n\| \mapsto 0$ for $1 \leq j \leq N$. Now it is easy to prove that the weak cluster points of the sequence $(x_n)_{n\geq 0}$ are in F. The proof for each fixed j is the same as in [7, Theorem 2.2, page 6]. We thus obtain the coherence of the sequence $(T_n)_{n\geq 0}$. □

Remark 18 *For $N = 1$ we recover [7, Theorem 2.1] for Algorithm 3 and [7, Theorem 2.2] for Algorithm 1.*

3.3 The case $\mathbf{H}_3$

Proposition 19 *In the case $\mathbf{H}_3$, the sequence $(R_n)_{n\geq 0}$, defined by $R_n \stackrel{def}{=} \Gamma_n^{(1)}$ with parameters satisfying $\mathbf{H}_\alpha$, satisfy the NST-condition(I) with $\mathcal{F} \stackrel{def}{=} Fix\{T^{(j)}(t)_{1\leq j\leq N, t\geq 0}\}$ and the sequence $T_n = (R_n + Id)/2$ is a $\mathcal{T}_C$-class and coherent sequence.*

Proof : As in the proof of Proposition 15 we obtain that for each bounded sequence $(x_n)_{n\geq 0}$ such that $\|x_n - R_n x_n\| \mapsto 0$ we also have $\|x_n - T^{(j)}(t_n)x_n\| \mapsto 0$ for $1 \leq j \leq N$. Now it is easy to prove that the weak cluster points of the sequence $(x_n)_{n\geq 0}$ are in F. The proof for each fixed j is the same as in [6, Theorem 4.1]. For each fixed j, it is a consequence of the inequality [6, Equation (8)] :

$$\|T^{(j)}(s)x_n - x_n\| \leq 2\|T_n^{(j)}x_n - x_n\| + \|T(s)(T_n^{(j)}x_n) - T_n^{(j)}x_n\| \qquad (13)$$

for every $0 \leq s < +\infty$ and $n \in \mathbb{N}$ with $T_n^{(j)}$ and the fact that the right hand side of the above inequality goes to zero as n goes to infinity for a bounded sequence $(x_n)_{n\geq 0}$ using [6, Lemma 2.1]. We thus obtain the coherence of the sequence $(T_n)_{n\geq 0}$. □

Remark 20 *For $N = 1$ we recover [6, Theorem 4.1] for Algorithm 1 and [9, Theorem 4.4] for Algorithm 3.*

4 Proof of Theorem 6

We prove here the strong convergence of Algorithm 3 for a $\mathcal{T}_C$-class sequence of coherent mappings. The proof follows the same steps as the proof of the convergence of Algorithm 1 in [1], we therefore give references to the original propositions.

The proof results from the next proposition and theorem in the following way. Let $(x_n)_{n\geq 0}$ be an arbitrary orbit of Algorithm 3 and let $F \stackrel{\text{def}}{=} \text{Fix}(\{T_n\}_{n\in\mathbb{N}})$. If $F \neq \emptyset$, then by Proposition 21 (iv) the sequence is defined. By Theorem 22 (ii) the sequence is bounded. Thus (v) is fulfilled and by the coherence property we have $\mathcal{M}(x_n)_{n\geq 0} \subset F$. Then, by Theorem 22 (iv), the sequence strongly converges to $P_F(x_0)$.

Proposition 21 *[1, Proposition 3.4] Let $(x_n)_{n\geq 0}$ be an arbitrary orbit of Algorithm 3. Then :*

(i) *If x_{n+1} is defined then $\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|$.*

(ii) *If x_n is defined then $x_0 = x_n \iff x_n = x_{n-1} = \cdots = x_0 \iff x_0 \in \bigcup_{k=0}^{n-1} Fix(T_k)$.*

(iii) *If $(x_n)_{n\geq 0}$ is defined then $(\|x_0 - x_n\|)_{n\in\mathbb{N}}$ is increasing.*

(iv) *$(x_n)_{n\geq 0}$ is defined if $F \stackrel{def}{=} Fix(\{T_n\}_{n\in\mathbb{N}}) \neq \emptyset$.*

Proof : (i) : If x_{n+1} is defined we have $x_{n+1} = P_{C_{n+1}}x_0$ and thus $x_{n+1} \in C_{n+1} \subset C_n$ and since $x_n = P_{C_n}x_0$ we have $\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|$. (ii) : The fist equivalence follows from (i). The second one is proved by induction. Note first that H is such that $y = P_{H(x,y)}x$. Now for $y \in C$, we obtain also that $y = P_{C\cap H(x,y)}x$. for $n = 1$, we have $x_1 = P_{C\cap H(x_0,T_0x_0)}x_0 = T_0x_0$ and thus $x_1 = x_0 \iff x_0 \in \text{Fix}(T_0)$. Now assume that the equivalence if fulfilled for n.

We have

$$x_{n+1} = x_n = \cdots = x_0 \iff \begin{cases} x_0 & \in \cup_{k=0}^{n-1} \operatorname{Fix}(T_k) \\ x_0 & = x_{n+1} = P_{C \cap \bigcap_{k=0}^{n} H(x_k, T_k x_k)} \\ & = P_{C \cap H(x_0, T_n x_0)} = T_n x_0 \,. \end{cases}$$

(iii) follows from (i). (iv) : The algorithm is defined if $C_n \neq \emptyset$ for all $n \in \mathbb{N}$. Thus it is enough to prove that $C \cap \left(\bigcap_{n\in\mathbb{N}} H(x_n, T_n x_n)\right) \neq \emptyset$. By definition of the $\mathcal{T}_C$ class we have $\operatorname{Fix}(T_n) \subset C \cap H(x_n, T_n x_n)$ and the result follows. $\square$

Theorem 22 *([1, Theorem 3.5]) Let $(x_n)_{n\geq 0}$ be an arbitrary orbit of Algorithm 3 and let $F \stackrel{def}{=} \bigcap_{n\in\mathbb{N}} Fix(T_n)$. Then*

- *(i) If $(x_n)_{n\geq 0}$ is defined then : $(x_n)_{n\geq 0}$ is bounded $\iff$ $(\|x_0 - x_n\|)_{n\in\mathbb{N}}$ converges.*
- *(ii) If $F \neq \emptyset$, then $(x_n)_{n\geq 0}$ is bounded and $(\forall n \in \mathbb{N}) x_n \in F \iff x_n = P_F(x_0)$.*
- *(iii) If $F \neq \emptyset$, then $(\|x_0 - x_n\|)_{n\in\mathbb{N}}$ converges and $\lim_n \|x_0 - x_n\| \leq \|x_0 - P_F x_0\|$.*
- *(iv) If $F \neq \emptyset$, then : $\lim_n x_n = P_F(x_0) \iff \mathcal{M}(x_n)_{n\in\mathbb{N}} \subset F$.*
- *(v) If $(x_n)_{n\geq 0}$ is defined and bounded then $\sum_{n\geq 0} \|x_{n+1} - x_n\|^2 < +\infty$ and $\sum_{n\geq 0} \|x_n - T_n x_n\|^2 < +\infty$.*

Proof : (i) follows from Proposition 21 (i). (ii) : If $F \neq \emptyset$ then by Proposition 21 (iv) the sequence is defined. We have $F \subset C \cap \left(\bigcap_{n\in\mathbb{N}} H(x_n, T_n x_n)\right)$ and thus $F \subset C_n$. Now, from $P_F(x_0) \in C_n$ and $x_n = P_{C_n} x_0$ we obtain $\|x_n - x_0\| \leq \|x_0 - P_F(x_0)\|$ and (ii) follows. (iii) follows from (i), (ii) and the previous inequality. (iv) : The forward implication is trivial. For the reverse implication, the proof exactly follows (iv) of [1, Theorem 3.5] since it does not involve C. (v) : From $x_n = P_{C_n} x_0$ and $x_{n+1} \in C_n$ we obtain :

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0 \,.$$

We thus have :

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &\leq \|x_{n+1} - x_n\|^2 + 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2 \,. \end{aligned} \tag{14}$$

Hence $\sum_{n\geq 0} \|x_{n+1} - x_n\|^2 \leq \sup_{n\in\mathbb{N}} \|x_0 - x_n\|^2 < +\infty$ since $(x_n)_{n\geq 0}$ is bounded. For all $n \in \mathbb{N}$ we have $x_{n+1} \in H(x_n, T_n x_n)$, which implies,

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - T_n x_n\|^2 - 2\langle x_{n+1} - T_n x_n, x_n - T_n x_n \rangle \\ &+ \|x_n - T_n x_n\|^2 \\ &\geq \|x_n - T_n x_n\|^2, \end{aligned} \tag{15}$$

and we therefore obtain $\sum_{n\geq 0} \|x_n - T_n x_n\|^2 < +\infty$. $\square$

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