SOME RESULTS ON SEPARATE AND JOINT CONTINUITY

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Abstract. Let $f : X \times K \to \mathbb{R}$ be a separately continuous function and $\mathcal{C}$ a countable collection of subsets of K. Following a result of Calbrix and Troallic, there is a residual set of points $x \in X$ such that f is jointly continuous at each point of $\{x\} \times Q$, where Q is the set of $y \in K$ for which the collection $\mathcal{C}$ includes a basis of neighborhoods in K. The particular case when the factor K is second countable was recently extended by Moors and Kenderov to any Čech-complete Lindelöf space K and Lindelöf α-favorable X, improving a generalization of Namioka's theorem obtained by Talagrand. Moors proved the same result when K is a Lindelöf p-space and X is conditionally σ-α-favorable space. Here we add new results of this sort when the factor X is $\sigma_{C(X)}$-β-defavorable and when the assumption "base of neighborhoods" in Calbrix-Troallic's result is replaced by a type of countable completeness. The paper also provides further information about the class of Namioka spaces.

1. Introduction

If K, X are topological spaces, a mapping $f : X \times K \to \mathbb{R}$ is said to be separately continuous if for every $x \in X$ and $y \in K$, the mappings $f(x, .) : K \to \mathbb{R}$ and $f(., y) : X \to \mathbb{R}$ are continuous, the reals being equipped with the usual topology. The spaces K and X satisfy the Namioka property $\mathcal{N}(X, K)$ if every separately continuous map $f : X \times K \to \mathbb{R}$ is (jointly) continuous at each point of a subset of $X \times K$ of the form $R \times K$, where R is a dense subset of X [20]. Following [8], the space X is called a Namioka space if the property $\mathcal{N}(X, K)$ holds for every compact K. It is well known that every Tychonoff Namioka space is a Baire space [24]. Following [9], a compact space K is said to be co-Namioka if $\mathcal{N}(X, K)$ holds for every Baire space X. The class of co-Namioka spaces contains several classes of compact spaces appearing in Banach spaces theory, like Eberlein or Corson compactums ([11], [10]); in this connection, the reader is referred to [18, 22, 3] and the references therein for more information. On the other hand, every σ-β-defavorable space (see below) is a Namioka space; this is Christensen-Saint Raymond's theorem [8, 24]. It is also well known that within the class of metrizable or separable spaces, Namioka spaces and σ-β-defavorable spaces coincide [24], a result that we will improve below by extending it to Grothendieck-Eberlein spaces (see also Proposition 5.5). Any Baire space which is a p-space (in Arhangel'skiĭ's sense) or K-analytic is σ-β-defavorable, hence a Namioka space; see respectively [5] and [9]. It should be noted that the

2000 *Mathematics Subject Classification.* 54C05,54C35, 54C45, 54C99.
Key words and phrases. separate continuity, joint continuity, Namioka spaces.

method of [9] can be used to extend this result of Debs to any Baire space which is dominated by the irrationals in the sense of [29]. In addition, a Baire space which is game determined in the sense of Kenderov and Moors in [15] is σ-β-defavorable. In particular, a Baire space which has countable separation is σ-β-defavorable.

The class of σ-β-defavorable spaces is defined in term of a topological game $\mathcal{J}$ introduced (in a strong form) by Christensen [8] and later modified by Saint Raymond in [24]. In the game $\mathcal{J}$ on the space X, two players α and β choose alternatively a decreasing sequence $V_0 \supseteq U_0 \supseteq \ldots \supseteq V_n \supseteq U_n \ldots$ of nonempty open subsets of X and a sequence $(a_n)_{n\in\mathbb{N}} \subset X$ as follows: Player β moves first and chooses V_0; then Player α gives $U_0 \subset V_0$ and $a_0 \in X$. At the $(n+1)$th step, Player β chooses an open set $V_{n+1} \subset U_n$ then Player α responds by giving $U_{n+1} \subset V_{n+1}$ and $a_{n+1} \in X$. The *play* $(V_n, (U_n, a_n))_{n\in\mathbb{N}}$ is won by Player α if

$$(\cap_{n\in\mathbb{N}} U_n) \cap \overline{\{a_n : n \in \mathbb{N}\}} \neq \emptyset.$$

The space X is said to be σ-β-defavorable if there is no winning strategy for Player β in the game $\mathcal{J}$.

The problem of knowing to what extent can we weaken the assumption of compactness on the factor K has interested several authors. In this work, we are interested in certain results obtained on this issue, that we describe now. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of K. In [7], Calbrix and Troallic have shown that there is a residual set $R \subset X$ such that the separately continuous mapping $f : X \times K \to \mathbb{R}$ is continuous at each point of $R \times Q$, where Q is the set of points $x \in K$ admitting a subsequence of $(U_n)_{n\in\mathbb{N}}$ as a neighborhoods basis. In particular, the property $\mathcal{N}(X, K)$ holds for every second countable space K and every Baire space X. A similar result has been proved previously by Saint Raymond [23] in the case where K and X are both Polish. In the same direction, Talagrand has demonstrated in [26] that $\mathcal{N}(X, K)$ holds when K is Čech-complete Lindelöf and X is compact (also announcing the same result for X Čech-complete complete). Mercourakis and Negrepontis suspected in their article [18] the possibility of extending these results in case where K is Lindelöf p-space, which has been established with success by Moors in a recent article [19] assuming X to be "conditionally" α-favorable. Shortly before that, Moors and Kenderov extended in [14] Talagrand's theorem to every α-favorable Lindelöf space X. As the class of σ-β-defavorable spaces encompasses so nicely different types of Namioka spaces, it seemed to us that it would be interesting to know if some results of this kind remain valid in the framework of this class.

The basic idea here is the reuse of the approach in [4], where a simplified proof is given for Christensen-Saint Raymond's theorem. In Theorem 3.2, the result of

Calbrix and Troallic is considered in a more general configuration, replacing the set Q by the set of $x \in K$ for which there is a subsequence of $(U_n)_{n\in\mathbb{N}}$ containing x and satisfying a sort of countable completeness (the precise definition is given in Section 3). This also concerns the above result by Moors. In Theorem 3.1, we shall examine the case where the sequence $(U_n)_{n\in\mathbb{N}}$ is a sequence of countable (not necessarily open) covers of K, which will allow us to unify the result of Talagrand (including the K-analytic variant of his theorem) and that of Kenderov and Moors. Concerning the factor X, we shall do a functional adjustment to the game of Christensen-Saint Raymond, thereby obtaining a class wider than that of σ-β-defavorable spaces whose members are still Namioka spaces. For instance, this new class contains all pseudocompact spaces. Related to this last result, a more general statement is proved in Proposition 5.5 in Section 5 which includes some additional results and observations.

2. Functional variants of Christensen-Saint Raymond's game

The game $\mathcal{J}_\Gamma$: Let $\Gamma \subset \mathbb{R}^X$. The game $\mathcal{J}_\Gamma$ differs from the game $\mathcal{J}$ only in the winning condition: Player α is declared to be the winner of the play $(V_n, (U_n, a_n))_{n\in\mathbb{N}}$ if for each $g \in \Gamma$ there exists $t \in \cap_{n\in\mathbb{N}} U_n$ such that

$$g(t) \in \overline{\{g(a_n) : n \in \mathbb{N}\}}.$$

The space X is said to be σ_Γ-β-*defavorable* if Player β has no winning strategy in the game $\mathcal{J}_\Gamma$. Using a terminology from [19], we shall say that X is *conditionally* σ_Γ-α-*favorable* if Player α has a strategy τ so that for any compatible play $(V_n, (U_n, a_n))_{n\in\mathbb{N}}$ satisfying $\cap_{n\in\mathbb{N}} U_n \neq \emptyset$, for every $g \in \Gamma$ there is $t \in \cap_{n\in\mathbb{N}} U_n$ such that

$$g(t) \in \overline{\{g(a_n) : n \in \mathbb{N}\}}.$$

It will be useful for our purpose to consider the closely related game $\mathcal{J}^*_\Gamma$ where Player α has not to produce the sequence $(a_n)_{n\in\mathbb{N}} \subset X$ but wins the play $((V_n, U_n))_{n\in\mathbb{N}}$ if (and only if) for each sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in U_n$ $(n \in \mathbb{N})$ and for each $g \in \Gamma$, there exists $t \in \cap_{n\in\mathbb{N}} U_n$ such that

$$g(t) \in \overline{\{g(a_n) : n \in \mathbb{N}\}}.$$

We make similar definitions with $\mathcal{J}_\Gamma$ replaced by $\mathcal{J}^*_\Gamma$; for instance, X is said to be σ^*_Γ-β-defavorable space if Player β has no winning strategy in the game $\mathcal{J}^*_\Gamma$.

Let $C(X)$ denote the algebra of real-valued continuous functions on X. It is clear that every σ-β-defavorable is $\sigma_{C(X)}$-β-defavorable. We shall show later that the converse is no longer true; however, in some situations it does as the following statement shows (the straightforward proof is omitted).

Proposition 2.1. *Let X be a normal space. Then X is $\sigma_{C(X)}$-β-defavorable if and only if X is σ-β-defavorable.*

Christensen-Saint Raymond's game $\mathcal{J}$ was invented to study the problem of the existence of continuity points for separately continuous mappings. As we shall see, it is quite possible to replace the game $\mathcal{J}$ by its variant $\mathcal{J}_\Gamma$ (with suitable Γ) and, in this connection, the next assertion tells us that these games are in a sense the appropriate ones. For a set $\Gamma \subset \mathbb{R}^X$, let X_Γ denote the space obtained when X is equipped with the topology generated by the functions in Γ ($C(X_\Gamma)$ stands for the algebra of real-valued continuous functions on X_Γ).

Proposition 2.2. *Let X be a topological space and $(K_n)_{n\in\mathbb{N}} \subset \mathbb{R}^X$. Let $\Gamma = \cup_{n\in\mathbb{N}} K_n$ and suppose that for each $n \in \mathbb{N}$, the set A_n of $x \in X$ such that K_n is equicontinuous at x is a residual subset of X. Then X is conditionally $\sigma^*_{C(X_\Gamma)}$-α-favorable. In particular, X is conditionally σ^*_Γ-α-favorable.*

Proof. We shall define a strategy τ for the Player α so that for each play which is compatible with τ, say $((V_n, U_n))_{n\in\mathbb{N}}$, the following holds: for every $t \in \cap_{n\in\mathbb{N}} U_n$, $a_n \in U_n$ ($n \in \mathbb{N}$) and $g \in \Gamma$, the sequence $(g(a_n))_{n\in\mathbb{N}}$ converges to $g(t)$; in other words, the sequence $(a_n)_{n\in\mathbb{N}}$ converges to t in X_Γ. Clearly, such a strategy for α is conditionally winning in the game $\mathcal{J}^*_\Gamma$.

Let $A = \cap_{n\in\mathbb{N}} A_n$ and let us fix a sequence $(G_n)_{n\in\mathbb{N}}$ of dense open subsets of X such that $\cap_{n\in\mathbb{N}} G_n \subset A$. Suppose that τ has been defined until stage n and denote by V_n the nth move of Player β. Let $\mathcal{E}_n$ be the collection of all nonempty open sets $U \subset V_n \cap G_n$ such that $|g(x) - g(y)| \leq 1/n$ for every $x, y \in U$ and $g \in \cup_{i\leq n} K_i$. Put $\tau(V_n) = V_n \cap G_n$ if $\mathcal{E}_n$ is empty; if not, choose $U_n \in \mathcal{E}_n$ and put $\tau(V_n) = U_n$.

Let $((V_n, U_n))_{n\in\mathbb{N}}$ be a play which is compatible with τ, $a_n \in U_n$ ($n \in \mathbb{N}$), $g \in \Gamma$ and $t \in \cap_{n\in\mathbb{N}} U_n$. We have $t \in \cap_{n\in\mathbb{N}} G_n$, which implies that all the collections $\mathcal{E}_n$, $n \in \mathbb{N}$, are nonempty. Let $p \in \mathbb{N}$ be such that $g \in K_p$; since $t, a_n \in U_n$, in view of the choice of the open set U_n, we have $|g(a_n) - g(t)| < 1/n$ for every $n \geq p$. Consequently, $\lim g(a_n) = g(t)$. □

The space X is called an Eberlein-Grothendieck space (EG-space for short) if X is Hausdorff and there is a compact set $\Gamma \subset C(X)$ such that $X = X_\Gamma$. The class of EG-spaces includes all metrizable spaces [1] (as suggested by the referee, it suffices to note that the functions $x \to d(x, y) - d(x_0, y)$, $y \in X$, lie in the pointwise compact set of the 1-Lipschitz functions that map the specified point $x_0 \in X$ to 0; d being a bounded compatible metric on X). Therefore, the next statement which is a consequence of the proof of Proposition 2.2 improves the result of Saint Raymond cited above.

Proposition 2.3. *Let X be an EG-space. Then the following are equivalent:*

1. *X is a Namioka space,*
2. *X is Baire and conditionally σ-α-favorable,*
3. *X is σ-β-defavorable.*

3. Main results

In what follows, including the statements of Theorems 3.1 and 3.2 and their respective Corollaries 3.3 and 3.4, $f : X \times K \to \mathbb{R}$ is a fixed separately continuous mapping and $\phi : K \to C_p(X)$ is the continuous mapping defined by $\phi(y)(x) = f(x, y)$. We denote by $C_p(X)$ the algebra $C(X)$ equipped with the pointwise convergence topology.

Let Γ be a nonempty subset of the product space $\mathbb{R}^X$. A decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of subsets of K is said to be *countably pair complete with respect to* (ϕ, Γ) if for any sequences $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ such that $y_n, z_n \in U_n$ for all $n \in \mathbb{N}$, the sequence $(\phi(y_n) - \phi(z_n))_{n\in\mathbb{N}}$ has at least one cluster point in the subspace Γ of the product space $\mathbb{R}^X$. A sequence $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$ $(n \in \mathbb{N})$ of covers of K is said to be *countably pair complete with respect to* (ϕ, Γ) if for each $\sigma \in \mathbb{N}^{\mathbb{N}}$, the sequence $(\cap_{i\leq n} U^i_{\sigma(i)})_{n\in\mathbb{N}}$ is countably pair complete with respect to (ϕ, Γ).

The main results are given in the next two statements. The first one should be compared with [26, Théorème 5.1]. The proofs are postponed to the next section.

Theorem 3.1. *Suppose that there exist a set $\Gamma \subset \mathbb{R}^X$ and a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of countable covers of K such that X is σ_Γ-β-defavorable and the sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is countably pair complete with respect to (ϕ, Γ). Then for every $\varepsilon > 0$, there is a residual subset R_ε of X such that for every $(x, y) \in R_\varepsilon \times K$ the following holds:*

(∗) *there are a finite sequence $F_i \in \mathcal{U}_i$, $i = 0, \ldots, k$, with $y \in \cap_{i\leq k} F_i$, and a neighborhood O of (x, y) in $X \times K$ such that:*

$$|f(x, y) - f(x', y')| < \varepsilon \text{ for every } (x', y') \in O \cap [X \times (\cap_{i\leq k} F_i)].$$

An important special case of Theorem 3.1 is when $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is a sequence of open covers of the space K; in this case, following a terminology from [14], condition (∗) says that the mapping f is ε-continuous at the point (x, y) of $R_\varepsilon \times K$.

Recall that a set $A \subset X$ is said to be everywhere of second category in X if for every nonempty open set $U \subset X$, the set $A \cap U$ is of the second category in U (equivalently, in X).

Theorem 3.2. *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of K, $\Gamma \subset \mathbb{R}^X$ and P the set of $y \in K$ for which there is $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $y \in \cap_{n\in\mathbb{N}} U_{\sigma(n)}$ and the sequence $(\cap_{i\leq n} U_{\sigma(i)})_{n\in\mathbb{N}}$ is countably pair complete with respect to (ϕ, Γ). Denote by $R_\varepsilon(P)$*

the set of $x \in X$ such that the mapping $f : X \times K \to \mathbb{R}$ is ε-continuous at each point of $\{x\} \times P$. Then

(1) *if X is conditionally σ_Γ-α-favorable, $R_\varepsilon(P)$ is a residual subset of X;*
(2) *if X is σ_Γ-β-defavorable, $R_\varepsilon(P)$ is everywhere of second category in X.*

To express some consequences of these results, we need to recall some terminology. Let Z be a topological space and $Y \subset Z$. The set Y is said to be bounded (or relatively pseudocompact) in Z if every continuous function $g : Z \to \mathbb{R}$ is bounded on Y; Z is pseudocompact if Z is Tychonoff (i.e., completely regular) and bounded in itself . The space Y is called Lindelöf in Z if every open cover of Z has a countable subcover of Y. The space Y is said to be relatively countably compact in Z if every sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ has a cluster point in Z.

Also let us remind that a Tychonoff space Y is called a p-space if there is a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of Y such that to each $x \in Y$ corresponds a sequence $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $x \in \cap_{n\in\mathbb{N}} U_n$ and the intersection $\cap_{n\in\mathbb{N}} U_n$ is a compact subset of Y for which the sequence $(\cap_{i\leq n} U_i)_{n\in\mathbb{N}}$ is an outer basis. Finally, a Tychonoff space Y is said to be Čech-complete if there is a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of Y which is complete in the sense that any closed filter basis $\mathcal{F}$ on Y has a nonempty intersection, provided that for each $n \in \mathbb{N}$ there are $F \in \mathcal{F}$ and $U \in \mathcal{U}_n$ such that $F \subset U$.

Corollary 3.3. *Suppose that X is a $\sigma_{C(X)}$-β-defavorable space. Then, there is a G_δ dense subset R of X such that f is jointly continuous at each point of $R \times K$, in each of the following cases:*

(1) *K is pseudocompact and every bounded subspace of $C_p(X)$ is relatively countably compact in $C_p(X)$.*
(2) *$K \times K$ is pseudocompact and every pseudocompact subspace of $C_p(X)$ is relatively countably compact in $C_p(X)$.*
(3) *$\phi(K)$ is relatively Lindelöf in a Čech-complete subspace of $C_p(X)$.*
(4) *K is Lindelöf Čech-complete.*

Proof. We apply Theorem 3.1 by taking $\Gamma = C(X)$ in each of these cases. For (1) and (2), let $\mathcal{U}_n = \{K\}$ for every $n \in \mathbb{N}$, and note that the set $L = \phi(K) - \phi(K)$ is bounded in $C_p(X)$. Indeed, L is a pseudocompact subspace of $C_p(X)$ in case (2); in case (1), the set L is the difference of two pseudocompact subspaces of the topological group $C_p(X)$, hence, according to a result of Tkačenko [27], it is bounded in $C_p(X)$.

For (3), let Z be a Čech-complete subspace of $C_p(X)$ such that $\phi(K)$ is Lindelöf in Z. Let $(\mathcal{W}_n)_{n\in\mathbb{N}}$ be a complete sequence of open covers of Z; since $\phi(K)$ is

Lindelöf in Z, for each $n \in \mathbb{N}$ there is a countable collection $\mathcal{V}_n \subset \mathcal{W}_n$ such that $\phi(K) \subset \cup\mathcal{V}_n$. The sequence $(\phi^{-1}(\mathcal{V}_n))_{n\in\mathbb{N}}$ fulfills the conditions of Theorem 3.1.

The proof of (4) is similar to (3). □

The point (4) in Corollary 3.3 is established in [14] for X Lindelöf α-favorable. The point (1) in the following is proved in [19] for X conditionally σ-α-favorable; the point (2) describes the situation in the "β-defavorable" case.

Corollary 3.4. *Suppose that K is a Lindelöf p-space and let $\varepsilon > 0$. Let R_ε be the set of $x \in X$ such that f is ε-continuous at each point of $\{x\} \times K$.*

(1) *If X is conditionally $\sigma_{C(X)}$-α-favorable, then R_ε is a residual subset of X.*
(2) *If X is $\sigma_{C(X)}$-β-defavorable, then R_ε is everywhere of second category in X.*

Proof. Since K is a Lindelöf p-space, letting $P = K$, Theorem 3.2 applies. □

4. The proofs

Lemma 4.1 below is used repeatedly in the proof of Theorem 3.1. The proof of Theorem 3.2 consists in adapting that of Theorem 3.1; Lemma 4.1 is not needed there, however, for the first item, it is replaced by the well-known characterization of residual sets in term of the Banach-Mazur game (see below). So we give the entire proof for Theorem 3.1 and only indicate the main changes to get Theorem 3.2.

Lemma 4.1 is an immediate consequence of the following well-known property [13]: "given a set $A \subset X$ which is of second category in X, there is a nonempty open subspace V of X such that $A \cap V$ is everywhere of second category in V". (The concept is recalled just before Theorem 3.2).

Lemma 4.1. *Let $A = \cup_{n\in\mathbb{N}} B_n$ be a set of the second category in the space Y. Then, there is a nonempty open set $V \subset Y$ and $n \in \mathbb{N}$ such that $B_n \cap V$ is everywhere of second category in V.*

Proof of Theorem 3.1 As said in the introduction, the main arguments follow the proof given–in French– in [4] for Christensen-Saint Raymond's theorem cited above.

The assumption. For $\varepsilon > 0$, $F \subset X$ and $L \subset K$, let $R_\varepsilon(F, L)$ (or simply $R(F, L)$) be the set of $x \in F$ such that the property $(*)$ is satisfied for all $y \in L$. We have to prove that $R(X, K)$ is a residual subset of X. Let us suppose to the contrary and show that X is σ_Γ-β-favorable. Thus, writing $D(F, L) = F \setminus R(F, L)$, our assumption says that the set $D(X, K)$ is of second category in X.

The strategy. Write $\mathcal{U}_n = \{F^n_k : k \in \mathbb{N}\}$. We are going to define a strategy σ for the Player β in the game $\mathcal{J}_\Gamma$ which produces parallel to each play $(V_n, (a_n, U_n))_{n\in\mathbb{N}}$ a set of sequences $(x_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}} \subset X$, $(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subset K$ and $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$, so that for every $n \in \mathbb{N}$:

(1) the set $D(V_n, \cap_{i\le n} F^i_{k_i})$ is everywhere of the second category in V_n;
(2) $y_{n+1}, z_{n+1} \in \cap_{i\le n} F^i_{k_i}$;
(3) $|f(a_i, z_{n+1}) - f(a_i, y_{n+1})| < 1/(n+1)$ for each $0 \le i \le n$;
(4) $V_{n+1} \subset \{t \in X : |f(t, z_{n+1}) - f(t_{n+1}, z_{n+1})| < \varepsilon/3\} \cap \{t \in X : |f(t, y_{n+1}) - f(x_{n+1}, y_{n+1})| < \varepsilon/3\}$;
(5) $|f(x_{n+1}, y_{n+1}) - f(t_{n+1}, z_{n+1})| \ge \varepsilon$.

Applying Lemma 4.1 to $Y = X$ and $A = D(X, K)$ gives a nonempty open set $V_0 \subset X$ and $k_0 \in \mathbb{N}$ such that $D(V_0, F^0_{k_0})$ is everywhere of second category in V_0. Let $y_0, z_0 \in F^0_{k_0}$, $x_0, t_0 \in X$ be arbitrary and define $\sigma(\emptyset) = V_0$. Assume that we are at stage p: Player α having produced $(a_0, U_0), \dots, (a_p, U_p)$, Player β his sequence $V_0, \dots, V_p$ and all terms of sequences above having been defined until p in accordance with (1)-(5). First, let $x_{p+1} \in U_p$ and $y_{p+1} \in \cap_{i\le p} F^i_{k_i}$ be so that the condition $(*)$ is not satisfied (the inductive hypothesis (1) ensures that $D(U_p, \cap_{i\le p} F^i_{k_i})$ is not empty). The set

$$A = \{t \in U_p : |f(t, y_{p+1}) - f(x_{p+1}, y_{p+1})| < \varepsilon/3\}$$

is a neighborhood of x_{p+1} in X and the set

$$B = \cap_{i\le p}\{z \in K : |f(a_i, z) - f(a_i, y_{p+1})| < \varepsilon/4\}$$

is a neighborhood of y_{p+1} in K; choose $(t_{p+1}, z_{p+1}) \in A \times [B \cap (\cap_{i\le p} F^i_{k_i})]$ such that $|f(t_{p+1}, z_{p+1}) - f(x_{p+1}, y_{p+1})| \ge \varepsilon$. The open set

$$O = A \cap \{t \in U_p : |f(t, z_{p+1}) - f(t_{p+1}, z_{p+1})| < \varepsilon/3\}$$

being nonempty ($t_{p+1} \in O$), the set $D(O, \cap_{i\le p} F^i_{k_i})$ is of the second category in O; since $F^p_{k_p} \subset \cup_{l\in\mathbb{N}} F^{p+1}_l$, Lemma 4.1 gives an integer k_{p+1} and a nonempty open set $V_{p+1} \subset O$ such that $D(V_{p+1}, \cap_{i\le p+1} F^i_{k_i})$ is everywhere of second category in V_{p+1}. Define

$$\tau((a_1, U_1), \dots, (a_p, U_p)) = V_{p+1}.$$

All items (1)-(5) are satisfied for $i \le p+1$. The definition of the strategy σ is complete.

Conclusion. We show that σ is a winning strategy for Player β. Suppose that $((a_n, U_n))_{n\in\mathbb{N}}$ is a winning play for α against the strategy σ. According to (2), there is a cluster point $g \in \Gamma$ of the sequence $(\phi(z_n) - \phi(y_n))_{n\in\mathbb{N}}$. According to (3), for every $m \in \mathbb{N}$, we have

$$\lim_n |f(a_m, z_n) - f(a_m, y_n)| = 0;$$

thus $g(a_m) = 0$ for every $m \in \mathbb{N}$. It follows that there is $t \in \cap_{n\in\mathbb{N}} U_n$ such that $g(t) = 0$; in particular, $|f(t, z_n) - f(t, y_n)| < \varepsilon/3$ for some $n \in \mathbb{N}$. It follows from (4) that

$$\begin{aligned} |f(x_n, y_n) - f(t_n, z_n)| &\le |f(x_n, y_n) - f(t, y_n)| + |f(t, y_n) - f(t, z_n)| \\ &\quad + |f(t, z_n) - f(t_n, z_n)| \\ &< \varepsilon, \end{aligned}$$

contrary to (5).

Remark 4.2. Suppose that X is σ-β-defavorable. Then, the argument in the conclusion step of the above proof also works if the assumption on K is weakened assuming that the sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is countably pseudo-complete with respect to ϕ in the following sense: For every $F_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, and every sequence $(y_n)_{n\in\mathbb{N}} \subset K$ such that $y_n \in \cap_{i\le n} F_i$, the set $\{\phi(y_n) : n \in \mathbb{N}\}$ is bounded in $C_p(X)$. Indeed, let $((a_n, U_n))_{n\in\mathbb{N}}$ be a winning play for α against the strategy σ and choose $t \in \cap_{n\in\mathbb{N}} U_n$ such that $t \in \overline{\{a_n : n \in \mathbb{N}\}}$. Write $A = \{t\} \cup \{a_n : n \in \mathbb{N}\}$ and let $r_A : C_p(X) \to C_p(A)$ be the map under which each $g \in C(X)$ is sent to its restriction to A. The sets $\{r_A(\phi(y_n)) : n \in \mathbb{N}\}$ and $\{r_A(\phi(z_n)) : n \in \mathbb{N}\}$ are bounded thus relatively compact in $C_p(A)$, since $C_p(A)$ is metrizable (see for instance Lemma III.4.7 in [1]); it follows that the sequence $\big(r_A(\phi(z_n)) - r_A(\phi(y_n))\big)_{n\in\mathbb{N}}$ has a cluster point $g \in C_p(A)$. By (3), $g(\{a_n : n \in \mathbb{N}\}) = \{0\}$, hence $g(t) = 0$ and the proof can be continued as above. Let us mention that the corresponding result (that is, the property $\mathcal{N}(X, K)$ holds) in case when $\phi(K)$ is bounded in $C_p(X)$ (and X is σ-β-defavorable) is due to Troallic [30]. Unfortunately, there is no hope to establish the same result in case when X is $\sigma_{C(X)}$-β-defavorable (see Example 5.3 below).

Before we pass to Theorem 3.2, let us recall the description of first category sets in term of the Banach-Mazur game. For a space Y and $R \subset Y$, a play in the game $BM(R)$ (on Y) is a sequence $((V_n, U_n))_{n\in\mathbb{N}}$ of pairs of nonempty open subsets of Y produced alternately by two players β and α as follows: β is the first to move and gives V_0, then Player α gives $U_0 \subset V_0$; at stage $n \ge 1$, the open set $V_n \subset U_n$ being chosen by β, Player α gives $U_n \subset V_n$. Player α wins the play if $\cap_{n\in\mathbb{N}} U_n \subset R$. It is well known that X is $BM(R)$-α-favorable (i.e., α has a winning strategy in the game $BM(R)$) if and only if R is a residual subset of Y. The reader is referred to [21].

Proof of Theorem 3.2. Denote by $\mathbb{N}^{<\mathbb{N}}$ the set of finite sequences of integers and let $\phi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be a bijective map such that $\phi(s) \geq |s|$ for every $s \in \mathbb{N}^{<\mathbb{N}}$, where $|s|$ stands for the length of s. For $n \in \mathbb{N}$, define

$$F_n = \bigcap_{i \leq |\phi^{-1}(n)|} U_{\phi^{-1}(n)(i)}.$$

We keep the notation $D(F, L)$ (for $F \subset X$ and $L \subset K$) used in the proof of Theorem 3.1 and write $R(P)$ for $R_\varepsilon(P)$.

1) Let τ_1 be a conditionally winning strategy for Player α in the game $\mathcal{J}_\Gamma$. We deduce from τ_1 a winning strategy τ_2 for Player α in the game $BM(R(P))$ as follows. Fix $* \notin K \cup X$. Let V_n be the nth move of β in the game $BM(R(P))$ and write $(W_n, a_n) = \tau_1(V_0, \ldots, V_n)$. If $D(W_n, F_n) = \emptyset$, define $\tau_2(V_0, \ldots, V_n) = W_n$ and $x_n = t_n = y_n = z_n = *$. If $D(W_n, F_n) \neq \emptyset$, first choose $x_n \in W_n$ and $y_n \in F_n$ such that f is not ε-continuous at the point (x_n, y_n). Then, considering the sets

$$A = \{t \in W_n : |f(t, y_n) - f(x_n, y_n)| < \varepsilon/3\}$$

and

$$B = \cap_{i \leq n}\{z \in K : |f(a_i, z) - f(a_i, y_n)| < 1/(n+1)\},$$

choose $t_n \in A$ and $z_n \in B \cap F_n$ such that $|f(x_n, y_n) - f(t_n, z_n)| \geq \varepsilon$; finally define

$$\tau_2(V_0, \ldots, V_n) = \{t \in W_n : |f(t, z_n) - f(t_n, z_n)| < \varepsilon/3\}.$$

The definition of τ_2 is complete.

Let us suppose for contradiction that Player β has a winning play $(V_n)_{n\in\mathbb{N}}$ against the strategy τ_2. Then $\cap_{n\in\mathbb{N}} V_n \not\subset R(P)$, that is, there are $a \in \cap_{n\in\mathbb{N}} V_n$ and $y \in P$ such that f is not ε-continuous at (a, y). Let $\sigma \in \mathbb{N}^{\mathbb{N}}$ be such that the sequence $(\cap_{i\leq n} U_{\sigma(i)})_{n\in\mathbb{N}}$ is countably pair complete with respect to (ϕ, Γ) and $y \in \cap_{i\in\mathbb{N}} U_{\sigma(i)}$. For $n \in \mathbb{N}$, let $k_n = \phi(\sigma_{|n})$; then $a \in W_{k_n}$ and $y \in F_{k_n}$, thus $D(W_{k_n}, F_{k_n}) \neq \emptyset$ which indicates that y_{k_n}, z_{k_n} have been selected in F_{k_n}. Since $F_{k_n} = \cap_{i\leq n} U_{\sigma(i)}$, the sequence $(\phi(y_{k_n}) - \phi(z_{k_n}))_{n\in\mathbb{N}}$ has at least a cluster point $g \in \Gamma$. Since $\cap_{n\in\mathbb{N}} V_n \neq \emptyset$ and the strategy τ_1 is conditionally winning, there is $t \in \cap_{n\in\mathbb{N}} V_n$ such that $g(t) \in \overline{\{g(a_n) : n \in \mathbb{N}\}}$ (note that the play $(V_n, (W_n, a_n))_{n\in\mathbb{N}}$ is compatible with τ_1). The argument from the "conclusion" step in the proof of Theorem 3.1 can now be used to get the required contradiction. Therefore, $R(P)$ is a residual subset of X.

2) We proceed as in (1), keeping the same notations. Suppose that there exists a nonempty open set Ω such that $R(P) \cap \Omega$ is of first category in X, that is, $R(P) \cap \Omega \subset \cup_{n\in\mathbb{N}} A_n$ where each A_n is a closed nowhere dense subset of X. We deduce from this a winning strategy σ for Player β in the game $\mathcal{J}_\Gamma$ as follows. To begin let $\sigma(\emptyset) = \Omega$. At step n, let $(V_0, a_0), \ldots, (V_n, a_n)$ be the first nth moves of

Player α and consider the nonempty open set $O_n = V_n \setminus A_n$. If $D(O_n, F_n) = \emptyset$, define $\sigma((V_0, a_0), \ldots, (V_n, a_n)) = O_n$ and $t_n = x_n = y_n = z_n = *$; if $D(O_n, F_n) \neq \emptyset$, define

$$\sigma((V_0, a_0), \ldots, (V_n, a_n)) = \{t \in O_n : |d(f(t, z_n), f(t_n, z_n)| < \varepsilon\},$$

the points x_n, t_n, y_n, z_n being chosen exactly as in (1).

Suppose that $((a_n, V_n))_{n\in\mathbb{N}}$ is a play for Player α which is compatible with σ and let us show that there is $g \in \Gamma$ so that $g(t) \notin \overline{\{g(a_n) : n \in \mathbb{N}\}}$ for every $t \in \cap_{n\in\mathbb{N}} V_n$. Since $\Gamma \neq \emptyset$, we may assume that $\cap_{n\in\mathbb{N}} V_n \neq \emptyset$. Let $a \in \cap_{n\in\mathbb{N}} V_n$; then $a \notin R(P)$ hence there is $y \in P$ such that f is not ε-continuous at the point (a, y). Since $y \in P$, there is $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $y \in \cap_{n\in\mathbb{N}} U_{\sigma(n)}$ and the sequence $(\cap_{i\leq n} U_{\sigma(i)})_{n\in\mathbb{N}}$ is countably pair complete with respect to (ϕ, Γ). As in (1), letting $k_n = \phi(\sigma_{|n})$ for $n \in \mathbb{N}$, we obtain that $a \in V_{k_n}$ and $y \in F_{k_n}$, hence $D(O_{k_n}, F_{k_n}) \neq \emptyset$ and, consequently, $\{y_{k_n}, z_{k_n}\} \subset F_{k_n}$ for every $n \in \mathbb{N}$. Take a cluster point $g \in \Gamma$ of the sequence $(\phi(y_{k_n}) - \phi(z_{k_n}))_{n\in\mathbb{N}}$; the assumption that $g(t) \in \overline{\{g(a_n) : n \in \mathbb{N}\}}$ for some $t \in \cap_{n\in\mathbb{N}} V_n$ leads to a contradiction as in (1).

To conclude this section, let us mention the following result which gives a description of the class of Namioka spaces and answers in a certain sense Question 1167 (or Question 8.2) in [3].

Proposition 4.3. *Let X be a Tychonoff space. Then the following are equivalent:*

(1) *X is a Namioka space,*
(2) *X is a Baire space and conditionally σ^*_Γ-α-favorable for every compact $\Gamma \subset C_p(X)$,*
(3) *X is a Baire space and conditionally σ_Γ-α-favorable for every compact $\Gamma \subset C_p(X)$,*
(4) *X is σ_Γ-β-defavorable for every compact $\Gamma \subset C_p(X)$.*

Proof. The fact that (1) implies (2) follows from Proposition 2.2 and Saint Raymond's theorem that every Tychonoff Namioka space is Baire [24]. The implications (2) → (3) and (3) → (4) are obvious. Finally, Theorem 3.1 shows that (4) implies (1). □

5. Some related results

Recall that a subspace X of a topological space Y is said to be C-embedded in Y if every $f \in C(X)$ has an extension $g \in C(Y)$. Suppose that X is dense in Y; it is well known that X is C-embedded in Y if and only if X is G_δ-dense in Y and

z-embedded in X, that is, every zero set of X is the intersection with X of a zero set of Y.

To establish the following proposition we note that the rule that Player α wins the play $((U_n, V_n))_{n\in\mathbb{N}}$ in the game $\mathcal{J}^*_{C(X)}$ (the strong version of $\mathcal{J}_{C(X)}$) can be formulated in an equivalent manner as follows: For every zero set $Z \subset X$ such that $Z \cap U_n \neq \emptyset$ for every $n \in \mathbb{N}$, we have $Z \cap (\cap_{n\in\mathbb{N}} U_n) \neq \emptyset$.

Proposition 5.1. *Let Y be a $\sigma^*_{C(Y)}$-β-defavorable space (respectively, $\sigma^*_{C(Y)}$-α-favorable). Then every C-embedded dense subspace X of Y is $\sigma^*_{C(X)}$-β-defavorable (respectively, $\sigma^*_{C(X)}$-α-favorable).*

Proof. We outline a proof of the β-defavorable case (the other case being similar). Let τ_X be a strategy for Player β in the game $\mathcal{J}^*_{C(X)}$ and let us show that it is not a winning one. Fix a map $V \to V^*$ under which each nonempty open subset of X is sent to an open subset V^* of Y such that $V = V^* \cap X$. Consider the following strategy τ_Y for Player β in the game $\mathcal{J}^*_{C(Y)}$. Write $V_0 = \tau_X(\emptyset)$ and put $\tau_Y(\emptyset) = V_0^*$. Suppose that τ_Y has been defined until stage n and write $V_{n+1} = \tau_X(U_0 \cap X, \ldots, U_n \cap X)$, where $U_0, \ldots, U_n$ are the first $n+1$ moves of Player α in the game $\mathcal{J}^*_{C(Y)}$. Define $\tau_Y(U_0, \ldots, U_n) = V^*_{n+1} \cap U_n$ (this open subset of U_n is nonempty because it contains V_{n+1}).

There is a winning play $(U_n)_{n\in\mathbb{N}}$ for Player α against the strategy τ_Y in the game $\mathcal{J}^*_{C(Y)}$. The corresponding sequence $(U_n \cap X)_{n\in\mathbb{N}}$ is a play with respect to the game $\mathcal{J}^*_{C(X)}$, which is compatible with τ_X. Let Z be a zero set of X such that $Z \cap U_n \neq \emptyset$ for every $n \in \mathbb{N}$. There is a zero set T of Y such that $Z = T \cap X$; the set $H = T \cap (\cap_{n\in\mathbb{N}} U_n)$ is a nonempty G_δ subset of Y; since X is G_δ-dense in Y, we obtain $Z \cap (\cap_{n\in\mathbb{N}} U_n) = H \cap X \neq \emptyset$. ☐

A standard example illustrating Proposition 5.1 is when X is pseudocompact and Y is its Stone-Čech-compactification βX. Clearly, βX (as any compact space) is $\sigma^*_{C(\beta X)}$-α-favorable; thus Proposition 5.1 leads to the following.

Corollary 5.2. *Every pseudocompact space X is $\sigma^*_{C(X)}$-α-favorable.*

It follows from Corollary 5.2 and Proposition 4.3 that every pseudocompact space is a Namioka space. Actually a stronger statement can be established (see Proposition 5.5 below). We are now ready to give an example of a $\sigma^*_{C(X)}$-α-favorable, hence $\sigma_{C(X)}$-β-defavorable, which is not σ-β-defavorable.

Example 5.3. It is shown by Shakhmatov in [25] that there exists a pseudocompact space P without isolated points, every countable subset of which is discrete. Such a space is $\sigma^*_{C(P)}$-α-favorable in view of Corollary 5.2. Using the fact that P has no isolated point and all its countable subspaces are closed, it is easy to check that P is σ-β-favorable. We have mentioned in Remark 4.2 that the property $\mathcal{N}(X,K)$ is generally false if K is pseudocompact and X is $\sigma_{C(X)}$-β-defavorable. Indeed, Shakhmatov's space P is such that the unit ball K of $C_p(P)$ is pseudocompact (see for instance Example I.2.5 in [1] or [28]) and since P has no isolated point, the evaluation mapping $e : P \times K \to [0,1]$ does not have any point of continuity.

Proposition 5.4. *Let X be a space such that every countable subspace of X is C-embedded in X. Then, the space $Y = C_p(X)$ is $\sigma^*_{C(Y)}$-α-favorable.*

Proof. For each cardinal number γ, the product space $\mathbb{R}^\gamma$ is $\sigma^*_{C(\mathbb{R}^\gamma)}$-$\alpha$-favorable; we refer to Christensen's paper [8] for a similar result about the product of τ-well α-favorable spaces (defined therein). Let νY stand for the realcompactification of Y; then $\nu Y = \mathbb{R}^X$ [28]. Since Y is C-embedded in νY, Proposition 5.1 shows that Y is $\sigma^*_{C(Y)}$-α-favorable. □

We should conclude under the assumption of Proposition 5.4 that the space $C_p(X)$ is a Namioka space, but Corollary 5.7 below provides a more general statement.

Proposition 5.5. *Let X be a Baire space with a dense σ-bounded subspace. Then, X is a Namioka space.*

Proof. Let Γ be a compact subset of $C_p(X)$ and let us show that X is σ_Γ-β-defavorable. Recall that every compact space L such that $C_p(L)$ contains a σ-compact subset separating the points of L is an Eberlein compactum ([1], p. 124). Let $e : X \to C_p(\Gamma)$ be the mapping $e(x)(y) = y(x)$. Since e is continuous and every bounded subset of $C_p(\Gamma)$ is relatively compact (by the generalization of Grothendieck's theorem in [1]), the closure of $e(X)$ in $C_p(\Gamma)$ contains a dense σ-compact space Y. Clearly, Y separates the points of Γ, hence Γ is an Eberlein compactum. By a result of Deville [11], every Eberlein compactum is co-Namioka; thus, following Proposition 2.2, the space X is σ_Γ-β-defavorable. □

Remark 5.6. Following [1], a space X is called k-primary Lindelöf if X is the continuous image of a closed subspace of a space of the form $K \times (L(\gamma))^\omega$, where K is a compact space and γ is cardinal number; $L(\gamma)$ stands for the one point Lindelöfication of the discrete space of cardinality γ. As suspected in [18], Remark 2.17, it can be proved that every Baire space with a dense k-primary Lindelöf subspace is a Namioka space. This can be established with the same method as in the proof of Proposition 5.5, replacing Deville's result by Debs's theorem that every Corson compactum is co-Namioka [10], and using the following theorem by

Bandlow [2]: If X is a k-primary Lindelöf space, then every compact subspace of $C_p(X)$ is a Corson compactum.

Recall that a space X is called b-discrete if every countable subspace A of X is discrete and C^*-embedded in X (every bounded continuous function on A has a continuous extension over X).

Corollary 5.7. *Let X be a space such that $C_p(X)$ is Baire. If X is b-discrete, then $C_p(X)$ is a Namioka space.*

Proof. Since X is b-discrete, the subspace $C^*(X) \subset C_p(X)$ of bounded continuous functions is σ-bounded [28]. Since $C^*(X)$ is dense in $C_p(X)$, Proposition 5.5 applies. □

The converse of 5.7 is not true as the following example shows.

Example 5.8. Example 7.2 in [17] exhibits a countable space X containing a non-C^*-embedded subspace, such that $C_p(X)$ is Baire. Since $C_p(X)$ is metrizable (and Baire) it is a Namioka space by the result of Saint Raymond mentioned in the introduction.

In view of Corollary 5.7 and Example 5.8, it seems likely that the space $C_p(X)$ (for a Tychonoff space X) is a Namioka space as soon as it is Baire.

Example 5.9. There is a Namioka space X which is $\sigma_{C(X)}$-β-favorable. (This is related to Proposition 4.3.) We give two examples of such spaces.

1) Let X be the reals equipped with the so-called density topology T_d [21]. The space X is a Namioka space, because it is a Baire space [16] and every compact subset of $C_p(X)$ is metrizable (see [12] for a general statement). To show that X is $\sigma_{C(X)}$-β-favorable, consider the strategy τ for Player β defined as follows: $\tau(\emptyset) = X$ and $\tau((a_0, U_0), \ldots, (a_n, U_n)) = V_{n+1}$, where V_{n+1} is a nonempty open subset of U_n such that $V_{n+1} \subset U_n \setminus \{a_0, \ldots, a_n\}$ and $|x - y| \leq 1/n$ for each $x, y \in V_n$ (recall that T_d is finer than the usual topology). Suppose that $((V_n, U_n, a_n))_{n\in\mathbb{N}}$ is a play which is compatible with τ. It is well known that every countable subset of X is closed [21]; thus, since the intersection $A = \cap_{n\in\mathbb{N}} V_n$ contains at most one point (and X is Tychonoff), there is a function $f \in C(X)$ such that $f_{|A} = 0$ and $f(a_n) = 1$ for every $n \in \mathbb{N}$. Thus τ is a winning strategy.

2) If the Continuum Hypothesis (CH) is assumed then there is a Namioka space X and a countably compact subspace Γ of $C_p(X)$ such that X is σ_Γ-β-favorable. Namely, under CH, Burke and Pol proved in [6] that the product $B = \{0,1\}^{\aleph_1}$ equipped with the so-called Baire topology, that is, the G_δ-modification of the

usual product $\{0,1\}^{\aleph_1}$, is a Namioka space. The subspace $\Gamma = \{f \in C(B) : f(B) \subset \{0,1\}\}$ of $C_p(B)$ is ω-compact, i.e., every countable subset of Γ is relatively compact in Γ. (See [6] or use Arhangel'skiĭ's result that for every P-space Y, the space $C_p(Y,[0,1])$ is ω-compact [1].) A winning strategy τ for Player β in the game $\mathcal{J}_\Gamma$ consists of producing clopen sets such that $\tau((a_0,U_0),\dots,(a_n,U_n)) \cap \{a_i : i \leq n\} = \emptyset$ (where $(U_0,a_0),\dots,(U_n,a_n)$ are the first nth moves of Player α). Such a strategy is indeed winning for if $(V_n,(U_n,a_n))_{n\in\mathbb{N}}$ is a compatible play, then the sequence $(1_{V_n})_{n\in\mathbb{N}} \subset \Gamma$ has a cluster point $f \in \Gamma$ (in fact, $(1_{V_n})_{n\in\mathbb{N}}$ converges to $1_{\cap_{n\in\mathbb{N}} V_n}$). Then, since $f(a_n) = 0$ for each $n \in \mathbb{N}$, there is no point $t \in \cap_{n\in\mathbb{N}} V_n$ for which $f(t) \in \overline{\{f(a_n) : n \in \mathbb{N}\}}$.

Acknowledgement. The authors would like to thank the referee for his valuable remarks and comments.

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