A few more functions that are not APN infinitely often
Yves Aubry, Gary McGuire, François Rodier

To cite this version:

HAL Id: hal-00415755
https://hal.archives-ouvertes.fr/hal-00415755v2
Submitted on 13 Nov 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
A Few More Functions That Are Not APN
Infinitely Often

Yves Aubry
Institut de Mathématiques de Toulon
Université du Sud Toulon-Var
France

Gary McGuire∗
School of Mathematical Sciences
University College Dublin
Ireland

François Rodier
Institut de Mathématiques de Luminy
C.N.R.S., Marseille
France

Abstract
We consider exceptional APN functions on $\mathbb{F}_{2^m}$, which by definition are functions that are APN on infinitely many extensions of $\mathbb{F}_{2^m}$. Our main result is that polynomial functions of odd degree are not exceptional, provided the degree is not a Gold number ($2^k + 1$) or a Kasami-Welch number ($4^k - 2^k + 1$). We also have partial results on functions of even degree, and functions that have degree $2^k + 1$.

1 Introduction

Let $L = \mathbb{F}_q$ with $q = 2^n$ for some positive integer $n$. A function $f : L \rightarrow L$ is said to be almost perfect nonlinear (APN) on $L$ if the number of solutions

∗Research supported by the Claude Shannon Institute, Science Foundation Ireland Grant 06/MI/006
in $L$ of the equation

$$f(x + a) + f(x) = b$$

is at most 2, for all $a, b \in L$, $a \neq 0$. Equivalently, $f$ is APN if the set \{ $f(x + a) + f(x) : x \in L$ \} has size at least $2^{n-1}$ for each $a \in L^*$. Because $L$ has characteristic 2, the number of solutions to the above equation must be an even number, for any function $f$ on $L$.

This kind of function is very useful in cryptography because of its good resistance to differential cryptanalysis as was proved by Nyberg in $[3]$. The best known examples of APN functions are the Gold functions $x^{2^k+1}$ and the Kasami-Welch functions $x^{4^k-2^k+1}$. These functions are defined over $\mathbb{F}_2$, and are APN on any field $\mathbb{F}_{2^m}$ where $\gcd(k, m) = 1$.

If $f$ is APN on $L$, then $f$ is APN on any subfield of $L$ as well. We will consider going in the opposite direction. Recall that every function $f : L \rightarrow L$ can be expressed as a polynomial with coefficients in $L$, and this expression is unique if the degree is less than $q$. We can “extend” $f$ to an extension field of $L$ by using the same unique polynomial formula to define a function on the extension field. With this understanding, we will consider functions $f$ which are APN on $L$, and we ask whether $f$ can be APN on an extension field of $L$. More specifically, we consider functions that are APN on infinitely many extensions of $L$. We call a function $f : L \rightarrow L$ exceptional if $f$ is APN on $L$ and is also APN on infinitely many extension fields of $L$. The Gold and Kasami-Welch functions are exceptional.

We make the following conjecture.

**Conjecture:** Up to equivalence, the Gold and Kasami-Welch functions are the only exceptional APN functions.

Equivalence here refers to CCZ equivalence; for a definition and discussion of this see $[1]$ for example.

We will prove some cases of this conjecture. It was proved in Hernando-McGuire $[2]$ that the conjecture is true among the class of monomial functions. Some cases for $f$ of small degree have been proved by Rodier $[6]$.

We define

$$\phi(x, y, z) = \frac{f(x) + f(y) + f(z) + f(x + y + z)}{(x + y)(x + z)(y + z)}$$

which is a polynomial in $\mathbb{F}_q[x, y, z]$. This polynomial defines a surface $X$ in the three dimensional affine space $A^3$.

If $X$ is absolutely irreducible (or has an absolutely irreducible component defined over $\mathbb{F}_q$) then $f$ is not APN on $\mathbb{F}_{q^n}$ for all $n$ sufficiently large. As
shown in [6], this follows from the Lang-Weil bound for surfaces, which guarantees many \( \mathbb{F}_{q^n} \)-rational points on the surface for all \( n \) sufficiently large.

Let \( \overline{X} \) denote the projective closure of \( X \) in the three dimensional projective space \( \mathbb{P}^3 \). If \( H \) is another projective hypersurface in \( \mathbb{P}^3 \), the idea of this paper is to apply the following lemma.

**Lemma 1.1** If \( \overline{X} \cap H \) is a reduced (no repeated component) absolutely irreducible curve, then \( \overline{X} \) is absolutely irreducible.

Proof: If \( \overline{X} \) is not absolutely irreducible then every irreducible component of \( \overline{X} \) intersects \( H \) in a variety of dimension at least 1 (see Shafarevich [4, Chap. I, 6.2, Corollary 5]). So \( \overline{X} \cap H \) is reduced or reducible.

\( \square \)

In particular, we will apply this when \( H \) is a hyperplane. In Section 2 we study functions whose degree is not a Gold number \( (2^k + 1) \) or a Kasami-Welch number \( (4^k - 2^k + 1) \). In Section 3 we study functions whose degree is a Gold number - this case is more subtle.

The equation of \( \overline{X} \) is the homogenization of \( \phi(x, y, z) = 0 \), which is \( \overline{\phi}(x, y, z, t) = 0 \) say. If \( f(x) = \sum_{j=0}^{d} a_j x^j \) write this as

\[
\overline{\phi}(x, y, z, t) = \sum_{j=3}^{d} a_j \phi_j(x, y, z) t^{d-j}
\]

where

\[
\phi_j(x, y, z) = \frac{x^j + y^j + z^j + (x + y + z)^j}{(x + y)(x + z)(y + z)}
\]

is homogeneous of degree \( j - 3 \). We will later consider the intersection of \( \overline{X} \) with the hyperplane \( z = 0 \), and this intersection is a curve in a two dimensional projective space with equation \( \overline{\phi}(x, y, 0, t) = 0 \). An affine equation of this surface \( \overline{X} \) is \( \overline{\phi}(x, y, z, 1) = \phi(x, y, z) = 0 \).

A fact we will use is that if \( f(x) = x^{2^k + 1} \) then

\[
\phi(x, y, z) = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} (x + \alpha y + (\alpha + 1)z).
\]

(1)

This can be shown by elementary manipulations (see Janwa, Wilson, [4, Theorem 4]).
Our definition of exceptional APN functions is motivated by the definition of exceptional permutation polynomials. A permutation polynomial \( f : \mathbb{F}_q \rightarrow \mathbb{F}_q \) is said to be exceptional if \( f \) is a permutation polynomial on infinitely many extensions of \( \mathbb{F}_q \). One technique for proving that a polynomial is not exceptional is to prove that the curve \( \phi(x, y) = (f(y) - f(x))/(y - x) \) has an absolutely irreducible factor over \( \mathbb{F}_q \). Then the Weil bound applied to this factor guarantees many \( \mathbb{F}_{q^n} \)-rational points on the curve for all \( n \) sufficiently large. In particular there are points with \( x \neq y \), which means that \( f \) cannot be a permutation.

The authors thank the referee for relevant suggestions.

2 Degree not Gold or Kasami-Welch

If the degree of \( f \) is not a Gold number \( 2^k + 1 \), or a Kasami-Welch number \( 4^k - 2^k + 1 \), then we will apply results of Rodier [6] and Hernando-McGuire [2] to prove our results.

Lemma 2.1 Let \( H \) be a projective hypersurface. If \( X \cap H \) has a reduced absolutely irreducible component defined over \( \mathbb{F}_q \) then \( X \) has an absolutely irreducible component defined over \( \mathbb{F}_q \).

Proof: Let \( Y_H \) be a reduced absolutely irreducible component of \( X \cap H \) defined over \( \mathbb{F}_q \). Let \( Y \) be an absolutely irreducible component of \( X \) that contains \( Y_H \). Suppose for the sake of contradiction that \( Y \) is not defined over \( \mathbb{F}_q \). Then \( Y \) is defined over \( \mathbb{F}_{q^t} \) for some \( t \). Let \( \sigma \) be a generator for the Galois group \( \text{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q) \) of \( \mathbb{F}_{q^t} \) over \( \mathbb{F}_q \). Then \( \sigma(Y) \) is an absolutely irreducible component of \( X \) that is distinct from \( Y \). However, \( \sigma(Y) \supseteq \sigma(Y_H) = Y_H \), which implies that \( Y_H \) is contained in two distinct absolutely irreducible components of \( X \). This means that a double copy of \( Y_H \) is a component of \( X \), which contradicts the assumption that \( Y_H \) is reduced.

Lemma 2.2 Let \( H \) be the hyperplane at infinity. Let \( d \) be the degree of \( f \). Then \( X \cap H \) is not reduced if \( d \) is even, and \( X \cap H \) is reduced if \( d \) is odd and \( f \) is not a Gold or Kasami-Welch monomial function.

Proof: Let \( \phi_d(x, y, z) \) denote the \( \phi \) corresponding to the function \( x^d \). In \( X \cap H \) we may assume \( \phi = \phi_d \).

If \( d \) is odd then the singularities of \( X \cap H \) were classified by Janwa-Wilson [3]. They show that the singularities are isolated (the coordinates
must be \((d - 1)\)-th roots of unity) and so the dimension of the singular locus of \(X \cap H\) is 0.

Suppose \(d\) is even and write \(d = 2^je\) where \(e\) is odd. In \(X \cap H\) we have
\[
(x + y)(x + z)(y + z)\phi_d(x, y, z) = x^d + y^d + z^d + (x + y + z)^d
\]
\[
= (x^e + y^e + z^e + (x + y + z)^e)^{2^j}
\]
\[
= ((x + y)(x + z)(y + z)\phi_e(x, y, z))^{2^j}.
\]

Therefore
\[
\phi_d(x, y, z) = \phi_e(x, y, z)^{2^j}((x + y)(x + z)(y + z))^{2^j-1}
\]
and is not reduced. \(\square\)

Here is the main result of this section.

**Theorem 2.3** If the degree of the polynomial function \(f\) is odd and not a Gold or a Kasami-Welch number then \(f\) is not APN over \(\mathbb{F}_{q^n}\) for all \(n\) sufficiently large.

Proof: By Lemma 2.2, \(X \cap H\) is reduced. Furthermore, we know by [2] that \(X \cap H\) has an absolutely irreducible component defined over \(\mathbb{F}_q\), which is also reduced. Thus, by Lemma 2.1, we obtain that \(X\) has an absolutely irreducible component defined over \(\mathbb{F}_q\). As discussed in the introduction, this enables us to conclude that \(f\) is not APN on \(\mathbb{F}_{q^n}\) for all \(n\) sufficiently large. \(\square\)

In the even degree case, we can state the result when half of the degree is odd, with an extra minor condition.

**Theorem 2.4** If the degree of the polynomial function \(f\) is \(2e\) with \(e\) odd, and if \(f\) contains a term of odd degree, then \(f\) is not APN over \(\mathbb{F}_{q^n}\) for all \(n\) sufficiently large.

Proof: As shown in the proof of Lemma 2.2 in the particular case where \(d = 2^je\) with \(e\) odd and \(j = 1\), we can write
\[
\phi_d(x, y, z) = \phi_e(x, y, z)^2(x + y)(x + z)(y + z).
\]

Hence, \(x + y = 0\) is the equation of a reduced component of the curve \(X_\infty = \overline{X} \cap H\) with equation \(\phi_d = 0\) where \(H\) is the hyperplane at infinity. The only absolutely irreducible component \(X_0\) of the surface \(\overline{X}\) containing...
the line $x + y = 0$ in $H$ is reduced and defined over $\mathbb{F}_q$. We have to show that this component doesn’t contain the plane $x + y = 0$.

The function $x + y$ doesn’t divide $\phi(x, y, z)$ if and only if the function $(x + y)^2$ doesn’t divide $f(x) + f(y) + f(z) + f(x + y + z)$. Let $x^r$ be a term of odd degree of the function $f$. We show easily that $(x + y)^2$ doesn’t divide $x^r + y^r + (x + y + z)^r$ by using the change of variables $s = x + y$ which gives:

$$x^r + y^r + (x + y + z)^r = s(x^{r-1} + z^{r-1}) + s^2 P$$

where $P$ is a polynomial.

Hence $\overline{X}$ has an absolutely irreducible component defined over $\mathbb{F}_q$ and then $f$ is not APN on $\mathbb{F}_{q^n}$ for all $n$ sufficiently large.

$\square$

**Remark:** This theorem is false if $2e$ is replaced by $4e$ in the statement. A counterexample is $x^{12} + cx^3$, where $c \in \mathbb{F}_4$ satisfies $c^2 + c + 1 = 0$, which is APN on $\mathbb{F}_{q^n}$ for any $n$ which is not divisible by 3, since it is CCZ-equivalent to $x^3$. Indeed this function is defined over $\mathbb{F}_4$, and is equal to $L \circ f$, where $f(x) = x^3$ and $L(x) = x^4 + cx$. Certainly $L$ is $\mathbb{F}_4$-linear, and it is not hard to show that $L$ is bijective on $\mathbb{F}_{q^n}$ if and only if $n$ is not divisible by 3. The graph of $x^3$ is $\{(x, x^3) \mid x \in \mathbb{F}_{q^n}\}$ and it is transformed in the graph of $x^{12} + cx^3$ which is $\{(x, x^{12} + cx^3) \mid x \in \mathbb{F}_{q^n}\}$ by the linear permutation $Id \times L$ where $Id$ is the identity function. So when $n$ is not divisible by 3, $L \circ f$ is APN on $\mathbb{F}_{q^n}$ because $f$ is APN. This example shows in particular that our conjecture has to be stated up to CCZ-equivalence.

### 3 Gold Degree

Suppose the degree of $f$ is a Gold number $d = 2^k + 1$. Set $d$ to be this value for this section. Then the degree of $\phi$ is $d - 3 = 2^k - 2$.

#### 3.1 First Case

We will prove the absolute irreducibility for a certain type of $f$.

**Theorem 3.1** Suppose $f(x) = x^d + g(x)$ where $\deg(g) \leq 2^{k-1} + 1$. Let $g(x) = \sum_{j=0}^{2^{k-1}+1} a_j x^j$. Suppose moreover that there exists a nonzero coefficient $a_j$ of $g$ such that $\phi_j(x, y, z)$ is absolutely irreducible. Then $\phi(x, y, z)$ is absolutely irreducible.
Proof: We must show that \( \phi(x, y, z) \) is absolutely irreducible. Suppose 
\( \phi(x, y, z) = P(x, y, z)Q(x, y, z) \). Write each polynomial as a sum of homogeneous parts:

\[
\sum_{j=3}^{d} a_j \phi_j(x, y, z) = (P_s + P_{s-1} + \cdots + P_0)(Q_t + Q_{t-1} + \cdots + Q_0)
\]

(2)

where \( P_j, Q_j \) are homogeneous of degree \( j \). Then from (1) we get

\[
P_s Q_t = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} (x + \alpha y + (\alpha + 1)z).
\]

In particular this implies that \( P_s \) and \( Q_t \) are relatively prime as the product is made of distinct irreducible factors.

The homogeneous terms in (2) of degree strictly less than \( d - 3 \) and strictly greater than \( 2^{k-1} - 1 \) are 0, by the assumed bound on the degree of \( g \). Equating terms of degree \( s + t - 1 \) in the equation (2) gives \( P_s Q_{t-1} + P_{s-1} Q_t = 0 \). Hence \( P_s \) divides \( P_{s-1} Q_t \) which implies \( P_s \) divides \( P_{s-1} \) because \( \gcd(P_s, Q_t) = 1 \), and we conclude \( P_{s-1} = 0 \) as \( \deg(P_{s-1}) < \deg(P_s) \). Then we also get \( Q_{t-1} = 0 \). Similarly, \( P_{s-2} = 0 = Q_{t-2}, P_{s-3} = 0 = Q_{t-3} \), and so on until we get the equation

\[
P_s Q_0 + P_{s-t} Q_t = 0
\]

where we suppose wlog that \( s \geq t \). (Note that when \( s \geq t \), one gets from \( s+t = d-3 \) that \( s \geq (d-3)/2 \) and \( t \leq (d-3)/2 \), and the bound on \( \deg(g) \) is chosen: \( \deg(g) < t + 3 \leq 2^{k-1} + 2 \).) This equation implies \( P_s \) divides \( P_{s-t} Q_t \), which implies \( P_s \) divides \( P_{s-t} \), which implies \( P_{s-t} = 0 \). Since \( P_s \neq 0 \) we must have \( Q_0 = 0 \).

We now have shown that \( Q = Q_t \) is homogeneous. In particular, this means that \( \phi_j(x, y, z) \) is divisible by \( x + \alpha y + (\alpha + 1)z \) for some \( \alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2 \) and for all \( j \) such that \( a_j \neq 0 \). We are done if there exists such a \( j \) with \( \phi_j(x, y, z) \) irreducible.

\[\square\]

Remark: The hypothesis that there should exist a \( j \) with \( \phi_j(x, y, z) \) is absolutely irreducible is not a strong hypothesis. This is true in many cases (see the next remarks). However, some hypothesis is needed, because the theorem is false without it. One counterexample is with \( g(x) = x^5 \) and \( k \geq 4 \) and even.

Remark: It is known that \( \phi_j \) is irreducible in the following cases (see [4]):
• $j \equiv 3 \pmod{4}$;
• $j \equiv 5 \pmod{8}$ and $j > 13$.

Remark: The theorem is true with the weaker hypothesis that there exists
a nonzero coefficient $a_j$ such that $\varphi_j(x, y, z)$ is prime to $\varphi_d$ (recall $d = 2^k+1$).
This is the case for
• $j = 2^r + 1$ is a Gold exponent with $r$ prime to $k$;
• $j$ is a Kasami exponent (see [3, Theorem 5]);
• $j = 2^e$ with $e$ odd and $e$ is in one of the previous cases.

Example: This applies to $x^{33} + g(x)$ where $g(x)$ is any polynomial of degree
$\leq 17$.

Remark: The proof did not use the fact that $f$ is APN. This is simply a
result about polynomials.

Remark: The bound $\deg(g) \leq 2^{k-1} + 1$ is best possible, in the sense that
there is an example with $\deg(g) = 2^{k-1} + 2$ in Rodier [4] where $\varphi$ is not
absolutely irreducible. The counterexample has $k = 3$, and $f(x) = x^9 + ax^6 + a^2x^3$. We discuss this in the next section.

3.2 On the Boundary of the First Case

As we said in the previous section, when $f(x) = x^{2^k+1} + g(x)$ with $\deg(g) = 2^{k-1} + 2$, it is false that $\varphi$ is always absolutely irreducible. However, the
polynomial $\phi$ corresponding to the counterexample $f(x) = x^9 + ax^6 + a^2x^3$
where $a \in \mathbb{F}_q$ factors into two irreducible factors over $\mathbb{F}_q$. We generalize this
to the following theorem.

Theorem 3.2 Let $q = 2^n$. Suppose $f(x) = x^d + g(x)$ where $g(x) \in \mathbb{F}_q[x]
and \deg(g) = 2^{k-1} + 2$. Let $k$ be odd and relatively prime to $n$. If $g(x)$ does
not have the form $ax^{2^k-1+2} + a^2x^3$ then $\varphi$ is absolutely irreducible, while if
$g(x)$ does have the form $ax^{2^k-1+2} + a^2x^3$ then either $\varphi$ is irreducible or $\varphi$
splits into two absolutely irreducible factors which are both defined over $\mathbb{F}_q$.

Proof: Suppose $\varphi(x, y, z) = P(x, y, z)Q(x, y, z)$ and let

$$g(x) = \sum_{j=0}^{2^{k-1}+2} a_j x^j.$$
Write each polynomial as a sum of homogeneous parts:

\[ \sum_{j=3}^{d} a_j \phi_j(x, y, z) = (P_s + P_{s-1} + \cdots + P_0)(Q_t + Q_{t-1} + \cdots + Q_0). \]

Then

\[ P_sQ_t = \prod_{\alpha \in \mathbb{F}_{2k} - \mathbb{F}_2} (x + \alpha y + (1 + \alpha)z). \]

In particular this means \( P_s \) and \( Q_t \) are relatively prime as in the previous theorem. We suppose wlog that \( s \geq t \), which implies \( s \geq 2^{k-1} - 1 \). Comparing each degree gives \( P_{s-1} = 0 = Q_{t-1} \), \( P_{s-2} = 0 = Q_{t-2} \), and so on until we get the equation of degree \( s + 1 \)

\[ P_sQ_1 + P_{s-t+1}Q_t = 0 \]

which implies \( P_{s-t+1} = 0 = Q_1 \). If \( s \neq t \) then \( s \geq 2^{k-1} \). Note then that \( a_{s+3} \phi_{s+3} = 0 \). The equation of degree \( s \) is

\[ P_sQ_0 + P_{s-t}Q_t = a_{s+3} \phi_{s+3} = 0. \]

This means that \( P_{s-t} = 0 \), so \( Q_0 = 0 \). We now have shown that \( Q = Q_t \) is homogeneous. In particular, this means that \( \phi(x, y, z) \) is divisible by \( x + \alpha y + (1 + \alpha)z \) for some \( \alpha \in \mathbb{F}_{2k} - \mathbb{F}_2 \), which is impossible. Indeed, since the leading coefficient of \( g \) is not 0, the polynomial \( \phi_{2^{k-1}+2} \) occurs in \( \phi \); as \( \phi_{2^{k-1}+2} = \phi_{2^{k-2}+1}(x+y)(y+z)(z+x) \), this polynomial is prime to \( \phi \), because if \( x + \alpha y + (1 + \alpha)z \) occurs in the two polynomials \( \phi_{2^{k-1}+2} \) and \( \phi_{2^{k-1}+1} \), then \( \alpha \) would be an element of \( \mathbb{F}_{2k} \cap \mathbb{F}_{2^{k-2}} = \mathbb{F}_2 \) because \( k \) is odd.

Suppose next that \( s = t = 2^{k-1} - 1 \) in which case the degree \( s \) equation is

\[ P_sQ_0 + P_0Q_s = a_{s+3} \phi_{s+3}. \]

If \( Q_0 = 0 \), then

\[ \phi(x, y, z) = \sum_{j=3}^{d} a_j \phi_j(x, y, z) = (P_s + P_0)Q_t \]

which implies that

\[ \phi(x, y, z) = a_d \phi_d(x, y, z) + a_{2^{k-1}+2} \phi_{2^{k-1}+2}(x, y, z) = P_sQ_t + P_0Q_t \]
and $P_0 \neq 0$, since $g \neq 0$. So one has $\phi_{2^{k-1}+2}$ divides $\phi_d(x, y, z)$ which is impossible as

$$\phi_{2^{k-1}+2} = \phi_{2^{k-2}+1}^2(x + y)(y + z)(z + x).$$

We may assume then that $P_0 = Q_0$, and we have $\phi_{2^{k-1}+2} = 0$. Then we have

$$\phi(x, y, z) = (P_s + P_0)(Q_s + Q_0) = P_s Q_s + P_0(Q_s + Q_s) + P_0^2. \quad (3)$$

Note that this implies $a_j = 0$ for all $j$ except $j = 3$ and $j = s + 3$. This means

$$f(x) = x^d + a_{s+3}x^{s+3} + a_3 x^3.$$

So if $f(x)$ does not have this form, this shows that $\phi$ is absolutely irreducible.

If on the contrary $\phi$ splits as $(P_s + P_0)(Q_s + Q_0)$, the factors $P_s + P_0$ and $Q_s + Q_0$ are irreducible, as can be shown by using the same argument.

Assume from now on that $f(x) = x^d + a_{s+3}x^{s+3} + a_3 x^3$ and that $(3)$ holds. Then $a_3 = P_0^2$, so clearly $P_0 = \sqrt{a_3}$ is defined over $F_q$. We claim that $P_s$ and $Q_s$ are actually defined over $F_2$.

We know from $(1)$ that $P_s Q_s$ is defined over $F_2$.

Also $P_0(P_s + Q_s) = a_{s+3} \phi_{s+3}$, so $P_s + Q_s = (a_{s+3}/\sqrt{a_3})\phi_{s+3}$. On the one hand, $P_s + Q_s$ is defined over $F_{2k}$ by $(1)$. On the other hand, since $\phi_{s+3}$ is defined over $F_2$ we may say that $P_s + Q_s$ is defined over $F_q$. Because $(k, n) = 1$ we may conclude that $P_s + Q_s$ is defined over $F_2$. Note that the leading coefficient of $P_s + Q_s$ is 1, so $a_{s+3}^2 = a_3$. Whence if this condition is not true, then $\phi$ is absolutely irreducible.

Let $\sigma$ denote the Galois automorphism $x \mapsto x^2$. Then $P_s Q_s = \sigma(P_s Q_s) = \sigma(P_s)\sigma(Q_s)$, and $P_s + Q_s = \sigma(P_s + Q_s) = \sigma(P_s) + \sigma(Q_s)$. This means $\sigma$ either fixes both $P_s$ and $Q_s$, in which case we are done, or else $\sigma$ interchanges them. In the latter case, $\sigma^2$ fixes both $P_s$ and $Q_s$, so they are defined over $F_4$. Because they are certainly defined over $F_{2k}$ by $(1)$, and $k$ is odd, they are defined over $F_2 \cap F_4 = F_2$.

Finally, we have now shown that $\mathcal{X}$ either is irreducible, or splits into two absolutely irreducible factors defined over $F_q$. \qed

### 3.3 Using the Hyperplane $y = z$

We study the intersection of $\phi(x, y, z) = 0$ with the hyperplane $y = z$. 
Lemma 3.3 \( \phi(x, y, y) \) is always a square.

Proof: It suffices to prove the result for \( f(x) = x^d \). This is equivalent to proving that \( \phi_d(x, 1, 1) \) is a square. This is equivalent to showing that its derivative with respect to \( x \) is identically 0. This is again equivalent to showing that the partial derivative with respect to \( x \) of \( \phi_d(x, y, 1) \), evaluated at \( y = 1 \), is 0. In Lemma 4.1 of [4] Rodier proves that \( y + z \) divides the partial derivative of \( \phi_d(x, y, z) \) with respect to \( x \), which is exactly what is required.

\[ \square \]

Lemma 3.4 Let \( H \) be the hyperplane \( y = z \). If \( X \cap H \) is the square of an absolutely irreducible component defined over \( \mathbb{F}_q \) then \( X \) is absolutely irreducible.

Proof: We claim that for any nonsingular point \( P \in X \cap H \), the tangent plane to the curve \( X \cap H \) at \( P \) is \( H \). The equation of the tangent plane is

\[
(x - x_0)\phi'_x(P) + (y - y_0)\phi'_y(P) + (z - z_0)\phi'_z(P) = 0
\]

where \( P = (x_0, y_0, z_0) \). Since \( P \in H \) we have \( y_0 = z_0 \). It is straightforward to show that \( \phi'_x(P) = 0 \) and \( \phi'_y(P) = \phi'_z(P) \), so this equation becomes

\[
(y + z)\phi'_y(P) = 0.
\]

But \( y + z = 0 \) is the equation of \( H \).

\[ \square \]

Corollary 3.1 If \( f(x) = x^d + g(x) \), and \( d = 2^k + 1 \) is a Gold exponent, and \( \phi(x, y, y) \) is the square of an irreducible, then \( X \) is absolutely irreducible.

Note that any term \( x^d \) in \( g(x) \) where \( d \) is even will drop out when we calculate \( \phi(x, y, y) \), because if \( d = 2e \) then

\[
\phi_d(x, y, z) = \frac{x^d + y^2 + z^d + (x + y + z)^d}{(x + y)(x + z)(y + z)} = \frac{(x^e + y^e + z^e + (x + y + z)^e)^2}{(x + y)(x + z)(y + z)} = \phi_e(x, y, z)(x^e + y^e + z^e + (x + y + z)^e)
\]

because the right factor vanishes on \( H \).
In order to find examples of where we can apply this Corollary, if we write
\[ \phi(x, y, y) = (x + y)^{2k-2} + h(x, y)^2 \]
then to apply this result we want to show that
\[ (x + y)^{2k-1} + h(x, y) \]
is irreducible. The degree of \( h \) is smaller than \( 2^{k-1} - 1 \). Letting \( t = x + y \) we want an example of \( h \) with \( t^{2k-1} + h(x, x + t) \) is irreducible.

**Example:** Choose any \( h \) so that \( h(x, x + t) \) is a monomial, and then \( t^{2k-1} + h(x, x + t) \) is irreducible.

**References**


