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High-gain observer with uniform in the initial condition
finite time convergence

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1 Introduction

In this short note, we show how the framework introduced in [1] allows to obtain an observer
with finite time convergence for globally Lipchitz upper triangular systems.

2 Finite time observer

we introduce an observer for systems of the form :

\[ \dot{x} = Sx + Bu + \delta(x,t) , \quad y = x_1 , \]  \hspace{1cm} (1)

where \( x = (x_1,\ldots,x_n) \) is in \( \mathbb{R}^n \) and \( \delta : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \) is a continuous function globally
Lipchitz in its first argument (uniformly in \( t \)).

The domination approach has been used to design observer for systems of the form (1). This approach has been popularized by high-gain observer [2]. These observers are given as:

\[ \dot{\hat{x}} = S\hat{x} + Bu + \delta(\hat{x},t) + L\mathcal{L}^{-1}K(\hat{x}_1 - y) \]  \hspace{1cm} (2)

where \( L \) is the extra high-gain parameter, \( \mathcal{L} = \text{diag}(1,L^{-1},L^{-2},\ldots,L^{-1-n}) \) and \( K \) is the
output injection which have to be designed to ensure that the state of the error system:

\[ \dot{\tilde{x}} = S\tilde{x} + \delta(\tilde{x},t) - \delta(\hat{x} - \tilde{x},t) + L\mathcal{L}^{-1}K(\tilde{x}_1) \]  \hspace{1cm} (3)

converges to the origin.

The error system (3) has the structure of a chain of integrators disturbed by nonlinear
terms which, assuming a global Lipschitz condition (as in [2]), is linearly bounded. In [2], the

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domination approach has been employed and a linear vector field $K$ in the observer (3) was introduced to ensure global and asymptotic convergence of the error $\hat{x}$ toward the origin.

Recently, this approach has been extended in (4) (see also (3)) to a homogeneous vector field $K$ with negative degree to allow semi-global and finite time estimation.

The homogeneous in the bi-limit vector field $K$ obtained from (1, Section 3) allows us to get a global observer with finite-time estimation and with an estimation time uniform in the initial condition:

**Corollary 1 (Finite time observer)** If for $(x, \hat{x})$ in $\mathbb{R}^{2n}$,

$$|\delta_i(x + \hat{x}, t) - \delta_i(x, t)| \leq c \sum_{j=1}^{i} |\hat{x}_j|$$

where $c$ is a positive real numbers, then there exist a continuous vector field $K : \mathbb{R} \to \mathbb{R}^n$ and a real number $L^* > 0$ such that for every $L$ in $[L^*, +\infty)$, the estimate given by the system (4) converges to the state of system (1) in finite time uniformly in the initial condition, i.e., there exists a positive real number $T$ such that for all initial state $x_0$ in $\mathbb{R}^n$, initial estimate $\hat{x}_0$ in $\mathbb{R}^n$ and all locally bounded continuous function $u : [0, T] \to \mathbb{R}$, we get:

$$x(T) = \hat{x}(T)$$

where $(x, \hat{x}) : \mathbb{R} \to \mathbb{R}^{2n}$ is a $C^1$ functions solution of systems (3) and (4) such that $x(0) = x_0$ and $\hat{x}(0) = \hat{x}_0$.

**Proof:** To construct the vector field $K$ we employ the homogeneous in the bi-limit framework and the procedure introduced in (1). We introduce two real numbers $d_0$ and $d_\infty$ (the degree of the homogeneous in the bi-limit vector field $K$) such that

$$-1 < d_0 < 0 < d_\infty < \frac{1}{n - 1}.$$  \hspace{1cm} (5)

As in (1), we introduce the associated weights vector $r_0$ and $r_\infty$ both in $\mathbb{R}^n_+$ defined as

$$r_{b,n} = 1, \quad r_{b,i} = r_{b,i+1} - d_b = 1 - d_b(n - i),$$

where the letter "b" stand for "0" or "\infty". Following the procedure (1, Section 3), we obtain a homogeneous in the bi-limit vector field $K : \mathbb{R} \to \mathbb{R}^n$ with associated triples $(r_0, d_0, K_0)$ and $(r_\infty, d_\infty, K_\infty)$ such that the origin of the systems with state $z = (z_1, \ldots, z_n)$ in $\mathbb{R}^n$:

$$\dot{z} = Sz + K(z_1),$$

$$\dot{z} = Sz + K_0(z_1),$$

$$\dot{z} = Sz + K_\infty(z_1),$$

is globally and asymptotically stable. Hence, we can employ (1, Corollary 2.22) to get a positive real number $c_G$ such that for all continuous function $R : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, satisfying

$$R_i(z, t) \leq c_G \left( \sum_{j=1}^{i} |z_j| \frac{1 - r_0(n - i - 1)}{1 - r_0(n - j)} + \sum_{j=1}^{i} |z_j| \frac{1 - r_\infty(n - i - 1)}{1 - r_\infty(n - j)} \right) i = 1, \ldots, n,$$ \hspace{1cm} (7)
where \( R(z, t) = (R_1(z, t), \ldots, R_n(z, t)) \), the origin of the system:

\[
\dot{z} = Sz + K(z_1) + R(z, t)
\]

is globally and asymptotically stable.

Note that since \( d_0 < 0 < d_\infty \), it follows from Young’s inequality that, given a continuous function \( R \) satisfying

\[
R_i(z, t) \leq c_G \sum_{j=1}^{i} |z_j|
\]

then the \( R_i \)'s satisfy also the bound (7) and in this case, the origin of system (8) is globally and asymptotically stable.

We introduce now the scaled coordinates defined as:

\[
e_i = L_1^{i-1} \hat{x}_i, \quad i = 1, \ldots, n,
\]

where \( L \), the high-gain parameter, is a positive real number which will be selected later. We can rewrite this change of coordinates in compact form as:

\[
E = \mathcal{L} \hat{x} , \quad \mathcal{L} = \text{diag}(1, L^{-1}, L^{-2}, \ldots, L^{1-n})
\]

We get along the trajectory of the error system (3):

\[
\dot{E} = L \left[ SE + \Delta(L, \hat{x}, \tilde{x}, t) + K(e_1) \right],
\]

where

\[
\Delta(L, \hat{x}, \tilde{x}, t) = L^{-1} \mathcal{L}[\delta(\hat{x}, t) - \delta(\hat{x} - \tilde{x}, t)]
\]

Moreover, due to (4), with \( L \geq 1 \), we get:

\[
|\Delta_i(L, \hat{x}, \tilde{x}, t)| \leq L^{-i}c \sum_{j=1}^{i} |\tilde{x}_j| \leq L^{-1}c \sum_{j=1}^{i} |e_j|
\]

Consequently with \( c_G \) defined in (4) and taking \( L^* > \frac{c}{c_G} \), we get that, for all \( L \) in \([L^*, +\infty)\), the origin of the system:

\[
\dot{E} = L \left[ SE + \Delta(L, \hat{x}, t) + K(e_1) \right],
\]

is globally and asymptotically stable. Hence, the estimate \( \hat{x} \) converges toward the state \( x \).

Moreover, the origin is also globally and asymptotically stable for the homogeneous approximations:

\[
\dot{E} = L \left( SE + K_0(e_1) \right),
\]

\[
\dot{E} = L \left( SE + K_\infty(e_1) \right),
\]

and with (6), we can apply [1, Corollary 2.24] to obtain the result. \[\square\]
References


