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Further Variations on the Six Exponentials Theorem

Michel Waldschmidt

Abstract. Let \( \mathcal{L} \) denote the set of linear combinations, with algebraic coefficients, of 1 and logarithms of algebraic numbers. The Strong Six Exponentials Theorem of D. Roy gives sufficient conditions for a \( 2 \times 3 \) matrix

\[
M = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23}
\end{pmatrix}
\]

whose entries are in \( \mathcal{L} \) to have rank 2.

Here we give sufficient conditions so that one at least of the three \( 2 \times 2 \) determinants

\[
\begin{vmatrix}
\lambda_{11} & \lambda_{12} \\
\lambda_{21} & \lambda_{22}
\end{vmatrix}, \quad \begin{vmatrix}
\lambda_{12} & \lambda_{13} \\
\lambda_{22} & \lambda_{23}
\end{vmatrix}, \quad \begin{vmatrix}
\lambda_{13} & \lambda_{11} \\
\lambda_{23} & \lambda_{21}
\end{vmatrix}
\]

is not in \( \mathcal{L} \).

1. Main result

We denote by \( \mathbb{Q} \) the field of rational numbers, by \( \overline{\mathbb{Q}} \) the field of algebraic numbers (algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \)), by \( \mathcal{L} \) the \( \mathbb{Q} \)-vector space of logarithms of algebraic numbers:

\[
\mathcal{L} = \{ \lambda \in \mathbb{C} ; \ e^\lambda \in \overline{\mathbb{Q}}^\times \} = \{ \log \alpha ; \ \alpha \in \overline{\mathbb{Q}}^\times \} = \exp^{-1}(\overline{\mathbb{Q}}^\times)
\]

and by \( \tilde{\mathcal{L}} \) the \( \mathbb{Q} \)-vector subspace of \( \mathbb{C} \) spanned by \( \{1\} \cup \mathcal{L} \). Hence \( \tilde{\mathcal{L}} \) is the set of linear combinations of 1 and logarithms of algebraic numbers with algebraic coefficients:

\[
\tilde{\mathcal{L}} = \left\{ \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n ; \right.
\]

\[
n \geq 0, \ (\alpha_1, \ldots, \alpha_n) \in (\overline{\mathbb{Q}}^\times)^n, \ (\beta_0, \beta_1, \ldots, \beta_n) \in \overline{\mathbb{Q}}^{n+1} \right\}.
\]

Here is the so-called strong six exponentials Theorem of D. Roy ( ([5] Corollary 2 §4 p. 38; see also [7] Corollary 11.16):

---

**Key words and phrases.** Transcendental numbers, logarithms of algebraic numbers, four exponentials Conjecture, six exponentials Theorem, algebraic independence.

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1
THEOREM 1.1. Let $M$ be a $2 \times 3$ matrix with entries in $\bar{\mathcal{L}}$: 

$$
M = \begin{pmatrix} 
\Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\
\Lambda_{21} & \Lambda_{22} & \Lambda_{23} 
\end{pmatrix}.
$$

Assume that the two rows of $M$ are linearly independent over $\overline{\mathbb{Q}}$ and also that the three columns are linearly independent over $\overline{\mathbb{Q}}$. Then $M$ has rank 2.

Consider the three $2 \times 2$ determinants

$$
\Delta_1 = \Lambda_{12} \Lambda_{23} - \Lambda_{13} \Lambda_{22}, \quad \Delta_2 = \Lambda_{13} \Lambda_{21} - \Lambda_{11} \Lambda_{23}, \quad \Delta_3 = \Lambda_{11} \Lambda_{22} - \Lambda_{12} \Lambda_{21}.
$$

From the relation

$$
\Delta_1 \begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \end{pmatrix} + \Delta_2 \begin{pmatrix} \Lambda_{12} \\ \Lambda_{22} \end{pmatrix} + \Delta_3 \begin{pmatrix} \Lambda_{13} \\ \Lambda_{23} \end{pmatrix} = 0,
$$

it follows from the assumptions of Theorem 1.1 that one at least of the three numbers $\Delta_1, \Delta_2, \Delta_3$ is transcendental. We want to prove that one at least of these three numbers is not in $\bar{\mathcal{L}}$.

If the five rows of the matrix

$$
\begin{pmatrix} 
M \\ I_3 
\end{pmatrix}
$$

(where $I_3$ is the $3 \times 3$ identity matrix) are linearly dependent over $\overline{\mathbb{Q}}$, which means that there exists $(\gamma_1, \gamma_2) \in \overline{\mathbb{Q}}^2 \setminus \{0\}$ such that the three numbers

$$
\delta_j = \gamma_1 \Lambda_{1j} + \gamma_2 \Lambda_{2j} \quad (j = 1, 2, 3)
$$

are algebraic, then the three numbers $\Delta_1, \Delta_2, \Delta_3$ are in $\bar{\mathcal{L}}$. Indeed, if $(j, h, k)$ denotes any of the triples $(1, 2, 3), (2, 3, 1), (3, 1, 2)$, then

$$
\gamma_1 \Delta_j = \delta_h \Lambda_{2k} - \delta_k \Lambda_{2h} \quad \text{and} \quad \gamma_2 \Delta_j = \delta_h \Lambda_{1h} - \delta_h \Lambda_{1k}.
$$

Here is the main result of this paper.

THEOREM 1.2. Let $M$ be a $2 \times 3$ matrix with entries in $\bar{\mathcal{L}}$:

$$
M = \begin{pmatrix} 
\Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\
\Lambda_{21} & \Lambda_{22} & \Lambda_{23} 
\end{pmatrix}.
$$

Assume that the five rows of the matrix

$$
\begin{pmatrix} 
M \\ I_3 
\end{pmatrix} = \begin{pmatrix} 
\Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\
\Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{pmatrix}
$$

are linearly dependent over $\overline{\mathbb{Q}}$, which means that there exists $(\gamma_1, \gamma_2) \in \overline{\mathbb{Q}}^2 \setminus \{0\}$ such that the three numbers $\Delta_1, \Delta_2, \Delta_3$ are in $\bar{\mathcal{L}}$. Indeed, if $(j, h, k)$ denotes any of the triples $(1, 2, 3), (2, 3, 1), (3, 1, 2)$, then

$$
\gamma_1 \Delta_j = \delta_h \Lambda_{2k} - \delta_k \Lambda_{2h} \quad \text{and} \quad \gamma_2 \Delta_j = \delta_h \Lambda_{1h} - \delta_h \Lambda_{1k}.
$$

Here is the main result of this paper.
are linearly independent over \( \mathbb{Q} \) and that the five columns of the matrix

\[
(I_2, M) = \begin{pmatrix}
1 & 0 & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\
0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23}
\end{pmatrix}
\]

are linearly independent over \( \mathbb{Q} \). Then one at least of the three numbers

\[
\Delta_1 = \begin{vmatrix}
\Lambda_{12} & \Lambda_{13} \\
\Lambda_{22} & \Lambda_{23}
\end{vmatrix}, \quad \Delta_2 = \begin{vmatrix}
\Lambda_{13} & \Lambda_{11} \\
\Lambda_{23} & \Lambda_{21}
\end{vmatrix}, \quad \Delta_3 = \begin{vmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{vmatrix}
\]

is not in \( \tilde{\mathbb{L}} \).

If \( M \) is a \( d \times \ell \) matrix of rank 1, with \( d \geq 2 \) and \( \ell \geq 2 \), whose columns are \( \mathbb{T} \)-linearly independent, then the \( d + \ell \) columns of the matrix \( (I_d \ M) \) are also \( \mathbb{T} \)-linearly independent. Hence on the one hand Theorem 1.2 generalizes Theorem 1.1. On the other hand, as noticed by G. Diaz, when one of the six numbers \( \Lambda_{ij} \) is algebraic, Theorem 1.2 reduces to the next consequence of Theorem 1.1 (further related results are given in [1] and [8]).

**Corollary 1.3.** Let \( \Lambda_1, \Lambda_2, \Lambda_3 \) be three elements of \( \tilde{\mathbb{L}} \) Assume that \( \Lambda_1 \) is transcendental and that the three numbers \( 1, \Lambda_2, \Lambda_3 \) are \( \mathbb{T} \)-linearly independent. Then one at least of the two numbers \( \Lambda_1 \Lambda_2, \Lambda_1 \Lambda_3 \) is not in \( \tilde{\mathbb{L}} \).

The simple example

\[
M = \begin{pmatrix}
0 & \Lambda_2 & \Lambda_3 \\
\Lambda_1 & 0 & 0
\end{pmatrix}
\]

shows that the assumptions of Theorem 1.2 are not sufficient to ensure that none of the three determinants is in \( \tilde{\mathbb{L}} \).

Here is a simple result which follows from Theorem 1.2: \textit{Let \( \Lambda_1, \Lambda_2, \Lambda_3 \) be three elements in \( \tilde{\mathbb{L}} \) such that \( 1, \Lambda_1, \Lambda_2, \Lambda_3 \) are linearly independent over \( \mathbb{T} \). Then one at least of the three numbers}

\[
\Lambda_1^2 - \Lambda_2 \Lambda_3, \quad \Lambda_2^2 - \Lambda_3 \Lambda_1, \quad \Lambda_3^2 - \Lambda_1 \Lambda_2
\]

\textit{is not in \( \tilde{\mathbb{L}} \).}

In §3 we shall deduce from Theorem 1.2 the following corollary.

**Corollary 1.4.** Let \( M \) be a \( 2 \times 3 \) matrix with entries in \( \mathbb{L} \):

\[
M = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23}
\end{pmatrix}
\]

Assume that the two rows of \( M \) are linearly independent over \( \mathbb{Q} \) and also that the three columns of \( M \) are linearly independent over \( \mathbb{Q} \). Then one at
least of the three numbers
(1.5) \( \lambda_{11} \lambda_{22} - \lambda_{12} \lambda_{21}, \lambda_{12} \lambda_{23} - \lambda_{13} \lambda_{22}, \lambda_{13} \lambda_{21} - \lambda_{11} \lambda_{23} \)

is not in \( \tilde{\mathbb{L}} \).

The six exponentials Theorem of S. Lang ([3], Chap. II § 1) and K. Ramachandra ([4] II § 4) states that, under the assumptions of Corollary 1.4, one at least of the three numbers (1.5) is not zero.

It is expected that a result similar to Theorem 1.2 holds when \( M \) is replaced by a \( 2 \times 2 \) matrix:

**Conjecture 1.6.** Let \( M \) be a \( 2 \times 2 \) matrix with entries in \( \tilde{\mathbb{L}} \):

\[
M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}.
\]

Assume that the four rows of the matrix

\[
\begin{pmatrix} M \\ I_2 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

are linearly independent over \( \overline{\mathbb{Q}} \) and that the four columns of the matrix

\[
\begin{pmatrix} I_2 & M \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Lambda_{11} & \Lambda_{12} \\ 0 & 1 & \Lambda_{21} & \Lambda_{22} \end{pmatrix}
\]

are linearly independent over \( \overline{\mathbb{Q}} \). Then the number

\[
\Delta = \begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}
\]

is not in \( \tilde{\mathbb{L}} \).

Conjecture 1.6 follows from the conjecture (see for instance [3], Historical Note of Chapter III, [2], Chap. 6 p. 259 and [7], Conjecture 1.15 and [8] Conjecture 1.1) that \( \mathbb{Q} \)-linearly independent logarithms of algebraic numbers are algebraically independent.

**2. A consequence of the Linear Subgroup Theorem**

Let \( n \) be a positive integer and \( Y \) a \( \overline{\mathbb{Q}} \)-vector subspace of \( \mathbb{C}^n \). We define

\[
\mu(Y, \mathbb{C}^n) = \min_{V \subset \mathbb{C}^n} \frac{\dim_{\mathbb{C}}(Y/Y \cap V)}{\dim_{\mathbb{C}}(\mathbb{C}^n/V)},
\]

where \( V \) runs over the set of \( \mathbb{C} \)-vector subspaces of \( \mathbb{C}^n \) with \( V \neq \mathbb{C}^n \).
For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ we denote by $x \cdot y$ the scalar product
\[ x \cdot y = x_1y_1 + \cdots + x_ny_n. \]
For $X$ and $Y$ two subsets of $\mathbb{C}^n$, we denote by $X \cdot Y$ the set of scalar products $x \cdot y$ where $x$ ranges over the set $X$ and $y$ over $Y$.

**Theorem 2.1.** Let $X$ and $Y$ be two $\overline{\mathbb{Q}}$-vector subspaces of $\mathbb{C}^n$. Assume $X$ has dimension $d$ with $d > n$. Assume further
\[ \mu(Y, \mathbb{C}^n) > \frac{d}{d-n}. \]
Then the set $X \cdot Y$ is not contained in $\mathcal{L}$.

**Proof.** This is essentially Proposition 6.1 of [6], where $\mathbb{Q}$ is replaced by $\overline{\mathbb{Q}}$ and the $\mathbb{Q}$-vector space $\mathcal{L}$ by the $\overline{\mathbb{Q}}$-vector space $\mathcal{L}$. Henceforth the proof runs as follows.

Like in Lemma 5.2 of [6], one checks that if $X$ and $Y$ are two vector subspaces of $\mathbb{C}^n$ over $\overline{\mathbb{Q}}$, of dimensions $d$ and $\ell$ respectively, then there exist a positive integer $n' \leq n$ and two vector subspaces $X'$ and $Y'$ of $\mathbb{C}^{n'}$, of dimensions $d'$ and $\ell'$ respectively, such that
\[ \mu(X', \mathbb{C}^{n'}) = \frac{d'}{n'} \geq \frac{d}{n}, \quad \mu(Y', \mathbb{C}^{n'}) = \frac{\ell'}{n'} \geq \mu(Y, \mathbb{C}^n) \]
and
\[ (2.2) \quad X' \cdot Y' \subset X \cdot Y. \]
This shows that for the proof of Theorem 2.1, there is no loss of generality to assume $\mu(X, \mathbb{C}^n) = d/n$ and $\mu(Y, \mathbb{C}^n) = \ell/n$. The assumption $\mu(Y, \mathbb{C}^n) > d/(d-n)$ reduces to $\ell d > n(\ell + d)$.

Following the argument of Lemma 5.4 in [6], one proves that if $X$ and $Y$ are two vector subspaces of $\mathbb{C}^n$ over $\overline{\mathbb{Q}}$, of dimensions $d$ and $\ell$ respectively, $X_1$ a subspace of $X$ of dimension $d_1$ and $Y_1$ a subspace of $Y$ of dimension $\ell_1$ such that $X_1 \cdot Y_1 = \{0\}$, then
\[ (2.3) \quad (d - d_1)\mu(Y, \mathbb{C}^n) + (\ell - \ell_1)\mu(X, \mathbb{C}^n) \geq n\mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n). \]
In Lemma 5.4 in [6] an extra assumption is required, namely
\[ \mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n) \geq \mu(X, \mathbb{C}^n) + \mu(Y, \mathbb{C}^n), \]
but we do not need it here, since our assumption $X_1 \cdot Y_1 = \{0\}$ is stronger than the assumption in Lemma 5.4 of [6] that $X_1 \cdot Y_1$ has rank $\leq 1$. 

Next we introduce the coefficient $\theta(M)$ attached to a $d \times \ell$ matrix $M$ with entries in $\mathbb{C}$. It is defined as follows:

$$\theta(M) = \min \frac{\ell'}{d'},$$

where $(d', \ell')$ ranges over the set of pairs of integers satisfying $0 \leq \ell' \leq \ell$, $1 \leq d' \leq d$, such that there exist a $d \times d$ regular matrix $P$ and a regular $\ell \times \ell$ regular matrix $Q$, both with entries in $\mathbb{Q}$, with

$$PMQ = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} d^* \\ \ell^* \end{pmatrix} \begin{pmatrix} d^* \\ \ell^* \end{pmatrix}$$

From (2.3) with $d_1 = d'$ and $\ell_1 = \ell^*$ it follows that if

$$X = \mathbb{Q}x_1 + \cdots + \mathbb{Q}x_d \quad \text{and} \quad Y = \mathbb{Q}y_1 + \cdots + \mathbb{Q}y_\ell$$

are again two vector subspaces of $\mathbb{C}^n$ over $\mathbb{Q}$, of dimensions $d$ and $\ell$ respectively, satisfying $\mu(X, \mathbb{C}^n) = d/n$, then the matrix

(2.4) $$M = (x_i \cdot y_j)_{1 \leq i \leq d, 1 \leq j \leq \ell}$$

has

$$\theta(M) \geq \frac{n}{d} \cdot \mu(Y, \mathbb{C}^n).$$

In particular if $\mu(X, \mathbb{C}^n) = d/n$ and $\mu(Y, \mathbb{C}^n) = \ell/n$, then $\theta(M) = \ell/d$.

Finally Theorem 4 in [5] (which is Proposition 11.19 or Theorem 12.19 in [7]) shows that the rank $r$ of a $d \times \ell$ matrix $M$ with entries in $\mathbb{Q}$ satisfies

$$r \geq \frac{d\theta}{1 + \theta},$$

where $\theta = \theta(M)$. Using this result for the matrix $M$ given by (2.4) whose rank $r$ is $\leq n$, one concludes that if $\mu(X, \mathbb{C}^n) = d/n$ and $\mu(Y, \mathbb{C}^n) = \ell/n$ with $X \cdot Y \subset \mathbb{C}$, then

$$n \geq \frac{\ell d}{\ell + d}.$$

Theorem 2.1 follows. \hfill \Box

Remark. Theorem 1.1 is equivalent with the case $n = 1$ of Theorem 2.1.
3. Proof of the main results

In this section we prove Theorem 1.2 and Corollary 1.4.

**Proof of Theorem 1.2.** Assume that the hypotheses of Theorem 1.2 are satisfied. Define elements \( v_1, \ldots, v_5 \) in \( \mathbb{C}^2 \) by

\[
v_1 = e_1, \quad v_2 = e_2, \quad v_{2+j} = (A_{1j}A_{2j}), \quad (j = 1, 2, 3),
\]

where \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). For \( v = (x, y) \in \mathbb{C}^2 \), set \( v' = (-y, x) \), so that \( v' \cdot v = 0 \). Consider the \( 5 \times 5 \) matrix

\[
A = (v_i \cdot v_j)_{1 \leq i, j \leq 5}.
\]

From its very definition, it is plain that \( A \) has rank 2. Explicitly one has

\[
A = \begin{pmatrix}
0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\
-1 & 0 & -\Lambda_{11} & -\Lambda_{12} & -\Lambda_{13} \\
-\Lambda_{21} & \Lambda_{11} & 0 & \Delta_3 & -\Delta_2 \\
-\Lambda_{22} & \Lambda_{12} & -\Delta_3 & 0 & \Delta_1 \\
-\Lambda_{23} & \Lambda_{13} & \Delta_2 & -\Delta_1 & 0
\end{pmatrix}.
\]

Let \( X \) be the \( \mathbb{Q} \)-vector space spanned by \( v_1, \ldots, v_5 \) in \( \mathbb{C}^2 \) and similarly let \( Y \) be the subspace of \( \mathbb{C}^2 \) spanned by \( v_1', \ldots, v_5' \) over \( \mathbb{Q} \). We claim

\[
\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) \geq 2.
\]

The equality \( \mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) \) follows from the fact that the map \( (x, y) \mapsto (-y, x) \) is an automorphism of \( \mathbb{C}^2 \).

Since the five columns of \( \begin{pmatrix} I_2 & M \end{pmatrix} \) are linearly independent over \( \mathbb{Q} \),

\[
\dim_{\mathbb{Q}} X = 5.
\]

Let \( V \) be a vector subspace of \( \mathbb{C}^2 \) of dimension 1 and let \( t_1z_1 + t_2z_2 = 0 \) be an equation of \( V \) in \( \mathbb{C}^2 \), with \( (t_1, t_2) \in \mathbb{C}^2 \setminus \{0\} \). Consider the linear map

\[
p : \mathbb{C}^2 \to \mathbb{C} \quad \quad (z_1, z_2) \mapsto t_1z_1 + t_2z_2
\]

whose kernel is \( V \). Since the five rows of \( \begin{pmatrix} M \\ I_3 \end{pmatrix} \) are \( \mathbb{Q} \)-linearly independent,

\[
\dim_{\mathbb{Q}}(X \cap V)/V = \dim_{\mathbb{Q}} p(X) \geq 2.
\]

This completes the proof of (3.1).

From (3.1) we deduce that the hypothesis \( \mu(Y, \mathbb{C}^2) > d/(d - n) \) of Theorem 2.1 is satisfied with \( d = 5 \) and \( n = 2 \), hence the set \( X \cdot Y \) is not contained in \( \mathcal{L} \). Consequently one at least of the three numbers \( \Delta_1, \Delta_2, \Delta_3 \) is not in \( \mathcal{L} \).
This completes the proof of the Main Theorem 1.2. □

Remark. In (3.1) we may have equality: for instance if $A_{22} = A_{23} = 0$ then $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) = 2$.

However the proof of Theorem 2.1 shows that in the case $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) < 5/2$, Theorem 1.2 should follow from Theorem 1.1. Indeed after a change of variables rational over $\overline{\mathbb{Q}}$ one needs only to consider a matrix

$$M = \begin{pmatrix}
0 & A_{12} & A_{13} \\
A_{21} & 0 & 0
\end{pmatrix},$$

which is the situation of Corollary 1.3. If $X$ is the $\overline{\mathbb{Q}}$-subspace of $\mathbb{C}^2$ spanned by

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (0, A_{21}), \quad v_4 = (A_{12}, 0), \quad v_4 = (A_{13}, 0)$$

and $Y$ the subspace spanned by

$$v'_1 = (0, 1), \quad v'_2 = (-1, 0), \quad v'_3 = (-A_{21}, 0), \quad v'_4 = (0, A_{12}), \quad v'_4 = (0, A_{13}),$$

then

$$X' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}A_{12} + \overline{\mathbb{Q}}A_{13} \quad \text{and} \quad Y' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}A_{21}$$

are $\overline{\mathbb{Q}}$-subspaces of $\mathbb{C}$ satisfying (2.2). Here $\mu(X', \mathbb{C}) = 3 > d/n = 5/2$ and $\mu(Y', \mathbb{C}) = 2 = \mu(Y, \mathbb{C}^2)$.

Proof of Corollary 1.4. From Baker’s Theorem it follows that if $Y_0$ is a $\mathbb{Q}$-vector subspace of $\mathcal{L}^n$ of dimension $\ell$, then the $\overline{\mathbb{Q}}$-vector subspace of $\overline{\mathbb{L}}^n$ spanned by $\overline{\mathbb{Q}}^n \cup Y_0$ has dimension $\ell + n$ (see Exercise 1.5 (iii) of [7]). Taking firstly $n = 2$, $\ell = 3$, and secondly $n = 3$, $\ell = 2$, we deduce that the matrix $M$ of corollary 1.4 satisfies the assumptions of Theorem 1.2. Corollary 1.4 follows. □

4. Erratum to [8]

We take the opportunity of this paper to point out a mistake in the statement of Corollary 2.12 p. 347 of [8]: the assumption that $A_{21}$ is not zero and $A_{11}/A_{21}$ is transcendental should be replaced by the assumption that the three numbers 1, $A_{11}$ and $A_{21}$ are linearly independent over the field of algebraic numbers. Otherwise a counterexample is obtained for instance with $A_{21} = 1$ and $A_{2j} = 0$ for $2 \leq j \leq 5$. 

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