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Improved intermediate asymptotics for the heat equation✩,✩,✩

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Abstract

This letter is devoted to results on intermediate asymptotics for the heat equation. We study the convergence towards a stationary solution in self-similar variables. By assuming the equality of some moments of the initial data and of the stationary solution, we get improved convergence rates using entropy / entropy-production methods. We establish the equivalence of the exponential decay of the entropies with new, improved functional inequalities in restricted classes of functions. This letter is the counterpart in a linear framework of a recent work on fast diffusion equations, see [8].

Results extend to the case of a Fokker-Planck equation with a general confining potential. The entropy method

Consider the heat equation in the euclidean space,

\[
\frac{\partial u}{\partial t} = \Delta u \quad t > 0, \quad x \in \mathbb{R}^d
\]

with an initial condition \(u_0 \in L^1(\mathbb{R}^d)\). By writing \(u = u_+ - u_-\) where \(u_+\) and \(u_-\) are respectively the positive and negative parts of \(u\) and solving (1) with initial data \((u_0)_+\) and \((u_0)_-\), we may reduce the problem to the case of a nonnegative function, corresponding to a nonnegative initial condition \(u_0\), without restriction. The heat equation being linear, we can assume without loss of generality that \(u_0\) is a probability measure so that in the sequel of this note \(\int_{\mathbb{R}^d} u_0 dx = 1 = \int_{\mathbb{R}^d} u(t,x) dx\) for any \(t \geq 0\). Getting decay rates and even an asymptotic expansion for large values of \(t\) is completely standard, see for instance [13]. However, a few details and some notations will be useful for later purpose.

First of all, as a straightforward consequence of the expression of the Green function, \(G(t,x,y) := (4\pi t)^{-d/2} e^{-|x-y|^2/4t}\), any solution \(u\) of (1) can be written as \(u(t,x) = \int_{\mathbb{R}^d} u_0(y) G(t,x,y) dy\) and therefore uniformly decays like \(O(t^{-d/2})\) since, as \(t \to \infty\), \(u(t,x) \sim G(t,x,0)\). It is also classical to estimate the decay of \(u(t, \cdot) - G(t, \cdot, 0)\) in various \(L^p(\mathbb{R}^d)\) norms. Such estimates are called intermediate asymptotics estimates. The point is to determine the first term of an asymptotic expansion of the solution as \(t \to \infty\). For instance, as we shall see below, it can be proved that \(\|u(t, \cdot) - G(t, \cdot, 0)\|_{L^1(\mathbb{R}^d)} = O(t^{-1/2})\) as \(t \to \infty\).

The entropy method can be used among various other approaches to obtain such an estimate. It relies on the logarithmic Sobolev inequality and goes as follows. First consider the time-dependent rescaling

\[
u(t,x) = R^{-d} v(\log R, x/R) \quad \text{with} \quad R = R(t) := \sqrt{1 + 2t}, \quad t > 0, \quad x \in \mathbb{R}^d.
\]

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If \( u \) is a solution of (3), then \( v \) solves the Fokker-Planck equation

\[
\frac{\partial v}{\partial t} = \Delta v + \nabla \cdot (x \cdot v)
\]

with same initial condition \( v(t=0, \cdot) = u_0 \). Let \( v_\infty(x) := (2\pi)^{-d/2} e^{-|x|^2/2} \) be the unique stationary solution of (3) with mass 1, and define \( d\mu := v_\infty \, dx \) as the Gaussian measure. We denote by \( L^p(\mathbb{R}^d) \) and \( L^p(\mathbb{R}^d, d\mu) \) the Lebesgue spaces corresponding respectively to Lebesgue’s measure and to the Gaussian measure. Understanding the intermediate asymptotics for \( u \) amounts to study the convergence of \( v \) to \( v_\infty \), as \( t \to \infty \). Define the entropy by \( \mathcal{E}_1[w] := \int_{\mathbb{R}^d} w \log w \, d\mu \). Let \( v \) be a solution of (3) and define \( w(t, \cdot) := v(t, \cdot)/v_\infty \), \( w_0 := w(t=0, \cdot) \). Then \( \frac{d}{dt} \mathcal{E}_1[w(t, \cdot)] = -I_1[w(t, \cdot)] \) where \( I_1 \) is the Fisher information defined by \( I_1[w] := \int_{\mathbb{R}^d} w \nabla \log w^2 \, d\mu \). Gross’ logarithmic Sobolev inequality exactly amounts to \( \mathcal{E}_1[v/v_\infty] \leq \frac{1}{2} I_1[v/v_\infty] \) and so, it follows that

\[
\mathcal{E}_1[w(t, \cdot)] \leq \mathcal{E}_1[w_0] e^{-\frac{1}{2} t} \quad \forall t \geq 0.
\]

By the Csiszár-Kullback inequality, see for instance [7,1], we get \( \|v(t, \cdot) - v_\infty\|_{L^1(\mathbb{R}^d)}^2 \leq \frac{1}{4} \mathcal{E}_1[w(t, \cdot)] \) and deduce that

\[
\|v(t, \cdot) - v_\infty\|_{L^1(\mathbb{R}^d)}^2 \leq \frac{1}{2} \mathcal{E}_1[w_0] e^{-\frac{1}{4} t} \quad \forall t \geq 0.
\]

Undoing the change of variables (2) and observing that \( u_\infty(t, x) := R(t)^{-d} v_\infty(x/R(t)) = G(t + 1/2, \cdot, 0) \), we finally get

\[
\|u(t, \cdot) - u_\infty(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq \frac{1}{2} \mathcal{E}_1[w_0] e^{-\frac{1}{4} t} \quad \forall t \geq 0,
\]

which establishes the claimed estimate, namely: \( \|u(t, \cdot) - G(t, x, 0)\|_{L^1(\mathbb{R}^d)} \leq O\left(\frac{1}{t^{1/2}}\right) \) as \( t \to \infty \). Such an estimate is quite classical. The above method is known as the Bakry-Emery method or entropy / entropy-production method and also provides a proof of the logarithmic Sobolev inequality. See [4, 3] for some references on this topic, in the context of partial differential equations.

By combining \( L^1(\mathbb{R}^d) \) and \( L^\infty(\mathbb{R}^d) \) estimates using Hölder’s inequality, we get that

\[
\|u(t, \cdot) - G(t, x, 0)\|_{L^p(\mathbb{R}^d)} \leq O\left(\frac{1}{t^{(1+ p - 1)d}}\right) \quad \text{as} \quad t \to \infty.
\]

In a \( L^2(\mathbb{R}^d) \) framework, a much more detailed description can be achieved using a spectral decomposition. If \( v \) is a solution of (3), then \( w = v/v_\infty \) is a solution of the Ornstein-Uhlenbeck equation

\[
\frac{\partial w}{\partial t} = \Delta w - x \cdot \nabla w
\]

with initial data \( w_0 = u_0/v_\infty \). Notice that \( \int_{\mathbb{R}^d} w_0 \, d\mu = 1 \) and, as a consequence, \( \int_{\mathbb{R}^d} w(t, \cdot) \, d\mu = 1 \) for all \( t \geq 0 \). Define by \( (H_k)_{k \in \mathbb{N}^d} \) the sequence of Hermite type polynomials (see for instance [19]) acting on \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \), such that \( H_k(x) := \prod_{j=1}^d h_j(x_j) \) where \( h_n(y) := (-1)^n (n!)^{-1/2} e^{y^2/2} \frac{d^n}{dy^n} (e^{-y^2/2}) \), \( y \in \mathbb{R} \) and \( k = (k_1, \ldots, k_d) \in \mathbb{N}^d \). These functions provide an orthonormal family of eigenfunctions in \( L^2(\mathbb{R}^d, d\mu) \) which spans the eigenspaces of the Ornstein-Uhlenbeck operator, that is \( -\Delta H_k - x \cdot \nabla H_k = \lambda_k H_k \), where \( |k| := \sum_{j=1}^d k_j \). Up to a scaling, \( (h_k)_{k \in \mathbb{N}^d} \) is the usual family of Hermite polynomials on \( \mathbb{R} \).

If \( w_0 \) satisfies the orthogonality condition

\[
\int_{\mathbb{R}^d} w_0 \, H_k \, d\mu = 0 \quad \forall k \in \mathbb{N}^d \text{ such that } 0 < |k| < n,
\]

then an improved rate of convergence follows, in the sense that

\[
\|w(t, \cdot) - 1\|_{L^2(\mathbb{R}^d, d\mu)} \leq e^{-nt} \|w_0 - 1\|_{L^2(\mathbb{R}^d, d\mu)} \quad \forall t \geq 0.
\]

If (5) initially holds, we indeed have \( \int_{\mathbb{R}^d} w(t, \cdot) \, H_k \, d\mu = 0 \) for any \( t \geq 0 \) and any \( k \in \mathbb{N}^d \) such that \( 0 < |k| < n \). Then, since \( \frac{d}{dt}\|w(t, \cdot) - 1\|_{L^2(\mathbb{R}^d, d\mu)}^2 = -2 \int_{\mathbb{R}^d} |\nabla w(t, \cdot)|^2 \, d\mu \), the conclusion holds using the following result.
Proposition 1 (Improved Poincaré inequality). Assume that \( w \in L^2(\mathbb{R}^d) \) is such that \( \int_{\mathbb{R}^d} w \, d\mu = 1 \) and the condition \( \int_{\mathbb{R}^d} w \, H_k \, d\mu = 0 \) holds for any \( k \in \mathbb{N}^d \) such that \( 0 < |k| < n \). Then the following inequality holds, with optimal constant:

\[
\|w - 1\|^2_{L^2(\mathbb{R}^d, d\mu)} \leq \frac{1}{n}\|\nabla w\|^2_{L^2(\mathbb{R}^d, d\mu)}.
\]

The proof is no more than a straightforward rewriting of the Rayleigh quotient of (4), then

\[
E_k[w] := \int_{\mathbb{R}^d} \frac{w^p - 1}{p - 1} \, d\mu.
\]

This definition is consistent with the definition of \( E_1 \) because, under the condition \( \int_{\mathbb{R}^d} w \, d\mu = 1 \), \( E_k[w] = \int_{\mathbb{R}^d} \frac{w^p - 1}{p - 1} \, d\mu \) for any \( p \to 1 \). The functional \( E_p \) controls the convergence in \( L^p(\mathbb{R}^d, d\mu) \) using a generalized Csiszár-Kullback inequality. In [3], it has been proved that \( \|w - 1\|^2_{L^2(\mathbb{R}^d, d\mu)} \leq \frac{2}{n} \left( 1 \right) \max \{ \|w\|^2_{L^2(\mathbb{R}^d, d\mu)} \} \} E_p[w], \) for any \( p \in [1, 2] \). Since \( \|w\|_{L^1(\mathbb{R}^d, d\mu)} = 1 \), we have \( 1 \leq \|w\|^p_{L^p(\mathbb{R}^d, d\mu)} + 1 + (p - 1) E_p[w] \), and so

\[
\|w - 1\|_{L^p(\mathbb{R}^d, d\mu)} \leq \mathcal{A}_p \left( E_p[w] \right)
\]

with \( \mathcal{A}_p(s) := \frac{2^{1/p}}{\sqrt[2/p]{s}} \left[ 1 + (p - 1) s \right]^{1/p} \). Next, assume that \( \int_{\mathbb{R}^d} w \, H_k \, d\mu = 0 \) for any \( k \in \mathbb{N}^d \) such that \( 0 < |k| < n \) and consider the generalized Poincaré inequalities, with \( p \in [1, 2] \), namely

\[
E_p[w] \leq B_{n,p} \int_{\mathbb{R}^d} \left| \nabla w \right|^{2/p} \, d\mu \quad \forall \, w \in H^1(\mathbb{R}^d, d\mu).
\]

Such inequalities have been established for \( n = 1 \) by W. Beckner in [3] with optimal constant \( B_{1,p} = 2/p \) for the Gaussian measure. By the same method, it has been shown in [4] that for a larger class of measures \( d\mu \), if \( \mathcal{P} \) holds for \( p = 1 \) and \( p = 2 \), for some positive constants \( B_{n,1} \) and \( B_{n,2} \), respectively, then it also holds for any \( p \in (1, 2) \) with

\[
B_{n,p} = \frac{1}{p - 1} \left[ 1 - ((2 - p)/p)^{B_{n,1}/(2B_{n,2})} \right] B_{n,2}.
\]

By the logarithmic Sobolev inequality and the improved Poincaré inequality, see Proposition [5], we know that \( B_{n,1} \leq 2 \) and \( B_{n,2} = 1/n \). Hence it follows that \( B_{n,p} \leq \frac{1}{p - 1} \left[ 1 - ((2 - p)/p)^p \right] \frac{1}{n} \). On the other hand, as in [3], if \( w \) is a solution of (4), then

\[
\frac{d}{dt} E_p[w(t, \cdot)] = - \frac{4}{p} \int_{\mathbb{R}^d} \left| \nabla w \right|^{2/p} \, d\mu.
\]

If (3) is satisfied, we conclude using (3) and (4) that any solution of (4) with initial data \( w_0 \) satisfies

\[
E_p[w(t, \cdot)] \leq E_p[w_0] e^{-2\lambda(n,p)t} \quad \text{and} \quad \|w(t, \cdot) - 1\|_{L^2(\mathbb{R}^d, d\mu)} \leq \mathcal{A}_p \left( E_p[w_0] \right) e^{-\lambda(n,p)t} \quad \forall \, t \geq 0,
\]

with \( \lambda(n,p) := \frac{2}{p} \left( n \right) (p - 1) \left[ 1 - ((2 - p)/p)^p \right] \) \( \lambda(n,p) \). The last estimate holds because, for any \( t \geq 0, \)

\[
\|w(t, \cdot) - 1\|_{L^2(\mathbb{R}^d, d\mu)} \leq \mathcal{A}_p \left( E_p[w(t, \cdot)] \right) \leq \mathcal{A}_p \left( E_p[w_0] \right) e^{-2\lambda(n,p)t} \leq \mathcal{A}_p \left( E_p[w_0] \right) e^{-\lambda(n,p)t}.
\]

Notice that \( \lambda(1, p) = 1 \) and \( \lambda(n, 2) = n \). Nothing is gained as \( p \to 1 \), since \( \lim_{p \to 1} \lambda(n, p) = 1 \) is independent of \( n. \)

On the other hand, by Hölder’s inequality, we have for free that \( \|w - 1\|_{L^2(\mathbb{R}^d, d\mu)} \leq \|w - 1\|_{L^2(\mathbb{R}^d, d\mu)}. \) Hence, if \( w \) is a solution of (4) with initial data \( w_0 \), we know that \( \|w(t, \cdot) - 1\|_{L^2(\mathbb{R}^d, d\mu)} \leq e^{-\sigma t} \|w_0 - 1\|_{L^2(\mathbb{R}^d, d\mu)} \) as \( t \to \infty \), for any
Consider first the case $p = 1$. An alternative approach is suggested by the method of [1], which applies to the fast diffusion equation $\frac{\partial u}{\partial t} = \Delta a^m$ for $m < 1$. By assuming some uniform bound on the initial data, which is preserved along the evolution, it is possible to relate the asymptotic rate for intermediate asymptotics with the spectrum of the linearized operator. We can indeed observe that $\|w_0 - 1\|_{L^2(\mathbb{R}^d, d\mu)} \leq \|w_0 - 1\|_{L^-(\mathbb{R}^d, d\mu)} = \frac{1}{\sqrt{\mathcal{E}_1[w_0]}} \|w_0 - 1\|_{L^2(\mathbb{R}^d)}$ using Hölder’s inequality and the Csizsár-Kullback inequality. This proves that

$$\|w(t, \cdot) - 1\|_{L^2(\mathbb{R}^d, d\mu)} \leq \frac{1}{\sqrt{\mathcal{E}_1[w_0]}} \|w_0 - 1\|_{L^2(\mathbb{R}^d)} e^{-nt} \quad \text{as } t \to \infty$$

if (5) is satisfied initially. Still, this provides neither an estimate of $\int_{\mathbb{R}^d} w(t, \cdot) \log w(t, \cdot) d\mu$ nor a functional inequality which improves upon the logarithmic Sobolev inequality. To prove such an inequality, we keep following the strategy of [1]. A simple but key idea is to observe that the functions defined for any $p \in [1, 2]$ by $h_p(0) = 1, h_p(1) = p/2$ and, for any $s \in (0, 1) \cup (1, \infty)$ by $h_p(s) := [s^p - 1 - p(s - 1)]/[p(p - 1)s - 1^2]$ if $p > 1$, $h_p(s) := [s \log s - (s - 1)]/[s - 1]^2$, are continuous, nonnegative, decreasing on $\mathbb{R}^+$ and achieve their maximum at $0$. Define on $L^\infty(\mathbb{R}^d)$ the functional

$$\mathcal{H}_p[w] := \|w\|_{L^p(\mathbb{R}^d)}^p \sup_{x \in \mathbb{R}^d} h_p(w(x)) = \|w\|_{L^{2p}(\mathbb{R}^d)}^p \inf_{x \in \mathbb{R}^d} h_p(w(x)).$$

**Theorem 2 (Improved logarithmic Sobolev inequality).** Assume that $w \in L^\infty(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} w d\mu = 1$ and satisfies the condition $\int_{\mathbb{R}^d} w H_k d\mu = 0$ for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. Then the following inequality holds, with optimal constant:

$$\int_{\mathbb{R}^d} w \log w d\mu \leq \frac{\mathcal{H}_1[w]}{n} \int_{\mathbb{R}^d} \frac{\|\nabla w\|^2}{w} d\mu.$$

**Proof.** We may indeed observe that by the Poincaré inequality and using the definition of $\mathcal{H}_1$, we get

$$\int_{\mathbb{R}^d} \frac{\|\nabla w\|^2}{w} d\mu \geq \frac{1}{\|w\|_{L^2(\mathbb{R}^d)}} \int_{\mathbb{R}^d} \|\nabla w\|^2 d\mu \geq \frac{n}{\|w\|_{L^2(\mathbb{R}^d)}} \int_{\mathbb{R}^d} \|w - 1\|^2 d\mu \geq \frac{n}{\mathcal{H}_1[w]} \int_{\mathbb{R}^d} w \log w d\mu.$$

The optimality of the constant can be checked by a lengthy but elementary computation using the functions $w_k^k := H_k(x) \chi(x e^k/2) + C_k^k$ for some smooth truncation function $\chi$ such that $0 \leq \chi \leq 1, \chi \equiv 1$ on $B(0,1)$ and $\chi \equiv 0$ in $\mathbb{R}^d \setminus B(0,2)$. Here for $k \in \mathbb{N}^d$ is such that $|k| = n$ and the constant $C_k^k$ is chosen so that $\int_{\mathbb{R}^d} w_k^k d\mu = 1$.

As a consequence of the Maximum Principle applied to the heat equation (1) and the fact that to $u_0 = \nu_\infty$ corresponds a self-similar solution of (2), namely $u(t, x) = G\left(t + \frac{1}{n}, x, 0\right)$, we have the estimate

$$\mathcal{H}_1[w(t, \cdot)] \leq \mathcal{H}_1[w_0] \quad \forall t \geq 0.$$

By applying Theorem 2, we obtain a new result of decay for $\mathcal{E}_1[w(t, \cdot)]$ with a constant which is exactly $\mathcal{E}_1[w_0]$, to the price of a rate which is less than $2n$.

**Corollary 3 (Improved decay rate of the entropy).** Let $w$ be a solution of (1) with a nonnegative bounded initial data $w_0 \in L^1(\mathbb{R}^d, d\mu)$ such that $\int_{\mathbb{R}^d} w_0 d\mu = 1$ and (5) is satisfied. Then

$$\mathcal{E}_1[w(t, \cdot)] \leq \mathcal{E}_1[w_0] e^{-nt/\mathcal{H}_1[w_0]} \quad \forall t \geq 0.$$

This result is actually equivalent to Theorem 2, as follows by differentiating the above inequality at $t = 0$ (for which equality is trivially satisfied) and using the fact that $-\int_{\mathbb{R}^d} \|\nabla w_0\|^2/w_0 d\mu = 2\mathcal{E}_1[w(t, \cdot)]|_{t=0} \leq \mathcal{E}_1[w_0] \int_{\mathbb{R}^d} w_0 \log w_0 d\mu$. What we have achieved is a global, improved exponential decay of the entropy $\mathcal{E}_1$ in a restricted class of functions. To simplify even further, for any $\varepsilon \in (0,1)$ and $n \in \mathbb{N}^+$, consider the set $X_n^\varepsilon := \{w \in L^1(\mathbb{R}^d, d\mu) :
$1 - \varepsilon \leq w \leq 1 + \varepsilon \text{ a.e. and the condition } \int_{R^d} w H_k \, d\mu = 0 \text{ holds for any } k \in \mathbb{N}^d \text{ such that } 0 < |k| < n$, which is appropriate to handle the optimality case corresponding to $\varepsilon \to 0$. The best constant in Theorem 3 is indeed asymptotically equivalent to the sharp rate of convergence in Corollary 3 in the sense that $\lim_{k \to 0} \inf_{w \in \mathcal{X}} n/H_k[w] = \lim_{n \to 0} n/(1 + \varepsilon) (1 - \varepsilon) = 2n$.

For simplicity, we consider only the case $p = 1$, but the method also applies to any $p \in (1, 2)$. We obtain an improved version of (3) under the restriction that $w \in L^1(R^d, \mu)$ is bounded nonnegative and the condition $\int_{R^d} w H_k \, d\mu = 0$ holds for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. With $B_{n,1} = 4H_k[w]/n$ and $B_{n,2} = 1/n$, we get $B_{n,p} \leq \mathcal{K}[n, p, w] : (n/(p - 1))^{-1} \left[1 - ((2 - p)/p)^2 \mathcal{H}_k[w]\right]$ by (3). Using the entropy/entropy-production identity (3), the fact that $\mathcal{K}[\mathcal{N}, p, w(t, \cdot)] \leq \mathcal{K}[n, p, w]$ and the generalized Csiszár-Kullback inequality (3), we obtain

$$E_p[w(t, \cdot)] \leq E_p[w_0] e^{-\frac{\mathcal{H}_p[w]}{n}} \quad \text{and} \quad \|w - 1\|_{L^p(R^d, d\mu)} \leq \mathcal{A}_p \left( E_p[w_0] \right) e^{-\frac{\mathcal{H}_p[w]}{n}} \quad \forall \ t \geq 0. \quad (10)$$

Alternatively, an elementary computation as in the proof of Theorem 3 gives a similar result:

$$\frac{4}{p^2} \int_{R^d} |\nabla w|^{p/2} \, d\mu = \int_{R^d} w^{p-2} |\nabla w|^2 \, d\mu \geq \frac{1}{p^2 w^{2p-2}} \int_{R^d} |\nabla w|^2 \, d\mu \geq \frac{n}{w^{2p-2}} \int_{R^d} |\nabla w|^2 \, d\mu \geq \frac{n}{\mathcal{H}_p[w]} E_p[w]$$

if $\int_{R^d} w \, d\mu = 1$ and the condition $\int_{R^d} w H_k \, d\mu = 0$ holds for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. This proves that

$$E_p[w] \leq \frac{4}{p^2} \frac{\mathcal{H}_p[w]}{n} \int_{R^d} |\nabla w|^{p/2} \, d\mu.$$ 

Using (3) and (4), this proves that any solution of (3) with initial data in $w_0 \in L^1 \cap L^\infty(R^d, \mu)$ satisfies

$$E_p[w(t, \cdot)] \leq E_p[w_0] e^{-\frac{\mathcal{H}_p[w]}{n}} \quad \text{and} \quad \|w - 1\|_{L^p(R^d, d\mu)} \leq \mathcal{A}_p \left( E_p[w_0] \right) e^{-\frac{\mathcal{H}_p[w]}{n}} \quad \forall \ t \geq 0. \quad (11)$$

Comparing the rates of (10) and (11) is a natural question. In the limit $\varepsilon \to 0$, $\inf_{w \in \mathcal{X}} \mathcal{H}_p[w] \sim \sup_{w \in \mathcal{X}} \mathcal{H}_p[w] \to p/2$ and it follows that $\lim_{\varepsilon \to 0} \frac{4}{p^2} \frac{n}{(p - 1)(1 - ((2 - p)/p)^2)} < 2n = \lim_{k \to 0} \mathcal{H}_p[w]/\mathcal{H}_k[w]$. Hence, at least in the regime $\varepsilon \to 0$, (11) is a better estimate in terms of rates than (10). Undoing the change of variables (3), we have achieved a detailed result on improved $u_0$.

**Corollary 4 (Improved intermediate asymptotics for the heat equation).** Let $p \in [1, 2]$ and assume that $u_0$ is a probability measure such that $w_0 = w_0/\|w_0\|_1$ is bounded and satisfies the condition $\int_{R^d} u_0 H_k \, dx = 0$ for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. If $u$ is the solution of (3) with initial condition $u_0$, then

$$\|u(t, \cdot) - u_0(t, \cdot)\|_{L^p(R^d)} \leq (2n)^{-\frac{1}{2}} \mathcal{A}_p \left( E_p[w_0] \right) (1 + 2t)^{-\frac{1}{2(1 + \frac{1}{p})}} \quad \forall \ t \geq 0.$$ 

The proof relies on the remark that $\|u(t, \cdot) - u_0(t, \cdot)\|_{L^p(R^d)} \leq \|u_0(t, \cdot)\|_{L^p(R^d)} \|w(t, \cdot) - 1\|_{L^p(R^d, d\mu)}$ where $u_0(t, \cdot) : = G(t + 1/2, \cdot, 0)$. The conclusion holds using $\|u_0(t, \cdot)\|_{L^p(R^d)} = (2\pi R^d)^{-d/2}$ with $R = \sqrt{1 + 2t}$.

Up to now, we have considered the simple case of the harmonic potential, $V(x) = \frac{1}{2}|x|^2$. As in [1], the previous results can be extended to more general potentials as follows. Consider $V \in W^{1, \infty}(\mathbb{R}^d) \cap W^{1, 2}(\mathbb{R}^d)$ such that $\int_{R^d} e^{-V(x)} \, dx = 1$, and define the probability measure $\mu(x) : = e^{-V(x)} \, dx$ in $R^d$, which generalizes the Gaussian measure. Under the above Gaussian measure on $V$, the logarithmic Sobolev inequality holds (resp. (1) for $p = 1$) for some positive constant (resp. for $B_{1,1} > 0$). The Ornstein-Uhlenbeck operator $N : = -\Delta + V V^*$ is essentially self-adjoint on $L^2(\mu)$, has a non-degenerate eigenvalue $\lambda_0 = 0$ and a spectral gap $\lambda_1 > 0$. According to [18 Theorem 2.1], $N$ has a pure point spectrum without accumulation points. Since $\lim_{k \to 0} \lambda_k = \infty$, then by [13, Theorem XIII.64], the eigenfunctions of $N$ form a complete basis of $L^2(R^d, \mu)$. We shall denote the eigenvalues by $\lambda_k$, $k \in \mathbb{N}$, and by $E_k$ the corresponding eigenspaces.

**Theorem 3** adapts without changes. Assume that $w \in L^\infty(\mathbb{R}^d)$ is such that $\int_{R^d} w \, d\mu = 1$. Then

$$\int_{R^d} w \log \frac{w}{\lambda_n} \, d\mu \leq \mathcal{H}_1[w] \int_{R^d} \frac{|\nabla w|^2}{w} \, d\mu.$$
under the orthogonality condition: \( w \in \left( \bigcup_{k=0}^{n-1} E_k \right)^\perp \), that is \( \int_{\mathbb{R}^d} w f_k \, d\mu = 0 \) for any \( f_k \in E_k, \ k = 1, \ldots, n - 1 \). Next, consider the solution \( w \) of the Ornstein-Uhlenbeck equation

\[
\frac{\partial w}{\partial t} = -N w = \Delta w - \nabla V \cdot \nabla w,
\]

with initial condition \( w_0 \in \left( \bigcup_{k=0}^{n-1} E_k \right)^\perp \cap L^\infty(\mathbb{R}^d) \) is such that \( \int_{\mathbb{R}^d} w_0 \, d\mu = 1 \). With the same definition as above for \( \mathcal{E}_p \), for any solution of \( (\mathcal{E}_2) \) with initial data \( w_0, (1) \) is now replaced by

\[
\mathcal{E}_p[ w(t, \cdot) ] \leq \mathcal{E}_p [ w_0 ] \ e^{-\lambda \sqrt{p} t / 2 \mathcal{H}_p [ w_0 ]} \quad \text{and} \quad \| w - 1 \|_{L^p(\mathbb{R}^d, d\mu)} \leq \mathcal{A}_p \left( \mathcal{E}_p [ w_0 ] \right) \ e^{\sqrt{p} \mathcal{H}_p [ w_0 ]} \quad \forall \ t \geq 0.
\]

Let us conclude this letter by some comments and open questions. It is standard in entropy / entropy-production methods that determining sharp rates of convergence in an evolution equation is equivalent to finding sharp constants in functional inequalities, as we have seen in the case of the heat equation: the rate of convergence in \( L^2(\mathbb{R}^d, d\mu) \) is given by the Poincaré inequality, while the rate of convergence in energy, which controls the \( L^1(\mathbb{R}^d, d\mu) \) norm, is related with the logarithmic Sobolev inequality. This is also true for nonlinear diffusion equations, see for instance \( (3) \). In this case, a breakthrough came from the observation that uniform norms can also be used, see \( (10, 7, 6) \), to the price of the restriction to minimum entropy. Unpublished, 2004.

As long as nonlinear evolution problems are concerned, only a few invariant quantities are usually available: the mass and the position of the center of mass of the solution, for instance. In linear evolution problems, we can impose an arbitrary number of orthogonality conditions, which are preserved along the evolution. Improved rates of convergence are then expected, even when measured with nonlinear quantities like the entropy. Various attempts have been done, see for instance \( (13) \), but the question has been left open for many years. Such ideas have been partially explored by R.J. McCann, including in the linear case (see \( (11) \)), based on considerations on an appropriate Hessian matrix. Our approach provides a simpler and elementary answer under restrictions which are natural in view of \( (6) \). It also raises a number of questions concerning the optimality of the new functional inequalities from a variational point of view, the convergence of minimizing sequences and the symmetry of the eventual minimizers.

References


