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HAL Id: hal-00405643
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Submitted on 20 Jul 2009

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Observation of the Anderson Metal-Insulator Transition with Atomic Matter Waves: Theory and Experiment

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(Dated: July 20, 2009)

Using a cold atomic gas exposed to laser pulses – a realization of the chaotic quasiperiodic kicked rotor with three incommensurate frequencies – we study experimentally and theoretically the Anderson metal-insulator transition in three dimensions. Sensitive measurements of the atomic wavefunction and the use of finite-size scaling techniques make it possible to unambiguously demonstrate the existence of a quantum phase transition and to measure its critical exponents. By taking proper account of systematic corrections to one-parameter scaling, we show the universality of the critical exponent ν = 1.59 ± 0.01, which is found to be equal to the one previously computed for the Anderson model.

PACS numbers: 03.75.-b, 72.15.Rn, 64.70.Tg, 05.45.Mt

I. INTRODUCTION

The interplay between quantum effects and disorder is a subject actively studied for many decades, both theoretically and experimentally. It plays a particularly important role in condensed matter physics, where, in a first approximation, a crystal is modeled as independent electrons interacting with a perfectly periodic lattice. The pioneering works of Bloch and Zener showed however that most predictions based on this model are not verified in real crystals. For example, the Bloch theory predicts fully delocalized wavefunctions implying a ballistic transport of the electrons through the crystal. Moreover, in the presence of a constant bias potential, Zener predicted an oscillatory motion (the Bloch-Zener oscillations) due to quantum interference effects. This contradicts well-known experimental facts at least in usual conditions.

An obvious possible explanation of these contradictions is the fact that there are no perfect crystals: In a real crystal some sites may be randomly occupied by ions of a different nature, thus breaking the periodicity of the lattice. In 1958, Anderson considered this approach and postulated that the dominant effect of the disorder is to change randomly the on-site energy. Starting from this assumption, he constructed a simple model of a single-electron interacting with a lattice in the tight-binding approximation:

\[ H_{\text{tb}} = \sum_{jn} \epsilon_{jn} |jn\rangle \langle jn| + \sum_{jn,k\mu} V_{jn,k\mu} |jn\rangle \langle k\mu|. \]  (1)

Here \( \epsilon_{jn} \) are the energies associated with the states labeled by \( n \) at the sites \( j \) of the lattice, and the non-diagonal elements \( V_{jn,k\mu} \) denote the matrix elements between these states. The diagonal part of the Hamiltonian corresponds to the potential energy and the non-diagonal part to the kinetic energy in a continuous space description. Disorder is introduced by giving the site energies \( \epsilon_{jn} \) a random distribution. Anderson thus showed that the electron wavefunctions can be localized by the disorder. This is naturally in sharp contrast with the prediction of the Bloch model.

The phenomenon of localization has its most striking manifestation in the transport properties of random media. If particle-particle interactions are negligible, exponentially localized states cannot contribute to transport at zero temperature since the coupling to phonons is negligible. Anderson localization as a consequence of the presence of disorder is one of the fundamental ingredients for the understanding of the existence of insulators and metals, and, in particular, the transition between the insulating and the metallic states of matter. An insulator is associated with localized states of the system while a metal generally displays diffusive transport associated with delocalized states.

It was later shown that the 3D Anderson model displayed a phase transition between a localized and a diffusive phase, the so-called Anderson metal-insulator transition: If the disorder is below a critical level, the localization disappears and one recovers a metallic (conductor) behavior. The link between the disorder-induced metal-insulator transition and second-order phase-transitions was established by reformulating the problem in terms of the renormalization group.

Based on Wegner’s work and the ideas of Thouless and Landauer, it was possible to formulate the so-called one-parameter scaling theory of localization, one of the most fruitful approaches to the disorder-induced metal-insulator transition. The essential hypothesis of the scaling theory is that, close to the transition, a single relevant scaling variable describes the critical behavior.

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An essential result of the one-parameter scaling theory is that the Anderson transition exists only in dimensions larger than two. In one dimension, all electronic states are localized, whatever the degree of randomness. In two dimensions, they are all localized, but only marginally, i.e. with a localization length exponentially large (thus possibly much larger than the sample size) for weak disorder.

In analogy to standard second-order phase-transitions, the localization length $\ell$ is assumed to diverge at criticality according to a power law:

$$\ell \sim (W - W_c)^{-\nu},$$

with $\nu$ the localization length critical exponent, $W$ the disorder strength and $W_c$ the critical disorder strength. The most important assumption of the theory, the one-parameter scaling hypothesis, was numerically validated using a finite-size scaling method developed in Ref. [13]. This technique, which implements a real space renormalization, allowed to establish numerically the existence of a scaling function for the localization length. However, the critical exponents measured using this method, $\nu \approx 1.57$ [11, 12], were not compatible with the result $\nu = 1$ obtained from a self-consistent approach of localization based on diagrammatic techniques, as developed in Ref. [14].

In the half-century since its birth, the Anderson model has become a paradigm for the studies of the interplay of quantum effects and disorder. Despite that, relatively few experimental results are available, for the following reasons: i) It is experimentally hard to finely tune the disorder in a real crystal; ii) the decoherence sources (collision with phonons, etc.) are difficult to master [14]; iii) electrons in a crystal present strong mutual interactions and iv) the wavefunction of the electrons in the crystal is not directly accessible, only transport properties can be directly measured [15].

It is thus interesting to search for other systems that display the Anderson transition, but are more favorable for experimental studies. Indeed, the concept of Anderson localization has progressively been extended from its original solid-state physics scope to a variety of systems where a wave propagates in a disordered medium for example electromagnetic radiation [21, 22] and sound waves [23, 24, 25]. Photons propagating in disordered materials revealed to be an excellent system to observe the effects of localization [26, 27]. However, in such systems, there is always some absorption, whose signature can be quite similar to the signature of localization. Also, the measured quantity is the transmission, and the wavefunction itself is not accessible. The recent experimental observation of Anderson localization [28] using ultra-cold atomic matter waves has been done in a 1D situation where states are always localized and no metal-insulator transition exists.

A very interesting Anderson-type system is obtained by combining the Anderson model with another paradigmatic system, the kicked rotor (KR), which has been theoretically studied for almost three decades. This system is well known to be classically chaotic [28, 29], and chaos plays here the role of a “dynamical” disorder. In the quantum case, the KR displays a localization phenomenon, called “dynamical localization” [30, 31] which is analogous to the 1D-Anderson localization [32]. Moreover, a quasi-periodic generalization of the kicked rotor, substantially equivalent to the 3D Anderson model, was numerically shown to display an Anderson-like phase transition [33]. Experimental studies of the quantum kicked rotor were boosted by the realization of such a system with laser-cooled atoms interacting with a standing wave by Raizen and co-workers, which observed, for the first time, the Anderson localization with matter waves [27].

In the present paper we describe in detail a realization of an atomic matter-wave system that allows us to observe the Anderson metal-insulator transition [28]. We report a full characterization of this phase transition which includes an experimental validation of the one parameter scaling hypothesis and the first non ambiguous experimental determination of the critical exponent $\nu$. Last but not least, we show numerically that the quantum chaotic system we consider has the same critical behavior as the true random 3D Anderson model. In particular, we show that the two models belong to the same universality class. Sec. II introduces the cold-atom realization of the periodic (standard) KR and its equivalence with the 1D Anderson model, as well as the quasi-periodic generalization of this system that is equivalent of the 3D-Anderson model. Sec. III describes the corresponding experimental setup, paying attention to its experimental limits (decoherence, stray effects, limited observation time). In sec. IV we report our direct experimental observation of the metal-insulator transition. In sec. V a scaling procedure is introduced that allows us to overcome experimental limitations and determine the critical exponent corresponding to the Anderson transition. Sec. VI is devoted to the universality of the critical behavior. Sec. VII concludes the paper.

II. THE ATOMIC KICKED ROTOR AND ITS RELATION TO THE ANDERSON MODEL

A. The atomic kicked rotor

Consider a two level atom interacting with a laser standing wave of frequency $\omega_L = k \cdot c$ detuned by $\Delta_L = \omega_L - \omega_0$ from the atomic transition of frequency $\omega_0$. It is well known that there are two kinds of interactions between the atom and the radiation: Firstly, the atom can absorb a photon from the laser and re-emit it spontaneously in a random direction. This is a dissipative process giving rise to radiation pressure force, whose rate is $\Gamma \Omega^2 / 4 \Delta_L^2$ where $\Gamma$ is the natural width and $\Omega$ the resonant Rabi frequency (we assume $|\Delta_L| \gg \Gamma$). Secondly, the atom can pick a photon in a laser mode and emit it in the same (or another) laser mode by stimulated emission.
This conservative process is associated with a potential acting on the atom’s center of mass motion, called the optical or dipole potential. For a standing wave this potential is:

$$V_{\text{opt}} = \frac{\hbar \Omega^2}{8 \Delta L} \cos (2k_L X)$$

where $X$ is the atom center of mass position along the standing wave. Clearly, this interaction is one dimensional, as momentum exchanges between the atom and the radiation are always along the standing wave: The atom absorbs a photon in one of the propagating beams and emits it in the counterpropagating beam, leading to a quantized momentum exchange of $2\hbar k_L$ along the $X$ axis. An important point is that the optical potential is the dominant contribution to the potential amplitude scales as $\Omega^2/\Delta L$ whereas the spontaneous emission rate scales as $\Gamma \Omega^2/\Delta^2 L$. In the regime $|\Delta_L| \gg \Gamma$, the optical potential is the dominant contribution to the dynamics, with spontaneous emission events being rare. Moreover, one can reduce the spontaneous emission rate by increasing the detuning $\Delta$.

The conservative process is associated with a potential which can be adjusted at will by modifying e.g. the period $T_1$. As shown in the following, the most interesting physics takes place in the momentum. The scaling Eqs. (6) is such that $P = 2 \hbar k_L$ corresponds to $p = k$.

If the atom is cold enough that its typical momentum is comparable to $2 \hbar k_L$ (the “quantum” of momentum exchange), quantum effects can be observed in the system. Fortunately, magneto-optical traps produce atoms with a typical momentum of a few $\hbar k_L$. It is customary to measure the atomic momentum $P$ in units of $2 \hbar k_L$, i.e. measure $p$ in units of $k$. We thus will use:

$$p = \frac{p}{k} = \frac{P}{2 \hbar k_L}. \quad (8)$$

For $K \gtrsim 5$, the classical KR is fully chaotic, and the dynamics, although perfectly deterministic, behaves like a pseudo-random diffusive process known as “chaotic diffusion”. For this reason, $K$ is usually called “stochasticity parameter”. The existence of classical chaos can be seen by integrating the classical equations of motion corresponding to Eq. (6) over a period, which leads to the so-called “Standard Map”:

$$x_{t+1} - x_t = pt \quad (9)$$

$$pt_{t+1} - pt = K \sin x_{t+1}. \quad (10)$$

If the stochasticity parameter $K$ is large enough, $\sin x_t$ generates random numbers for successive $t$ values. The momentum then performs a random (though deterministic) walk and the kinetic energy (averaged over the initial conditions) increases linearly with time. If $\tau$ is short enough, the distribution changes from a Gaussian shape characteristic of a diffusive process to a localized, exponential shape.

$$\langle p^2 \rangle(t) = Dt,$$  

with $D \approx K^2/2k^2$ being the diffusion constant.

In the quantum case, a chaotic diffusion is observed for times shorter than a characteristic “localization time” $\tau_{\text{loc}} = D/2$, after which quantum interferences build-up in the system that eventually “freeze” the dynamics, suppressing the diffusion. The mean kinetic energy then tends to a constant $\langle p^2 \rangle(t \rightarrow \infty) \rightarrow 2\ell^2$ with $\ell \approx K^2/4k^2$. At the same time, the momentum distribution changes from a Gaussian shape characteristic of a diffusive process to a localized, exponential shape.

One can thus realize an atomic kicked rotor with $p = \hbar k$. The above Hamiltonian is associated with the Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H} \psi. \quad (7)$$

$k$ plays the crucial role of an effective Planck constant, which can be adjusted at will by modifying e.g. the period $T_1$. As shown in the following, the most interesting physics takes place in the momentum. The scaling Eqs. (6) is such that $P = 2 \hbar k_L$ corresponds to $p = k$. If the atom is cold enough that its typical momentum is comparable to $2 \hbar k_L$ (the “quantum” of momentum exchange), quantum effects can be observed in the system. Fortunately, magneto-optical traps produce atoms with a typical momentum of a few $\hbar k_L$. It is customary to measure the atomic momentum $P$ in units of $2 \hbar k_L$, i.e. measure $p$ in units of $k$. We thus will use:

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B. Equivalence with the 1D-Anderson model

Let us consider the KR quantum dynamics. From a stroboscopic point of view, the motion is determined by the evolution operator over one period:

\[ U = e^{-iK \cos x / \hbar} e^{-i p^2 / 2 \hbar}, \]

(12)

whose eigenstates form a basis set allowing to calculate the temporal evolution. These Floquet states \( |\phi \rangle \) are fully characterized by their quasienergy \( \omega \), defined modulo \( 2\pi \):

\[ U |\phi_\omega \rangle = e^{-i \omega} |\phi_\omega \rangle. \]

(13)

The Hamiltonian, Eq. (6), is \( 2\pi \)-periodic in position \( x \), and so is the evolution operator, Eq. (12). The Bloch theorem tells us that a Floquet eigenstate is a product of a periodic function of \( x \) by a plane wave \( e^{i\beta x} \) with \( 0 \leq \beta < 1 \), a constant, \( \beta \hbar \) being usually called the “quasi-momentum”. A trivial transformation shows that one can equivalently consider periodic functions of \( x \) governed by the Hamiltonian, Eq. (3), where \( p \) is replaced by \( p + \beta \hbar \).

In the following discussion, we will omit for simplicity the quasi-momentum, although it is straightforward to take it into account. Note that in all numerical simulations shown hereafter, we perform an averaging over the quasi-momentum, although it is straightforward to take it into account. Note that in all numerical simulations shown hereafter, we perform an averaging over the quasi-momentum, to follow the experimental conditions where an incoherent sum of all quasi-momenta is prepared.

At this point, contact with a 1D Anderson tight-binding model can be made by reformulating Eq. (13) for the Floquet states \[ |\phi \rangle \equiv |\phi_\omega \rangle \]. Firstly, we rewrite the (unitary) kick operator:

\[ e^{-iK \cos x / \hbar} = \frac{1 + iW(x)}{1 - iW(x)}, \]

(14)

with

\[ W(x) = \tan(K \cos x / 2\hbar). \]

(15)

The periodic function \( W(x) \) can be expanded in a Fourier series:

\[ W(x) = \sum_r W_r e^{i r x}. \]

(16)

Similarly, for the kinetic part, one gets:

\[ e^{-i(\rho^2 / 2\hbar - \omega)} = \frac{1 + iV}{1 - iV}, \]

(17)

with \( V \) diagonal in the momentum eigenbasis \( |m \rangle \equiv |p = km \rangle \). Secondly, we make the following expansion in the momentum eigenbasis:

\[ \frac{1}{1 - iW(x)} |\phi_\omega \rangle = \sum_m \Phi_m |m \rangle. \]

(18)

Then, the eigen-equation for the Floquet state can be rewritten:

\[ \epsilon_m \Phi_m + \sum_{r \neq 0} W_r \Phi_{m-r} = -W_0 \Phi_m, \]

(19)

with \( \epsilon_m = \tan \left[ \frac{1}{2} (\omega - m^2 \hbar^2) \right] \).

This is the equation for a tight-binding model with hopping elements \( W_r \) to the \( r \)-th neighbor, with eigenenergy \( W_0 \), and with on-site energy \( \epsilon_m \). The hopping elements are not restricted to nearest-neighbors, but they decrease exponentially with \( r \). In the original Anderson model, a random distribution is assigned to \( \epsilon_m \). Here, the sequence \( \epsilon_m \), although not satisfying the most stringent mathematical tests of randomness, is nevertheless pseudo-random. These two conditions are sufficient for the Anderson localization to take place. The hopping integrals \( W_r \) increase with the kick strength \( K \), which thus plays the role of a control parameter in the Anderson model (14). Note that if \( K \) is a rational multiple of \( 2\pi \), the \( \epsilon_m \) are periodic in \( m \). This leads to the quantum resonances of the kicked rotor, where the states are extended.

When \( K \) is incommensurate with \( 2\pi \), the Floquet states are found to be exponentially localized, and this property accounts for dynamical localization. As shown in Ref. (13), the localization length observed at long times for a wavepacket is essentially identical to the localization length of individual Floquet states.

Many references discuss the detailed correspondence between quantum behavior of this dynamical system and Anderson localization: In Ref. (4) an analogy between the KR and band random matrices was pointed out; the latter have been reduced to a 1D nonlinear \( \sigma \) model (3) similar to those employed in the localization theory (6).

In Ref. (3) the direct correspondence between the KR and the diffusive supersymmetric nonlinear \( \sigma \) model was demonstrated. A diagrammatic approach (3) to the dynamical localization in the Kicked Rotor was reported in Refs. (3).

C. The quasi-periodic Kicked Rotor and its analogy to the 3D-Anderson model

As the Anderson transition exists only in three (or more) dimensions, one must generalize the KR to obtain a system analogous to a 3D Anderson model.

Different generalizations of the KR have been theoretically considered as analogs of the 3D-Anderson model (3). Here we use the convenient three-incommensurate-frequencies generalization introduced in Refs. (3, 13):

\[ \mathcal{H}_{qp} = \frac{p^2}{2} + K(t) \cos x \sum_n \delta(t - n), \]

(20)

obtained simply by modulating the amplitude of the standing wave pulses with two new frequencies \( \omega_2 \) and \( \omega_3 \):

\[ K(t) = K [1 + \varepsilon \cos (\omega_2 t + \varphi_2) \cos (\omega_3 t + \varphi_3)]. \]

(21)

One can legitimately ask: where is the three dimensional aspect in the latter Hamiltonian? An answer can be given...
by drawing a formal analogy between the quasiperiodic kicked rotor and a 3D kicked rotor with an initial condition taken as "plane source" [21] in momentum space.

We start from the Hamiltonian of a 3D periodically kicked rotor:

$$\mathcal{H} = \frac{p_1^2}{2} + \omega_2 p_2 + \omega_3 p_3$$
$$+ K \cos x_1 \left[1 + \varepsilon \cos x_2 \cos x_3 \right] \sum_n \delta(t - n), \quad (22)$$

let us consider the evolution of a wavefunction $\Psi$ with the initial condition:

$$\Psi(x_1, x_2, x_3, t = 0) \equiv \Xi(x_1, t = 0) \delta(x_2 - \varphi_2) \delta(x_3 - \varphi_3) \quad (23)$$

The initial state being perfectly localized in $x_2$ and $x_3$, it is entirely delocalized in the conjugate momenta $p_2$ and $p_3$, and can thus be seen as a "plane source" [21] in momentum space.

From a stroboscopic point of view, the time-evolution of $\Psi$ is determined by the evolution operator over one period:

$$U = e^{-iK \cos x_1 (1 + \varepsilon \cos x_2 \cos x_3) / k} \times e^{-i(p_1^2/2 + \omega_2 p_2 + \omega_3 p_3) / k}. \quad (24)$$

It is then straightforward to see that the 3D-wave function $\Psi$ at time $t$ is related to its initial condition as:

$$\Psi(x_1, x_2, x_3, t) = U(t) \Psi(x_1, x_2, x_3, t = 0)$$
$$= \Xi(x_1, t) \delta(x_2 - \varphi_2 - \omega_2 t) \delta(x_3 - \varphi_3 - \omega_3 t), \quad (25)$$

with:

$$\Xi(x_1, t) \equiv \prod_{t' = 1}^{t} e^{-iK \cos x_1 (1 + \varepsilon \cos \varphi_2 + \omega_2 t') \cos \varphi_3 + \omega_3 t')} / k e^{-ip_1^2/2k} \Xi(x_1, t = 0). \quad (26)$$

On the other hand, consider now the evolution of an initial wave function $\psi(x, t = 0)$ with the Hamiltonian $H_{\text{qp}}$ of the quasiperiodic kicked rotor. It is also determined by an evolution operator from kick to kick, but now this evolution operator $U_{\text{qp}}(t; t - 1)$ depends on time, since the Hamiltonian $H_{\text{qp}}$, Eq. (22), is not time-periodic:

$$U_{\text{qp}}(t; t - 1) = e^{-iK \cos x (1 + \varepsilon \cos \varphi_2 + \omega_2 t) \cos \varphi_3 + \omega_3 t) / k e^{-ip_1^2/2k}. \quad (27)$$

The wave-function $\psi(t)$ at time $t$ is obtained by applying successively $U_{\text{qp}}(t'; t' - 1)$ for $t'$ from 1 to $t$:

$$\psi(x, t) = \prod_{t' = 1}^{t} U_{\text{qp}}(t'; t' - 1) \psi(x, t = 0). \quad (28)$$

From Eqs. (23), (27), (28) and (28), it follows that $\psi(x, t)$ and $\Xi(x_1, t)$ follow exactly the same evolution. Consequently, the dynamics of the quasiperiodic kicked rotor is strictly equivalent to that of a 3D kicked rotor with a plane source. Our experiment with the quasiperiodic kicked rotor can be seen as a localization experiment in a 3D disordered system, where localization is actually observed in the direction perpendicular to the plane source [21]. In other words, the situation is thus comparable to a transmission experiment where the sample is illuminated by a plane wave and the exponential localization is only measured along the wave vector direction. Therefore, the behavior of the wave function $\psi$ subjected to the quasiperiodic kicked rotor Hamiltonian $H_{\text{qp}}$, Eq. (22), depicts all the properties of the dynamics of the quantum 3D kicked rotor, Eq. (23).

The Hamiltonian $\mathcal{H}$, Eq. (22), is invariant under the following transformation, product of time-reversal with parity:

$$T : t \rightarrow -t, x \rightarrow -x, p \rightarrow p, \quad (29)$$

which is relevant for dynamical localization [15, 16]. The evolution of the states according to the Hamiltonian, Eq. (21), is governed by the operator $\mathcal{U}$, Eq. (24), which belongs to the Circular Orthogonal Ensemble class [14, 18], with the additional constraint at $t = 0$ Eq. (24). Of course, the transformation (29) amounts to changing $(\varphi_2, \varphi_3)$ to $(-\varphi_2, -\varphi_3)$ into the constraint (23), i.e. starting from a different wavefunction. On the other hand, from Eq. (24), one clearly sees that after $t$ steps, the constraint reads $\delta(x_2 - \varphi_2) \delta(x_3 - \varphi_3)$, with

$$\varphi_2 = \varphi_2 + \omega_2 t \quad \varphi_3 = \varphi_3 + \omega_3 t. \quad (30)$$

Since the frequencies $\omega_2$ and $\omega_3$ are incommensurate, the preceding equation immediately tells us that, along the time evolution, the constraint on the wavefunction can be arbitrary close to any phases $(\varphi_2', \varphi_3')$ [18]. This way, the time evolution results in an average over (almost) all possible phases, showing thus that the localization properties are independent of a particular choice $(\varphi_2, \varphi_3)$, but only depend on the operator $\mathcal{U}$. Therefore, the dynamical properties of the present quasiperiodic kicked rotor also belong to the orthogonal ensemble.

It should be noted that the 3D aspect comes from the presence of 3 frequencies in the dynamical system: the usual "momentum frequency" $k$ present in the standard
kicked rotor Eq. \(\text{[2]}\), and two additional time-frequencies \(\omega_2\) and \(\omega_3\). Thus, increasing the number of incommensurate frequencies allows one to tune the effective dimensionality of the system.

Let us now consider the conditions for the observation of Anderson localization with the quasiperiodic kicked rotor. As for the standard kicked rotor, the Floquet states of the time-periodic 3D Hamiltonian \(\mathcal{H}\), Eq. (\(22\)), can be mapped onto a 3D Anderson-like model:

\[
\epsilon_m \Phi_m + \sum_{r \neq 0} W_r \Phi_{m-r} = -W_0 \Phi_m ,
\]

where \(m \equiv (m_1, m_2, m_3)\) and \(r\) label sites in a 3D cubic lattice, the on-site energy \(\epsilon_m\) is

\[
\epsilon_m = \tan \left( \frac{1}{2} \left[ \omega - \left( m_1^2 + \omega_2 m_2 + \omega_3 m_3 \right) \right] \right) ,
\]

and the hopping amplitudes \(W_r\) are coefficients of a three-fold Fourier expansion of

\[
W(x_1, x_2, x_3) = \tan \left[ K \cos x_1 (1 + \varepsilon \cos x_2 \cos x_3) / 2 \right] .
\]

An obvious necessary condition for the observation of localization effects is that \(\epsilon_m\) is not periodic. This is achieved if \((k, \omega_2, \omega_3, \pi)\) are incommensurate. Of course, the presence of disorder in the diagonal energy \(\epsilon_m\) is crucial to observe Anderson localization. When \(k\) is incommensurate with \(2\pi\), due to the presence of a nonlinear dispersion in the \(m_1\) direction, the classical dynamics can become chaotic with diffusive spreading in all \(m\) directions \([2, 50]\). A typical numerical simulation is shown in Fig. 1: the classical motion is almost perfectly diffusive along the three \(p_i\) coordinates with a characteristic Gaussian shape in each direction. From Eq. (\(32\)) it is clear that hopping along the directions \("2"\) and \("3"\) is diminished by a factor \(\varepsilon\) compared to hopping along direction \"1". Not surprisingly, diffusion along \(p_1\) is slightly faster than along \(p_2\) and \(p_3\). The quasi-periodically kicked rotor is thus analogous to an anisotropic Anderson model \([2, 24, 25]\).

When those conditions are verified, localization effects as predicted for the 3D Anderson model are expected, namely either a diffusive or a localized regime. Localized states would be observed if the disorder strength is large as compared to the hopping. In the case of the model Eq. (\(32\)), the amplitude of the disorder is fixed, but the hopping amplitudes can be controlled by changing the stochasticity parameter \(K\) (and/or the modulation amplitude \(\varepsilon\)): \(W_r\) is easily seen to increase with \(K\). In other words, the larger \(K\), the smaller the disorder. One thus expects to observe diffusive regime for large stochasticity \(K\) or modulation amplitude (small disorder) and localized regime for small \(K\) or \(\varepsilon\) (large disorder).

III. EXPERIMENTAL REALIZATION WITH ATOMIC MATTER-WAVES

A. Experimental setup

Our experimental setup has been described in detail in previous publications \([54, 55, 56, 57]\) and was used in

Figure 1: (Color online) Classical diffusive motion for the 3D kicked rotor Eq. (\(22\)). The initial state is localized around the origin. After 1000 kicks, the classical momentum distribution (black and red curves) has the Gaussian shape characteristic of a diffusive motion. The blue and green curves are fits by a Gaussian which do not show any statistically significant deviation. The black (resp. red) curve is the momentum distribution along \(p_1\) (resp. \(p_2\)). The distribution along \(p_3\) is identical that along \(p_2\). The anisotropic diffusion happens because the hopping along the directions \"2" and \"3" is diminished by a factor \(\varepsilon\) compared to hopping along direction \"1\". Parameters are \(K = 10, k = 2.85, \varepsilon = 0.8, \omega_2/2\pi = \sqrt{5}, \omega_3/2\pi = \sqrt{13} \).
various investigations on the quasiperiodic kicked rotor \cite{58, 59, 60, 61}. Briefly, our experiments are performed with cesium atoms produced in a standard magnetooptical trap (MOT). A long Sisyphus-molasses phase (25 ms) allows us to obtain $10^9$ atoms at a measured temperature of 3.2 $\mu$K. The velocity distribution of the atoms is well modeled by an incoherent sum of plane waves forming a Gaussian of full width at half maximum (FWHM) equal to $8\hbar k_L$, which is much narrower than the expected localization length. The MOT beams and magnetic field are turned off and the sequence of kicks is applied to the atoms. The beam forming the standing wave passes through an acousto-optical modulator driven by RF pulse synthesizers, which generates the kicks at a typical frequency of $1/T_1 = 36$ kHz (which corresponds to $k = 2.89$), of duration $\tau = 900$ ns and with a raising time of 50 ns. The modulation is thus an almost perfect square, at the time scale of the atomic motion, and its duration and period can be set by a microcomputer. The beam is then injected in an optical fiber that brings it to the interaction region, and the standing wave is obtained simply by back-reflection of this beam. The standing wave has a typical power 160 mW, its profile intensity has a FWHM of 1.5 mm, and it is far off-resonant (7.3 GHz to red, or $1.4 \times 10^8$), in order to reduce spontaneous emission. The corresponding stochasticity parameter is $K \approx 15$.

A very interesting property of our system (as compared to solid-state systems) is that the wave function is accessible (or at least its square modulus). We measure the atomic velocity distribution by velocity-selective Raman stimulated transitions, which are sensitive to the atomic velocity via Doppler effect, allowing an optimal velocity resolution of about 2 mm/s. A Raman pulse detuned of $\delta_R$ with respect to the Raman resonance transfers the atoms in the velocity class $v = \delta_R/(2k_R) - v_R$ with $v_R = \hbar k_R/M$ ($k_R$ is the wave number of the Raman beams) from the $F_g = 4$ to the $F_g = 3$ ground-state hyperfine sublevel. A beam resonant with the $F_g = 4 \rightarrow F_e = 5$ transition is then applied to push the remaining atoms out of the interaction region. The $F_g = 3$ atoms are then optically pumped to the $F_g = 4$ sublevel and interact with a resonant probe beam: The absorption signal is thus proportional to the population of the $F_g = 4$ level, thus to the population of the selected velocity class. The whole sequence then starts again with a different value of the Raman detuning to probe a new velocity class, allowing a reconstruction of the velocity distribution \cite{56, 57}.

### B. Decoherence sources

Any quantum experiment must consider decoherence sources that destroy quantum interference effects (in our case, localization) reestablishing a diffusive dynamics. The most important sources of decoherence in our experiment are (i) atomic collisions, (ii) spontaneous emission, and (iii) the deviation of the standing wave from strict horizontality.

For an isolated system described by a single wavefunction, phase coherence between different positions is “perfect”. When the system is weakly coupled to an external bath, it cannot be any longer described by a single wavefunction; the most convenient description usually involves a density matrix $\rho$. Non-diagonal matrix elements of the type $\langle x|\rho|x'\rangle$ quantify the degree of coherence of the system between position $x$ and $x'$. As a general rule, the effect of the external bath is to make the non-diagonal elements of the density matrix to decay relatively rapidly, more rapidly than the diagonal elements: this is decoherence (not to be confused with dissipation) \cite{62}. Effects like Anderson localization are due to subtle destructive interference amongst various components of the wavefunction, which inhibit the classically allowed transport: they are thus very sensitive to decoherence. One usually quantify the strength of decoherence effects by defining a phase coherence time, the characteristic time over which the non-diagonal elements of the density matrix decay because of coupling to the external bath. In our case, the non-diagonal element of interest are between eigenstates $|p\rangle$ and $|p'\rangle$ located at a typical distance $|p-p'|$ comparable to the localization length in momentum space.

Localisation effects can be observed only for times shorter than the phase coherence time \cite{63}. Beyond the phase coherence time, interference effects are killed and classical-like diffusive dynamics sets in. In the following, we shall express the characteristic times of the decoherence processes (i), (ii) and (iii), as functions of the experimental parameters to show that they can be set large enough for localization effects to be observable.

In atom-atom collisions, the dominant effect is that of collisions between cold atoms, the density of the cloud being around 8 orders of magnitude larger than the density of the background hot gas. A cloud density of $10^{12}$ cm$^{-3}$ with a mean velocity 1 cm/s and a collision cross-section of $6 \times 10^{-11}$ cm$^2$ gives a collision rate of $\approx 60$ s$^{-1}$, or $1.6 \times 10^{-3}$ per kick; the collision phase coherence time is thus $\sim 600$ kicks.

In order to have a better idea of the decoherence effect induced by spontaneous emission, let us consider the temporal evolution of an initial plane-wave function: $\psi(p, t = 0) = \delta(p-p_0)$ evolving with the KR Hamiltonian Eq. (8). After dynamical localisation sets in, the momentum distribution ceases to expand because of destructive interference between the various components of the wavefunction. Spontaneous emission brings a random recoil to the atomic momentum which is not an integer multiple of $2\hbar k_L$. Thus, the quasi-momentum $\beta$ performs a random jump. As the phase factors involved in the free evolution depend on the quasi-momentum, the relative phases between interfering paths are scrambled, resulting in a new transient diffusive behavior for another duration of $\tau_{loc}$. DL is thus expected to be destroyed if spontaneous emission is regularly repeated. Note that a single spontaneous emission event completely breaks the...
The dynamics of an initial thermal state is simulated and the corresponding mean kinetic energy is plotted versus time. For angles larger than 0°, the slow drift of momentum induces a diffusive behavior clearly visible on the time-scale of the experiment.

We now discuss the conditions that must be satisfied in order to observe localization effects experimentally.

Firstly, the system must present some kind of disorder: As discussed in section 114, this means that $k$, $\omega_2$ and $\omega_1$ and $\pi$ must be incommensurate. This is achieved if we take $k = 2.89$, $\omega_2 = 2\pi\sqrt{5}$ and $\omega_1 = 2\pi\sqrt{13}$. A more detailed discussion concerning the choice of these parameters will be given in section 118.

Secondly, in order to observe dynamical localization effects instead of trivial classical localization, we must be in a regime where the classical system has no KAM barriers which can prevent the classical diffusive transport. For the standard, periodic KR, full chaos is obtained for $K \gtrsim 4$. In order to determine the corresponding thresholds, various values of the stochasticity parameter. The dynamics is found to be fully diffusive for $K \gtrsim 2$, a considerably smaller value than for the standard KR. In particular, no classical localization effects due to KAM barriers are observed for $K \gtrsim 2$. In any case, the experiments and the numerical simulations in the following are all performed for $K > 4$, where the classical dynamics is diffusive, see Fig. 3.

Thirdly, short enough pulses must be used that they can be considered as delta pulses. Numerical simulations of the quasiperiodic kicked rotor with a finite pulse duration $\tau = 0.9\mu s$ and a thermal initial momentum distribution show that less than 1% of the atoms are due to KAM barriers are observed for $K \gtrsim 2$. In any case, the experiments and the numerical simulations in the following are all performed for $K > 4$, where the classical dynamics is diffusive, see Fig. 3.
sensitive to the duration of the pulses. Only atoms in the tails of the momentum distribution have sufficiently large atomic velocity to move by a significant fraction of $\lambda_L$ during the pulse, thus feeling a smaller effective kick.

Fourthly, a sufficiently narrow initial state must be prepared in order to observe dynamical localization, i.e. the freezing of the initial diffusive expansion of the wavefunction into an exponentially localized state. A sufficient condition is that the initial width of the momentum distribution be smaller than the localization length. In our system, we have an initial momentum distribution of half-width $2\bar{k}$. This is comparable to the shortest localization length at the lowest $K = 4$ value, as experimentally proved, see inset of Fig. 5. A consequence is that, in this regime, the exponential shape of the wavefunction after dynamical localization is established is slightly rounded at the tip. For higher values – say $K > 5$ – the initial width of the atomic wavefunction can be safely neglected.

Finally, decoherence processes must be kept small during the experiment. The large detuning of the standing wave allows to keep the spontaneous emission rate very small, i.e. the corresponding phase-coherence large as compared to the duration of the experiment. A good control on the horizontality of the standing wave insures that gravity do not lead to a destruction of localization effects on the time-scale of the experiment.

IV. EXPERIMENTAL OBSERVATION OF THE DISORDER INDUCED METAL-INSULATOR TRANSITION

In a typical experimental run, we apply a sequence of kicks to the atomic cloud and measure its dynamics. In the localized regime, the evolution of its momentum distribution is “frozen” after the localization time (typically of the order of 12 kicks at low $K$) into an exponential curve $\exp\left(-|p|/\ell\right)$. In the diffusive regime, the initial Gaussian shape is preserved and the distribution gets broader as kicks are applied, corresponding to a linear increase of the average kinetic energy. Figure 4 shows the experimentally observed momentum distributions, an exponentially localized distribution for small $K$ and $\epsilon$ (blue curve), characteristic of dynamical localization, and a broad, Gaussian-shaped distribution for large $K$ and $\epsilon$ (red curve), characteristic of the diffusive regime.

Measuring the whole momentum distribution takes too much time: one must repeat the whole sequence (from the preparation of a new atom cloud up to the Raman

Figure 3: Classical chaotic diffusion for the quasiperiodic kicked rotor Eq. (22). The dynamics of an initial thermal distribution of classical particles is simulated and the corresponding mean kinetic energy is plotted versus time (number of kicks). The stochasticity parameter $K$ (the modulation amplitude $\epsilon$) varies linearly between 4 and 9 (0.1 and 0.8), following the experimental path, Fig. 6. The dashed line of slope 1 demonstrates the linear increase of $\langle p^2 \rangle$ vs. time $t$. No classical localization effects are observed. The chaotic diffusion is characteristic of the presence of pseudo-disorder in the quasiperiodic kicked rotor, leading to a pseudo-random walk in momentum space.

Figure 4: (Color online) Experimentally measured momentum distributions after 150 kicks, exponentially localized in the insulator region (blue) and Gaussian in the diffusive (metallic) region (red). (a) linear scale, (b) log scale. For both curves $\bar{k} = 2.89$, for the localized distribution (blue) $K = 5.0$ and $\epsilon = 0.24$, for the Gaussian distribution (red) $K = 9.0$ and $\epsilon = 0.8$. 

measurement of the velocity distribution) for each velocity class. Moreover, for each time step, a complete momentum distribution must be measured. Fortunately, it is sufficient, and much easier, to measure the population $\Pi_0(t)$ of the zero-momentum class as a function of time (number of kicks) and plot the quantity $\Pi_0^{-2}(t) \propto \langle p^2 \rangle(t)$. Clearly, it tends to a constant in the localized regime (blue lower curve corresponding to $K = 4$ and $\varepsilon = 0.1$) and increases linearly with time in the diffusive regime (red upper curve corresponding to $K = 9$ and $\varepsilon = 0.8$). The inset shows the behavior close to the localization time. $\bar{k} = 2.89$.

Figure 5: (Color online) Temporal dynamics of the quasi-periodic kicked rotor. We experimentally measure the population $\Pi_0(t)$ of the zero-momentum class as a function of time (number of kicks) and plot the quantity $\Pi_0^{-2}(t) \propto \langle p^2 \rangle(t)$. For each run, a value of $\Pi_0^{-2}(t)$ is recorded after a given number of kicks, then the measurement sequence starts again with the next number of kicks. We also record the background signal obtained by not applying the Raman detection sequence, and the total number of atoms in the cold-atom cloud. These signals are used to correct the experimental data from background signals and long-term drifts of the cloud population.

Figure 6: (Color online) Phase diagram of the quasiperiodic kicked rotor, from numerical simulations. The localized (insulator) region is shown in blue, the diffusive (metallic) region is shown in red. The experimental parameters are swept along the diagonal dash-dotted line.

Figure 5 shows the experimentally measured $\Pi_0^{-2}(t)$ in the localized and diffusive regimes. It clearly shows the initial diffusive phase and the freezing of the quantum dynamics in the localized regime (blue curve in Fig. 5). Along with the observation of an exponential localization of the wave-function in Fig. 3, this constitutes a clear-cut proof of the observation of dynamical localization. In the diffusive regime, $\Pi_0^{-2}(t)$ is seen to increase linearly with time (red curve in Fig. 5), corresponding to the Gaussian red curve in Fig. 4.

After having observed Anderson localization for strong effective disorder strength and diffusive transport for small effective disorder, the next step is to walk the way between these two regimes, and explore the phase transition expected (numerically) to take place along a critical line in the plane $(K, \varepsilon > 0)$ (Fig. 6). In order to confine the transition to a narrow range of parameters, we choose a path that cross the critical curve (Fig. 6) “at a right angle”: we thus vary simultaneously $K$ and $\varepsilon$ along a line going from $K = 4$, $\varepsilon = 0.1$ in the localized region to $K = 9$, $\varepsilon = 0.8$ in the diffusive region; the critical line is then crossed at $K = K_c = 6.6$.

A simple way to investigate the phase transition is the following [42]. In the localized regime, wait for a time longer than the localization time so that a localized frozen wave-function is observed, then measure its localization length. One can in such a way study the behavior of the localization length vs. disorder: at criticality, it should diverge as $\ell \sim (K - K_c)^{-\nu}$. This would give the critical stochasticity parameter $K_c$ and the critical exponent $\nu$. However, we cannot proceed that way in our case, because when one approaches the critical point from the insulator side, the localization time diverges as $\tau_l \sim \ell^3 \sim (K - K_c)^{-3\nu}$ in three dimensions (see below). In our system, a localized momentum distribution would be observable in the vicinity of the transition only for prohibitively large numbers of kicks, which are, in practice, limited to 150, essentially because of decoherence effects and because the free fall of the atom cloud takes it out of the standing wave. Consequently, it is vain to investigate experimentally the Anderson transition only
from static properties such as the divergence of the localization length at criticality, which could be obtained only for \( t \gg \tau_{\text{loc}} \). Fortunately, there is another way to observe the Anderson transition, which we shall present in the following sections.

V. CHARACTERIZATION OF THE ANDERSON PHASE-TRANSITION

Finite-time effects act as finite-size effects do on finite-size samples subjected to phase-transitions. Numerical simulations of the Anderson transition on the standard 3D-Anderson model are necessarily performed on finite-size samples of finite size \( L \). In the vicinity of the transition, the localization length \( \ell \) [see Eq. (33)] diverges and thus can greatly exceed \( L \). In this regime, \( L \) acts as an upper bound for the effectively observed localization length \( \ell_L \). This smooths the transition, no divergence of the localization length can be directly observed on a finite-size sample. In order to overcome this limitation, a powerful real-space renormalization method, called finite-size scaling [2], [3], was introduced. This method is based on a single parameter scaling hypothesis [8] and allows to extrapolate from the scaling behavior of \( \ell_L \) versus \( L \) the asymptotic value of the localization length \( \ell \) corresponding to \( L \to \infty \). We can generalize static scaling laws to cover our time-dependent problem (see [2] for a similar approach in percolation theory). The single parameter scaling theory [8], successfully used for the standard 3D Anderson model [1], [4], can be applied to analyze our experimental and numerical data, and especially to determine the critical properties of the Anderson transition that we observe, i.e. the critical exponents.

A. Scaling law at finite time

Knowing the asymptotic behavior when \( t \to \infty \) is not enough, an additional time-dependent property is needed, too, which we shall investigate now. For \( K \) far above \( K_c \) one observes normal diffusion, \( \langle p^2 \rangle \propto t \), whereas for \( K \) far below \( K_c \), the quantum dynamics freezes, at sufficiently long times. Following the standard analysis of the Anderson transition, we make the hypothesis that the transition that we observe for the quasi-periodically kicked rotor follows a one-parameter scaling law [2] (the validity of this scaling hypothesis will of course be checked at the end of the analysis). At the critical point, a third kind of dynamics, namely anomalous diffusion, with \( \langle p^2 \rangle \sim t^k \) \( k \neq 1 \), is expected. Let us consider in greater detail the behavior very close to \( K_c \) where these three different laws merge.

In the localized regime, for sufficiently long times, the behavior depends only on the localization length which diverges as \( K \) goes to \( K_c \):

\[
\langle p^2 \rangle \sim \ell^2 \sim (K_c - K)^{-2\nu} \quad \text{(for } K < K_c \text{)} , \tag{36}
\]

with \( \nu \) the localization length critical exponent.

For \( K > K_c \), the mean kinetic energy increases linearly with time, and the proportionality constant is the diffusion coefficient \( D(K) \). For \( K < K_c \), \( \langle p^2 \rangle \) is bounded by Eq. (34) and there is no diffusion. Thus \( D(K) \) vanishes below \( K_c \). A different critical exponent \( s \) is used to describe how \( D(K) \) goes to zero above threshold:

\[
D(K) \sim (K - K_c)^s \quad \text{(for } K > K_c \text{)} . \tag{37}
\]

We shall now find a single expression presenting these two limit behaviors and also displaying anomalous diffusion at the critical point. We note that, according to the theory of phase-transitions in finite-size samples, a scaling can be applied to \( \langle p^2 \rangle \) depending on the two variables \( 1/t \) and \( (K - K_c) \), both going to zero. We thus use the general scaling law:

\[
\langle p^2 \rangle = t^{k_1} F \left[ (K - K_c) t^{k_2} \right] , \tag{38}
\]

with \( F(x) \) an unknown scaling function. The exponents \( k_1 \) and \( k_2 \) can be determined as follows.

In the diffusive regime, for long enough times, we must recover the diffusion law with \( D \sim (K - K_c)^s \) [Eq. (35)]: hence, for \( x \gg 1 \), the scaling function \( F(x) \) should scale as \( x^s \):

\[
\langle p^2 \rangle \sim t^{k_1 + s k_2} (K - K_c)^s . \tag{39}
\]

As in the diffusive regime, \( \langle p^2 \rangle \sim t \), we must have \( k_1 + s k_2 = 1 \).

In the localized regime, on the other hand, one must recover \( \langle p^2 \rangle \sim (K_c - K)^{-2\nu} \) [Eq. (34)] for sufficiently long times. Thus, for \( x \to -\infty \), \( F(x) \to (-x)^{-2\nu} \), giving:

\[
\langle p^2 \rangle = t^{k_1 - 2\nu k_2} (K_c - K)^{-2\nu} \tag{40}
\]

which is compatible with Eq. (38) only if \( k_1 = 2\nu k_2 \).

These two relations determine \( k_1 \) and \( k_2 \) in terms of the physically more meaningful critical exponents \( s \) and \( \nu \).

\[
k_1 = \frac{2\nu}{s + 2\nu} , \quad k_2 = \frac{1}{s + 2\nu} .
\]

In the standard Anderson model, the critical exponents are related by Wegner’s scaling law [1]:

\[
s = (d - 2)\nu \ , \tag{41}
\]

with \( d \) being the dimensionality of the system. For our system, one obtains:

\[
k_1 = 2/3 ; \quad k_2 = 1/3\nu . \tag{42}
\]

We therefore expect at the critical point anomalous diffusion with \( \langle p^2 \rangle = t^{k_1} F(0) \sim t^{2/3} \). We present in the next sub-section a numerical and experimental validation of this prediction.
plies that the Wegner’s scaling law slope is very close to the theoretical prediction $2/3$ im-
also that the fact that the numerically measured critical phase transition and not a smooth cross-over. Note
that the critical behavior changes is related to the critical exponent of the
behavior for large enough time. The rate at which the
point since the localized dynamics close to criticality will
ized case for example, for which a characteristic time can
ior remains the same at all times (opposite to the local-
point [16]. It is a fixed point because the critical behav-
ual behavior shows the existence of a fixed hyperbolic point [16]. It is a fixed point because the critical behavior
remains the same at all times (opposite to the localized
case for example, for which a characteristic time can be
defined, the localization time), and it is a hyperbolic
point since the localized dynamics close to criticality will
follow only for a finite time the anomalous diffusion with
exponent $2/3$ and will progressively tend to a localized behavior for large enough time. The rate at which the
behavior changes is related to the critical exponent of the
phase transition $\nu$.

An efficient way to observe the departing of the dynamics from the critical anomalous diffusion is to consider the

\begin{align*}
\Pi_0^{-2}(t) &\sim \langle p^2 \rangle
\end{align*}

Figure 7: (Color online) Numerically simulated time-
evolution of $\langle p^2 \rangle$ for the quasiperiodic kicked rotor. At the
critical point $K = K_c \approx 6.4$ (purple middle curve), anomalous
diffusion $\langle p^2 \rangle \sim t^{2/3}$ is clearly observed, as expected from the-
torical arguments (cf. text). The log-log plot of the critical
curve is very well fitted by a straight line of slope $0.664$ (black
dashed line). In the vicinity of the transition, the dynamics
departs from the anomalous diffusion to tend gradually ei-
ther to a diffusive dynamics (red upper curves corresponding
to $K > K_c$ bending upwards for large $t$) or to a localized
dynamics (blue lower curves corresponding to $K < K_c$ bend-
ing downwards for large $t$). Other parameters are $k = 2.85$,
$\omega_2 = 2\pi\sqrt{3}$ and $\omega_3 = 2\pi\sqrt{13}$.

\begin{align*}
\Pi_0^{-2} &\sim \langle p^2 \rangle
\end{align*}

Figure 8: (Color online) Experimentally observed time-
evolution of $\Pi_0^{-2} \sim \langle p^2 \rangle$ for the quasiperiodic kicked ro-
tor. Close to the critical point $K = K_c \approx 6.4$ (purple mid-
dle curve), anomalous diffusion $\Pi_0^{-2}(t) \sim t^{2/3}$ is clearly ob-
served. (a) The critical anomalous curve is well fitted by
$\Pi_0^{-2}(t) = A + Bt^{2/3}$ (black dashed line). The red upper curve
evidence the far-above-criticality diffusive behavior ($K = 9.0$) and
and the blue lower curve the far-below-criticality ($K = 4.0$) localized behavior. (b) These experimental results show a
clear algebraic behavior, with exponent $\approx 0$ (blue lower curve,
localized regime), $2/3$ (purple middle curve, critical regime) and 1 (red upper curve, diffusive regime), slightly perturbed
by decoherence processes responsible for the residual increase
in the localized regime. Other parameters are the same as in Fig. 7.

\begin{align*}
\Pi_0^{-2} &\sim \langle p^2 \rangle
\end{align*}

B. Critical anomalous diffusion

We verified numerically that the critical behavior, cor-
responding to the anomalous diffusion in $t^{2/3}$ is observed
up to a very large number of kicks ($t = 10^6$). The (pur-
ple) middle curve of Fig. 7 displays the time-evolution of
$\langle p^2 \rangle$ from numerical simulations for the stochastici-
ty parameter $K = 6.4$. Anomalous diffusion $\langle p^2 \rangle \sim t^{2/3}$ is
clearly seen from the log-log plot over 4 orders of magni-
tude, which is very well fitted by a straight line of slope $0.664$.
Other curves, for different $K$, tend at long times to bend either horizontally (below $K_c$) or towards slope
unity (above $K_c$). This is a clear proof that we face here a
true phase transition and not a smooth cross-over. Note
also that the fact that the numerically measured critical
slope is very close to the theoretical prediction $2/3$ im-
plies that the Wegner’s scaling law $s = \nu$ is valid at an
accuracy better than 1%.

Fig. 7 displays the experimental evolution of $\Pi_0^{-2}(t) \sim
\langle p^2 \rangle$ versus time. The critical curve (middle curve corre-
sponding to $K \approx 6.4$) in purple is well fitted by the rela-
tion $\Pi_0^{-2}(t) = A + Bt^{2/3}$, see Fig. 7a. Fig. 7b displays in
log-log scale the experimental data $\Pi_0^{-2}(t)$ vs $t$. The alge-
braic dependence (with exponent $2/3$) of the critical dy-
namics is again clearly visible. In all plots in Figs. 7 the
red upper curves evidence the above-criticality diffusive
behavior and the blue lower curves the below-criticality
localized behavior.

From renormalization theory, we know that the cr-

quantity
\[ \Lambda = \langle p^2 \rangle t^{2/3}, \] (43)

or, equivalently, in the case of the experimental data:
\[ \Lambda_0 = \frac{1}{\Pi^2(t) t^{2/3}}, \] (44)
as a function of time. This is illustrated in Fig. 9, which displays \( \ln \Lambda \) vs. \( t \). The critical behavior can be easily pin-pointed: The corresponding (purple) curve has a zero slope, as the quantity \( \Lambda \) is (asymptotically) constant at criticality. In the diffusive regime, the quantity \( \Lambda \) increases with time (red curves), whereas it decreases in the localized regime (blue curves). In the far localized regime, we observe an algebraic dependence \( \Lambda(t) \sim t^{-2/3} \) as \( \langle p^2 \rangle(t) = 2t^2 \) for \( t > t_{oc} \). In the far diffusive regime, the algebraic dependence is \( \Lambda(t) \sim t^{1/3} \) as \( \langle p^2 \rangle(t) \sim t \).

The above numerical and experimental observations validate the theoretical prediction for the critical behavior: \( \langle p^2 \rangle \sim t^{2/3} \). Such critical behavior for the quasi-periodic kicked rotor was predicted using another scaling approach and numerically verified in [4]. It was also numerically observed for a spatially-3D kicked rotor [5] and in the standard 3D Anderson model [6], and put on firm theoretical grounds in [7].

C. Finite-time scaling

We shall now explain the procedure used to verify the scaling of our numerical and experimental data according to the law deduced above:
\[ \langle p^2 \rangle = t^{2/3} F \left[ (K - K_c) t^{1/3} \right], \] (45)

Our method is similar to the finite-size scaling procedure used by MacKinnon and Kramer [8] [9], and Pichard and Sarma [10] to numerically study the Anderson transition on finite-samples of the 3D Anderson model, but we apply it here to the temporal behavior of the data, thus the name “finite-time scaling”.

We assume the quantity \( \Lambda(K,t) = \langle p^2 \rangle t^{-2/3} \) to be an arbitrary function
\[ \Lambda(K,t) = f \left( \xi(K) t^{-1/3} \right), \] (46)

where the scaling parameter \( \xi(K) \) depends only on \( K \), which is the parameter appearing in the one-parameter scaling hypothesis. This scaling assumption is less restrictive than Eq. (45) since no assumption on the dependence of \( \xi \) on \( K \) is made. We must thus show that the resulting scaling parameter \( \xi(K) \) is compatible with Eq. (45).

In the left part of Fig. 10, we display plots of \( \ln \Lambda(K,t) \) vs. \( \ln t^{-1/3} \) for different values of \( K \). For most values of \( \ln \Lambda \), several values of \( \ln t^{-1/3} \) correspond to the same \( K \) value. The only way to conform with the condition (42) is to shift each curve horizontally by a different quantity \( \ln \xi(K) \) such that curves corresponding to different values of \( K \) overlap. This can be achieved by minimizing the variance of the values \( \ln \xi(K) t^{-1/3} \) corresponding to each value of \( \ln \Lambda \). The function \( \xi(K) \) can be determined by applying a least square fit to the data.

This minimization procedure does not allow one to compute the absolute scale of \( \xi(K) \), as the shifting procedure (see Fig. 10) is invariant under a global shift of the origin. We can thus set the scaling parameter \( \xi(K) \) to be equal to the localization length in the strongly localized regime where the duration of the experiment is much
Figure 11: (Color online) Finite-time scaling applied to the results of numerical simulations of the quasiperiodic kicked rotor. The time-evolution of $\langle p^2 \rangle$ is computed as a function of time, from $30$ to $10^4$ kicks, for several values of $K$ between $K = 4$ and $K = 9$. The finite-time scaling procedure allows us to determine both the scaling function $f(a)$, clearly displaying an upper branch (red) associated with the diffusive regime, and a lower branch (blue) associated with the localized regime. The dependence of the scaling parameter $\xi$ on $K$ displays a divergent behavior around the critical point $K_c = 6.4$, which is the signature of the Anderson phase transition. The dashed line is a fit using Eq. (48). The resulting critical exponent is $\nu = 1.6 \pm 0.1$. Other parameters as in Fig. 11.

Figure 12: (Color online) Finite-time scaling applied to the experimental results (from $30$ to $150$ kicks). The scaling procedure is the same as in Fig. 11. (a) The fact that all experimental points lie on a single curve, with a diffusive (red) and a localized (blue) branch, is a proof of the relevance of the one-parameter scaling hypothesis. (b) The maximum displayed by the scaling parameter $\xi$ in the vicinity of $K_c = 6.4$ is a clear-cut proof of the Anderson transition. Phase-breaking mechanisms (cf. text) smooth the divergence at the critical point. When these effects are properly taken into account, one obtains a critical exponent $\nu = 1.4 \pm 0.3$, [the dashed line is a fit with Eq. (48)] compatible with the numerical results. This plot corresponds to 48 experimental runs. Other parameters as in Fig. 11.

The scaling parameter $\xi(K)$ is plotted in Figs. 11(b) and 12(b), for numerical and experimental data respectively. As stated above, this parameter can be identified to the localization length in the localized regime. In the diffusive regime, it scales as the inverse of the diffusive constant. Indeed, in the far diffusive regime one has $\langle p^2 \rangle = D(K)t$, which implies

$$\Lambda(K,t) = f\left(\xi(K)t^{-1/3}\right) = 2\ell^2 t^{-2/3},$$

which implies, if we identify the scaling parameter with the localization length, $\xi(K) \sim t$, $f(x) = 2x^2$.

Figures 11(a) and 12(a) show the results of the fitting procedure applied to the numerical data and to the experimental data, respectively. In both cases, the procedure groups all points in a single curve, within the accuracy of the data. The resulting curve clearly displays two branches, a diffusive (red) and a localized (blue) one, with the critical point being at the tip joining the two branches; this is a signature of the Anderson transition. It also justifies a posteriori the scaling hypothesis Eq. (48) used for analyzing the data.

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$$\Lambda(K,t) = D(K)t^{1/3}$$

$$f(x) = x^{-1/3},$$

so that $\xi(K) = 1/D(K)$ in the far diffusive regime.

One notes that $\xi(K)$ increases rapidly in the vicinity of

$$\ln \Lambda \quad \ln(\xi/t^{1/3}) \quad \ln(\xi/t^{1/3})$$

$$\xi \quad \xi \quad 2000$$

$$0 \quad 0 \quad 0$$

$$2 \quad 2 \quad 2$$

$$4 \quad 4 \quad 4$$

$$6 \quad 6 \quad 6$$

$$K \quad K \quad K$$

$$\xi(K,t) = f(\xi(K)t^{-1/3}) = 2\ell^2 t^{-2/3},$$

which implies, if we identify the scaling parameter with the localization length, $\xi(K) \sim t$, $f(x) = 2x^2$.

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$$K \quad K \quad K$$

$$\xi(K,t) = f(\xi(K)t^{-1/3}) = 2\ell^2 t^{-2/3},$$

which implies, if we identify the scaling parameter with the localization length, $\xi(K) \sim t$, $f(x) = 2x^2$.

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$$K \quad K \quad K$$

$$\xi(K,t) = f(\xi(K)t^{-1/3}) = 2\ell^2 t^{-2/3},$$

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One notes that $\xi(K)$ increases rapidly in the vicinity of

$$\ln \Lambda \quad \ln(\xi/t^{1/3}) \quad \ln(\xi/t^{1/3})$$

$$\xi \quad \xi \quad 2000$$

$$0 \quad 0 \quad 0$$

$$2 \quad 2 \quad 2$$

$$4 \quad 4 \quad 4$$

$$6 \quad 6 \quad 6$$

$$K \quad K \quad K$$

$$\xi(K,t) = f(\xi(K)t^{-1/3}) = 2\ell^2 t^{-2/3},$$

which implies, if we identify the scaling parameter with the localization length, $\xi(K) \sim t$, $f(x) = 2x^2$. Larger than the localization time, and $\langle p^2 \rangle$ converges to its asymptotic value $2\ell^2$. Thus

$$\Lambda(K,t) = f\left(\xi(K)t^{-1/3}\right) = 2\ell^2 t^{-2/3},$$

which implies, if we identify the scaling parameter with the localization length, $\xi(K) \sim t$, $f(x) = 2x^2$.
the critical value $K_c$, on both sides of the transition. This corresponds to a divergence of the localization length and to a vanishing of the diffusion constant at criticality (in practice smoothed by decoherence, see below). This constitutes a clear experimental evidence of the Anderson phase transition.

D. Experimental determination of the critical exponent

The behavior of $\xi(K)$ gives a fundamental information about the transition, namely the value of the localization length critical exponent $\nu$. There is a discrepancy in the literature between the theoretical predictions $\nu = 1 \pm 0.2$ [3], and the result of numerical simulations $\nu = 1.57 \pm 0.02$ [2], which stresses even more the importance of an experimental determination. In this section, we present the first unambiguous experimental determination of the critical exponent of the Anderson transition in 3 dimensions.

The finite-time scaling procedure allows us to extract from finite-time experimental data the localization length $\ell$ (corresponding to $t \rightarrow \infty$), which is the order parameter of the Anderson transition. It is given by the scaling parameter $\xi(K)$ and predicted to diverge at criticality with the power law

$$\ell \sim |K - K_c|^{-\nu} . \quad (47)$$

We thus expect that the singularity in $\xi(K)$ can be described by Eq. (17), and to be able to extract the value of the critical exponent $\nu$. This is of primary importance, as there is presently no unambiguous accurate experimental determination of $\nu$ for non-interacting particles, and there is a discrepancy in the literature between the theoretical predictions $\nu = 1 \pm 0.2$ [3], and the result of numerical simulations $\nu = 1.57 \pm 0.02$ [2].

When the slope of ln $\Lambda$ vs ln $t^{-1/3}$ is small, as it is near the critical point, the scaling procedure tends to round off the singularity in $\xi(K)$. Moreover, decoherence in the experiment produces a cut-off the algebraic divergence. If the system has a finite phase-coherence time $\tau_\varphi$, a new characteristic length $\xi_\varphi = [D\tau_\varphi]^{1/2}$ appears in the problem, which sets an upper bound for the observable localization length $\ell$ and thus smoothes its divergence at criticality. In practice, we model such smoothing by introducing a small cut-off on the divergence of $\xi(K)$, which takes into account both the finite-time scaling procedure itself and decoherence effects:

$$\frac{1}{\xi(K)} = \alpha|K - K_c|^{\nu} + \beta . \quad (48)$$

The experimental data have been fitted with this formula [23] [dashed curve in Fig. 13(b)], which gives $K_c \approx 6.4 \pm 0.2$, and a critical exponent $\nu = 1.4 \pm 0.3$. In order to compare these results to the ideal case of the perfectly coherent quasiperiodic kicked rotor, Eq. (19), we also fitted the curve in Fig. 13(b) with Eq. (19); in this case, the cutoff $\beta$ accounts for limitations of the finite-time scaling procedure. The model Eq. (19) fits very well to the numerical data [dashed curve in Fig. 1(b)] and gives the critical stochasticity $K_c \approx 6.4 \pm 0.1$ and the critical exponent $\nu = 1.6 \pm 0.2$. The good agreement between the numerical simulations and the experimental results proves that spurious effects (such as decoherence) are well under control. Moreover, the experimental value we obtained $\nu = 1.4 \pm 0.3$ is compatible with the value found in numerical simulations of the true random 3D Anderson model [11, 12]. We emphasize that there are no adjustable parameters in our procedure, all parameters are determined using the atoms themselves as probes.

VI. UNIVERSALITY OF THE ANDERSON TRANSITION

At this point, a reasonable question is: Does the quasiperiodic kicked rotor exhibits the same critical phenomena – i.e. belongs to the same (orthogonal) universality class – as the true 3D-Anderson model? Can this simple three-frequency dynamical system exactly mimic the critical behavior of 3D disordered electronic conductors? In this section, we show that the answer is positive: The 3-frequency quasiperiodic kicked rotor and the true 3D-Anderson model belong to the same universality class. This is a strong claim that relies on a very precise determination of the critical exponent $\nu$. The accuracy of this determination is comparable to that obtained in the most sophisticated numerical studies of the 3D Anderson model [13, 14]. Within numerical uncertainties, the critical exponent is found to be universal and identical to the one found for the 3D-Anderson model [12]. The technical details of the calculation have already been reported in [15]. We here just discuss the essential ingredients proving universality.

A. Precise estimate of the critical exponent

Reliably distinguishing the different universality classes of the Anderson transition requires a very precise determination of the critical exponent; for instance, the value $\nu = 1.43 \pm 0.04$ for the unitary symmetry class is close to the one for the orthogonal symmetry class $\nu = 1.57 \pm 0.02$.

The main uncertainty in our experimental determination of the critical exponent is due to statistical errors on $\Pi_0$ and to the limited duration of the experiment. However, numerical simulations are not limited to 150 kicks but can be ran for several thousands of kicks, and statistical uncertainties on $\langle p^2 \rangle$ can be sharply reduced by averaging over initial conditions. The numerical inaccuracy in the finite-time scaling determination of $\nu$ from the numerical data is thus mainly due to the procedure failing to reproduce the singular behavior of the scaling
can be determined from the slope of $\ln \Lambda$ at $K$.

The interesting feature of Eq. (50) is that the critical exponent $\nu$ can be evaluated by fitting directly the raw data $\ln \Lambda(K,t)$ vs. $K$. All curves intersect, to a very good approximation, at a single point $(K_c, \ln \Lambda_c \approx 1.6)$. This multiple crossing indicates the occurrence of the metal-insulator transition. Small deviations from crossing are due to the existence of an irrelevant scaling parameter at finite time and residual correlations in the disordered potential (see text). $K$ and $\epsilon$ are swept along the straight line drawn in Fig. 6. Parameters are $K = 2.85$, $\omega_2 = 2\pi \sqrt{t}$, $\omega_3 = 2\pi \sqrt{13}$.

How can one improve the accuracy on the determination of the critical exponent $\nu$? This can be achieved by fitting directly the raw data $\ln \Lambda(K,t)$ vs. $K$. The starting point of our analysis is the behavior of the scaling function $\mathcal{F} \equiv \ln F$ in the vicinity of the critical point:

$$\ln \Lambda = \ln F \left[ (K - K_c) t^{1/3 \nu} \right] = \mathcal{F} \left[ (K - K_c) t^{1/3 \nu} \right].$$

As $\ln \Lambda(K,t)$ is an analytical function for finite $t$ (Fig. 13), the scaling function $\mathcal{F}$ can be expanded around $K_c$:

$$\ln \Lambda(t) \simeq \ln \Lambda_c + (K - K_c) t^{1/3 \nu} \mathcal{F}_1 + \ldots,$$

where $\ln \Lambda_c \equiv \mathcal{F}[0]$ and $\mathcal{F}_1 = d\mathcal{F}(x)/dx|_{x=0}$.

A remarkable feature of Eq. (51) is that when $\ln \Lambda$ is plotted against $K$, the curves for different times $t$ should intersect at a common point $(K_c, \ln \Lambda_c)$, and this crossing, indicates the occurrence of the metal-insulator transition. This is clearly visible in Fig. 14. Another interesting feature of Eq. (51) is that the critical exponent $\nu$ can be determined from the slope of $\ln \Lambda$ at $K_c$:

$$(\ln \Lambda)'(K_c,t) \equiv \frac{\partial \ln \Lambda(K,t)}{\partial K} \bigg|_{K=K_c} \propto t^{1/3 \nu}.$$  (51)

This is the simplest procedure to evaluate the critical exponent: $(\ln \Lambda)'(K_c,t)$ is evaluated by linear regression of $\ln \Lambda$ vs $K$ in a small interval near $K_c$, giving an exponent $\nu \approx 1.61 \pm 0.10$ (see Fig. 14). The linear regime has nevertheless very small size: $(K - K_c)^{1/3 \nu} \ll 1$, and neglecting non-linear corrections lead to systematic errors on the estimation of $\nu$. This is why the error $\pm 0.1$ refers to systematic errors and not to the uncertainty on the fitting parameters, which is much smaller as easily seen in Fig. 14.

In practice, there are small systematic deviations from Eq. (50). Such deviations can have different sources:

- the presence of an irrelevant scaling variable, that is when, in addition to $(K - K_c)^{1/3 \nu}$, $(\ln \Lambda)$ depends also on another scaling variable which vanishes in the limit $t \to \infty$, but still plays a role at short time;
- non-linear dependence of the scaling variables in the stochasticity parameter $K$;
- resonances due to the periods being well approximated by a ratio of small integers.

The latter one is specific to our three-frequency dynamical system, but the former two also play an important role in the standard Anderson model [11, 12]. These small corrections can be taken into account following the method devised in [12] for the Anderson model – by slightly modifying the basic scaling law, Eq. (50), in two ways: introduce a non-linear of the argument of the $\mathcal{F}$ function with $K - K_c$ in Eq. (51) on the one hand, and allow to subtract irrelevant scaling corrections to $(\ln \Lambda)$ on the other hand. To minimize the the effect of resonances, we only retain data for sufficiently long times and average over different initial conditions, i.e. different quasi-momenta $\beta$ and phases $\varphi_2$ and $\varphi_3$.

We computed $\ln \Lambda$ for times up to $t = 10^6$ kicks with an accuracy of 0.15%, for which more than 1000 initial conditions are required. We analyze data over the full range of times $t \in [10^3, 10^6]$.
by minimizing the deviation

$$\chi^2 = \sum_{K,t} \left[ \frac{\ln \Lambda(K,t) - \mathcal{F}(K,t)}{\sigma(K,t)} \right]^2,$$

(52)

where $\sigma(K,t)$ is the numerical uncertainty (one standard deviation) of the computed quantities $\ln \Lambda(K,t)$.

In Fig. 15 we plot the scaling function corrected from the irrelevant scaling variable, as a function of $\xi(K,t)^{1/3}$. All data collapse almost perfectly on the scaling function deduced from the model.

Since the measurement errors in the data introduce some uncertainty in the determination of the fitted parameters, the confidence intervals for the fitted parameters were estimated using the bootstrap method which yields Monte-Carlo estimates of the errors in the fitted parameters [77]. The fitted parameters presented below are given with the corresponding 68.2% confidence intervals (standard errors).

A key property of the Anderson transition is that the critical behavior can be described [74, 75] in a framework of universality classes. This means that the critical behavior should not be sensitive to the microscopic details but should depend only on the underlying symmetries of the system (e.g., time-reversal symmetry). Irrelevant parameters become negligible for sufficiently long times/large system size, whereas the relevant parameter behavior is universal. This brings the universality of the critical exponents. When considering a system with pseudo-random disorder such as the quasi-periodic kicked rotor, one could ask whether the universality is broken or not due to correlations in the disorder potential. To answer the question, we changed some parameters that govern the microscopic details of the disorder potential of the quasi-periodic kicked rotor, namely $k$, $\omega_2$ and $\omega_3$ and the path along which we cross the transition.

The computer time required in those sophisticated numerical studies is very long. Therefore we chose to restrict ourselves to the detailed study of only four different cases, see Table I.

The estimated critical parameters and their confidence intervals are given in Table I. A typical scaling function is drawn in Fig. 13.

The most important point to be drawn from Table I is that the estimates of the exponent $\nu$ for the four different sets are in almost perfect agreement with each other and with the estimate of $\nu$ based on numerical studies of the true random Anderson model $\nu = 1.57 \pm 0.02$ of the orthogonal symmetry class [44]. Note also that in the case of the quasiperiodic kicked rotor, the critical stochasticity $K_c$ and $\ln \Lambda_c$ depend on: (i) the anisotropy governed by the parameter $\varepsilon$ and (ii) $k$, $\omega_2$ and $\omega_3$. The dependence (i) of the critical disorder and critical $\ln \Lambda$ on anisotropy is a typical feature of the Anderson transition in anisotropic solids [51, 52, 53]. The quasiperiodic kicked rotor may indeed be seen to correspond to a model of random chains (coupled by terms scaling like $\varepsilon$ in the two transverse directions) considered in [51], see Eq. (43). The dependence (ii) follows from the relation between the initial “classical” diffusion constant (see section II) and the parameters $k$, $\omega_2$ and $\omega_3$. Such a dependence was observed both numerically and experimentally for the stan-

<table>
<thead>
<tr>
<th>$K_c$</th>
<th>$\ln \Lambda_c$</th>
<th>$\nu$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$6.36 \pm 0.02$</td>
<td>$1.60 \pm 0.04$</td>
<td>$1.58 \pm 0.01$</td>
</tr>
<tr>
<td>$B$</td>
<td>$7.33 \pm 0.03$</td>
<td>$1.08 \pm 0.09$</td>
<td>$1.60 \pm 0.03$</td>
</tr>
<tr>
<td>$C$</td>
<td>$7.00 \pm 0.03$</td>
<td>$1.19 \pm 0.15$</td>
<td>$1.60 \pm 0.02$</td>
</tr>
<tr>
<td>$D$</td>
<td>$8.09 \pm 0.03$</td>
<td>$1.64 \pm 0.03$</td>
<td>$1.59 \pm 0.01$</td>
</tr>
</tbody>
</table>

Table II: Best fit estimates of the critical parameters $K_c$ and $\ln \Lambda_c$, the critical exponent $\nu$ together with their uncertainty (one standard deviation). $\nu$ is expected to be universal whereas $\ln \Lambda_c$ and $K_c$ do depend on anisotropy [24] and $k$, $\omega_2$ and $\omega_3$. Irrelevant parameters are sensitive to microscopic details, therefore $y$ is strictly positive and not universal.

B. Universality of the critical exponent

Figure 15: (Color online) $\ln \Lambda_c$, after subtraction of corrections due to the irrelevant scaling variable, plotted versus $\ln (\xi(K,t)^{1/3})$ and the scaling function deduced from the model (black curve). The parameters are that of the set $D$ (see Table I). The best fit estimates of the critical stochasticity and the critical exponent are in this case: $K_c = 8.09 \pm 0.01$, $\ln \Lambda_c = 1.64 \pm 0.03$ and $\nu = 1.59 \pm 0.01$.

<table>
<thead>
<tr>
<th>Set</th>
<th>$k$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$K$</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$2.85 \times 10^{-2}$</td>
<td>$2\pi \sqrt{3}$</td>
<td>$2\pi \sqrt{3}$</td>
<td>$6.24 \rightarrow 6.58$</td>
<td>$0.413 \rightarrow 0.462$</td>
</tr>
<tr>
<td>$B$</td>
<td>$2.85 \times 10^{-2}$</td>
<td>$2\pi \sqrt{5}$</td>
<td>$2\pi \sqrt{7}$</td>
<td>$5.49 \rightarrow 5.57$</td>
<td>$0.499 \rightarrow 0.514$</td>
</tr>
<tr>
<td>$C$</td>
<td>$2.516 \times 10^{-2}$</td>
<td>$1/\eta$</td>
<td>$1/\eta$</td>
<td>$4.98 \rightarrow 5.05$</td>
<td>$0.423 \rightarrow 0.436$</td>
</tr>
<tr>
<td>$D$</td>
<td>$3.599 \times 10^{-2}$</td>
<td>$k/\eta$</td>
<td>$k/\eta^2$</td>
<td>$7.9 \rightarrow 8.3$</td>
<td>$0.425 \rightarrow 0.485$</td>
</tr>
</tbody>
</table>

Table I: The four sets of parameters considered: $k$, $\omega_2$ and $\omega_3$ control the microscopic details of the disorder, while $\varepsilon$ drives the anisotropy of the hopping amplitudes.
standard kicked rotor \cite{7,80}, and was accounted for in terms of correlations between the kicks by Shepelyansky in his early work \cite{25}.

The Anderson transition with the quasiperiodic kicked rotor is a robust feature: we observed that, for certain mutually incommensurate triplets \((\kappa, \omega_2, \omega_3)\), systematic deviations to scaling (such as resonances) can occur for intermediate times, but eventually vanish.

\section*{VII. CONCLUSION}

We discussed in detail in the present work the first unambiguous evidence of the Anderson transition in 3D with atomic matter waves with atomic matter waves by realizing experimentally a quasiperiodic kicked rotor. This allowed us to put into evidence the existence of the transition and to measure its critical exponent thanks to a finite-time scaling procedure. Our numerical result \(\nu = 1.59 \pm 0.01\) is in perfect agreement with the current value for the Anderson model, and is compatible with our experimental determination \(1.4 \pm 0.3\). We have also shown that the quasiperiodic kicked rotor exhibits the same critical phenomena as the truly random Anderson model, and therefore that both systems belong to the same (orthogonal) universality class.

These results are particularly relevant since they show that it is possible to explore a system like the Anderson model, that played an important role in many areas of physics but resisted thorough experimental investigations. One can guess that this kind of analogy will be extended to other models in the near future, as evidenced by the work of Wang and Gong \cite{25} concerning the analogy of a quantum kicked rotor and the Harper model. This shall open new and exciting tracks in cold-atom physics. These analog models can even prove more flexible and more powerful than the original ones, as, for example, our Anderson-equivalent system can very easily be extended to higher dimensions by introducing new incommensurate frequencies. Intermediate situations like a 2D kicked rotor with two or three incommensurate frequencies might be a convenient solution from the experimental point of view. This can hardly be done in condensed-matter systems or even in the ultracold atom realization of the 1D Anderson model \cite{2}. The theoretical study of quantum phase transitions in high dimensions will most probably be boosted as experimental results become available. We are presently working in this direction: Numerical and experimental determinations of the critical exponents in four dimensions seems feasible.

\section*{Acknowledgments}

The authors are pleased to acknowledge H. Lignier for fruitful discussions.
When the quasi-momentum is taken into account, $m$ should simply be replaced by $m + \beta$, leaving the conclusions unaffected.

$W(\theta)$ is singular for $K/k > \pi$ and the behavior of $W_r$ is complicated. However, this non-physical singularity can be avoided by a mapping on a different tight-binding model.


