Foliated Structure of The Kuranishi Space and Isomorphisms of Deformation Families of Compact Complex Manifolds

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Abstract. Consider the following uniformization problem. Take two holomorphic (parametrized by some analytic set defined on a neighborhood of 0 in $\mathbb{C}^p$, for some $p > 0$) or differentiable (parametrized by an open neighborhood of 0 in $\mathbb{R}^p$, for some $p > 0$) deformation families of compact complex manifolds. Assume they are pointwise isomorphic, that is for each point t of the parameter space, the fiber over t of the first family is biholomorphic to the fiber over t of the second family. Then, under which conditions are the two families locally isomorphic at 0?

In this article, we give a sufficient condition in the case of holomorphic families. We show then that, surprisingly, this condition is not sufficient in the case of differentiable families. We also describe different types of counterexamples and give some elements of classification of the counterexamples. These results rely on a geometric study of the Kuranishi space of a compact complex manifold.

Introduction

This article deals with the problem of giving a useful criterion to ensure that two holomorphic (respectively differentiable) deformation families are isomorphic as families. This takes the form of the following uniformization problem. Let

$$i = 1, 2 \qquad \pi_i \ : \ \mathcal{X}_i \to U \quad \text{respectively} \quad \pi_i \ : \ \mathcal{X}_i \to V$$

be two holomorphic (respectively differentiable) families of compact complex manifolds parametrized by some analytic set U defined on a neighborhood of 0 in $\mathbb{C}^p$, for some $p > 0$ (respectively an open neighborhood V of 0 in $\mathbb{R}^p$, for some $p > 0$). Assume that they are *pointwise isomorphic*, that is, for all $t \in U$ (respectively $t \in V$), the fiber $X_1(t) = \pi_1^{-1}(\{t\})$ is biholomorphic to the fiber $X_2(t) = \pi_2^{-1}(\{t\})$. Then the question is

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Question 1. *Under which hypotheses are the families $\mathcal{X}_1$ and $\mathcal{X}_2$ locally isomorphic at* 0*?*

By *locally isomorphic*, we mean that there exists an open neighborhood W of $0 \in U$ (respectively in V), and a biholomorphism Φ (respectively a CR-isomorphism) between $\mathcal{X}_1(W) = \pi_1^{-1}(W)$ and $\mathcal{X}_2(W) = \pi_2^{-1}(W)$ such that the following diagram is commutative.

$$\begin{tikzcd} \mathcal{X}_1(W) \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & \mathcal{X}_2(W) \arrow[d, "\pi_2"] \\ W_1 \arrow[r, "\text{Identity}"'] & W_2 \end{tikzcd}$$

We are also interested in the following broader problem.

Question 2. *Under which hypotheses are the families $\mathcal{X}_1$ and $\mathcal{X}_2$ locally equivalent at* 0*?*

By *locally equivalent*, we mean that there exists open neighborhood W_1 and W_2 of $0 \in U$ (respectively in V), a biholomorphism ϕ between W_1 and W_2 (respectively a diffeomorphism) and a biholomorphism Φ (respectively a CR-isomorphism) between $\mathcal{X}_1(W_1) = \pi_1^{-1}(W_1)$ and $\mathcal{X}_2(W_2) = \pi_2^{-1}(W_2)$ such that the following diagram is commutative.

$$\begin{tikzcd} \mathcal{X}_1(W_1) \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & \mathcal{X}_2(W_2) \arrow[d, "\pi_2"] \\ W_1 \arrow[r, "\phi"'] & W_2 \end{tikzcd}$$

In other words, $\mathcal{X}_1$ and $\mathcal{X}_2$ are locally equivalent at 0 if $\phi^*\mathcal{X}_2$ and $\mathcal{X}_2$ are locally isomorphic for some local biholomorphism ϕ of U (respectively V) fixing 0.

Fix a family $\mathcal{X}_1$. In this paper, we shall say that this family has the *local isomorphism property* (at 0), respectively has the *local equivalence property* (at 0) if every other family $\mathcal{X}_2$ which is pointwise isomorphic to it is locally isomorphic to it (at 0), respectively locally equivalent (at 0).

It is known since Kodaira-Spencer (see [K-S2], [We] and Section V.1 of this article) that there exist pointwise isomorphic families of primary Hopf surfaces which are not locally isomorphic, both in the differentiable and the holomorphic cases.

On the other hand, the classical Fischer-Grauert Theorem [F-G], can be restated as follows. Let X be a compact complex manifold and U be an open neighborhood of 0 in some $\mathbb{C}^p$. Then every trivial family $X \times U$ has the local isomorphism property. This works also for differentiable families. Indeed the proof given in [F-G] for holomorphic families is easily adapted to the differentiable case, the core of the proof being Theorem 6.2 of [K-S1] which is valid both for differentiable and holomorphic families.

Moreover, J. Wehler proved in [We] that, over a smooth base, holomorphic families of compact complex tori (in any dimension) as well as holomorphic families of compact manifolds with negatively curved holomorphic curvature (and hence

Kobayashi hyperbolic) have the local isomorphism property. This time, the proofs do not adapt to the differentiable case.

Observe that in the two previous cases, the function $h^0(t)$, that is the dimension of the cohomology group $H^0(X_t, \Theta_t)$ (where Θ_t is the sheaf of holomorphic vector fields along X_t) is constant for all $t \in U$. Wehler writes in the introduction of [We] that it is not clear if this condition is sufficient to have the local isomorphism property.

In this paper, we prove that this is the case, even over a singular basis. Namely,

Theorem 3. *If U is reduced and if the function h^0 is constant for all the fibers of a holomorphic deformation family $\pi : \mathcal{X} \to U$, then $\mathcal{X}$ has the local isomorphism property.*

We then give examples (both in the differentiable and holomorphic setting) of families not having the local equivalence property, as well as of locally equivalent but not locally isomorphic families. We classify these counterexamples into two types, and we give in Theorem 4 a complete classification of 1-dimensional families of type II not having the local equivalence property.

Coming back to the search for a criterion, we prove that, surprisingly, things are completely different in the differentiable case.

Theorem 5. *There exist differentiable families of 2-dimensional compact complex tori parametrized by an interval that are pointwise isomorphic but not locally isomorphic at a given point.*

To solve the uniformization problems stated above, we first study the geometry of the Kuranishi space K of a compact complex manifold X. We show in Theorem 1 that it has a natural holomorphic foliated structure: two points belonging to the same leaf correspond to biholomorphic complex structures. More precisely, K admits an analytic stratification such that each piece of the induced decomposition (see Section III for more details) is foliated. The leaves are complex manifolds, but the transverse structure of the foliation may be singular (this happens when the Kuranishi space is singular).

The foliation may be of dimension or of codimension zero. In Theorem 2, we prove that there exists leaves of positive dimension (that is the foliation has positive dimension on some piece of the decomposition) if and only if the function h^0 is not constant in the neighbourhood of 0 in K (0 representing the central point X). In particular, in many examples, the foliation is a foliation by points.

Although Theorems 3, 4 and 5 on the uniformization problems are not strictly speaking a consequence of Theorems 1 and 2 on the foliated structure of K, the geometric picture of K they bring played an essential role in the understanding and resolution of the problem. The key ingredients to prove the Theorems are some trivial remarks on diffeomorphisms of the Kuranishi space (see Section II), the Fischer-Grauert Theorem [F-G], a result of Namba [Na] and a fundamental proposition proved by Kuranishi in [Ku2].

We end the article with a discussion of the relationship between the uniformization problem and the universality of the Kuranishi space.

I. Notations and background

Let X be a compact complex manifold. We denote by X^{diff} the underlying smooth manifold and by J the corresponding complex operator.

In this paper, a *(holomorphic) deformation family of* X is a $\mathbb{C}$-analytic space $\mathcal{X}$ (possibly non-reduced) together with a holomorphic projection $\pi : \mathcal{X} \to U$ over an analytic set U defined on an open neighborhood of 0 in $\mathbb{C}^p$ for some $p > 0$ such that

(i) The projection π is flat and proper.
(ii) The central fiber $X_0 = \pi^{-1}(\{0\})$ is biholomorphic to X.

In the particular case where U is smooth, then π is a holomorphic submersion so $\mathcal{X}$ is locally biholomorphic to a product and by Ehresmann's Lemma it is globally *diffeomorphic* to a product $X \times U$. In the general case, by [G-K], it is still locally isomorphic to a product, and, shrinking if necessary, every fiber $X_t = \pi^{-1}(\{t\})$ is diffeomorphic to X^{diff}.

In some cases, we will consider *marked* deformation families of X, that is a family $\pi : \mathcal{X} \to U$ in the previous sense together with a holomorphic identification $i : X \to X_0$.

We also consider in this paper *differentiable deformation families* of X in the sense of [K-S1]. This means a smooth manifold $\mathcal{X}$ endowed with a Levi-flat integrable almost CR-structure, a smooth submersion $\pi : \mathcal{X} \to V$ over an open neighborhood V of 0 in some $\mathbb{R}^p$ whose level sets coincide with the foliation by complex manifolds induced by the CR-structure; and *marked differentiable families* that is families in the previous sense coming with a holomorphic identification $i : X \to X_0$.

Finally, we consider *differentiable deformation families of almost-complex structures* of (X, J), that is families satisfying the previous definition except that the almost CR-structure on $\mathcal{X}$ is not supposed to be integrable.

We denote by K the Kuranishi space of X and by $\mathcal{K}$ the Kuranishi family. We consider a particular representant of these objects rather than the germs that are usually used. Nevertheless, by abuse, we still call them *the* Kuranishi space and *the* Kuranishi family since the representant we use has no particular property. Let us recall the construction of K following [Ku1].

The complex operator J defined on X^{diff} induces a decomposition into eigenspaces of the complexified tangent bundle

$$T_{\mathbb{C}}X^{diff} := TX^{diff} \otimes_{\mathbb{R}} \mathbb{C} = T^{0,1} \oplus T^{1,0}$$

where $T^{0,1}$ (respectively $T^{1,0}$) is associated to the eigenvalue i (respectively $-i$). Notice that $T^{1,0}$ and $T^{0,1}$ are conjugated.

A different almost complex structure J' induces another decomposition of the bundle $T_{\mathbb{C}}X^{diff}$. If J' is close enough to J then the eigenspace $T^{0,1}(J')$ (the knowledge of which is enough to have the complete decomposition) may be seen as the graph of a linear map from $T^{0,1}$ to $T^{1,0}$. In other words, the set of almost complex structures close to J is identified with a neighbourhood A of 0 in the space

A^1 of $(0,1)$-forms on X with values in $T^{1,0}$. In particular, 0 represents the complex structure J we started with.

Remark. Here and in the sequel, the topology used on spaces of sections of a vector bundle over X is induced by some Sobolev norms; see the appendix for more details.

The diffeomorphism group of X^{diff} acts on the set of almost complex structures on X^{diff} identifying isomorphic structures. Put a hermitian metric h on X. Then we have a $\bar{\partial}$-operator on A^p, the space of $(0,p)$-forms with values in $T^{1,0}$; and a formal adjoint operator δ with respect to the induced hermitian product on A^p

$$\langle \omega_1, \omega_2 \rangle = \int_X h(\omega_1(x), \omega_2(x)) dV$$

for some volume form dV on X.

Let SH^1 denote the set of δ-closed forms in A^1. A neighborhood B of 0 in it contains all almost complex structures sufficiently close to 0 by the following Proposition.

Proposition K1. *For A small enough, there exists an application Ξ from A to B mapping an almost complex structure α onto a δ-closed representant $\Xi(\alpha)$. Moreover, if $\alpha(t)$ is a smooth family of almost complex structures, then so is $\Xi(\alpha(t))$.*

By representant, we mean that $\Xi(\alpha)$ and α induce isomorphic almost complex structures on X^{diff}. This key proposition proved in [Ku1] is a direct application of the implicit function theorem.

On the other hand, we may define a Laplace-like operator $\square$ on each A^p by putting

$$\square = \delta\bar{\partial} + \bar{\partial}\delta$$

and thus define harmonic forms. The set H^1 of harmonic forms of A^1 is finite-dimensional by standard theory.

Then Kuranishi defines a holomorphic map G from A^1 to A^1 and proves, using the inverse mapping theorem, that it is a biholomorphism between a special subset of A^1 (containing in particular all integrable almost complex structures close enough to 0) and a neighborhood W of 0 in H^1.

Using the Dolbeault isomorphisms, one has that $W \subset H^1$ can be identified with a neighborhood of 0 in the first cohomology group of X with values in the sheaf of holomorphic tangent vectors, classically denoted by $H^1(X, \Theta)$. In the sequel, by abuse of notation, we still denote by W this neighborhood and we identify an element of $H^1(X, \Theta)$ with the corresponding almost complex structure of H^1.

Remark that to W is naturally associated a smooth family $\mathcal{W}$ of almost complex structures. Namely, put on $X^{diff} \times W$ the almost CR-structure tangent to the fibers $X^{diff} \times \{Cst\}$ and equal to α on the fiber $X^{diff} \times \{\alpha\}$. Choose an inclusion $j : X \to W_0$ if a marked family is needed. Proposition K1 implies that W is *complete* for differentiable deformation families of X and even for marked families. This just means that, given $\pi : \mathcal{X} \to V$, a differentiable deformation family of X, then, shrinking V if necessary, there exists a smooth map of pairs $f : (V, 0) \to (W, 0)$ such that $\mathcal{X}$ is locally isomorphic to the pull-back of $\mathcal{W}$ by f, as follows:

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\Phi} & f^*W & \xrightarrow{F} & \mathcal{W} \\ {\scriptstyle\pi}\downarrow & & \downarrow & & \downarrow \\ V & \xrightarrow[\text{Identity}]{} & V & \xrightarrow[f]{} & W \end{array}$$

By abuse of notation, we write $\mathcal{X} = f^*\mathcal{W}$. If $\mathcal{X}$ has a marking i, then we ask (f, F) to respect the markings, that is we ask $F \circ \Phi \circ i \equiv j$.

Finally, one shows that the set of integrable almost complex structures of W is an analytic set defined as the zero set of a quadratic form

$$Q \ : \ H^1(X, \Theta) \times H^1(X, \Theta) \longrightarrow H^2(X, \Theta)$$

restricted to W.

This analytic set K is *the Kuranishi space* (it would be more correct to say that it is a representant of the germ of Kuranishi space) of X. The deformation family $\mathcal{K}$ obtained from $\mathcal{W}$ by restriction to $K \subset W$ is called *the Kuranishi family.* Moreover, it is a marked family, that is we have an identification j between X and the central fiber K_0 of $\mathcal{K}$.

The Kuranishi family has the following properties.

(i) it is *complete* for smooth deformation families as well as for holomorphic deformation families (unmarked and marked), not only at 0 but also at all points of K (shrinking K if necessary). For smooth deformation families, this can be derived from Proposition K1; but for holomorphic deformation families, it requires an extra-argument (see [Ku1]).

(ii) it is *versal* at 0, i.e. complete in the previous sense with the additional property that its Zariski tangent space has dimension equal to the dimension of $H^1(X, \Theta)$. This last property may not be true at points different from 0.

Remark. The versality property is equivalent to the following. Given a holomorphic *marked* deformation family $\pi : \mathcal{X} \to U$, let $f : U \to K$ such that $\mathcal{X}_{|U} = f^*\mathcal{K}$ (which exists by completeness, shrinking U if necessary). Then, f may not be unique, but its differential at 0 is. That is: let $g : U \to K$ be another marked family with $\mathcal{X}_{|U} = g^*\mathcal{K}$, then we have $f'(0) = g'(0)$.

It must be stressed that this property is related to the markings (recall that the notation $\mathcal{X}_{|U} = f^*\mathcal{K}$ for a marked family means that the marking is respected). It is usually lost when dealing with unmarked families. The marking is necessary in order to prevent from reparametrizing the family by an automorphism which acts non-trivially in the central fiber.

On the other hand, the same property holds for smooth deformation families.

Remark. In the setting of the previous remark, if $f \equiv g$, then K is called *universal.* The universality property does not hold for any Kuranishi space, see [Wa] and Section V.6.

Remark. The versality property of K at 0 implies its unicity (as a germ). To be more precise, let $\mathcal{K}' \to K'$ be another deformation family of X with identification map j'. Assume it is versal at 0. Then there exists a family isomorphism (f, F) fixing the central fibers such that, shrinking the domains if necessary, we have $\mathcal{K}' = f^*\mathcal{K}$ and $F \circ \Phi \circ j' \equiv j$. We give a quick proof of this fact following [Ca, Proposition 5.3] since we will use it in the sequel. The versality of both K and K' at 0 implies the existence of pull-back maps $\mathcal{K}' = f^*\mathcal{K}$ and $\mathcal{K} = g^*\mathcal{K}'$. But it also implies that $\mathcal{K} = (f \circ g)^*\mathcal{K}$. Hence the differential of $f \circ g$ at 0 is the identity and the inverse mapping theorem implies that $g = f^{-1}$.

Observe that this argument applies to the case of smooth deformation families. So if a deformation family is versal for smooth families, then it is diffeomorphic to the Kuranishi space of the marked manifold X.

Observe also that this unicity property justifies the expressions "the" Kuranishi space and/or family.

Remark. The Kuranishi space may be not reduced at every point [Mu]. This causes some trouble when applying some classical results of algebraic geometry.

Finally, we will make an intensive use of the following Proposition

Proposition K2. *If h^0 is constant on K, there exists a neighborhood U of 0 in K and a neighborhood $\mathcal{U}$ of the identity in the group of diffeomorphisms of X^{diff} such that, for all couples (t, t') of distinct points of U, we have*

$$\alpha_{t'} \equiv f_*\alpha_t \quad \textit{for some diffeomorphism } f \Longrightarrow f \notin \mathcal{U}$$

Here we have $\alpha_t = G^{-1}(t)$ (respectively $\alpha_{t'} = G^{-1}(t')$). To fully understand this statement, recall that the Kuranishi family is constructed as a families of complex operators. Hence every fiber K_t is naturally defined as (X^{diff}, α_t). This crucial Proposition is proved by Kuranishi in [Ku2] and used to show that h^0 constant implies the universality of K. We will discuss this at the end of the article.

II. Preliminary remarks on diffeomorphisms of the Kuranishi family

Let us begin with some definitions.

Definition. A *diffeomorphism* of the Kuranishi space K is a bijective map ϕ from some open neighborhood of 0 in K onto some open neighborhood of 0 in K such that

(i) It sends a complex structure onto an isomorphic complex structure.

(ii) Both ϕ and ϕ^{-1} are restrictions to $K \cap W' \subset W' \subset W \subset H^1(X, \Theta)$ of a smooth map of $W' \subset W$ for some W'.

Such a diffeomorphism is generally, but not always, assumed to fix 0. Notice that such a map is smooth in the sense of [Ku1].

Definition. A *diffeomorphism* of the Kuranishi family $\mathcal{K}$ is a continuous map F from some open neighborhood of K_0 in $\mathcal{K}$ to $\mathcal{K}$ such that

(i) F descends as a diffeomorphism f of K.
(ii) the restriction of F to any fiber of $\mathcal{K} \to K$ is a biholomorphism.

(iii) F is CR in the following sense. Since $\mathcal{K} \to K$ is flat morphism, it is locally isomorphic at each point to an open set of $\mathbb{D}^{\dim K_0} \times K$. Representing F locally as a map between two such charts, we ask it to be holomorphic in the $\mathbb{D}^{\dim K_0}$-variables, and smooth in the other variables (in the sense of the previous definition).

Notice that, even when F fixes the central fiber, *we do not ask it to respect the markings.*

In the same way, we define *automorphisms* of K as isomorphisms of some open neighborhood of 0 in the analytic space K generally fixing 0 (and thus as restrictions to K of local isomorphisms of W at 0); and *automorphisms* of $\mathcal{K}$ as local isomorphisms of $\mathcal{K}$ descending as automorphisms of K. Finally, all these definitions apply with trivial changes to other deformation families of X than the Kuranishi family.

Remark. In the definition of a diffeomorphism (and an automorphism) of K, we consider K as an analytic space of a $\mathbb{C}$-vector space, and not as a set of complex operators. This explains why a diffeomorphism of K may not lift to a diffeomorphism of $\mathcal{K}$. This lifting problem is very close to the local isomorphism problem. Indeed, we will give a criterion for lifting an automorphism of K in Corollary 4, as a consequence of the criterion to have the local isomorphism property. And we also give in Lemma 5 an example of an automorphism of K which does not lift.

In the second part of this Section, we deal with the problem of extending a diffeomorphism (respectively an automorphism) of the central fiber K_0 to a diffeomorphism (respectively an automorphism) of $\mathcal{K}$. Let us make first the following trivial remark. Let ϕ be an automorphism of X. Via the marking of $\mathcal{K}$, we consider it as an automorphism of K_0. The family $(\mathcal{K}, K)$ with the new identification $\phi \circ i$ is a new versal family for X, hence there exists an automorphism Φ of $\mathcal{K}$ fixing 0 and extending ϕ.

The two following Lemmas are trivial but of fundamental importance for the sequel. Part (i) of the first one is even weaker than the previous statement but it has the advantage to admit slight generalizations stated in Lemma 2 and Proposition 1.

Lemma 1.
(i) Let ϕ be an automorphism of X. Then there exists a diffeomorphism Φ of $\mathcal{K}$ extending ϕ.
(ii) Let ϕ be a diffeomorphism of X^{diff} such that ϕ_*0 belongs to the set A of Proposition K1. Then there exists a diffeomorphism Φ of $\mathcal{K}$ extending ϕ.

Proof. In both cases, see ϕ as a diffeomorphism of X^{diff}. Then it satisfies $\phi_*0 = 0$ (case (i)) or ϕ_*0 close to 0 (case (ii)). In other words, this diffeomorphism induces a map

$$\phi_* \ : \ \alpha \in A^1 \longmapsto \phi_*\alpha \in A^1$$

with 0 as fixed point or with 0 sent close to 0. Consider the following composition

$$W' \subset W \subset H^1 \xrightarrow{G^{-1}} A^1 \xrightarrow{\phi_*} A^1 \xrightarrow{\Xi} SH^1 \xrightarrow{G} H^1$$

taking W' small enough to have $\phi_*(W') \subset A$.

This gives $\tilde{\phi}$, a map from $W' \subset W$ to H^1. This $\tilde{\phi}$ respects the almost complex structures, that is sends an almost complex structure onto one which is isomorphic. Hence, it induces naturally a smooth map from $\mathcal{K}$ to $\mathcal{K}$ that we denote by Φ.

Assume that $\tilde{\phi}$ is not a local diffeomorphism at 0. This implies that Φ maps $\mathcal{K}$ to a family $\mathcal{K}' \to K'$ whose Zariski tangent space at 0 is of strictly smaller dimension (recall that the Zariski tangent space of K at 0 has same dimension as H^1). But, since Φ respects the complex structures, the family $\mathcal{K}' \to K'$ is complete at 0 and even versal for differentiable families of complex structures. Now, by unicity of the versal deformation space, we have that K' must have same Zariski dimension at 0 as K. Contradiction. We conclude that $\tilde{\phi}$ is a local diffeomorphism at 0. So is Φ for $\mathcal{K}$. $\square$

Lemma 2. *Let ϕ be an automorphism of X isotopic to the identity. Then there exists a diffeomorphism Φ of $\mathcal{K}$ extending ϕ isotopic to the identity. Moreover, if we fix an isotopy ϕ_t between ϕ and the identity on X, then the isotopy between Φ and the identity of $\mathcal{K}$ can be assumed to be equal to ϕ_t when restricted to X.*

Proof. Apply the proof of Lemma 1 to each member of the isotopy ϕ_t. We thus obtain a family Φ_t of diffeomorphisms of $\mathcal{K}$ extending ϕ_t for all t. By compacity of $[0, 1]$, all the Φ_t can be defined on a same open neighborhood of 0. Finally smoothness in t comes from Proposition K1. $\square$

Of course, the important point in Lemma 2 is that Φ is isotopic to the identity. We draw from these lemmas an important consequence.

Proposition 1. *Assume that the function h^0 is not constant at $0 \in K$. Then, we can find ϕ, an automorphism of X isotopic to the identity, such that*

(i) it extends as a diffeomorphism Φ of $\mathcal{K}$ isotopic to the identity.
(ii) the projection of every such extension on K gives a diffeomorphism of K whose germ at 0 is not the germ of the identity.

Proof. The first step of the proof is a very classical argument. The function h^0 is known to be upper semi-continuous. The assumption that it is not constant in a neighborhood of 0 implies then that it has a strict maximum at 0. Take a basis of $H^0(K_0, \Theta_0)$. If every vector field of this basis could be extended to a vector field of the family $\mathcal{K}$ (tangent to the fibers), at a point $t \in K$ close enough to 0, they all would form a free family of dimension $h^0(0)$ of $H^0(K_t, \Theta_t)$, contradicting the inequality $h^0(t) < h^0(0)$.

Hence, there exists a global vector field ξ on K_0 which cannot be extended as a vector field of $\mathcal{K}$ tangent to the fibers. Let ϕ be the corresponding automorphism isotopic to the identity obtained by exponentiation for small time. By Lemma 2, there exists a diffeomorphism Φ of $\mathcal{K}$ extending ϕ isotopic to the identity, proving (i).

Now, for every such choice of Φ, the induced diffeomorphism of K cannot be the identity, even in germ, otherwise the global vector field of $\mathcal{K}$ obtained by differentiating Φ would be tangent to the fibers and would extend ξ. Contradiction which proves the Proposition. $\square$

This proposition must be compared with the following classical result of [K-S1].

Theorem. *Let $\pi : \mathcal{X} \to V$ be a smooth deformation family of X. If the function h^0 is constant along V, then every automorphism of X_0 isotopic to the identity can be extended as a diffeomorphism of the family $\mathcal{X}$ which is the identity on V.*

Automorphisms of X which does not extend as automorphisms of $\mathcal{K}$ that are the identity on K are usually called *obstructed automorphisms*. Proposition 1 tells us that obstructed automorphisms extend as diffeomorphisms of $\mathcal{K}$ with non-trivial projection on K, but isotopic to the identity.

III. Foliated structure of the Kuranishi space

1. Local submersions.
Let t be a point of K corresponding to the complex manifold $X_t = (X^{diff}, J_t)$. Denote by $(K)_t$ the space K but with base point t and not 0 and by $(\mathcal{K})_t$ the corresponding deformation family of X_t (choosing some identification maps). The family $(\mathcal{K})_t \to (K)_t$ is complete at t, but not always versal. On the other hand, let $K(t)$ be the Kuranishi space of X_t, and $\mathcal{K}(t)$ the corresponding versal family. We thus have a sequence of pointed analytic spaces

$$(\mathcal{S}) \qquad (K(t), 0) \xrightarrow{i_t} ((K)_t, t) \xrightarrow{s_t} (K(t), 0)$$

which lifts to a sequence of maps between families

$$\mathcal{K}(t) \xrightarrow{I_t} (\mathcal{K})_t \xrightarrow{S_t} \mathcal{K}(t)$$

And $(\mathcal{S})$ is the restriction of the sequence (defined on neighborhoods of the base points)

$$(H^1(X_t, \Theta_t), 0) \xrightarrow{\tilde{\imath}_t} (H^1(X_0, \Theta_0), \alpha_t) \xrightarrow{\tilde{s}_t} (H^1(X_t, \Theta_t), 0)$$

Lemma 3. *The map $\tilde{s}_t$ is a submersion at α_t.*

Proof. Since $K(t)$ is versal at 0, the composition $\imath_t \circ s_t$ is a local isomorphism at 0. So is $\tilde{\imath}_t \circ \tilde{s}_t$. Hence $\tilde{s}_t$ is a submersion at J_t. $\square$

Apply the submersion Theorem to $\tilde{s}_t$. This gives a diagram

$$\begin{tikzcd} V \subset (H^1(X_0, \Theta_0), J_t) \arrow[r, "\tilde{s}_t"] \arrow[d, "\text{local biholomorphism}"'] & W \subset (H^1(X_t, \Theta_t), 0) \\ W \times B \arrow[r, "\text{identity}"'] & W \times B \arrow[u, "\text{natural projection}"'] \end{tikzcd}$$

where B is the unit euclidean ball of $\mathbb{C}^p$ for $p = h^1(0) - h^1(t)$ and $h^1(t)$ denotes the dimension of $H^1(X_t, \Theta_t)$.

This submersion allows to locally foliate $H^1(X_0, \Theta_0)$ in a neighborhood of α_t. The leaves correspond to deformation families of almost complex structures which are pull-back by $\tilde{s}_t$ of a constant family. In other words, the points of a same leaf all define the same almost complex structure up to isomorphism.

When we restrict to K, we obtain the diagram of analytic spaces

$$\begin{tikzcd}
(K)_t \arrow[r, "s_t"] \arrow[d, "\text{local biholomorphism}"'] & K(t) \\
K(t)\times B \arrow[r, "\text{identity}"'] & K(t)\times B \arrow[u, "\text{natural projection}"']
\end{tikzcd}$$

that defines a local foliated structure of K.

We aim at gluing those local foliations together into a global foliation. This brings some problems since the induced foliations in two arbitrarily close points may be of different dimensions. For example, assuming that the dimension of K is strictly greater than the dimension of $K(t)$ for $t \neq 0$ (which is the case for e.g. the Hirzebruch surface $\mathbb{F}_2$; one has $K = \mathbb{C}$ and $K(t) = \{0\}$ for $t \neq 0$, corresponding to the rigid $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$), then the previous diagram for $t = 0$ gives a foliation of K by points, whereas for $t \neq 0$, it gives a foliation of positive dimension of $(K)_t \setminus \{0\}$.

To overcome this problem, it is necessary to decompose the space K.

2. Decomposition of K.

Decompose K as a disjoint union

$$(\mathcal{D}) \qquad K = K^{min} \sqcup \ldots \sqcup K^{max}$$

where K^i denotes the set of points t of K such that the dimension of $K(t)$ is i. Observe that the completeness of K *at each point* implies that 0 belongs to K^{max}.

We want to show that this decomposition is analytic in the following sense.

Definitions. Let X be an analytic space. Then a finite set $\mathcal{D} = (X_i)_{1\leq i\leq k}$ of X is a *decomposition* of X if X is the disjoint union of the X_i.

Set $Z_j = \sqcup_{i\leq j} X_i$. Then the sequence $Z_1 \subset \ldots \subset Z_k = X$ is called the *stratification associated to* $\mathcal{D}$.

Finally, $\mathcal{D}$ or equivalently its associated stratification is called *analytic* if for all $i < k$, the set Z_i is an analytic subset of Z_{i+1} and thus of X.

Notice that a piece X_i of an analytic decomposition is a quasi-analytic open set of the corresponding stratum Z_i. We have now

Lemma 4. *The decomposition* $(\mathcal{D})$ *of* K *is analytic.*

Proof. Assume first that K is reduced. Define

$$min \leq c \leq max \qquad E_c = \{t \in K \quad | \quad h^1(t) \geq c\}$$

These sets forms an analytic stratification of K, see [Gr]. Call $\mathcal{G} = (G_c)$ the associated decomposition.

On the other hand, denoting by $\mathcal{G}_c$ the family of complex structures with base G_c, the results of [Gr] show that the group $H^1(\mathcal{G}_c, \Theta)$ is isomorphic to a locally free sheaf over G_c whose stalk at t is $H^1(X_t, \Theta_t)$. The set of integrable structures in this sheaf is given as the zero set Z_t of a field of quadratic forms in the fibers which is analytic in t.

This allows us to analytically stratify each piece G_c by

$$d \leq c \qquad G_{c,d} = \{t \in G_c \quad | \quad \text{codim } Z_t \leq d\}$$

Then the E_c respectively the $G_{c,d}$ are analytic sets of K respectively of G_c, whereas $G_c \subset E_c$ is a quasi-analytic open set.

Observe now that the completeness property of a Kuranishi space *at each point* close enough from the base point implies that the function

$$t \in K \longmapsto \dim K(t)$$

is upper semi-continuous for the standard topology. Since $G_{c,d}$ is an analytic set of G_c and since the Zariski closure of G_c, that is E_c, is equal to its closure for the standard topology, this proves that the Zariski closure $\overline{G_{c,d}}$ of each $G_{c,d}$ in $\overline{G_c} = E_c$ is just

$$\overline{G_{c,d}} = \{t \in E_c \quad | \quad \text{codim } Z_t \leq d\}$$

We now just have to set

$$min \leq i \leq max \qquad F_i = \cup_{c-d \geq i} \overline{G_{c,d}}$$

to obtain an analytic stratification of K whose associated decomposition $F_i \setminus F_{i+1}$ $(i > min)$ and F_{min} is the decomposition $\mathcal{D}$.

If K is not reduced, we just perform the previous stratification on its reduction, then we obtain the decomposition $(\mathcal{D})$ by putting on each F_i the multiplicity induced from K. $\square$

Let $t \in K^i$ for some i. By definition, i_t and s_t respect the decompositions of K and $K(t)$, that is

$$i_t(K(t)^{max}) = K^i \qquad \text{et} \qquad s_t(K^i) = K(t)^{max}$$

It is thus possible to restrict the submersions s_t to a piece of the decomposition to obtain the following diagram

$$\begin{tikzcd}
(K^i)_t \arrow[r, "s_t"] \arrow[d, "\text{local biholomorphism}"'] & K(t)^{max} \\
K(t)^{max} \times B \arrow[r, "\text{identity}"'] & K_t^{max} \times B \arrow[u, "\text{natural projection}"']
\end{tikzcd}$$

In the sequel, s_t will always denote the restricted submersion from K^i to the piece $K(t)^{max}$, for $t \in K^i$.

3. Foliated structure of K.

Using the submersion s_t, we will define a foliated structure on each piece K^i.

Definition. Let X be an analytic space. A *transversally singular foliation* of dimension p on X is given by an open covering (U_α) of X and local isomorphisms $\phi_\alpha : U_\alpha \to B \times Z_\alpha$ (for B the unit ball of $\mathbb{C}^p$ and Z_α an analytic space) such that the changes of charts $\phi_{\alpha\beta} \equiv \phi_\beta \circ (\phi_\alpha)^{-1}$ preserve the plaques $B \times \{pt\}$.

Thus a transversally singular foliation is a lamination with a transverse structure of an analytic space. The choice for this somewhat unusual name (rather than analytic lamination or something analogous) comes from the fact that we find more judicious to reserve the word lamination to a situation where the total space has no analytic structure.

Now, given such a foliation, one may define the leaves as in the classical case by gluing the plaques. Hence the leaves are holomorphic manifolds. Remark also that the germ of the analytic space Z_α is the same along a fixed leaf.

We may state

Theorem 1. *Let K be the Kuranishi space of a manifold X. Consider the decomposition $\mathcal{D}$ of K.*

Then each piece K^i admits a transversally singular foliation $\mathcal{F}^i$ locally defined by the submersions s_t of Section 2.

Notice that two points belonging to the same leaf correspond to the same complex manifold (up to biholomorphism). Notice also that the Kuranishi family $\mathcal{K}$ admits an induced decomposition in pieces $\mathcal{K}^i$, each of these pieces being foliated. By [F-G], the leaf of $\mathcal{K}^i$ corresponding to $t \in K^i$ is a locally trivial fibre bundle with fibre X_t over the leaf of K^i through t. And the foliation of K (respectively $\mathcal{K}$) extends to a holomorphic foliation, respectively an almost-complex foliation (this time in the classical sense, that is with smooth transverse structure) of W and respectively $\mathcal{W}$ using the submersions S_t. To simplify the exposition, we will always consider the foliation of K.

Proof. Take a covering of a piece K^i by open sets where a submersion s_t is well defined. Using the submersion Theorem, we obtain foliated charts modelled on a product of a ball of dimension $(\dim K - i)$ with an analytic space of dimension i. Now, the changes of charts respect the leaves, since they have an intrinsic geometric definition: the leaf through $t \in K^i$ is the maximal connected subset of K^i of points corresponding to the complex manifold X_t up to biholomorphism. □

Remark that this proof adapts immediately to the case of W, with the only difference that X_t may just be an almost complex manifold.

Notation and Definition. We denote by $\mathcal{F}$ the global foliation of Theorem 1, that is the union of the foliations $\mathcal{F}^i$.

We say that the foliation $\mathcal{F}$ is *trivial* if each foliation $\mathcal{F}^i$ is a foliation by points.

The foliation $\mathcal{F}$ is trivial if and only if the decomposition $(\mathcal{D})$ has a single piece, that is if and only if the dimension of $K(t)$ is constant near 0. On the other hand, observe that $\mathcal{F}$ may be of codimension 0 (that is each $\mathcal{F}^i$ has codimension 0 in K^i) with a non-trivial decomposition (see the examples below).

4. Examples.

Let us look at the foliation of some well-known examples.

(i) Let X be a complex torus of dimension n. Following [K-S2], the Kuranishi space of X may be represented by a neighborhood of 0 in $\mathbb{C}^{n^2}$ and it is versal at each point. Therefore, the decomposition $(\mathcal{D})$ has a unique piece and the foliation $\mathcal{F}$ is trivial.

However, observe that given a well-chosen compact complex torus, there exists an infinite sequence of points of K corresponding to this torus [K-S2, p. 413]. This shows that two points belonging to different leaves of $\mathcal{F}$ may nevertheless define the same complex structure.

(ii) Let X be the Hirzebruch surface $\mathbb{F}_2$. Its Kuranishi space may be represented by a unit 1-dimensional disk whose non-zero points all correspond to $\mathbb{P}^1 \times \mathbb{P}^1$, see [Ca]. The decomposition is

$$K = K^1 \sqcup K^0 = \{0\} \sqcup \mathbb{D}^*$$

and both foliations have codimension zero.

(iii) Let X be the Hopf surface obtained from $\mathbb{C}^2 \setminus \{(0,0)\}$ by taking the quotient by the group $\langle 2Id\rangle$ generated by the homothety $(z,w) \mapsto 2\cdot(z,w)$. Its Kuranishi space is described in [K-S2]. It may be represented by a neighborhood of the matrix $2Id$ in

$$K = \{A \in \mathrm{GL}_2(\mathbb{C}) \quad | \quad |\mathrm{Tr}\ A| > 3,\ |\Delta(A)| = |(\mathrm{Tr}\ A)^2 - 4\det A| < 1\}$$

A point A of K corresponds to the Hopf surface $\mathbb{C}^2 \setminus \{(0,0)\}/\langle A\rangle$. If A is a multiple of the identity, then the corresponding Kuranishi space $K(A)$ has dimension four; in other words K is versal along the set

$$\Delta = \{\lambda Id \quad | \quad |\lambda| > \frac{3}{2}\}$$

However, if A is not a multiple of the identity, the dimension of $K(A)$ drops to 2. Thus we decompose K into two pieces

$$K^4 = \Delta \quad \text{and} \quad K^2 = K \setminus \Delta$$

On the other hand, consider the map

$$\phi \ :\ A \in K \longmapsto (\mathrm{Tr}\ A, \Delta(A)) \in \mathbb{C}^2$$

Let (σ,δ) be a point of $\mathbb{C}^2$ with $|\sigma| > 3$ and $|\delta| < 1$. If δ is different from zero, all points of $\phi^{-1}(\sigma,\delta)$ correspond to the same Hopf surface. If δ is zero, the same is true for all points of $\phi^{-1}(\sigma,\delta)$ except for $\sigma/2\cdot Id$, which corresponds to a different Hopf surface. Notice that in this case, the level set $\phi^{-1}(\{(\sigma,\delta)\})$ is singular at $\sigma/2\cdot Id$.

As a consequence of all that, the foliation $\mathcal{F}^4$ is a foliation by points, whereas the foliation $\mathcal{F}^2$ is a non-trivial one, which is given by the level sets of the submersion ϕ restricted to K^2. It has dimension and codimension two.

IV. Non-triviality criterion for $\mathcal{F}$

The aim of this section is to prove the following result.

Theorem 2. *Let K be the Kuranishi space of X and let $\mathcal{F}$ be the foliation of K constructed in section III.*

Then $\mathcal{F}$ is trivial if and only if h^0 is a constant function on K.

This leads to the following corollary.

Corollary 1. *The Kuranishi space K is versal at all points if and only h^0 is a constant function on K.*

Proof of Corollary 1. Combine Theorem 2 and the remark after the definition of triviality for $\mathcal{F}$. □

Let us proceed to the proof of Theorem 2.

Proof of Theorem 2. Assume h^0 constant on K and assume at the same time that $\mathcal{F}$ is non-trivial. Thus there exists a piece $K^i \subset K$ whose foliation $\mathcal{F}^i$ has positive-dimensional leaves. So there exist non-constant smooth paths $c : [0,1] \to K$ such that the induced family $\mathcal{C} = c^*\mathcal{K}$ has all fibers biholomorphic. Choose such a non-constant path staying inside the neighborhood U appearing in Proposition K2. Now Fischer-Grauert Theorem [F-G] implies that $\mathcal{C}$ is the trivial family; in other words there exists $(\phi_t)_{t\in[0,1]}$ an isotopy such that

(i) $\phi_0 \equiv Id$.
(ii) For all $t \in [0,1]$, we have $(\phi_t)_*\alpha_{c(0)} = \alpha_{c(t)}$.

For t small enough, we have ϕ_t in $\mathcal{U}$, violating Proposition K2. Contradiction. The foliation is trivial.

Reciprocally, assume h^0 non-constant. Then by Proposition 1, there exists an automorphism ϕ of X isotopic to the identity all of whose extensions as a diffeomorphism of $\mathcal{K}$ does not project onto the identity of K on any neighborhood of 0. Let Φ be one of these extensions; still by Proposition 1, recall that Φ may be chosen isotopic to the identity. Let Φ_t be the isotopy. All that means that there exist points $x \in K$ arbitrary close to 0 such that the path $t \in [0,1] \mapsto \Phi_t(x) \in K$ is a non-constant path. But Fischer-Grauert Theorem may be geometrically reformulated as: the Kuranishi space of a compact complex manifold does not contain a non-constant path passing through 0 all of whose points correspond to X (use the versality at 0). As a consequence, for such a point x, the space $(K)_x$ cannot be the Kuranishi space $K(x)$. That is it is not versal at x. But we already noticed in III.3 that this is enough to prove that the foliation is non-trivial. □

Finally, note that:

Corollary 2. *The stratum K^{max} is the set of points $t \in K$ such that $h^0(t) = h^0(0)$.*

Proof. Let

$$H = \{t \in K \quad | \quad h^0(t) = h^0(0)\}$$

This is an analytic space by [Gr] (recall that $h^0(t) \geq h^0(0)$ implies equality). Arguing exactly as in the first part of the proof of Theorem 2, we show that $\mathcal{F}$ is trivial on H. So H is included in K^{max}.

Conversely, assume that h^0 is not constant on K^{max}. Then arguing as in the second part of the proof of Theorem 2, we show that $\mathcal{F}^{max}$ is non trivial. Contradiction. $\square$

Remark. Indeed, although Proposition K2 is stated for the complete Kuranishi space K, a quick look at the proof shows that it is valid in restriction to any subset $V \subset K$ where h^0 is constant equal to $h^0(0)$.

V. The isomorphism and equivalence problems

Let us recall these problems. Let

$$i = 1, 2 \qquad \pi_i \ : \ \mathcal{X}_i \to U \quad \text{respectively} \quad \pi_i \ : \ \mathcal{X}_i \to V$$

be two holomorphic (respectively differentiable) families of compact complex manifolds parametrized by an analytic set U defined on an open neighborhood of 0 in some $\mathbb{C}^p$ (respectively by an open neighborhood V in some $\mathbb{R}^p$). Assume that they are pointwise isomorphic. Then the question is to decide if they are locally isomorphic at 0, or at least locally equivalent at 0, as defined in the introduction. Observe that there are in fact two different problems, the differentiable one (when the parameter space is $V \subset \mathbb{R}^p$) and the holomorphic one (when the parameter space is an analytic set U).

Recall also that given such a family $\mathcal{X}$, we say that it has the local isomorphism property (respectively local equivalence property) if every family pointwise isomorphic to $\mathcal{X}$ is locally isomorphic to $\mathcal{X}$ (respectively locally equivalent) at 0. Finally recall that in these problems, we are dealing with *unmarked* families. Let us give two more definitions.

Definitions. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two families which are pointwise isomorphic but not locally isomorphic at 0. Then we say that they form *a type (II) counterexample (to the isomorphism property)* if there exist

$$f, \ g \ : \ (U, 0) \longrightarrow (K, 0)$$

such that

(i) We have $\mathcal{X}_1 = f^*\mathcal{K}$ and $\mathcal{X}_2 = g^*\mathcal{K}$.
(ii) There exist $U_1 \subset U$ and $U_2 \subset U$ such that $f(U_1)$ and $g(U_2)$ are equal.

And we say that they form a *type (I) counterexample* if we cannot find f and g as above.

Same definitions are valid for the equivalence problem. Roughly speaking, a type (II) counterexample is a counterexample obtained by reparametrization, whereas a type (I) counterexample relies on the particular geometric structure of the Kuranishi space.

1. Counterexamples.

A counterexample of type (II) for the two problems (in both differentiable and holomorphic cases) can be found in [K-S2] (and explained in [We]). Start with the Hopf surface $\mathbb{C}^2 \setminus \{(0,0)\}/\langle 2Id \rangle$. We use the notations of Section III.4.(iii). Define $\mathcal{X}_1$ as the family corresponding to an embedding disk (respectively an interval) in the closure of the two-dimensional leaf $\phi^{-1}(4,0)$. This is an example of a jumping family. Let us give a precise definition.

Definition. A holomorphic (respectively differentiable) *jumping family* is a family $\pi : \mathcal{X} \to U$ (respectively $\pi : \mathcal{X} \to V$) such that

(i) it is trivial outside 0, but the central fiber X_0 is not biholomorphic to the generic fiber.
(ii) The Kodaira-Spencer map at 0 is not zero.

Notice that the second condition is automatically satisfied if the map from the parameter space to the Kuranishi space K is an embedding at 0. This follows from the versality property of K at 0.

In particular the Kodaira-Spencer map ρ_1 of our family $\mathcal{X}_1$ is not zero at 0. Consider now the ramified covering

$$z \in \mathbb{D} \longmapsto z^2 \in \mathbb{D} \quad \text{or more generally } z^n \in \mathbb{D}$$

or respectively $t \in I \longmapsto t^3 \in I$. Then define $\mathcal{X}_2$ as the pull-back of $\mathcal{X}_1$ by this application. By the "chain-rule for the Kodaira-Spencer map", we have at 0

$$\rho_2\left(\frac{\partial}{\partial z}\right) = \rho_1\left(\text{Jac}\ _0(z \mapsto z^n) \cdot \frac{\partial}{\partial z}\right) = \rho_1(0) = 0$$

$$\text{respectively } \rho_2\left(\frac{\partial}{\partial z}\right) = \rho_1\left(\text{Jac}\ _0(t \mapsto t^3) \cdot \frac{\partial}{\partial z}\right) = \rho_1(0) = 0$$

so $\mathcal{X}_2$ is not isomorphic, nor equivalent, to $\mathcal{X}_1$.

Of course, this construction can be generalized starting from any jumping family (for example, one can take the jumping family with the Hirzebruch surface $\mathbb{F}_2$ as central fiber and $\mathbb{P}^1 \times \mathbb{P}^1$ as generic fiber; this shows that such counterexamples exist even for projective manifolds). So we state:

Proposition 2. *A (holomorphic or differentiable) jumping family has neither the local isomorphism, nor the local equivalence property.*

It is important for the sequel to observe that the function h^0 is not constant in a jumping family (cf [Gri]).

We give now a type (I) counterexample for the two problems, in both differentiable and holomorphic cases. Although it can be obtained easily from the treatment of Hopf surfaces in [K-S2], we do not know of any reference where it is described.

Once again, we use the results and the notations of Section III.4.(iii). Consider

$$\begin{cases} \mathcal{X}_1 = \mathbb{C}^2 \setminus \{(0,0)\} \times \mathbb{D} / \left\langle \begin{pmatrix} 2+t & t \\ 0 & 2+t \end{pmatrix},\ t \right\rangle \\ \\ \mathcal{X}_2 = \mathbb{C}^2 \setminus \{(0,0)\} \times \mathbb{D} / \left\langle \begin{pmatrix} 2+t & t^3 \\ 0 & 2+t \end{pmatrix},\ t \right\rangle \end{cases}$$

Replacing $\mathbb{D}$ by I in the definition of $\mathcal{X}_1$ and $\mathcal{X}_2$, one obtains a differentiable counterexample.

We claim that $\mathcal{X}_1$ and $\mathcal{X}_2$ are pointwise isomorphic but not locally equivalent at $2Id$, and finally that they have distinct image in K. The last point is a direct consequence of the fact that, since the families are embedded, same image would imply locally isomorphic.

Now, an elementary computation shows that for $t\neq 0$, the fibers $(X_1)_t$ and $(X_2)_t$ are biholomorphic and conjugated by

$$P(t)=\begin{pmatrix}\pm t & q\\ 0 & \pm t^{-1}\end{pmatrix}$$

where q is any complex number (we assume without loss of generality that P has determinant one). Since $(X_1)_0=(X_2)_0$ and since none of these conjugating matrices extend at 0, we are done for the isomorphism problem. Finally, for the equivalence problem, just observe that if t and t' are distinct and both different from 0, then $(X_1)_t$ and $(X_2)_t$ are not biholomorphic (look at the traces). Hence, in this case, there is no difference between the isomorphism problem and the equivalence problem.

Notice that h^0 is not constant along these families, dropping from 4 (at 0) to 2.

Let us give now examples of locally equivalent but not locally isomorphic families. We still use the Kuranishi space of the Hopf surface described in Section III, 4, (iii). The key point is given by the following Lemma.

Lemma 5. *The map $A\in K\to {}^tA\in K$ is an automorphism of K fixing $2Id$ which does not lift to an automorphism of $\mathcal{K}$.*

Proof. This is clear for K using the fact that A and ${}^tA$ are conjugated. On the other hand, assume that this automorphism lifts to an automorphism of $\mathcal{K}$. Then, in a neighborhood of $2Id$, it would be possible to find a family of invertibles matrices $P(A)$ depending holomorphically on A such that

$${}^tA=P^{-1}(A)\cdot A\cdot P(A)$$

where we assume without loss of generality that $P(A)$ has determinant equal to one. Straightforward computations show that we must have

$$P(A)=\begin{pmatrix}\alpha & \pm i\\ \pm i & 0\end{pmatrix}\qquad\text{for}\qquad A=\begin{pmatrix}2 & t\\ 0 & 2\end{pmatrix}\qquad\qquad t\in\mathbb{C}$$

where α is any complex number and can be chosen independently of t. And we must also have

$$P(A)=\begin{pmatrix}\alpha & 0\\ 0 & 1/\alpha\end{pmatrix}\qquad\text{for}\qquad A=\begin{pmatrix}2 & 0\\ 0 & 2+t\end{pmatrix}\qquad\qquad t\in\mathbb{C}$$

where α is any non-zero complex number and can be chosen independently of t.

Since these two families do not have any common limit where t goes to zero, we are done. $\square$

Now let $\mathcal{X}_1$ be the Kuranishi family and let $\mathcal{X}_2$ be obtained by pull-back by the transposition map. So the two families are locally equivalent by definition. Now, by Lemma 5, they are not locally isomorphic.

Observe that this trick gives only type (II) counterexamples.

2. Holomorphic families.

In this section, we prove

Theorem 3. *Let $\pi : \mathcal{X} \to U$ be a holomorphic family of deformations. If U is reduced and if h^0 is constant in a neighborhood of 0, then it has the local isomorphism property.*

Notice the immediate Corollary.

Corollary 3. *Let X be a compact complex manifold such that h^0 is constant on its Kuranishi space X. Then any holomorphic deformation family of X with reduced base has the local isomorphism property at 0.*

Proof of Theorem 3. Assume h^0 constant in a neighborhood of 0. Assume first that π is a 1-dimensional family parametrized by the unit disk. Let $\pi' : \mathcal{X}' \to \mathbb{D}$ be a pointwise isomorphic family. Let

$$f, g \ : \ \mathbb{D} \longrightarrow K$$

such that $\mathcal{X} = f^*\mathcal{K}$ and $\mathcal{X}' = g^*\mathcal{K}$. We may assume without loss of generality that

(i) The maps f and g are defined on the whole disk (otherwise shrink and uniformize).
(ii) The families $\mathcal{X}$ and $\mathcal{X}'$ are *equal* to $f^*\mathcal{K}$ and $g^*\mathcal{K}$, not only isomorphic (otherwise replace).

Call D the image of f, and D' that of g. If D or D' is reduced to a point, then all the fibers of $\mathcal{X}$ and of $\mathcal{X}'$ are biholomorphic and Fischer-Grauert Theorem implies that *both* D and D' are reduced to a point. Both families are locally trivial, hence locally isomorphic. So we may assume that D and D' are embedded disks.

Choose $(\phi_t)_{t\in\mathbb{D}}$ a family of pointwise biholomorphisms

$$\phi_t \ : \ K_{f(t)} \longrightarrow K_{g(t)}$$

Lemma 6. *For each t in a dense subset of points of $\mathbb{D}$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that ϕ_{t_n} converges to ϕ_t in Sobolev norms as n goes to infinity.*

Proof. It is inspired from [F-G]. Since the set $\mathbb{D}$ is uncountable, the sequence $(\phi_t)_{t\in\mathbb{D}}$ contains an accumulation point for the Sobolev topology. This fact is proven in the appendix. Moreover, given any open set U' of $\mathbb{D}$ the same is true for the subset $(\phi_t)_{t\in U'}$. Hence the claim. □

As a consequence, fix a neighborhood V_0 of 0 in K. Then Lemma 6 implies that there exists a sequence $(t_n)_{n\in\mathbb{N}} \in V_0$ converging to some point t_∞ of V_0 with ϕ_{t_n} converging to ϕ_{t_∞} in Sobolev norms.

Assume that $f(t_\infty)$ is equal to $g(t_\infty)$ and that ϕ_{t_∞} is the identity. By Proposition K2, assuming V_0 small enough, this would mean that we must have

$$n \geq n_0 \qquad\qquad f(t_n) = g(t_n)$$

for n_0 big enough. Now, this implies that $f - g$ is a holomorphic function on the disk with a non-discrete set of zeros, hence $f \equiv g$ and we are done. Observe that K

is naturally embedded in the vector space $H^1(X,\Theta)$ as an analytic set, hence the difference $f-g$ is meaningful as holomorphic map from $\mathbb{D}$ to $H^1(X,\Theta)$.

In the general case, things become more complicated, but the previous pattern can be used as a guideline to proceed. Consider the embedded family $\mathcal{K}_{|D'}\to D'$ and write

$$\mathcal{K}_{|D'}=(D'\times X^{diff},\alpha)$$

where the complex operator $\alpha_t\in A^1$ turns $\{t\}\times X^{diff}$ into the complex manifold $K(t)$.

Remark that we have a diffeomorphism

$$\phi_{t_\infty}^{-1}\ :\ K_{g(t_\infty)}\longrightarrow K_{f(t_\infty)}$$

Remark also that, by Corollary 2, the set K is versal at both $f(t_\infty)$ and $g(t_\infty)$. So by Lemma 1, there exists a diffeomorphism (Ψ,ψ) of $\mathcal{K}$ defined on a neighborhood of $f(t_\infty)$ which extends $\phi_{t_\infty}^{-1}$. To simplify the notations, we identify in this proof a point t of K and the integrable almost-complex operator $\alpha_t=G^{-1}(t)$ defining K_t. With this convention, ψ is constructed as a composition of

$$(\phi_{t_\infty}^{-1})_*\ :\ A^1\longrightarrow A^1$$

with the map Ξ of Proposition K1. This gives us a new realization

$$h\equiv\psi\circ g\ :\ U'\subset\mathbb{D}\longrightarrow K$$

defined on a neighborhood U' of t_∞ such that $\mathcal{X}'$ is locally isomorphic to h^*K at t_∞.

But now we can make use of Proposition K2. The sequence

$$\Psi\circ\phi_{t_n}\ :\ K_{f(t_n)}\longrightarrow K_{h(t_n)}$$

converges in Sobolev norms to

$$\Psi\circ\phi_{t_\infty}\ :\ K_{f(t_\infty)}\xrightarrow{\text{Identity}}K_{h(t_\infty)}=K_{f(t_\infty)}$$

hence f and h take the same values not only at t_∞ but also at every t_n for n big enough. Moreover, still by Proposition K2, the map h is the unique map such that $\mathcal{X}'$ is locally isomorphic to $h^*\mathcal{K}$ at t_∞ (provided K is based and marked at $f(t_\infty)$ and provided a marking of $\mathcal{X}'$ is fixed at t_∞ and asked to be preserved). Since the family $\mathcal{X}'$ is a holomorphic family, h must be holomorphic. So as before we have $f\equiv h$.

Remark. This is just another way of saying that K is universal with respect to families with h^0 constant equal to $h^0(0)$. Proposition K2 was proved by Kuranishi to have this type of result.

We claim that $\mathcal{X}'$ is isomorphic to $h^*\mathcal{K}$ over the whole disk $\mathbb{D}$, and not only over a neighborhood of t_∞ in $\mathbb{D}$.

This can be proven as follows. Let $U' \subset \mathbb{D}$ be the maximal subset of $\mathbb{D}$ such that $\mathcal{X}'_{|U'}$ is isomorphic to $h^*_{|U'}\mathcal{K}$. Let $t \in \mathbb{D}$ and let c be a path in $\mathbb{D}$ joining $c(0) = t_\infty$ to $c(1) = t$. We will prove that t is in U'.

The problem that could appear is that $(\phi_{t_\infty}^{-1})_* f(c)$, which is a path in A^1, is not fully included in the domain of definition A of Ξ. Let $K' \subset A^1$ be the Kuranishi space of $X'_{f(c(1))}$ based at $(\phi_{t_\infty}^{-1})_* f(c)$. Let Ξ' be the map of Proposition K1 defined in a neighborhood A' of $(\phi_{t_\infty}^{-1})_* f(c(1))$ in A^1. For simplicity, assume that the whole path $(\phi_{t_\infty}^{-1})_* f(c)$ is included in $A \cup A'$. Take a point $s \in [0,1]$ such that $(\phi_{t_\infty}^{-1})_* f(c(s))$ lies in the intersection of A and A'. Then there exists a local isomorphism between the pointed analytic sets $(K, \Xi((\phi_{t_\infty}^{-1})_* f(c(s)))$ and $(K', \Xi'((\phi_{t_\infty}^{-1})_* f(c(s)))$ since these two spaces are versal for $X'_{f(c(s))}$. And this isomorphism can be chosen in such a way that the image of $\Xi((\phi_{t_\infty}^{-1})_* f(c))$ is sent to $\Xi'((\phi_{t_\infty}^{-1})_* f(c))$ in a neighborhood of s, still by the universality property.

Let us sum up. We can glue K and K' to obtain an analytic space $\tilde{K}$ such that h extends as $\tilde{h}$ along c in such a way that $\mathcal{X}'$ is isomorphic to $\tilde{h}^*\tilde{\mathcal{K}}$ along the full path c. Still by universality, in our case, $\tilde{h}$ must be equal to h, so that t is in U. In particular, observe that the image of $\tilde{h}$ stays in $K \subset \tilde{K}$. So the claim is proved.

Now, we have

$$\mathcal{X}' \simeq h^*\mathcal{K} = f^*\mathcal{K} \simeq \mathcal{X}$$

on the whole disk (the symbol $\simeq$ meaning isomorphic). In other words, $\mathcal{X}$ and $\mathcal{X}'$ are locally isomorphic in a neighborhood of 0. This proves the Theorem for 1-dimensional families.

Let us now assume that the families $\mathcal{X}$ and $\mathcal{X}'$ are p-dimensional. We will use general arguments (already used in [We], though not exactly in the same way) to pass from the one-dimensional to the general case.

By a Theorem of Namba [Na, Theorem 2], the union $\mathcal{H}$ of pointwise holomorphic maps from X_t to X'_t for all t can be endowed with a structure of a reduced analytic space such that the natural projection map $p : \mathcal{H} \to U$ is holomorphic and surjective. Moreover, the topology of $\mathcal{H}$ is that of uniform convergence.

Let $\mathcal{S} \subset \mathcal{H}$ be the subset of pointwise isomorphisms. It is an open set of $\mathcal{H}$ so a reduced analytic space with a holomorphic (still surjective in this particular case) projection map p. This openness property can be shown as follows. Given ϕ, an isomorphism between X_t and X'_t for a fixed t, every ψ close enough from ϕ in the topology of uniform convergence is a local isomorphism at each point. We just have to prove now that ψ must be bijective. Forgetting the complex structures we can see ϕ and ψ as maps of X^{diff}, using differentiable trivializations. Since X^{diff} is compact and ψ locally bijective, ψ is surjective. Besides, still by compacity, there exists a finite open covering of X^{diff} such that any map close enough from ϕ is injective when restricted to any member of this covering. Assume ψ is not globally injective. Then, we could find a sequence of non-injective maps ψ_n converging uniformly onto ϕ. So there would be two sequences of points (x_n) and (y_n) such that

$$n \in \mathbb{N} \qquad\qquad x_n \neq y_n \qquad \psi_n(x_n) = \psi_n(y_n)$$

By compacity of X^{diff}, they will converge to some points x and y such that $\phi(x) = \phi(y)$, hence $x = y$. This clearly contradicts the previous property of local injectivity of all ϕ_n on a fixed covering.

To finish the proof of Theorem 3, it is enough to show that $p : S \to U$ has a local holomorphic section at 0. But now, we conclude from what we did for 1-dimensional families that p has local holomorphic sections at 0 along every embedded disk $\mathbb{D}$ in U. Fix one of these local sections, say σ. Take another such section σ'. Then σ and σ' differ by composition (at the source) by an automorphism of X_0 and by composition (at the target) by an automorphism of X'_0. Since U is reduced and h^0 is constant, these two automorphisms extend locally as automorphisms of the nearby fibers [Gr]. But this means exactly that, composing σ' with these extensions, one may assume without loss of generality that σ' takes the same value at 0 as σ. Using this trick, we see that there exist local sections with the same value at 0 for every disk embedded in U passing through 0.

Now, by a Proposition of Grauert and Kerner [G-K], there exists an analytic embedding of a neighborhood S of $\sigma(0)$ in $\mathcal{S}$

$$i \ : \ S \longrightarrow \mathbb{D}^{\dim p^{-1}(\{0\})} \times U$$

such that the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "p"'] & \mathbb{D}^{\dim p^{-1}(\{0\})} \times U \arrow[d, "\text{2nd proj.}"] \\ U \arrow[r, "\text{Identity}"'] & U \end{tikzcd}$$

Observe that the dimension of $p^{-1}(\{0\})$ is $h^0(0)$ and that, since h^0 is constant, $i(p^{-1}(s))$ is an open set of $\mathbb{D}^{h^0(0)} \times \{s\}$. On the other hand, by what preceeds, $p(S)$ must be equal to an open neighborhood of 0 in U (because U is reduced). As a consequence, i is a local isomorphism, which exactly means that $\mathcal{X}$ and $\mathcal{X}'$ are locally isomorphic at 0. □

Remark. The last strategy (using Namba's Theorem and so on) cannot be used directly to obtain the result for 1-dimensional families. Indeed, it is not possible to exclude the case of $p^{-1}(\{0\})$ being isolated from the other fibers, so that in the diagram above, the image $p(S)$ reduces to 0. The only fact that can be proven directly is that, if we know that there exists a sequence

$$\phi_{t_n} \ : \ X_{t_n} \longrightarrow X'_{t_n}$$

converging uniformly to some $\phi_0 : X_0 \to X'_0$, then the two families are locally isomorphic at 0. This is just because, in this case, $p(S)$ is an analytic set of $\mathbb{D}$ (we are in the 1-dimensional case) containing an infinite sequence (t_n) accumulating on 0. So $p(S)$ must contain an open neighborhood of 0. Now, we obtained the same conclusion using Proposition K2.

Remark. In the non-reduced case, Theorem 3 is false, as shown by the following easy example. Consider the upper half-plane $\mathbb{H}$ of $\mathbb{C}$ as the parameter space of elliptic curves. Let $\mathcal{H} \to \mathbb{H}$ the versal (at each point) associated family. Choose a point $\tau \in \mathbb{H}$. Take U to be the double point

$$U = \{t^2 = 0 \quad | \quad t \in \mathbb{C}\}$$

Let $\pi : \mathcal{X}_1 \to U$ be the constant family obtained by pull-back by a constant map from U to $\mathbb{H}$ (with value τ). Now, since U is not reduced, there exists also non-constant morphisms from U to $\mathbb{H}$. Let f be the unique such morphism sending the single point of U to τ and the vector $\partial/\partial t$ of the Zariski tangent space of U to the horizontal unit vector of $\mathbb{H}$ based at τ. Define $\mathcal{X}_2$ as $f^*\mathbb{H}$. Then $\mathcal{X}_1$ and $\mathcal{X}_2$ are obviously pointwise isomorphic, but they are not locally isomorphic, by computation of their Kodaira-Spencer map. It is 0 for $\mathcal{X}_1$, and not zero for $\mathcal{X}_2$.

3. Application to automorphisms of a holomorphic family.

As a consequence of Theorem 3, we state:

Corollary 4. *Let $\pi : \mathcal{X} \to U$ be a holomorphic family. Assume that h^0 is constant. Then*

(i) Every diffeomorphism of $\mathcal{X}$ fixing 0 is an automorphism.
(ii) If moreover U is reduced, then every automorphism of U lifts to an automorphism of $\mathcal{X}$.

Proof. The first statement is a direct consequence of the universality property as explained in the proof of Theorem 3. A diffeomorphism of $\mathcal{X}$ must send every analytic subspace of U onto an analytic subspace of U, that is must be holomorphic.

The second statement follows from Theorem 3. Let f be an automorphism of U, then $\mathcal{X}$ and $f^*\mathcal{X}$ are pointwise isomorphic. So are locally isomorphic by Theorem 3. And this means that f lifts. □

On the other hand, recall that we gave in Lemma 5 an example of an automorphism of a reduced Kuranishi space which does not lift as an automorphism of the Kuranishi family.

4. Type (II)-counterexamples to the equivalence problem.

We derive now a characterization of type (II) counterexamples to the equivalence problem in the one-dimensional case.

Theorem 4. *The following statements are equivalent.*

(i) The one-dimensional holomorphic families $\pi : \mathcal{X} \to \mathbb{D}$ and $\pi' : \mathcal{X}' \to \mathbb{D}$ form a type (II) counterexample to the equivalence problem.
(ii) Both are obtained from the same jumping family $\pi'' : \mathcal{X}'' \to \mathbb{D}$ by pull-backs by some maps. Moreover, the degrees of these maps (as ramified coverings of $\mathbb{D}$) are different.

Proof. Assume that $\mathcal{X}$ (respectively $\mathcal{X}'$) are obtained from the Kuranishi space of X by pull-back by some map f (respectively h). Call D the image of f and D' that of h. Shrinking the domains of definition if necessary to have the same image $D'' \subset D \cap D'$ and uniformizing at the source and at the target by unit disks, we obtain the following diagram

$$\begin{array}{ccccccc}
\mathbb{D} & \xrightarrow{\text{uniform.}} & f^{-1}(D'') \subset \mathbb{D} & \xrightarrow{f} & D'' & \xrightarrow{\text{uniform.}} & \mathbb{D} \\
 & & & & {\scriptstyle Id}\big\downarrow & & \big\downarrow{\scriptstyle Id} \\
\mathbb{D} & \xrightarrow{\text{uniform.}} & h^{-1}(D'') \subset \mathbb{D} & \xrightarrow{h} & D'' & \xrightarrow{\text{uniform.}} & \mathbb{D}
\end{array}$$

To simplify, we still denote by f (respectively by h) the composition of the top arrows (respectively of the bottom arrows). Moreover, we denote by $\pi'' : \mathcal{X}'' \to \mathbb{D}$ the target family and replace $\mathcal{X}$ (respectively $\mathcal{X}'$) by $f^*\mathcal{X}''$ (respectively $h^*\mathcal{X}''$).

We may assume without loss of generality that f and g are unramified coverings over $\mathbb{D}^*$ of respective degrees n and m. So we have [Fo, Theorem 5.11]

$$z \in \mathbb{D} \qquad\qquad f(z) = z^n \qquad g(z) = z^m$$

changing the uniformizing maps at the source by a rotation if necessary.

If m and n are equal, then $\mathcal{X}$ and $\mathcal{X}'$ are locally equivalent at 0. So assume $n > m$.

Now, from the one hand, by definition of the pull-back, for all $t \in \mathbb{D}$, the fibers X_t and X''_{t^n} are biholomorphic, as well as X'_t and X''_{t^m}. And from the other hand, the assumption for the families of being pointwise isomorphic means in this new setting that there exists

$$\Phi \ : \ (U_1 \subset \mathbb{D}, 0) \longrightarrow (U_2 \subset \mathbb{D}, 0)$$

a biholomorphism such that X_t and $X'_{\phi(t)}$ are biholomorphic. Hence by transitivity, X_t and $X_{\phi^{-1}(t^{n/m})}$ are biholomorphic for every choice of a determination of $t^{n/m}$. Observe that this is valid for t belonging to a sufficiently small neighborhood U'_1 of 0 in $\mathbb{D}$. Set

$$C = \{t \in \mathbb{D} \quad | \quad |t| = \lambda\}$$

for λ a fixed real number, which is supposed small enough to have $C \subset U'_1$.

Lemma 7. *Let $t_0 \in C$. Then the closure of the set*

$$E_{t_0} = \{t \in \mathbb{D} \quad | \quad X_{t_0} \textit{ is biholomorphic to } X_t\}$$

contains C.

Proof of Lemma 7. Assume first that ϕ is equal to $a \cdot Id$ for a non-zero. Defining α_k for $k \in \mathbb{N}$ by induction

$$\begin{cases} \alpha_0 = a^{-1} \\ \alpha_{k+1} = \alpha^{-1} \cdot \alpha_k^{n/m} \end{cases}$$

(we choose a determination of $\alpha_k \mapsto \alpha_k^{n/m}$ for each k), we have that X_t and $X_{\alpha_k \cdot (t^{n/m})^k}$ are biholomorphic. In particular, all the points of

$$\{t_0 \exp(2i\pi l(m/n)^k) \quad | \quad k > 0, \ l \in \mathbb{Z}\}$$

correspond to X_{t_0} proving the density of E_{t_0} in C.

Now, if ϕ is not a homothety, it admits a Taylor expansion

$$\phi(t) = at + \text{ higher order terms}$$

Besides, one has that $t \in E_{t_0}$ as soon as $t^{n/m} = t_0^{n/m}$, or

$$(\phi^{-1}(t^{n/m}))^{n/m} = (\phi^{-1}(t_0^{n/m}))^{n/m}$$

or more generally

$$\Big(\phi^{-1}\big(\dots(\phi^{-1}(t^{n/m}))^{n/m}\dots\big)^{n/m}\Big)^{n/m} = \Big(\phi^{-1}\big(\dots(\phi^{-1}(t_0^{n/m}))^{n/m}\dots\big)^{n/m}\Big)^{n/m}$$

Using the Taylor expansion of ϕ together with the fact that $n/m > 1$, we obtain that the sequence

$$\frac{\Big(\phi^{-1}\big(\dots(\phi^{-1}(t^{n/m}))^{n/m}\dots\big)^{n/m}\Big)^{n/m}}{\alpha_k \cdot (t^{n/m})^k}$$

tends to 1 as k goes to infinity. In this expression, the determinations of the n/m-th power are chosen at each step according to the choices made for α_k.

This means that, given

$$t = t_0 \exp(2i\pi l(m/n)^k)$$

for some fixed $k > 0$ and $l \in \mathbb{Z}$, and given any $\epsilon > 0$, there exists $t' \in \mathbb{D}$ which is ϵ-close to t such that t' belongs to E_{t_0}. This is enough to conclude that the closure of E_{t_0} contains C. $\Box$

Hence, there exists a dense subset of points corresponding to X_{t_0} in any annulus around the circle $|z| = |t_0|$. Following [Gr], the function h^1 is constant on a Zariski open subset of $\mathbb{D}$. So we may assume that it is constant on $\mathbb{D}^*$. That means that the differentiable family of deformations parametrized by $|z| = |t_0|$ is a regular one.

Proposition 3. *All points of the circle $C = \{|z| = |t_0|\}$ in $\mathbb{D}$ correspond to the same complex manifold X_{t_0}.*

Proof. This is a step by step adaptation of the proof of [F-G]. We will prove that the Kodaira-Spencer map is zero for a dense subset of points, hence by regularity for all points, so that all points correspond to the same manifold X_{t_0}. We will explain in details how to modify the proof of [F-G] so that it generalizes to this case, but will refer freely to [F-G] for the common parts.

Choose $\epsilon > 0$. Choose also an annulus A around C. Choose finally a differentiable trivialization

$$T \ : \ \pi^{-1}(\{s \in A \quad | \quad |s - t_0| < \epsilon\}) \longrightarrow X_{t_0}$$

with $T_{t_0} \equiv Id$.

For every t in C, define a diffeomorphism $\tilde{\alpha}_t$ from X_{t_0} to X_t as follows. First, choose some $t' \in A$ such that

(i) We have $|t'-t_0| < \min(|t-t_0|, \epsilon)$.
(ii) The parameter t' belongs to the set

$$E_t \cap \pi^{-1}(\{s \in A \quad | \quad |s-t_0| < \epsilon\})$$

This is possible by Lemma 7. By what preceeds, there exists a biholomorphism β_t between $X_{t'}$ and X_t. Define

$$\tilde{\alpha}_t \equiv \beta_t \circ T_{t'}^{-1}$$

First notice that the set

$$E = \{t \in C \quad | \quad \exists (t_n)_{n\in\mathbb{N}} \in C \text{ such that } (\tilde{\alpha}_{t_n}) \text{ uniformly converges to } \tilde{\alpha}_t\}$$

is dense in C. This comes from the fact that the set of continuous maps from X to $\pi^{-1}(C)$ is of countable type, see [F-G] and the appendix.

Let $t \in E$. Without loss of generality, we may assume that there exists a finite set of submersion charts

$$\begin{array}{ccc}
U_i \subset \pi^{-1}(\{s \in A \quad | \quad |s-t| < \epsilon\}) & \xrightarrow{\psi_i} & \mathbb{C}^{\dim X} \times \{s \in A \quad | \quad |s-t| < \epsilon\} \\
\pi \Big\downarrow & & \Big\downarrow \text{2nd projection} \\
\{s \in A \quad | \quad |s-t| < \epsilon\} & \xrightarrow[\text{Identity}]{} & \{s \in A \quad | \quad |s-t| < \epsilon\}
\end{array}$$

covering $\pi^{-1}(\{s \in A \quad | \quad |s-t| < \epsilon\})$.

Set $\alpha_n \equiv \tilde{\alpha}_{t_n} \circ \alpha_t^{-1}$. Then the sequence (α_n) converges uniformly to the identity of X_t. Let (V_j) be a covering of $\pi^{-1}(\{s \in A \quad | \quad |s-t| < \epsilon\})$ by relatively compact open sets with smooth boundaries such that there exists a refining map r and an integer n_0 satisfying

$$\forall n \geq n_0, \quad \alpha_n(V_j) \subset U_{r(j)}$$

First, assume for simplicity that $h^0(t) = 0$. Let $x \in V_i \cap X_t$ and let (z,t) be the coordinates in the chart $\psi_{r(i)}$. For $n \geq n_0$, define

$$\begin{aligned}
\xi_i^n(x) &= (\psi_{r(i)}^{-1})_*(\psi_{r(i)} \circ \alpha_n(x) - \psi_{r(i)}(x)) \\
&= (\psi_{r(i)}^{-1})_*(z(\alpha_n(x)) - z(x), t(\alpha_n(x)) - t(x)) \\
&= (\psi_{r(i)}^{-1})_*(z(\alpha_n(x)) - z(x), t_n - t)
\end{aligned}$$

where $(\psi_{r(i)}^{-1})_*$ denotes the pushforward of a vector field by the differential of $\psi_{r(i)}^{-1}$. This gives a smooth vector field on $V_i \cap X_t$ which is transverse to X_t (since the t-coordinate is non-zero). Now, let

$$M_n = \max_i \sup_{x \in V_i \cap X_t} \|(\psi_{r(i)})_* \xi_i^n(x)\|$$

for some choice of a norm on $\mathbb{C}^{\dim X_t} \times \mathbb{R}$. Notice that M_n is positive since it is bigger than $|t_n - t|$; and that it is finite because of the finiteness of the number of charts and because of the relative compactness of the V_i.

Lemma 8. *The sequence* $1/M_n(\xi_i^n)$ *converges uniformly to a holomorphic vector field* ξ_i *on* $V_i \cap X_t$.

Proof of Lemma 8. Let $\eta_i^n = 1/M_n(\psi_{r(i)})_*(\xi_i^n)$. It is a uniformly bounded sequence of functions on $D_i = \psi_{r(i)}(V_i \cap X_t)$. If we prove that is an equicontinuous sequence, then Ascoli's Theorem will ensure the uniform convergence.

Now, for all j between 1 and $\dim X$, the sequence

$$\bar{\partial}_j \eta_i^n \equiv \frac{\partial}{\partial \bar{z}_j} \eta_i^n$$

is uniformly convergent to zero since we have

$$\bar{\partial}_j \eta_i^n = \frac{1}{M_n}(\bar{\partial}_j(z(\alpha_n \circ \psi_{r(i)}^{-1})) - \bar{\partial}_j z, \bar{\partial}_j(t(\alpha_n \circ \psi_{r(i)}^{-1})))$$

and since α_n tends uniformly to the identity, hence $z(\alpha_n \circ \psi_{r(i)}^{-1})$ tends uniformly to z and $t(\alpha_n \circ \psi_{r(i)}^{-1}))$ to t.

On the other hand, writing down the Bochner-Martinelli formula for η_i^n, one obtains, for $k = \dim X_t$,

$$\eta_i^n(z) = \frac{(k-1)!}{(2i\pi)^k}\Big(\int_{\partial D_\subset} \sum_{\nu=1}^{k} ((-1)^{\nu-1}\eta_i^n)(\zeta) \frac{(\bar{\zeta}_\nu - \bar{z}_\nu)}{|\zeta - z|^{2k}} d\bar{\zeta}[\nu] \wedge d\zeta \\ - \int_{D_i} \sum_{\nu=1}^{k} ((-1)^{\nu-1}\bar{\partial}_\nu \eta_i^n)(\zeta) \frac{(\bar{\zeta}_\nu - \bar{z}_\nu)}{|\zeta - z|^{2k}} d\bar{\zeta}[\nu] \wedge d\zeta\Big)$$

where $d\bar{\zeta}[\nu] = d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_{\nu-1} \wedge d\bar{\zeta}_{\nu+1} \wedge \ldots \wedge d\bar{\zeta}_n$. Deriving with respect to z_j gives

$$\partial_j \eta_i^n(z) = \frac{k!}{(2i\pi)^k}\Big(\int_{\partial D_i} \sum_{\nu=1}^{k} ((-1)^{\nu}\eta_i^n)(\zeta) \frac{(\bar{\zeta}_\nu - \bar{z}_\nu)(\bar{\zeta}_j - \bar{z}_j)}{|\zeta - z|^{3k+1}} d\bar{\zeta}[\nu] \wedge d\zeta \\ + \int_{D_i} \sum_{\nu=1}^{k} ((-1)^{\nu}\bar{\partial}_\nu \eta_i^n)(\zeta) \frac{(\bar{\zeta}_\nu - \bar{z}_\nu)(\bar{\zeta}_j - \bar{z}_j)}{|\zeta - z|^{3k+1}} d\bar{\zeta}[\nu] \wedge d\zeta\Big)$$

Since (η_i^n) is a uniformly bounded sequence and since $(\bar{\partial}_j \eta_i^n)$ is uniformly convergent to zero, we obtain that $(\partial_j \eta_i^n)$ is also a uniformly bounded sequence. So (η_i^n) is Lipschitz with a Lipschitz constant independant of n. Therefore it is equicontinuous.

Finally, since $(\bar{\partial}_j \eta_i^n)$ is uniformly convergent to zero, the limit is automatically holomorphic. □

Following [F-G], it is easy to prove that these ξ_i glue together to define a global non-zero holomorphic vector field ξ on X_t. This vector field must be transverse to X_t for we assumed $h^0(t) = 0$. Hence, the Kodaira-Spencer map at t is zero.

If $h^0(t)$ is not zero, one has first to modify each α_n by composition with a finite number of well-chosen automorphisms of X_t. The construction of the holomorphic vector field ξ is then exactly the same. And finally one uses the special properties of this new sequence of (α_n) to prove that ξ cannot be tangent to X_t. Details are exactly the same as in [F-G].

As a consequence, one obtains that the family over C has zero Kodaira-Spencer map on a dense subset of points, hence on C as it is a regular family. And applying Theorem 6.2 of [K-S1], one has that this family is locally trivial at every point. In particular, all the fibers correspond to the same compact complex manifold up to biholomorphism. □

But the existence of such a circle of biholomorphic fibers forces the foliation of Section III to be non-trivial. From the previous proof, we deduce that all the circles $z = |t|$ of $\mathcal{X}$ correspond to a unique complex structure, say X_t. Fix such a t different from 0. This implies that the intersection of $\mathbb{D}$ with the leaf of the foliation passing through t contains a circle of points. Since the foliation is holomorphic, this means that a neighborhood of this circle corresponds to X_t. Let s be in the boundary of this neighborhood. Then the same argument shows that a neighborhood of the circle $|z| = |s|$ lies in the leaf passing through s. Now, the two previous neighborhoods must have non-empty intersection which implies that X_s and X_t are biholomorphic.

We conclude from that that all the points of $\mathbb{D}^*$ correspond to X_t. Hence, by Fischer-Grauert Theorem, $\mathcal{X}''$ must be a jumping family.

To prove the converse, we need to refine the argument given in the proof of Proposition 2. Consider the *local* Kodaira-Spencer map of $\mathcal{X}$ at 0

$$0 \in U \subset \mathbb{D} \qquad\qquad H^0(U, \Theta) \xrightarrow{\rho_{\mathcal{X}}} H^1(\mathcal{X}_{|U}, \Theta)$$

which represents the obstruction to lifting a holomorphic vector field in the base $U \subset \mathbb{D}$ to the family $\mathcal{X}_{|U} = \pi^{-1}(U)$. The direct limit of $\rho_{\mathcal{X}}$ for U smaller and smaller gives the pointwise Kodaira-Spencer map used in the proof of Proposition 2 and which represents the pointwise first-order obstruction to this lifting problem.

But we can also define a pointwise $(p+1)$-th order obstruction for any $p \in \mathbb{N}$ and any $\xi \in H^0(U, \Theta)$ by taking the p-jet of $\rho_{\mathcal{X}}(\xi)$ at 0 (jet as local sections of Θ) and passing to the direct limit. This defines a $(p+1)$-th order Kodaira-Spencer map

$$J_0^p(T\mathbb{D}) \xrightarrow{\rho_{\mathcal{X}}^{(p)}} H^1(X_0, \Theta^{(p)})$$

where $J_0^p(T\mathbb{D})$ is the vector space of p-jets at 0 of holomorphic vector fields of $\mathbb{D}$ defined in a neighborhood of 0 and $\Theta^{(p)}$ is the bundle of p-jets of holomorphic sections of Θ (cf [Wa]).

Since the local Kodaira-Spencer map satisfies a chain-rule property, so does $\rho_{\mathcal{X}}^{(p)}$, Hence, starting from $\mathcal{X}$, $\mathcal{X}'$ pull-backs of $\mathcal{X}''$ by maps f and g, we obtain the following equality

$$\rho_{\mathcal{X}}^{(p)}\left(\frac{\partial}{\partial t}\right) = \rho_{\mathcal{X}''}^{(p)}\left(f_*(j_o^p(\frac{\partial}{\partial t}))\right) \qquad \text{and} \qquad \rho_{\mathcal{X}'}^{(p)}\left(\frac{\partial}{\partial t}\right) = \rho_{\mathcal{X}''}^{(p)}\left(g_*(j_0^p((\frac{\partial}{\partial t}))\right)$$

with f_* (respectively g_*) denoting the action of f (respectively g) on p-jets of vector fields. Now if f has degree n and g degree m, the above $(p+1)$-th obstruction of $\mathcal{X}$ vanishes for $p < n$ and does not vanish for $p = n$, whereas that of $\mathcal{X}'$ vanishes for

$p < m$ and does not vanish for $p = m$. Hence, if m and n are different, the families $\mathcal{X}$ and $\mathcal{X}'$ are not locally isomorphic at 0. $\square$

Of course, this is no more true for higher-dimensional families. Starting from two pointwise isomorphic but not locally isomorphic one-dimensional jumping families, one can take their products with a fixed family and obtain type (II) counterexamples which are not coming from jumping families.

5. Differentiable families.

Things are completely different for differentiable families. In fact, we have

Theorem 5.
(i) Let $\pi : \mathcal{X} \to V$ be a real analytic family. If h^0 is constant along the family, then it has the local isomorphism property.
(ii) Some differentiable families $\pi : \mathcal{X} \to I$ of 2-dimensional compact complex tori do not have the local isomorphism property.

Moreover, there exist counterexamples of type (I) among families of 2-dimensional compact complex tori.

Proof.
(i) This is exactly the same proof as that of Theorem 3. For the 1-dimensional part, we observe that the only properties of holomorphic maps used are properties of analytic functions. For the passage to higher dimension, it is enough to embed pointwise isomorphic families $\mathcal{X}$ and $\mathcal{X}'$ in holomorphic families $\mathcal{X}_{\mathbb{C}} \to U$ and $\mathcal{X}'_{\mathbb{C}} \to U$ with constant h^0. For example, one may take for U the reduction of the stratum K^{max}. Then the only difference is that the map $p : \mathcal{S} \to U$ given by Namba's Theorem may not be surjective. But the same argument shows that it has a holomorphic section at 0 defined on an analytic subspace of U containing V.

Remark. The same proof shows that if two differentiable families over V are pointwise isomorphic *and* locally isomorphic along each path of V containing 0, then they are locally isomorphic.

(ii) Because of (i), a smooth family of tori not having the local isomorphism property at 0 must be flat at 0.

Recall [K-S2] that the open set

$$M = \{A \in M_2(\mathbb{C}) \quad | \quad \det(\Im A) > 0\}$$

is a versal (and even universal) deformation space for every 2-dimensional compact complex torus. A point $A = (A_1, A_2)$ of M corresponds to the quotient of $\mathbb{C}^2$ by the lattice generated by

$$(1,0) \quad (0,1) \quad A_1 \quad A_2$$

Notice that every torus can be obtained as such a quotient. Two different points A and B of M define the same torus up to biholomorphism if and only if there exists

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in SL_4(\mathbb{Z}) \quad \text{such that} \quad B = A \cdot \gamma = (\gamma_{11} + A\gamma_{21})^{-1}(\gamma_{12} + A\gamma_{22})$$

Finally, h^0 is constant equal to 4 (given by the translations), so the condition of Theorem 3 is satisfied.

Let

$$\Omega_0 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad \Omega(t) = \begin{pmatrix} i+t & b(t) \\ c(t) & i+t \end{pmatrix} \qquad t \in \mathbb{R}$$

and let X_t be the corresponding tori. The smooth functions b and c satisfy

1. They are smoothly flat at zero, i.e. all their derivatives at zero are zero.
2. We have $b(0) = c(0) = 0$ and $b(t) > c(t) > 0$ for t different from zero.

The path Ω in M defines a differentiable family of 2-dimensional compact complex tori centered at X_0. Define $\Omega_1 \equiv \Omega$ and

$$\Omega_2(t) = \begin{cases} \Omega_1(t) & \text{if } t \leq 0 \\ {}^t\Omega_1(t) & \text{if } t \geq 0 \end{cases}$$

Remark that conditions 1 and 2 imply that Ω_2 is also a smooth path.

We claim that the corresponding families $\mathcal{X}_1 \to \Omega_1$ and $\mathcal{X}_2 \to \Omega_2$ are pointwise isomorphic but not locally isomorphic at 0.

First note that, for all t,

$${}^t\Omega_1(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \Omega_1(t) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_1(t) \cdot \gamma$$

for

$$\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in SL_4(\mathbb{Z})$$

That implies that, for $t > 0$, the map

$$(z, w) \in \mathbb{C}^2 \longmapsto (w, z) \in \mathbb{C}^2$$

descends as a biholomorphism between $X_1(t)$ and $X_2(t)$. So the families are pointwise isomorphic.

On the other hand, for a generic lattice, it is well-known that the automorphism group of a torus is generated by translations and by $-Id$. Indeed, for this particular choice of matrices $\Omega(t)$, it is straightforward that this is the case if the numbers $i+t$, $b(t)$, $c(t)$, their squares and all the products of two of them are linearly independent over $\mathbb{Q}$. Hence, for generic t, the tori $X_1(t)$ and $X_2(t)$ have no other automorphisms than these ones. This allows to find sequences $(t'_n)_{n\in\mathbb{N}}$ of negative numbers and $(t''_n)_{n\in\mathbb{N}}$ of positive numbers converging to 0 such that

(i) For each n, up to translations, the only biholomorphisms between $X_1(t'_n)$ and $X_2(t'_n)$ are the projection of $\pm Id$ on $\mathbb{C}^2$.
(ii) For each n, up to translations, the only biholomorphisms between $X_1(t'_n)$ and $X_2(t'_n)$ are the projection of $\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{C}^2$.

Suppose now that $\mathcal{X}_1$ and $\mathcal{X}_2$ are locally isomorphic at 0. Then there would exist a family (Φ_t) of biholomorphisms of $\mathbb{C}^2$ (for t in a neighborhood of 0) such that

(i) It is smooth in t.
(ii) Every Φ_t descends as a biholomorphism between $X_1(t)$ and $X_2(t)$.

But, by what preceeds, at t'_n the map Φ_t must be $\pm Id$ up to a translation factor, whereas at t''_n, it must be $\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ up to a translation factor. Since these two sequences do not converge to the same type of limit when t goes to infinity, we arrive to a contradiction. The families Ω_1 and Ω_2 are not locally isomorphic at 0.

On the other hand, the previous family still has the local isomorphism property when restricted to $(-\infty, 0]$ and $[0, \infty)$. Nevertheless, it is easy to modify it in order to have a counterexample even when restricted to $(-\infty, 0]$ and $[0, \infty)$.

Start with the same path Ω as before, but this time assume that the functions b and c satisfy

1. There exists a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers converging to 0 such that b and c are zero and flat at all t_n.
2. We have b and c even.
3. We have $b(t) \neq c(t)$ for t positive and not belonging to the sequence (t_n).

For example, let

$$h(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \exp(-1/t) & \text{otherwise} \end{cases}$$

and

$$f : t \in \mathbb{R} \longmapsto \sum_{p\in\mathbb{Z}} h(t+p) \cdot h(-t-p+1) \in \mathbb{R}$$

and finally

$$b \equiv \alpha h(|-|) \cdot f(\log|-|) \qquad b \equiv \beta h(|-|) \cdot f(\log|-|)$$

for $\beta \neq \alpha$. In this case, we have $(t_n) = \exp(-n)$.

The path Ω in M defines a differentiable family of 2-dimensional compact complex tori centered at X_0. Define $\Omega_1 \equiv \Omega$ and

$$\Omega_2(t) = \begin{cases} \Omega_1(t) & \text{if } |t| \in [t_{2n}, t_{2n+1}] \text{ for some } n \\ {}^t\Omega_1(t) & \text{if } |t| \in [t_{2n-1}, t_{2n}] \text{ for some } n \end{cases}$$

That implies that, for $t \in [t_{2n-1}, t_{2n}]$ for some n, the map

$$(z, w) \in \mathbb{C}^2 \longmapsto (w, z) \in \mathbb{C}^2$$

descends as a biholomorphism between $X_1(t)$ and $X_2(t)$. In particular, it defines an automorphism of X_0 and of $X_1(t_n) = X_2(t_n)$ for all n. This proves the pointwise isomorphism between the fibers.

On the other hand, as in the previous example, one can find sequences $(t'_n)_{n\in\mathbb{N}}$ and $(t''_n)_{n\in\mathbb{N}}$ of positive numbers converging to 0 such that

(i) For each n, we have $t'_n \in [t_{2n}, t_{2n+1}]$ and, up to translations, the only biholomorphisms between $X_1(t'_n)$ and $X_2(t'_n)$ are the projection of $\pm Id$ on $\mathbb{C}^2$.
(ii) For each n, we have $t'_n \in [t_{2n-1}, t_{2n}]$ and, up to translations, the only biholomorphisms between $X_1(t'_n)$ and $X_2(t'_n)$ are the projection of $\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{C}^2$.

This is enough to prove that these two families, when seen as families over $[0, \infty)$, are not locally isomorphic at 0. Since the functions b and c are even, the same is true over $(-\infty, 0]$. □

In the differentiable case, it seems difficult to give a sufficient condition to have the local isomorphism property, except for the following trivial one.

Proposition 4. *Let X be a compact complex manifold. Suppose that K is a local moduli space for X (that means that two different points of X corresponds to two non-biholomorphic manifolds). Then every holomorphic (over a reduced base) as well as differentiable deformation family of X has the local isomorphism property.*

Proof. In this case, given any deformation family $\mathcal{X}$ of X, the map from the parameter space of $\mathcal{X}$ to K is uniquely determined by the pointwise complex structure of the fibers. □

6. Universality.

Let us finish this section by a comparison between our uniformization problems and the problem of universality of the Kuranishi space for differentiable families.

Proposition 5. *The Kuranishi space K of some X is universal for differentiable families if and only if h^0 is constant on K.*

Notice the immediate corollary.

Corollary 5. *Let K be the Kuranishi space of some compact complex manifold X. Then the following statements are equivalent.*

(i) The space K is universal for differentiable families.
(ii) The foliation of K described in Section III is trivial.
(iii) The function h^0 is constant on K.

Proof of Proposition 5. The "if" part is an immediate consequence of Proposition K2. Indeed, it is used in [Ku2] to prove that.

For the converse, assume h^0 non-constant. Then, by Proposition 1, there exists an automorphism ϕ of X isotopic to the identity such that any extension as a diffeomorphism of $\mathcal{K}$ does not project onto the identity of K. Let Φ be such an extension. Still by Proposition 1, we may assume that Φ is isotopic to the identity. Let $(\Phi_t)_{t\in[0,1]}$ be such an isotopy, Φ_0 being the identity map. Now set $\Psi(-) = \Phi_{\lambda(-)}(-)$, for some smooth function $\lambda : K \to [0, 1]$ satisfying

(i) $\lambda_0(0) = 0$.
(ii) $\det \mathrm{Jac}_0\ \lambda \neq 0$.

For a good choice of λ, the map Ψ is a local diffeomorphism at 0. Indeed, a direct computation shows that

$$\mathrm{Jac}_0\ \Psi = Id + \mathrm{Jac}_0\ \lambda \cdot \frac{\partial \Phi_t}{\partial t}_{|t=0}$$

so it is enough to take $\|\mathrm{Jac}_0 \lambda\|$ very small.

Recall now that Φ_t may be chosen so that, for all t, its germ at 0 does not project as the germ of the identity (see the proof of Lemma 2 and Proposition 1). From this, we deduce that the germ of Ψ at X_0 is not the identity, even if $\Psi_{|X_0}$ is the identity of X_0. In other words, one can find a path c in K passing through 0 whose image by Ψ is different from c. But this means that the family corresponding to c is locally isomorphic to the family corresponding to $\Psi(c)$, with the same identification at 0. Hence K is not universal for differentiable families. $\square$

We do not know if Proposition 5 is true for *holomorphic* universality property. Of course the "if" part is still true in the holomorphic setting using Proposition K2; and on the other hand, the converse is true if K is reduced (see [Wa]).

Observe that Theorems 3 and 5 compared to Corollary 4 show that, surprisingly, the local isomorphism problem is fundamentally different from the universality problem.

Appendix: countability properties of Sobolev spaces

Let X be a compact complex manifold and X^{diff} be the underlying smooth manifold. Let E be a $\mathbb{R}$ or $\mathbb{C}$ vector bundle over X^{diff}. Choose an atlas (U_i, ϕ_i) on X^{diff}. For any k, we can define a Sobolev norm on the space $C^\infty(X^{diff}, E)$ of smooth sections in E by putting

$$\|f\|^2 = \sum_i \sum_D \int_{\phi_i(U_i)} \langle D(f \circ \phi_i^{-1}), D(f \circ \phi_i^{-1})\rangle dV$$

where

(i) D runs over all derivatives of order less than or equal to k.
(ii) $\langle -, - \rangle$ is a riemannian metric on E.
(iii) dV is a volume form on X^{diff}.

This induces a distance and thus a topology on $C^\infty(X^{diff}, E)$, but also on $Diff(X^{diff})$, the group of diffeomorphisms of X^{diff}: just embed X^{diff} in some $\mathbb{R}^N$ and take E to be the trivial bundle of rank N.

We claim that every *uncountable* sequence $\mathcal{F} = (f_t)_{t\in T}$ of $Diff(X^{diff})$ has an accumulation point for the Sobolev topology.

Assume not. Let $\mathcal{F}$ be such an uncountable sequence without accumulation point. In particular, there cannot exist an infinite sequence $(t_n)_{n\in\mathbb{N}}$ whose corresponding elements of $\mathcal{F}$ all represent the same function. Hence we may assume without loss of generality that f_t is different from $f_{t'}$ as soon as t is different from t'.

Consider now $\mathcal{F}$ as a sequence in

$$\mathcal{C}^k = C^k(X^{diff}, \mathbb{R}^N)$$

endowed with the supremum norm $\| - \|_{\sup}$ on all the derivatives up to order k. Then $\mathcal{F}$ has no accumulation point in $\mathcal{C}^k$, because there exists some constant C such that

$$\forall g \in \mathcal{C}^k \qquad \|g\| \leq C \cdot \|g\|_{\sup}$$

Moreover, the map

$$f \in \mathcal{C}^k \longmapsto j^k f \in \mathcal{C} = C^0(X^{diff}, \mathbb{R}^{N(k)})$$

(where $j^k f$ is the k-jet of f and $N(k)$ is the corresponding dimension) embeds continuously $(\mathcal{C}^k, \| - \|_{\sup})$ in $\mathcal{C}$. So arguing as above, we conclude that $\mathcal{F}$ has no accumulation point in $\mathcal{C}$.

In other words, for all $t \in T$, there exists an open neighborhood V_t of f_t in $\mathcal{C}$ such that

$$V_t \cap V_{t'} \neq \emptyset \iff t = t'$$

(recall that the f_t are supposed to be all distinct).

On the other hand, since X^{diff} is compact and metrizable, it follows from [Bourbaki, Topologie Générale, Chapitre 10, Théorème 3.1] that $\mathcal{C}$ endowed with the topology of uniform convergence is of countable type, that is contains a *countable* dense sequence $(g_n)_{n\in\mathbb{N}}$. For each $t \in T$, there must be some $g_{n(t)}$ in V_t, with the property that $g_{n(t)}$ is different from $g_{n(t')}$ as soon as t is different from t'. So the sequence $(g_n)_{n\in\mathbb{N}}$ is uncountable. Contradiction. The claim is proved.

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