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Abstract

Radiation by solid or fluid bodies can be characterized by resonance modes. They are complex, as well as resonance frequencies, because of the energy loss due to radiation. For ducts, they can be computed from the knowledge of the radiation impedance matrix. For the case of a flanged duct of finite length radiating on one side in an infinite medium, the expression of this matrix was given by Zorumski, using a decomposition in duct modes. In order to calculate the resonance frequencies, the formulation used in Zorumski's theory must be modified as it is not valid for complex frequencies. The analytical development of the Green's function in free space used by Zorumski depends on the integrals of Bessel functions which become divergent for complex frequencies. This paper proposes first a development of the Green's function which is valid for all frequencies. Results are applied to the calculation of the complex resonance frequencies of a flanged duct, by using a formulation of the internal pressure based upon cascade impedance matrices. Several series of resonance modes are found, each series being shown to be related to a dominant duct mode. Influence of higher order duct modes and the results for several fluid densities is presented and discussed.

Keywords: Acoustics, radiation impedance, cylindrical pipe, resonance frequencies
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1. Introduction

For the problem of duct radiation, many works have been done concerning the calculation of radiation for a given duct mode, but to the author's knowkedge no study has been done concerning the resonance modes. We will treat the problem, in order to get a better insight of the coupling of the duct and the

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surrounding space by radiation. We choose the case of a duct with infinite flange, because of its relative simplicity. One difficulty is due to the fact that resonance frequencies (and modes) are complex, because radiation is a form of dissipation. Notice that in the literature, resonance modes are also called eigenmodes: *they must be distinguished from the duct modes* used in the present paper for the purpose of the calculation.

Green's functions are widely used in many physical situations and notably in acoustics, for the calculation of the pressure field radiated by physical sources (*e.g.* speakers, musical instruments, vibrating structures,...). For instance, the solution given by Rayleigh [1] to the classical problem of a plane piston radiating into an infinite flange involves the Green's function in free space. Zorumski [2] extended this result to know the radiation of a semi-infinite flanged duct in the form of matrix impedance, giving the coupling between duct modes and used the Sonine's infinite integral (Ref. [3], p.416, Eq.4) to develop the Green's function in free space. In Refs [4, 5], formulations based on the Zorumski's method of this radiation impedance are obtained for a larger class of problems. However, the development fails when the frequency becomes complex: the corresponding infinite integral, involving a Bessel function whose argument is a product of the frequency and the dummy argument, becomes divergent for complex frequencies. In many studies, the Zorumski's radiation matrix is used as a boundary condition at the end of the duct in order to calculate input impedances, length corrections or reflection coefficients (see e.g. Refs. [6] or [7]). It is worth noting that complex resonance frequencies can occur in various situations (*e.g.* dissipative fluid, radiation, complex impedance wall boundary conditions such as in Refs. [8] or [9]). In section 2 of this paper, we present a new expression for the Green's function in free space for complex frequencies. In section 4, an application of this result is devoted to the determination of the complex resonance frequencies of a cylindrical duct, closed at its input, considering the influence of higher order duct modes. In the same secion, some results are given and discussed. For this purpose, the internal Green's function is previously calculated in section 3 with a method of cascade impedance.

2. Calculations of Green's function for the Helmholtz equation in free space

Many studies on sound radiation by cylindrical ducts can be found in the literature. For the case of an infinite flange, Norris and Sheng [10] or Nomura [11] used a Green's function integral to find an appropriate formulation for the external field. We can also cite the classical work by Levine and Schwinger [12] for the case of an unflanged pipe. Zorumski [2] extended the results for the planar mode to obtain a multimodal radiation impedance which is a combination of the duct modes present in the duct. In this section, these calculations are briefly recalled, exhibiting the difficulty related to complex frequencies. Thus, a new analytical formula for the Green's function, valid for a dissipative problem, is presented.

2.1. Zorumski's radiation impedance

We consider the radiation of sound into an infinite half space from a circular duct (with radius b and length L), with an infinite flange at $z_0 = 0$ (the index 0 corresponds to the cross section S_0 at the end of the duct) and we have chosen to work with circular coordinates where the vector $\mathbf{r}$ is denoted (z, r, θ) as shown by Fig. 1.

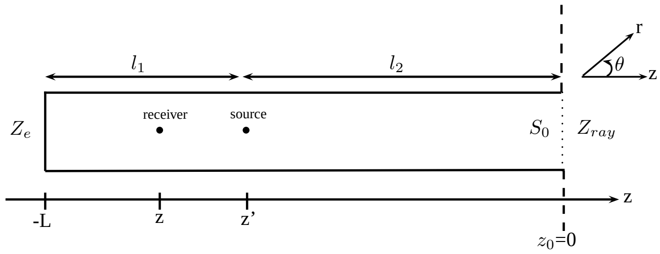

Figure 1: Schema and coordinates of the duct.

The acoustic pressure in the infinite medium ($z \geq 0$) is given by a Helmholtz integral (the time factor $\exp(-i\omega t)$ is omitted throughout this paper):

$$p(\mathbf{r}) = -\frac{i\omega\rho}{2\pi} \int_0^{2\pi} \int_0^b r_0 v(r_0, \theta_0) \frac{e^{ikh}}{h} dr_0 d\theta_0, \tag{1}$$

where $h = [r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) + z^2]^{\frac{1}{2}}$, ρ the ambient density and the wavenumber $k = \omega/c$ (with ω the circular frequency and c the speed of sound). The pressure p and the velocity v inside the duct ($z < 0$) are expressed as a series of duct eigen modes, so in $z = 0$:

$$p(r, \theta, z = 0) = \rho c^2 \sum_m \sum_n \psi_{mn}(kr) e^{im\theta} P_{mn}, \tag{2}$$

$$v(r, \theta, z = 0) = c \sum_m \sum_n \psi_{mn}(kr) e^{im\theta} V_{mn}, \tag{3}$$

where $\psi_{mn}(kr)e^{im\theta}$ is the transverse function for the mode mn with $\psi_{mn}(kr) = J_0(kr)/N_{mn}$. The λ_{mn} are the eigenvalues, solutions of $J'(\lambda_{mn}kb) = 0$, and the norm N_{mn} is chosen similarly to that used by Zorumski [2].

Substituting Eq. (3) into Eq. (1) gives the pressure for $z \geq 0$ in terms of the modal velocity amplitudes V_{mn}:

$$p(r, \theta, z) = -\frac{i\omega\rho c}{2\pi} \sum_m \sum_n V_{mn} \int_0^{2\pi} e^{im\theta_0} \int_0^b r_0 \frac{e^{ikh}}{h} \psi_{mn}(kr_0) dr_0 d\theta_0. \tag{4}$$

Zorumski expressed the free space Green's function in equation (4) in terms of a Sonine's infinite integral ([3], p. 416, Eq. 4) and wrote for $z = 0$ in the expression of h:

$$\frac{e^{ikh}}{h} = k \int_0^\infty \tau(\tau^2 - 1)^{-\frac{1}{2}} J_0(\tau kh) d\tau. \tag{5}$$

Next, he introduced a concept of "generalized radiation impedance matrix $\mathbf{Z_{ray}}$" for a semi-infinite duct with an infinite flange to describe the relation between the modal pressure and velocity amplitudes:

$$P_{mn} = \sum_{l=1}^{\infty} Z_{mnl} V_{ml}, \tag{6}$$

where m, l and n are, respectively, the orders of circumferential, radial incident and reflected modes. The element Z_{mnl} of the radiation impedance matrix gives the contribution of the velocity mode ml to the pressure mode mn (for reasons of symmetry, the coupling is possible only for a duct mode with the same azimuthal dependance). The expression of the radiation impedance is obtained as:

$$Z_{mnl} = -i \int_0^{\infty} \tau(\tau^2 - 1)^{-\frac{1}{2}} D_{mn}(\tau, k) D_{ml}(\tau, k) d\tau, \tag{7}$$

with, for a hard wall condition:

$$D_{mn}(\tau, k) = kb \frac{\tau \psi_{mn}(kb) J'_m(\tau kb)}{\lambda_{mn}^2 - \tau^2}. \tag{8}$$

2.2. Green's function for the Helmholtz equation in free space for complex frequencies

The following asymptotic form (see Ref. [13], Eq. 9.2.1, p. 364) occurs when ν is fixed and $|\kappa| \to \infty$:

$$J_\nu(\kappa) = \sqrt{\frac{2}{\pi\kappa}} [cos(\kappa - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + e^{|\Im(\kappa)|} O(|\kappa|^{-1})], \tag{9}$$

with $|\arg \kappa| < \pi$ (in this paper, the real part and imaginary part are represented, respectively, by the symbols $\Re$ and $\Im$). As a consequence, for $\tau \to \infty$ with $k \in \mathbb{C}$ we have $J_0(\tau k h) \to \infty$ when $\Im(k) \neq 0$, thus relation (5) and the radiation impedance (7) given by Zorumski are divergent integrals for all non real frequencies.

In order to have a Green's function for the Helmholtz equation in free space valid for complex frequencies, we use another form of the Sonine's infinite integral to develop this Green's function, expressed by Watson [3] (p. 416, Eq. 4):

$$\frac{e^{ikh}}{h} = \int_0^{\infty} \tau(\tau^2 - k^2)^{-\frac{1}{2}} J_0(\tau h) d\tau. \tag{10}$$

This integral remains convergent even for k complex. A difficulty occurs since k is a branch point of the square root. For a time factor $\exp(-i\omega t)$, the integration path on the real axis must remain below k. However the complex resonance frequencies $\omega = ck$ have a negative imaginary part (see explanation in section 4), so the previous formula must be adapted because of the branch cut. The integration path below k is classically deformed, as shown in Fig. 2:

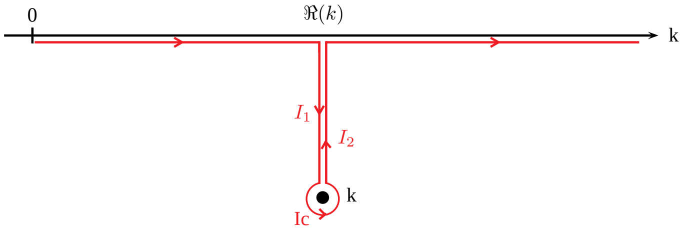

Figure 2: Deformation of the integration contour.

Now, the integral in Eq. (10) is written as:

$$\frac{e^{ikh}}{h} = \int_0^{|\Re(k)|} J_0(\tau h)\tau(\tau^2-k^2)^{-\frac{1}{2}}d\tau + I_1(k,h) + I_c + I_2(k,h) + \int_{|\Re(k)|}^{\infty} J_0(\tau h)\tau(\tau^2-k^2)^{-\frac{1}{2}}d\tau, \quad (11)$$

with

$$I_1(k,h) = I_2(k,h) = -\int_{|\Re(k)|}^{k} J_0(\tau h)\tau(\tau^2-k^2)^{-\frac{1}{2}}d\tau$$

and I_c is zero according to Jordan's lemma.
After calculations (similar to those developed by Morse and Feshbach [14], p.410), the following five cases can be distinguished:
i) $\Re(k) \leq 0$ and $\Im(k) > 0$

$$\frac{e^{ikh}}{h} = \int_0^{\infty} J_0(\tau h)\tau(\tau^2-k^2)^{-\frac{1}{2}}d\tau. \quad (12)$$

ii) $\Re(k) \leq 0$ and $\Im(k) \leq 0$

$$\begin{aligned}\frac{e^{ikh}}{h} &= -i\int_0^{|\Re(k)|} J_0(\tau h)\frac{\tau}{\sqrt{k^2-\tau^2}}d\tau + \int_{|\Re(k)|}^{\infty} J_0(\tau h)\frac{\tau}{\sqrt{\tau^2-k^2}}d\tau \\ &\quad -2\int_{|\Re(k)|}^{k} J_0(\tau h)\frac{\tau}{\sqrt{\tau^2-k^2}}d\tau. \end{aligned} \quad (13)$$

iii) $\Re(k) > 0$ and $\Im(k) > 0$

$$\frac{e^{ikh}}{h} = +i\int_0^{|\Re(k)|} J_0(\tau h)\frac{\tau}{\sqrt{k^2-\tau^2}}d\tau + \int_{|\Re(k)|}^{\infty} J_0(\tau h)\frac{\tau}{\sqrt{\tau^2-k^2}}d\tau. \quad (14)$$

iv) $\Re(k) > 0$ and $\Im(k) = 0$

$$\begin{aligned}\frac{e^{ikh}}{h} &= J_0(kh)k\sqrt{2\epsilon}(1-i) + i\int_0^{|k|(1-\epsilon)} J_0(\tau h)\frac{\tau}{\sqrt{k^2-\tau^2}}d\tau \\ &\quad + \int_{|k|(1+\epsilon)}^{\infty} J_0(\tau h)\frac{\tau}{\sqrt{\tau^2-k^2}}d\tau, \end{aligned} \quad (15)$$

with $\epsilon << 1$.
v) $\Re(k) > 0$ and $\Im(k) < 0$

$$\begin{aligned}\frac{e^{ikh}}{h} &= +i\int_0^{|\Re(k)|} J_0(\tau h)\frac{\tau}{\sqrt{k^2-\tau^2}}d\tau + \int_{|\Re(k)|}^{\infty} J_0(\tau h)\frac{\tau}{\sqrt{\tau^2-k^2}}d\tau \\ &\quad -2\int_{|\Re(k)|}^{k} J_0(\tau h)\frac{\tau}{\sqrt{\tau^2-k^2}}d\tau. \end{aligned} \tag{16}$$

Contrary to the original Zorumski's formulation, these results involve convergent integrals when the frequency is complex.

2.3. New formulation of generalized impedance of a flanged circular duct for real frequencies

Initially, the previous result will be checked for the real case, in order to compare the original Zorumski's result, noted in Eq. (7), and that obtained using the expansion of $\exp(ikh)/h$ for the case (iv) where $\Im(k) = 0$. Eq. (15) leads to the following results:

$$\begin{aligned} Z_{mnl} &= -ik[\tilde{D}_{mn}(k)\tilde{D}_{ml}(k)k\sqrt{2\epsilon}(1-i) + i\int_0^{|k|(1-\epsilon)} \frac{\tau}{\sqrt{k^2-\tau^2}}\tilde{D}_{mn}(\tau)\tilde{D}_{ml}(\tau)d\tau \\ &\quad + \int_{|k|(1+\epsilon)}^{\infty} \frac{\tau}{\sqrt{\tau^2-k^2}}\tilde{D}_{mn}(\tau)\tilde{D}_{ml}(\tau)d\tau], \end{aligned} \tag{17}$$

where

$$\tilde{D}_{mn}(\tau) = b\frac{\tau\psi_{mn}(b)J'_m(\tau b)}{\lambda_{mn}^2-\tau^2}. \tag{18}$$

The radiation impedance for the planar mode ($m = n = 0$ with $l = 0$) with Zorumski's formulation (7) and formulation (17) (with $\epsilon = 10^{-6}$) are very similarly and thus, confirms the validity of formula (17) with the identical computational cost. This formula is used in Ref. [15] to calculate an approximation of the reflection coefficient and of the length correction, taking into account the effect of the higher order duct modes below the first cut-off frequency.
The comparison between complex Zorumski's formulation for the planar mode (m=n=0 and l=0) and Rayleigh's radiation impedance of a flanged plane piston confirm the validity of Z_{000} for all the frequencies (see subsection 4.1), because Rayleigh's radiation impedance is by definition the same quantity as Z_{000}.
In what follows, we show the interest of the radiation impedance valid for complex frequencies when calculating the complex resonance frequencies of a flanged finite length duct terminating in a Zorumski's radiation condition. In a first instance, the internal Green's function must be determined, with a method of cascade impedances.

3. Calculation of the finite flanged duct internal Green's function with a method of cascade impedances

3.1. Definition of the internal Green's function

We search for the internal Green's function $G(M, M', \omega)$ at a point $M(r, \theta, z)$ with a source in $M'(r', \theta', z')$, satisfying:

$$(\Delta_M + k^2)G(M, M', \omega) = -\frac{1}{2\pi r}\delta(r - r')\delta(\theta - \theta')\delta(z - z'), \qquad (19)$$

with $\partial_n G(M, M', \omega) = 0$ on the walls. It is classically (see e.g. Morse and Feshbach [14]) expanded in a series of duct modes (the boundary conditions for the variable z will be given later on):

$$G(M, M', \omega) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \phi_{mn}(r, \theta)\phi_{mn}(r', \theta')g_{mn}(z, z', \omega), \qquad (20)$$

where $\phi_{mn}(r, \theta) = \psi_{mn}(r)e^{im\theta}$ is the transverse function for the mode mn and $g_{mn}(z, z', \omega)$ is the longitudinal function for the mode mn. In what follows, the dependance in ω of $g_{mn}(z, z', \omega)$ is omitted for simplicity. It is worth noting that now we have the transverse function with respect to r and not kr, as per Zorumski [2]. Therefore, $\psi_{mn}(r) = \frac{J_m(\lambda_{mn} r)}{N_{mn}}$ where the λ_{mn} are solutions of $J'_m(\lambda_{mn} b) = 0$, with $\lambda_{mn} = \gamma_{mn}/b$. The γ_{mn} are the $(n+1)^{th}$ zeros of the first derivative of the Bessel's function J_m. The following norm N_{mn} is chosen (Ref. [2] with respect to b and not kb):

$$N_{mn} = b\sqrt{\pi}\sqrt{1 - \frac{m^2}{\lambda_{mn}^2 b^2}}J_m(\lambda_{mn} b). \qquad (21)$$

The transverse modes $\phi_{mn}(r, \theta)$ satisfy:

$$(\Delta_\perp + \lambda_{mn}^2)\phi_{mn}(r, \theta) = 0,$$

with $\Delta_\perp = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, thus: $\Delta_\perp\phi_{mn}(r, \theta) = -\lambda_{mn}^2\phi_{mn}(r, \theta)$. Studies such as Refs. [6] or [7] show that a small finite number of terms is necessary for a numerical estimate of the summation and it can be truncated: M_m is defined as the number of circumferential modes m and N_n as the number of radial modes n. Introducing the previous expression (20) in (19) gives:

$$\sum_{m=0}^{M_m}\sum_{n=0}^{N_n}(\frac{\partial^2}{\partial z^2} + k_{mn}^2)\phi_{mn}(r, \theta)\phi_{mn}(r', \theta')g_{mn}(z, z') = -\frac{1}{2\pi r}\delta(r - r')\delta(\theta - \theta')\delta(z - z'), \qquad (22)$$

where $k_{mn}^2 = k^2 - \lambda_{mn}^2$. Then, multiplying the left and right sides of (22) by $\phi_{m'l}^*(r, \theta)$ and integrating the resulting equality on the surface S, the orthogonality condition ($\int_S \phi_{mn}\phi_{m'l}^* dS = \delta_{mm'}\delta_{nl}$), leads to:

$$(\frac{\partial^2}{\partial z^2} + k_{mn}^2)g_{mn}(z, z') = -\delta(z - z'). \qquad (23)$$

With a conservative Neumann or Dirichlet boundary condition at the extremities of the duct, the resonance wavenumbers can be easily computed. But with a dissipative boundary condition like those occurring for the sound radiation of a flanged cylinder, there is no simple analytical solution. Therefore, in the next section, the elements $g_{mn}(z, z')$ are calculated with a method of cascade impedances presented in Refs. [6],[16] or [17].

3.2. Presentation of the method of cascade impedances

The calculation of the duct impedance at an abscissa z_1 with respect to another abscissa z_2 is based on the following transfer matrix relationship (see e.g. Ref. [17]):

$$\begin{pmatrix} P_{mn}(z_1) \\ V_{mn}(z_1) \end{pmatrix} = (M_T) \begin{pmatrix} P_{mn}(z_2) \\ V_{mn}(z_2) \end{pmatrix}, \tag{24}$$

$$M_T = \begin{pmatrix} \cosh(ik_{mn}(z_2 - z_1)) & Z_{c,mn}\sinh(ik_{mn}(z_2 - z_1)) \\ \dfrac{\sinh(ik_{mn}(z_2 - z_1))}{Z_{c,mn}} & \cosh(ik_{mn}(z_2 - z_1)) \end{pmatrix},$$ where $Z_{c,mn}$ is an element of the diagonal matrix of characteristic impedance $\mathbf{Z_c}$:

$$Z_{c,mn} = \frac{k\rho c}{k_{mn}}.$$

Moreover, matrices formulation is chosen: $\mathbf{\Phi}(r, \theta)$ is a column vector constituted by the $M_m N_n$ elements ϕ_{mn}, verifying $\int_S \mathbf{\Phi}(r, \theta)\mathbf{\Phi}^T(r, \theta)dS = \mathbf{I}$, $\mathbf{P}(z)$ is a column vector constituted by the $M_m N_n$ elements P_{mn} and $\mathbf{V}(z)$ is a column vector constituted by the $M_m N_n$ elements V_{mn}. Thus, relation (24) is now written:

$$\begin{pmatrix} \mathbf{P}(z_1) \\ \mathbf{V}(z_1) \end{pmatrix} = \begin{pmatrix} \mathbf{C} & \mathbf{Z_c S} \\ \mathbf{Z_c^{-1} S} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{P}(z_2) \\ \mathbf{V}(z_2) \end{pmatrix}, \tag{25}$$

where $\mathbf{C}$ and $\mathbf{S}$ are diagonal matrices constituted by the elements $\cosh(ik_{mn}(z_2 - z_1))$ and $\sinh(ik_{mn}(z_2 - z_1))$, respectively.
The transfer matrix formulation (25) is now transformed into an impedance matrix formulation. The calculation, presented in Ref. [17], is recalled in Appendix A. This gives the following matrix equation:

$$\begin{pmatrix} \mathbf{P}(z_1) \\ \mathbf{P}(z_2) \end{pmatrix} = \begin{pmatrix} \mathbf{Z_{11}} & -\mathbf{Z_{12}} \\ \mathbf{Z_{21}} & -\mathbf{Z_{22}} \end{pmatrix} \begin{pmatrix} \mathbf{V}(z_1) \\ \mathbf{V}(z_2) \end{pmatrix}, \tag{26}$$

where $\mathbf{Z_{11}} = \mathbf{Z_c S^{-1} C}$, $\mathbf{Z_{12}} = \mathbf{Z_c S^{-1}}$, $\mathbf{Z_{21}} = \mathbf{Z_c S^{-1}}$ and $\mathbf{Z_{22}} = \mathbf{Z_c S^{-1} C}$.

*3.3. Calculation of **P**(z')*

First, the pressure vector at the source position z' is calculated. For this purpose, we calculate a right-side matrix impedance $\mathbf{Z}^+(z')$ at z' with respect to $\mathbf{Z_{ray}}$, a left-side matrix impedance $\mathbf{Z}^-(z')$ at z' with respect to $\mathbf{Z}_e$ and finally the connection between these two matrices at z', the abscissa of the source.

Step 1: Right-side matrix impedance $\mathbf{Z^+(z')}$
Let us denote $\mathbf{C_s}$ and $\mathbf{S_s}$ the diagonal matrix constituted by the elements $\cosh(ik_{mn}l_2)$ and $\sinh(ik_{mn}l_2)$, respectively, where l_2 (see Fig. 1) is the distance between the abscissa z' and the extremity ($z = 0$). The radiation impedance $\mathbf{Z_{ray}}$, constituted by the elements Z_{mnl} calculated in the previous section, verifies $\mathbf{P}(z' + l_2) = \mathbf{Z_{ray}}\mathbf{V}(z' + l_2)$. Thus Eq. (25) leads to:

$$\mathbf{V}(z' + l_2) = (\mathbf{Z_{ray}} + \mathbf{Z_{22}})^{-1}\mathbf{Z_{21}}\mathbf{V}(z'), \tag{27}$$

and, with Eqs. (26) and (27), to:

$$\mathbf{P}(z') = \mathbf{Z_{11}}\mathbf{V}(z') - \mathbf{Z_{12}}(\mathbf{Z_{ray}} + \mathbf{Z_{22}})^{-1}\mathbf{Z_{21}}\mathbf{V}(z').$$

With $\mathbf{Z}^+$ verifying $\mathbf{P}(z') = \mathbf{Z}^+(z')\mathbf{V}(z')$, we have finally:

$$\mathbf{Z}^+(z') = \mathbf{Z_{11}} - \mathbf{Z_{12}}(\mathbf{Z_{ray}} + \mathbf{Z_{22}})^{-1}\mathbf{Z_{21}}, \tag{28}$$

or:

$$\mathbf{Z}^+(z') = \mathbf{Z_c}\mathbf{S_s^{-1}}\mathbf{C_s} - \mathbf{Z_c}\mathbf{S_s^{-1}}[\mathbf{Z_c^{-1}}\mathbf{Z_{ray}} + \mathbf{S_s^{-1}}\mathbf{C_s}]^{-1}\mathbf{S_s^{-1}}. \tag{29}$$

Step 2: Left-side matrix impedance $\mathbf{Z^-(z')}$
We choose a Neumann condition for $z = -L$ (thus $\mathbf{V}(z' - l_1) = 0$). Here, $\mathbf{C_e}$ and $\mathbf{S_e}$ are the diagonal matrices constituted by the elements $\cosh(ik_{mn}l_1)$ and $\sinh(ik_{mn}l_1)$, respectively. l_1 (see Fig. 1) is the distance between the point $z = -L$ and the point z'. The impedance $\mathbf{Z_e}$ is calculated at $z = -L$.
Relation (26) is written for the present case as:

$$\mathbf{P}(z') = \mathbf{Z_c}\mathbf{S_e^{-1}}\mathbf{V}(z' - l_1) - \mathbf{Z_c}\mathbf{S_e^{-1}}\mathbf{C_e}\mathbf{V}(z'). \tag{30}$$

Thus, with the Neumann condition in $z = -L$ and with Eq. (30), using $\mathbf{P}(z') = \mathbf{Z}^-(z')\mathbf{V}(z')$, the following result is obtained:

$$\mathbf{Z}^-(z') = -\mathbf{Z_c}\mathbf{S_e^{-1}}\mathbf{C_e}. \tag{31}$$

Step 3: Connection between the impedance matrices $\mathbf{Z^+}$ and $\mathbf{Z^-}$ at $\mathbf{z'}$
Let us denote $P^{\pm}_{mn}(z') = [g_{mn}(z, z')]_{z=z'\pm\epsilon}$. Using the continuity of the Green's function at $z = z'$, leads to when $\epsilon \to 0$:

$$P^+_{mn}(z') = P^-_{mn}(z') = P_{mn}(z'), \tag{32}$$

Integrating relation (23) on an interval of width 2ε between $z' + \varepsilon$ and $z' - \varepsilon$ gives:

$$\int_{z'-\varepsilon}^{z'+\varepsilon} (\frac{\partial^2}{\partial_{z^2}} + k^2_{mn})g_{mn}(z, z')dz = -1 \tag{33}$$

and, with the pressure continuity, when $\varepsilon \to 0$:

$$\partial_z P^+_{mn}(z') - \partial_z P^-_{mn}(z') = -1, \tag{34}$$

with $\partial_z P^{\pm}_{mn}(z') = [\partial_z g_{mn}(z,z')]_{z=z'\pm\epsilon}$.
Euler's dimensionless equation $\frac{1}{\rho c}V_{mn}(z') = -\frac{i}{\omega\rho}\partial_z P_{mn}(z')$ implies $\partial_z P_{mn}(z') = ikV_{mn}(z')$, thus, using Eq. (34):

$$V^{+}_{mn}(z') - V^{-}_{mn}(z') = -\frac{1}{ik}. \tag{35}$$

We introduce a column vector $\mathbf{W}$ of $M_m N_n$ lines whose elements equal 1. Equation (35) may be expressed as:

$$(\mathbf{Z}^{+}(z'))^{-1}\mathbf{P}^{+}(z') - (\mathbf{Z}^{-}(z'))^{-1}\mathbf{P}^{-}(z') = -\frac{1}{ik}\mathbf{W},$$

and using Eq. (32), the pressure at the source is written as follows:

$$\mathbf{P}(z') = -\frac{1}{ik}\left[(\mathbf{Z}^{+}(z'))^{-1} - (\mathbf{Z}^{-}(z'))^{-1}\right]^{-1}\mathbf{W}. \tag{36}$$

3.4. Expression of the function ***g****(z,z')*

We introduce a column vector $\mathbf{g}(z,z')$ constituted by the $M_m N_n$ elements $g_{mn}(z,z')$. They are two possible configurations with respect to the relative positions of receiver at z and source at z':

First configuration: $\mathbf{z > z'}$
Let $l_r = z - z'$ be the distance between the receiver and the source (see Fig. 1), $\mathbf{C_{l_r}}$ the diagonal matrix constituted by the elements $\cosh(ik_{mn}l_r)$, and $\mathbf{S_{l_r}}$ the diagonal matrix constituted by the elements $\sinh(ik_{mn}l_r)$. Relation (24) gives for $z > z'$:

$$\mathbf{g}(z,z') = \mathbf{C_{l_r}}\mathbf{P}(z') - \mathbf{Z_c}\mathbf{S_{l_r}}\mathbf{V}(z'),$$

then, with $\mathbf{P}(z') = \mathbf{Z}^{+}(z')\mathbf{V}(z')$:

$$\mathbf{g}(z,z') = \mathbf{S_{l_r}}\left[\mathbf{S_{l_r}^{-1}}\mathbf{C_{l_r}} - \mathbf{Z_c}(\mathbf{Z}^{+}(z'))^{-1}\right]\mathbf{P}(z'). \tag{37}$$

Second configuration: $\mathbf{z' > z}$
Let $l_l = z' - z$ be the distance between the receiver and the source(see Fig. 1), $\mathbf{C_{l_l}}$ the diagonal matrix constituted by the elements $\cosh(ik_{mn}l_l)$, and $\mathbf{S_{l_l}}$ the diagonal matrix constituted by the elements $\sinh(ik_{mn}l_l)$. Relation (24) gives for $z' > z$:

$$\mathbf{g}(z,z') = \mathbf{C_{l_l}}\mathbf{P}(z') + \mathbf{Z_c}\mathbf{S_{l_l}}\mathbf{V}(z'),$$

then, with $\mathbf{P}(z') = \mathbf{Z}^{-}(z')\mathbf{V}(z')$:

$$\mathbf{g}(z,z') = \mathbf{S_{l_l}}\left[\mathbf{S_{l_l}^{-1}}\mathbf{C_{l_l}} + \mathbf{Z_c}(\mathbf{Z}^{-}(z'))^{-1}\right]\mathbf{P}(z'). \tag{38}$$

Finally, the Green's function of a finite duct with an infinite flange is given as:

$$\mathbf{G}(M,M') = \mathbf{\Phi}(r,\theta)\mathbf{\Phi}(r',\theta')\mathbf{g}(z,z'). \tag{39}$$

4. Application to complex resonance frequencies of a flanged, finite length duct

Resonances of a flanged, finite length duct are interesting as they contain important information about the coupling between internal and external fluids. Their calculation is based on the fact that the internal pressure becomes infinite at each resonance. Newton's method is used to compute the zeros of the inverse of the pressure. Since the resonances of a dissipative problem are complex, a complex formulation of the impedance radiation is needed. As a time dependence $\exp(-i\omega t)$ has been chosen, the imaginary part needs to be negative for resonance frequencies in order to ensure that the amplitude remains bounded for all times t. Using the integrals (13) and (16), for $\Re(k) > 0$ and $\Im(k) < 0$, Z_{mnl} becomes:

$$\begin{aligned} Z_{mnl} \quad = \quad & -ik[i \int_0^{|\Re(k)|} \frac{\tau}{\sqrt{k^2 - \tau^2}} \tilde{D}_{mn}(\tau)\tilde{D}_{ml}(\tau)d\tau + \int_{|\Re(k)|}^{+\infty} \frac{\tau}{\sqrt{\tau^2 - k^2}} \tilde{D}_{mn}(\tau)\tilde{D}_{ml}(\tau)d\tau \\ & -2 \int_{|\Re(k)|}^{k} \frac{\tau}{\sqrt{\tau^2 - k^2}} \tilde{D}_{mn}(\tau)\tilde{D}_{ml}(\tau)d\tau]. \end{aligned} \tag{40}$$

This expression is used as a radiation condition $\mathbf{Z_{ray}}$ at the end of the finite length duct.

4.1. Resonance wavenumbers for the planar mode without influence of higher order duct modes

In a first instance, we consider the resonance wavenumbers of the planar duct mode (m=n=0) without the influence of higher order duct modes (l=0). With radiation, the j^{th} resonance wavenumbers of duct mode mn are denoted $k^j_{mn,r}$ and denoted k^j_{mn} without radiation.
In order to validate the complex formulation of the radiation impedance, we compare the resonances obtained with radiation impedance given by relation (40) for $m = n = 0$ and $l = 0$, with the resonances obtained with the radiation impedance of a flanged plane piston given by Rayleigh's formulation as (see Ref. [14] p.1458):

$$Z_0 \simeq \rho c[1 - \frac{1}{kb} J_1(2kb) - \frac{i}{kb} S_1(2kb)] \tag{41}$$

The radiation impedance for m=n=l=0 calculated with relation (40) and that of a flanged plane piston (41) are identical. Thus, resonance wavenumbers calculated with these two formulations are so identical, as observed in Fig. 3.

It is worth noting in Table 1 that without radiation, the resonance frequencies are those obtained by the usual longitudinal resonances of a cylindrical duct with one side "closed" and the other side "open" (Neumann/Dirichlet problem) for:

$$k^j_{00} = \frac{(2j+1)\pi}{2L}, j = 0, 1, 2, ...$$

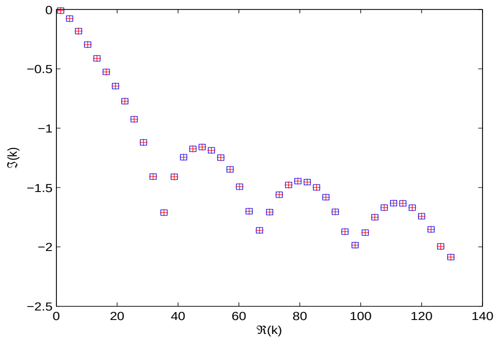

Figure 3: Evolution of resonance wavenumbers $k^j_{00,r}$ depending on wavenumber k (with $b = 0.1m$ and $L = 1m$) with the radiation impedance defined by Eq. (40) for $m = n = 0$ and $l = 0$ (+) and with the radiation impedance of a flanged plane piston (Eq. 41) (□).

We can observe in Table 1 of Appendix B that the real part of resonance frequency decreases when radiation is taken into account: this is normal behavior because the reactive effect of radiation can roughly be described as an increase in the duct length. The semi-infinite duct length correction is estimated by the following formula (see Ref. [10] with the modification given in private communication):

$$\frac{\triangle L}{b} = 0.82159 \frac{1 + \frac{kb}{1.2949}}{1 + \frac{kb}{1.2949} + (\frac{kb}{1.2949})^2}. \quad (42)$$

The difference between the real part of the j first resonance wavenumbers $k^j_{00,r}$ of a finite radiating cylindrical duct (below the first cut off wavenumber $k_{cut01} = 3.83/b$) and those estimated using the length correction defined by Eq. (42) ($k^j_{00,\triangle L} = (2j+1)\pi/[2(L + \triangle L)]$), tends to zero when the ratio L/b increases. The values agree well ($\leq 1\%$ of difference) for $L/b \geq 3$ with several values of L.

4.2. Resonance wavenumbers with influence of higher order duct modes

The principal interest of the complex Zorumski's formulation is that we can observe the influence of higher order duct modes (also denoted by H.M afterward). In this paper, we taken into account only the axisymmetric modes ($m = 0$) (but the non-axisymmetric are very easy to calculate with the same method). Figure 4 shows resonance wavenumbers $k^j_{00,r}$, $k^j_{01,r}$, $k^j_{02,r}$. We observe three series of resonances. Each series starts at the cutoff frequency of a duct mode. The first one corresponds to a domination of the planar duct mode, the second one to a domination of duct mode 01 and the third one to a domination of duct mode 02 (see Appendix B, the Green's function profile is plotted around k^{14}_{00} and k^7_{01}). Fig. 5 shows the influence of the two first higher order duct modes ($m = 0, l = 1$ and $m = 0, l = 2$) on the resonance wavenumbers of the first series. It is worth noting that below the first cut off wavenumber $k_{cut01} = 3.83/b$, only one higher order duct mode is sufficient to accurately describe the resonances (see some values in Table 1); between the first and second cut off wavenumber $k_{cut02} = 7.02/b$, only two higher order duct modes are enough and similarly to the higher order duct modes: between the n^{th} and $(n+1)^{th}$ cut off, only $(n+1)$ higher order duct modes are enough.

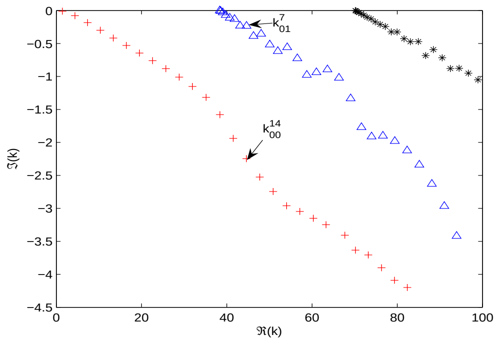

Figure 4: Resonance wavenumbers $k^j_{00,r}$ $(+)$, $k^j_{01,r}$ $(\triangle)$, $k^j_{02,r}$ $(*)$ for $L = 1m$ and $b = 0.1m$. Figure B.8 in Appendix C shows the mode profiles around the two wavenumbers indicated by an arrow.

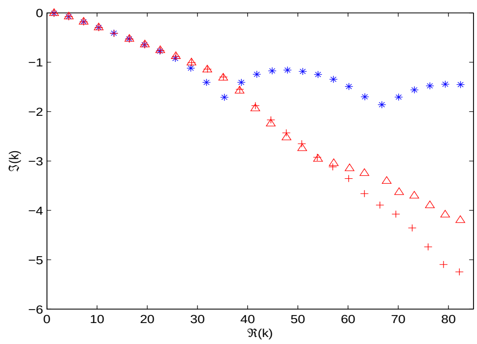

Figure 5: Resonance wavenumbers $k^j_{00,r}$ of the first series ($m = 0$,$n = 0$) with an influence of 0 ($l = 0$: $*$), 1 ($l = 1$: $\triangle$) and 2 ($l = 2$: $+$) H.M for $L = 1m$ and $b = 0.1m$

j	k_{00}^j	$k_{00,r}^j$ with 0 H.M	$k_{00,r}^j$ with 1 H.M
0	1.5708	1.449 - 0.0095i	1.451 - 0.0096i
1	4.712	4.369-0.077i	4.375-0.0776i
2	7.854	7.333-0.182i	7.345-0.183i
3	10.996	10.336-0.296i	10.345-0.299i
4	14.137	13.365-0.412i	13.4-0.415i
5	17.279	16.409-0.526i	16.45-0.53i
6	20.42	19.463-0.645i	19.523-0.644i
7	23.562	22.523-0.773i	22.608-0.76i
8	26.7	25.59 -0.924i	25.707-0.881i
9	29.85	28.674-1.12i	28.824-1.01i
10	32.99	31.822-1.408i	31.965-1.152i
11	36.13	35.377-1.711i	35.149-1.317i
12	39.27	38.747-1.409i	38.395-1.579i

Table 1: Values of the j first resonance wavenumbers without radiation (k_{00}^j) and with radiation ($k_{00,r}^j$) for 0 and 1 H.M, with $b = 0.1m$ and $L = 1m$.

4.3. Evolution of the j first resonance wavenumbers with respect to radiation

In the present section, we show the evolution of the j first resonance wavenumbers when only the planar duct mode propagates with respect to radiation and as shown in the previous section, we take into account the effect of one higher order duct mode. For this purpose, we introduce a multiplicative coefficient on the radiation impedance. Physically, this coefficient η_ρ can be regarded as the ratio between the external fluid density ρ_{ext} and the internal fluid density ρ_{int}, such as:

$$\eta_\rho = \frac{\rho_{ext}}{\rho_{int}}, \tag{43}$$

the density of the external fluid ρ_{ext} varying from a vacuum to water density, the sound celerity, $340m.s^{-1}$, remaining constant.

Figures 6 and 7 show that when the parameter η_ρ increases, a Neumann/Neumann problem is obtained: the real part tends to $j\pi/L$ and the imaginary part tends to zero. This behavior corresponds to a system without losses, the external fluid becoming a perfectly reflective surface.

Fig. 7 shows that the absolute value of the imaginary part of the resonance frequency goes through a maximum, corresponding to the maximum of the radiation. Similarly results have been observed for non planar modes. So, we can conclude that energy radiation losses evolves with the densities of internal and external fluid and goes through a maximum for a specific densities ratio.

5. Conclusion

A development of the Green's function for the Helmholtz equation in a free space valid for complex frequencies is possible and leads to a new formula in Zo-

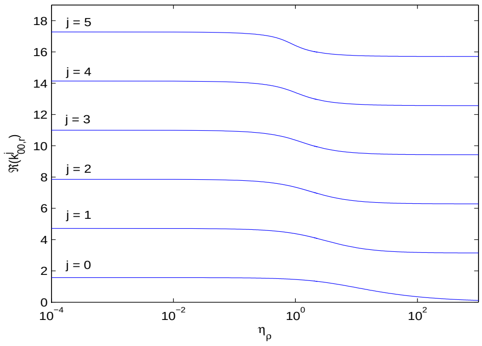

Figure 6: Evolution of the real part of the j first longitudinal resonance wavenumbers $k^j_{00,r}$ with respect to η_ρ, for $L = 1m$ and $b = 0.1m$.

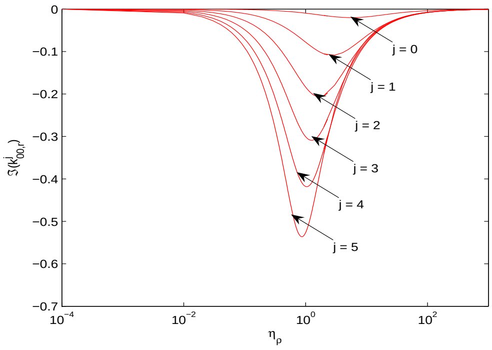

Figure 7: Evolution of the imaginary part of the j first longitudinal resonance wavenumbers $k^j_{00,r}$ with respect to η_ρ, for $L = 1m$ and $b = 0.1m$.

rumski's radiation impedance. The interest has been shown for an application example, dedicated to the calculations of the complex resonance frequencies of a radiating flanged cylindrical duct. It has been shown that length correction calculated for a semi-infinite duct is a good estimate of a finite duct radiation when the ratio $L/b \geq 2$ and for frequencies sufficiently below the first cut off frequency. The study of the influence of higher order duct modes has shown that below the first cut off frequency, only one higher order duct mode is needed to accurately describe the influence of the external fluid on the resonances and between the n^{th} and $(n+1)^{th}$ cut off, only $(n+1)$ higher order duct modes are enough. In the last part, it has been observed a maximum of radiation for a specific densities ratio. In the future, it will be interesting to use a BEM method to study more complicated geometries and to observe the resonances with an experimental method. This work is a first step to study the relation between radiation and several parameters in order to optimize geometry for minimizing (e.g. for noise pollution) or maximizing the sound radiation (e.g. for wind instruments).

Acknowledgements

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Appendix A. Matricial calculation

The main steps required to obtain the results of section 3.2 are presented here (see Ref. [17]).

For a cylinder, the general solutions at point z_1 can be described with respect to the values of P and V at point z_2 such as described by relation (24). For all the modes mn, the following matrix problem is obtained:

$$\mathbf{P}(z_1) = \mathbf{C}\mathbf{P}(z_2) + \mathbf{Z_c}\mathbf{S}\mathbf{V}(z_2), \tag{A.1}$$

$$\mathbf{V}(z_1) = \mathbf{Z_c^{-1}}\mathbf{S}\mathbf{P}(z_2) + \mathbf{C}\mathbf{V}(z_2), \tag{A.2}$$

$\mathbf{C}$ being a diagonal matrix constituted by the elements $\cosh(ik_{mn}(z_2 - z_1))$ et $\mathbf{S}$ a diagonal matrix constituted by the elements $\sinh(ik_{mn}(z_2 - z_1))$. Eq. (A.2) implies:

$$\mathbf{P}(z_2) = \mathbf{Z_c}\mathbf{S^{-1}}\mathbf{V}(z_1) - \mathbf{Z_c}\mathbf{S^{-1}}\mathbf{C}\mathbf{V}(z_2). \tag{A.3}$$

Introducing Eq. (A.3) in Eq. (A.1) and using the commutativity of the diagonal matrices, we obtain:

$$\mathbf{P}(z_1) = \mathbf{Z_c}\mathbf{S^{-1}}\mathbf{C}\mathbf{V}(z_1) - [\mathbf{Z_c}\mathbf{S^{-1}}\mathbf{C}\mathbf{C} - \mathbf{Z_c}\mathbf{S}]\mathbf{V}(z_2),$$

with

$$\begin{aligned}\mathbf{Z_c}\mathbf{S^{-1}}\mathbf{C}\mathbf{C} - \mathbf{Z_c}\mathbf{S} &= \mathbf{Z_c}\mathbf{S^{-1}}(\mathbf{I} + \mathbf{S}\mathbf{S}) - \mathbf{Z_c}\mathbf{S} \\ &= \mathbf{Z_c}\mathbf{S^{-1}} + \mathbf{Z_c}\mathbf{S} - \mathbf{Z_c}\mathbf{S} \\ &= \mathbf{Z_c}\mathbf{S^{-1}},\end{aligned}$$

thus:

$$\mathbf{P}(z_1) = \mathbf{Z_c}\mathbf{S^{-1}}\mathbf{C}\mathbf{V}(z_1) - \mathbf{Z_c}\mathbf{S^{-1}}\mathbf{V}(z_2). \tag{A.4}$$

Therefore, with Eqs. (A.3) and (A.4), we obtain the relation (26).

Appendix B. Green's function profile in the duct around resonance frequencies

Figure B.8 shows that around the resonance frequency k_{00}^{14}, the profile of Green's function corresponds to the profile of the planar duct mode even if duct mode 01 is propagating and similarly around the resonance frequency k_{01}^{7}, the profile of Green's function corresponds to the profile of the first non planar duct mode even if planar duct mode is propagating. The same comportment is observed around other resonance frequencies. Therefore, it is worth noting that each series observed in Fig. 4 corresponds to a predominant duct mode even if other duct modes are propagating. Notice that the evanescent duct modes exist mainly near to the source (at $z = -0.5$): this clearly appears on the upper figure.

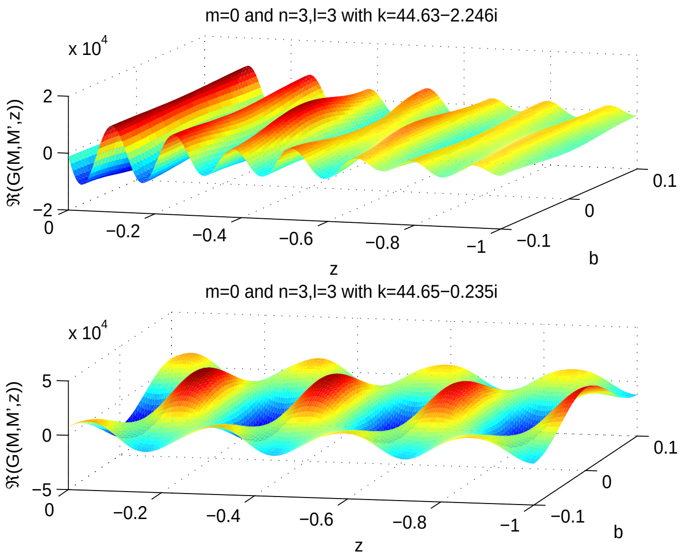

Figure B.8: Profile in the duct of the real part of Green's function around k_{00}^{14} with 3 H.M (upper figure) and around k_{01}^{7} with 3 H.M (lower figure)

[1] L. Rayleigh, The theory of sound (volume 2), Dover, New York, 1945.

[2] W. Zorumski, Generalized radiation impedances and reflection coefficients of circular and annular ducts, J. Acoust. Soc. Am. 54 (1973) 1667–1673.

[3] G. Watson, A Treatise on the Theory of Bessel Functions, Cambridge U. P., 1962.

[4] W.Shao, C. Mechefske, Analyses of radiation impedances of finite cylindrical ducts, Journal of Sound and Vibration 286 (2005) 363–381.

[5] A. Kuijpers, S. Rienstra, G. Verbeek, J. Verheij, The acoustic radiation of baffled finite ducts with vibrating walls, Journal of Sound and Vibration 216 (1998) 461–493.

[6] V. Pagneux, N. Amir, J. Kergomard, A study of wave propagation in varying cross-section waveguides by modal decomposition. part ii. results, J. Acoust. Soc. Am. 101 (1997) 2504–2517.

[7] N. Amir, H. Matzner, The acoustics of flanged cylindrical pipe, examining the influence of higher order modes, Acustica 90 (2004) 1–7.

[8] L. Campos, J. Oliveira, On the acoustic modes in a cylindrical duct with an arbitrary wall impedance distribution, J. Acoust. Soc. Am. 116 (2004) 3336–3347.

[9] W. Zorumski, J. Mason, Multiple eigenvalues of sound-absorbing circular and annular ducts, J. Acoust. Soc. Am. 55 (1974) 1158–1165.

[10] A. Norris, I. Sheng, Acoustic radiation from a circular pipe with an infinite flange, Journal of Sound and Vibration 135 (1) (1989) 85–93.

[11] Y. Nomura, I. Yamamura, S. Inawashiro, On the acoustic radiation from a flanged circular pipe, Journal of the Physical Society of Japan 15 (1960) 510–517.

[12] H. Levine, J. Schwinger, On the radiation of sound from an unflanged circular pipe, Physical Review.

[13] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1968.

[14] M. Morse, H. Feshbach, Methods of Theoretical Physics (Part I), McGraw-Hill, New York, 1953.

[15] F. Silva, P. Guillemain, J. Kergomard, B. Mallaroni, A. N. Norris, Approximation formulae for the acoustic radiation impedance of a cylindrical pipe, submitted to Journal of Sound and Vibration 322 (2009) 255–263.

[16] V. Pagneux, N. Amir, J. Kergomard, A study of wave propagation in varying cross-section waveguides by modal decomposition. part i. theory and validation, J. Acoust. Soc. Am. 100 (1996) 2034–2048.

[17] J. Kergomard, Calculation of discontinuities in waveguides using mode-matching method: an alternative to the scattering matrix approach, J. Acoustique 4 (1991) 111–138.