Some almost sure results for unbounded functions of intermittent maps and their associated Markov chains

Jerome Dedecker, Sébastien Gouëzel, Florence Merlevede

Some almost sure results for unbounded functions of intermittent maps and their associated Markov chains

J. Dedecker[*], S. Gouëzel[†] and F. Merlevède[‡]

June 22, 2009

Abstract

We consider a large class of piecewise expanding maps T of $[0,1]$ with a neutral fixed point, and their associated Markov chain Y_i whose transition kernel is the Perron-Frobenius operator of T with respect to the absolutely continuous invariant probability measure. We give a large class of unbounded functions f for which the partial sums of $f \circ T^i$ satisfy both a central limit theorem and a bounded law of the iterated logarithm. For the same class, we prove that the partial sums of $f(Y_i)$ satisfy a strong invariance principle. When the class is larger, so that the partial sums of $f \circ T^i$ may belong to the domain of normal attraction of a stable law of index $p \in (1,2)$, we show that the almost sure rates of convergence in the strong law of large numbers are the same as in the corresponding i.i.d. case.

Mathematics Subject Classifications (2000): 37E05, 37C30, 60F15.
Key words: Intermittency, almost sure convergence, law of the iterated logarithm, strong invariance principle.

1 Introduction and main results

1.1 Introduction

The Pomeau-Manneville map is an explicit map of the interval $[0,1]$, with a neutral fixed point at 0 and a prescribed behavior there. The statistical properties of this map are very well known when one considers Hölder continuous observables, but much less is known for more complicated observables.

[*]Université Paris 6-Pierre et Marie Curie, Laboratoire de Statistique Théorique et Appliquée.
[†]Université Rennes 1, IRMAR and CNRS UMR 6625.
[‡]Université Paris Est-Marne la Vallée, LAMA and CNRS UMR 8050.

Our goal in this paper is twofold. First, we obtain optimal bounds for the behavior of functions of bounded variation with respect to iteration of the Pomeau-Manneville map. Second, we use these bounds to get a bounded law of the iterated logarithm for a very large class of observables, that previous techniques were unable to handle.

Since we use bounded variation functions, our arguments do not rely on any kind of Markov partition for the map T. Therefore, it turns out that our results hold for a larger class of maps, that we now describe.

Definition 1.1. *A map $T : [0,1] \to [0,1]$ is a generalized Pomeau-Manneville map (or GPM map) of parameter $\gamma \in (0,1)$ if there exist $0 = y_0 < y_1 < \cdots < y_d = 1$ such that, writing $I_k = (y_k, y_{k+1})$,*

1. *The restriction of T to I_k admits a C^1 extension $T_{(k)}$ to $\overline{I_k}$.*
2. *For $k \geq 1$, $T_{(k)}$ is C^2 on $\overline{I_k}$, and $|T'_{(k)}| > 1$.*
3. *$T_{(0)}$ is C^2 on $(0, y_1]$, with $T'_{(0)}(x) > 1$ for $x \in (0, y_1]$, $T'_{(0)}(0) = 1$ and $T''_{(0)}(x) \sim cx^{\gamma-1}$ when $x \to 0$, for some $c > 0$.*
4. *T is topologically transitive.*

The third condition ensures that 0 is a neutral fixed point of T, with $T(x) = x + c'x^{1+\gamma}(1 + o(1))$ when $x \to 0$. The fourth condition is necessary to avoid situations where there are several absolutely continuous invariant measures, or where the neutral fixed point does not belong to the support of the absolutely continuous invariant measure.

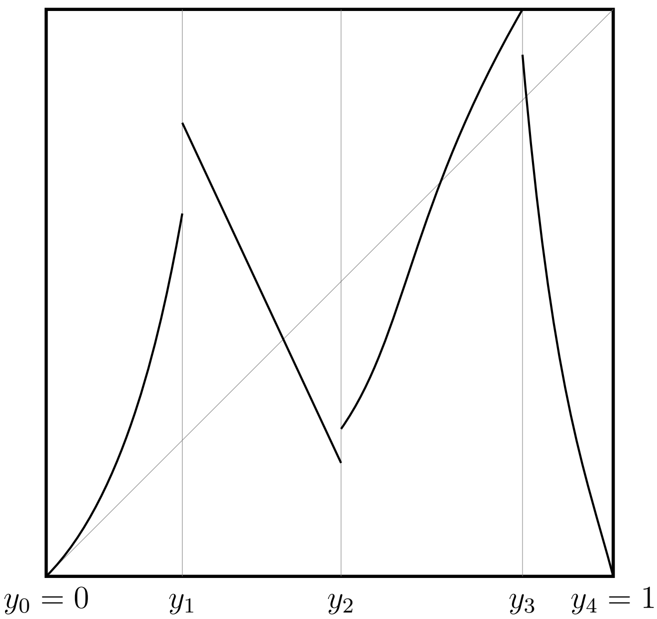

Figure 1: The graph of a GPM map, with $d = 4$

A well known GPM map is the original Pomeau-Manneville map (1980). The Liverani-Saussol-Vaienti (1999) map

$$T_\gamma(x) = \begin{cases} x(1+2^\gamma x^\gamma) & \text{if } x \in [0,1/2] \\ 2x-1 & \text{if } x \in (1/2,1] \end{cases}$$

is also a much studied GPM map of parameter γ. Both of them have a Markov partition, but this is not the case in general for GPM maps as defined above.

Theorem 1 in Zweimüller (1998)[1] shows that a GPM map T admits a unique absolutely continuous invariant probability measure ν, with density h_ν. Moreover, it is ergodic, has full support, and $h_\nu(x)/x^{-\gamma}$ is bounded from above and below.

From the ergodic theorem, we know that $S_n(f) = n^{-1}\sum_{i=0}^{n-1}(f \circ T^i - \nu(f))$ converges almost everywhere to 0 when the function $f : [0,1] \to \mathbb{R}$ is integrable. If f is Hölder continuous, the behavior of $S_n(f)$ is very well understood, thanks to Young (1999) and Melbourne-Nicol (2005): these sums satisfy the almost sure invariance principle for $\gamma < 1/2$ (in particular, the central limit theorem and the law of the iterated logarithm hold). For the Liverani-Saussol-Vaienti map, Gouëzel (2004a) shows that, when $\gamma \in (1/2, 1)$ and f is Lipschitz continuous, $S_n(f)$ suitably renormalized converges to a gaussian law (resp. a stable law) if $f(0) = \nu(f)$ (resp. $f(0) \neq \nu(f)$).

On the other hand, when f is less regular, much less is known. If f has finitely many discontinuities and is otherwise Hölder continuous, the construction of Young (1999) could be adapted to obtain a tower avoiding the discontinuities of f – the almost sure invariance principle follows when $\gamma < 1/2$. However, functions with countably many discontinuities are not easily amenable to the tower method, and neither are very simple unbounded functions such as $g(x) = \ln|x - x_0|$ or $g_a(x) = |x - x_0|^a$ for any $x_0 \neq 0$. This is far less satisfactory than the i.i.d. situation, where optimal moment conditions for the invariance principle or the central limit theorem are known, and it seems especially interesting to devise new methods than can handle functions under moment conditions as close to the optimum as possible.

For the Liverani-Saussol-Vaienti maps, using martingale techniques, Dedecker and Prieur (2009) proved that the central limit theorem holds for a much larger class of functions (including all the functions of bounded variation and several piecewise monotonic unbounded discontinuous functions, for instance the functions g and g_a above up to the optimal value of a) – our arguments below show that their results in fact hold for all GPM maps, not only markovian ones. Our main goal in this article is to prove the bounded law of the iterated logarithm for the same class of functions. We shall also make use of martingale techniques, but we will also need a more precise control on the behavior of bounded variation functions under the iteration of GPM maps.

[1] This theorem does not apply directly to our maps since they do not satisfy its assumption (A). However, this assumption is only used to show that the jump transformation $\tilde{T}$ satisfies (AFU), and this follows in our setting from the distortion estimates of Lemma 5 in Young (1999).

The main steps of our approach are the following:

1. *The main probabilistic tool.* Let $(Y_1, Y_2, \dots)$ be an arbitrary stationary process. We describe in Paragraph 1.3 a coefficient α which measures (in a weak way) the asymptotic independence in this process, and was introduced in Rio (2000). It is weaker than the usual mixing coefficient of Rosenblatt (1956), since it only involves events of the form $\{Y_i \leq x_i\}$, $x_i \in \mathbb{R}$. In particular, it can tend to 0 for some processes that are not Rosenblatt mixing (this will be the case for the processes to be studied below). Thanks to its definition, α behaves well under the composition with monotonic maps of the real line. This coefficient α contains enough information to prove the maximal inequality stated in Proposition 1.11, by following the approach of Merlevède (2008). In turn, this inequality implies (a statement more precise than) the bounded law of the iterated logarithm given in Theorem 1.13, for processes of the form $(f(Y_1), f(Y_2), \dots)$ where $(Y_1, Y_2, \dots)$ has a well behaved α coefficient, and f belongs to a large class of functions.

2. *The main dynamical tool.* Let K denote the Perron-Frobenius operator of T with respect to ν, given by
$$Kf(x) = \frac{1}{h(x)} \sum_{T(y)=x} \frac{h(y)}{|T'(y)|} f(y), \tag{1.1}$$
where h is the density of ν. For any bounded measurable functions f, g, it satisfies $\nu(f \cdot g \circ T) = \nu(K(f)g)$. Since ν is invariant by T, one has $K(1) = 1$, so that K is a Markov operator. Following the approach of Gouëzel (2007), we will study the operator K on the space BV of bounded variation functions, show that its iterates are uniformly bounded, and estimate the contraction of K^n from BV to L^1 (in Propositions 1.15 and 1.16).

3. Let us denote by $(Y_i)_{i\geq 1}$ a stationary Markov chain with invariant measure ν and transition kernel K. Since the mixing coefficient α involves events of the form $\{Y_i \leq x_i\}$, it can be read from the behavior of K on BV. Therefore, the previous estimates yield a precise control of the coefficient α of this process. With Theorem 1.13, this gives a bounded law of the iterated logarithm for the process $(f(Y_1), f(Y_2), \dots)$.

4. It is well known that on the probability space $([0,1], \nu)$, the random variable $(f, f \circ T, \dots, f \circ T^{n-1})$ is distributed as $(f(Y_n), f(Y_{n-1}), \dots, f(Y_1))$. Since there is a phenomenon of time reversal, the law of the iterated logarithm for $(f(Y_1), f(Y_2), \dots)$ does not imply the same result for $(f, f \circ T, \dots)$. However, the technical statement of Theorem 1.13 is essentially invariant under time reversal, and therefore also gives a bounded law of the iterated logarithm for $S_n(f)$.

In the next three paragraphs, we describe our results more precisely. The proofs are given in the remaining sections.

Remark 1.2. The class of maps covered by our results could be further extended, as follows. First, we could allow finitely many neutral fixed point, instead of a single one (possibly with different behaviors). Second, we could allow infinitely many monotonicity branches for T if, away from the neutral fixed points, the quantity $|T''|/(T')^2$ remains bounded, and the set $\{T(Z)\}$, for Z a monotonicity interval, is finite (this is for instance satisfied if all branches but finitely many are onto). Finally, we could drop the topological transitivity.

The ergodic properties of this larger class of maps is fully understood thanks to the work of Zweimüller (1998): there are finitely many invariant measures instead of a single one, and the support of each of these measures is a finite union of intervals. Our arguments still apply in this broader context, although notations and statements become more involved. For the sake of simplicity, we shall only consider the class of GPM maps (which is already quite large).

1.2 Statements of the results for intermittent maps

Definition 1.3. *A function H from $\mathbb{R}_+$ to $[0,1]$ is a tail function if it is non-increasing, right continuous, converges to zero at infinity, and $x \to xH(x)$ is integrable.*

Definition 1.4. *If μ is a probability measure on $\mathbb{R}$ and H is a tail function, let $\mathrm{Mon}(H,\mu)$ denote the set of functions $f : \mathbb{R} \to \mathbb{R}$ which are monotonic on some open interval and null elsewhere and such that $\mu(|f| > t) \leq H(t)$. Let $\mathcal{F}(H,\mu)$ be the closure in $\mathbb{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^{L} a_\ell f_\ell$, where $\sum_{\ell=1}^{L} |a_\ell| \leq 1$ and $f_\ell \in \mathrm{Mon}(H,\mu)$.*

Note that a function belonging to $\mathcal{F}(H,\mu)$ is allowed to blow up at an infinite number of points. Note also that any function f with bounded variation (BV) such that $|f| \leq M_1$ and $\|df\| \leq M_2$ belongs to the class $\mathcal{F}(H,\mu)$ for any μ and the tail function $H = \mathbb{1}_{[0,M_1+2M_2)}$ (here and henceforth, $\|df\|$ denotes the variation norm of the signed measure df). Moreover, if a function f is piecewise monotonic with N branches, then it belongs to $\mathcal{F}(H,\mu)$ for $H(t) = \mu(|f| > t/N)$. Finally, let us emphasize that there is no requirement on the modulus of continuity for functions in $\mathcal{F}(H,\mu)$

Our first result is a bounded law of the iterated logarithm, when $0 < \gamma < 1/2$.

Theorem 1.5. *Let T be a GPM map with parameter $\gamma \in (0,1/2)$ and invariant measure ν. Let H be a tail function with*

$$\int_0^\infty x(H(x))^{\frac{1-2\gamma}{1-\gamma}} \, dx < \infty \,. \tag{1.2}$$

Then, for any $f \in \mathcal{F}(H,\nu)$, the series

$$\sigma^2 = \nu((f-\nu(f))^2) + 2\sum_{k>0} \nu((f-\nu(f)) f \circ T^k)$$

converges absolutely to some nonnegative number. Moreover,

1. *There exists a nonnegative constant A such that*

$$\sum_{n=1}^{\infty} \frac{1}{n} \nu\Big(\max_{1\leq k\leq n} \Big| \sum_{i=0}^{k-1} (f\circ T^i - \nu(f)) \Big| \geq A\sqrt{n \ln(\ln(n))} \Big) < \infty \,, \tag{1.3}$$

and consequently[2]

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n\ln(\ln(n))}} \Big| \sum_{i=0}^{n-1} (f\circ T^i - \nu(f)) \Big| \leq A \,, \ \textit{almost everywhere.}$$

2. *Let $(Y_i)_{i\geq 1}$ be a stationary Markov chain with transition kernel K and invariant measure ν, and let $X_i = f(Y_i) - \nu(f)$. Enlarging if necessary the underlying probability space, there exists a sequence $(Z_i)_{i\geq 1}$ of i.i.d. gaussian random variables with mean zero and variance σ^2 such that*

$$\Big| \sum_{i=1}^{n} (X_i - Z_i) \Big| = o(\sqrt{n\ln(\ln(n))}) \,, \textit{almost surely.} \tag{1.4}$$

In particular, we infer that the bounded law (1.3) holds for any BV function f provided that $\gamma < 1/2$. Note also that (1.2) is satisfied provided that $H(x) \leq Cx^{-2(1-\gamma)/(1-2\gamma)}(\ln(x))^{-b}$ for x large enough and $b > (1-\gamma)/(1-2\gamma)$. Let us consider two simple examples. Since the density h_ν of ν is such that $h_\nu(x) \leq Cx^{-\gamma}$ on $(0,1]$, one can easily prove that:

1. If f is positive and non increasing on (0, 1), with

$$f(x) \leq \frac{C}{x^{(1-2\gamma)/2}|\ln(x)|^b} \quad \text{near 0, for some } b > 1/2,$$

then (1.3) and (1.4) hold.

2. If f is positive and non decreasing on (0, 1), with

$$f(x) \leq \frac{C}{(1-x)^{(1-2\gamma)/(2-2\gamma)}|\ln(1-x)|^b} \quad \text{near 1, for some } b > 1/2,$$

then (1.3) and (1.4) hold.

In fact, if $f \in \mathcal{F}(H,\nu)$ for some H satisfying (1.2) then the central limit theorem and the weak invariance principle hold. This can be easily deduced from the proof of Theorem 4.1 in Dedecker and Prieur (2009) and by using the upper bound for the coefficient $\alpha_{1,\mathbf{Y}}(k)$ given in Proposition 1.17 (which improves on the corresponding bound in Dedecker and Prieur (2009)). Hence, if f is as in Item 1 above, both the central limit theorem and the bounded law of the iterated logarithm hold.

[2] see e.g. Stout (1974), Chapter 5.

An open question is: can we obtain the almost sure invariance principle (1.4) for the sequence $(f\circ T^i)_{i\geq 0}$ instead of $(f(Y_i))_{i\geq 1}$? According to the discussion in Melbourne and Nicol (2005), this appears to be a rather delicate question. Indeed, to obtain Item 2 of Theorem 1.5, we use first a maximal inequality for the partial sums $\sum_{i=1}^k f(Y_i)$ and next a result by Volný and Samek (2000) on the approximating martingale. As pointed out by Melbourne and Nicol (2005, Remark 1.1), we cannot go back to the sequence $(f\circ T^i)_{i\geq 0}$, because the system is not closed under time reversal. Using another approach, going back to Philipp and Stout (1975) and Hofbauer and Keller (1982), Melbourne and Nicol (2005) have proved the almost sure invariance principle for $(f\circ T^i)_{i\geq 0}$ when $\gamma < 1/2$ and f is any Hölder continuous function, with a better error bound $O(n^{1/2-\epsilon})$ for some $\epsilon > 0$. As a consequence, their result imply the functional law of the iterated logarithm for Hölder continuous function, which is much more precise than the bounded law. However, our approach is clearly distinct from that of Melbourne and Nicol (2005), for we cannot deduce the control (1.3) from an almost sure invariance principle.

In the next theorem, we give rates of convergence in the strong law of large numbers under weaker conditions than (1.2), which do not imply the central limit theorem.

Theorem 1.6. *Let $1<p<2$ and $0<\gamma<1/p$. Let T be a GPM map with parameter γ and invariant measure ν. Let H be a tail function with*

$$\int_0^\infty x^{p-1}(H(x))^{\frac{1-p\gamma}{1-\gamma}}\,dx < \infty\,. \tag{1.5}$$

Then, for any $f\in\mathcal{F}(H,\nu)$ and any $\varepsilon>0$, one has

$$\sum_{n=1}^\infty \frac{1}{n}\nu\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^k (f\circ T^i-\nu(f))\Big|\geq n^{1/p}\varepsilon\Big) < \infty\,. \tag{1.6}$$

Consequently, $n^{-1/p}\sum_{k=1}^n (f\circ T^i-\nu(f))$ converges to 0 almost everywhere.

Note that (1.5) is satisfied provided that $H(x)\leq Cx^{-p(1-\gamma)/(1-p\gamma)}(\ln(x))^{-b}$ for x large enough and $b>(1-\gamma)/(1-p\gamma)$. For instance, one can easily prove that, for $1<p<2$ and $0<\gamma<1/p$,

1. If f is positive and non increasing on $(0,1)$, with
$$f(x)\leq \frac{C}{x^{(1-p\gamma)/p}|\ln(x)|^b}\quad \text{near } 0\text{, for some } b>1/p,$$
then (1.6) holds.

2. If f is positive and non decreasing on $(0,1)$, with
$$f(x)\leq \frac{C}{(1-x)^{(1-p\gamma)/(p-p\gamma)}|\ln(1-x)|^b}\quad \text{near } 1\text{, for some } b>1/p,$$

then (1.6) holds.

The condition (1.5) of Theorem 1.6 means exactly that the probability $\mu_{H,p,\gamma}$ on $\mathbb{R}_+$ such that $\mu_{H,p,\gamma}((x,\infty)) = (H(x))^{\frac{1-p\gamma}{1-\gamma}}$ has a moment of order p. Let us see what happen if we only assume that $\mu_{H,p,\gamma}$ has a weak moment of order p.

Theorem 1.7. *Let $1 < p \leq 2$ and $0 < \gamma < 1/p$. Let T be a GPM map with parameter γ and invariant measure ν. Let H be a tail function with*

$$(H(x))^{\frac{1-p\gamma}{1-\gamma}} \leq Cx^{-p} \,. \tag{1.7}$$

Then, for any $f \in \mathcal{F}(H,\nu)$, any $b > 1/p$ and any $\varepsilon > 0$, one has

$$\sum_{n=1}^{\infty} \frac{1}{n} \nu\Big(\max_{1\leq k\leq n} \Big| \sum_{i=0}^{k-1} (f \circ T^i - \nu(f)) \Big| \geq n^{1/p} (\ln(n))^b \varepsilon \Big) < \infty \,. \tag{1.8}$$

Consequently, $n^{-1/p}(\ln(n))^{-b} \sum_{k=0}^{n-1}(f \circ T^i - \nu(f))$ converges to 0 almost everywhere.

Applying Theorem 1.7, one can easily prove that, for $1 < p \leq 2$ and $0 < \gamma < 1/p$,

1. If f is positive and non increasing on $(0,1)$, with $f(x) \leq Cx^{-(1-p\gamma)/p}$ then (1.8) holds.
2. If f is positive and non decreasing on $(0,1)$, with $f(x) \leq C(1-x)^{-(1-p\gamma)/(p-p\gamma)}$ then (1.8) holds.

This requires additional comments. Gouëzel (2004a) proved that if f is exactly of the form $f(x) = x^{-(1-p\gamma)/p}$ for $1 < p < 2$ and $0 < \gamma < 1/p$, then $n^{-1/p} \sum_{i=0}^{n-1}(f \circ T^i - \nu(f))$ converges in distribution on $([0,1],\nu)$ to a centered one-sided stable law of index p, that is a stable law whose distribution function $F^{(p)}$ is such that $x^p F^{(p)}(-x) \to 0$ and $x^p(1 - F^{(p)}(x)) \to c$, as $x \to \infty$, with $c > 0$. Our theorem shows that $n^{-1/p}(\ln(n))^{-b}(\sum_{i=1}^{n}(f \circ T^i - \nu(f)))$ converges almost everywhere to zero for $b > 1/p$. This is in total accordance with the i.i.d. situation, as we describe now. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. centered random variables satisfying $n^{-1/p}(X_1 + \cdots + X_n) \to F^{(p)}$. It is well known (see for instance Feller (1966), page 547) that this is equivalent to $x^p\mathbb{P}(X_1 < -x) \to 0$ and $x^p\mathbb{P}(X_1 > x) \to c$ as $x \to \infty$. For any nondecreasing sequence $(b_n)_{n\geq 1}$ of positive numbers, either $(X_1 + \cdots + X_n)/b_n$ converges to zero almost surely or $\limsup_{n\to\infty} |X_1 + \cdots + X_n|/b_n = \infty$ almost surely, according as $\sum_{n=1}^{\infty} \mathbb{P}(|X_1| > b_n) < \infty$ or $\sum_{n=1}^{\infty} \mathbb{P}(|X_1| > b_n) = \infty$ – this follows from the proof of Theorem 3 in Heyde (1969). If one takes $b_n = n^{1/p}(\ln(n))^b$ we obtain the constraint $b > 1/p$ for the almost sure convergence of $n^{-1/p}(\ln(n))^{-b}(X_1 + \cdots + X_n)$ to zero. This is exactly the same constraint as in our dynamical situation.

Let us comment now on the case $p = 2$. In his (2004a) paper, Gouëzel also proved that if f is exactly of the form $f(x) = x^{-(1-2\gamma)/2}$ then the central limit theorem holds with the normalization $\sqrt{n\ln(n)}$. As mentioned above such an f belongs to the class $\mathcal{F}(H,\nu)$ for some

H satisfying (1.7) with $p = 2$, which means that $\mu_{H,2,\gamma}$ has a weak moment of order 2. This again is in accordance with the i.i.d. situation. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. centered random variables such that $x^2\mathbb{P}(X_1 < -x) \to c_1$ and $x^2\mathbb{P}(X_1 > x) \to c_2$ as x tends to infinity, with $c_1 + c_2 = 1$. Then $(n\ln(n))^{-1/2}(X_1 + \cdots + X_n)$ converges in distribution to a standard gaussian distribution, but according to Theorem 1 in Feller (1968),

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n\ln(n)\ln(\ln(n))}} \sum_{i=1}^{n} X_i = \infty \,.$$

Moreover, if $(b_n)_{n\geq 1}$ is a non decreasing sequence such that $b_n/\sqrt{n\ln(n)\ln(\ln(n))} \to \infty$ (plus the mild conditions (2.1) and (2.2) in Feller's paper), then either $(X_1+\cdots+X_n)/b_n$ converges to zero almost surely or $\limsup_{n\to\infty} |X_1 + \cdots + X_n|/b_n = \infty$ almost surely, according as $\sum_{n=1}^{\infty} \mathbb{P}(|X_1| > b_n) < \infty$ or $\sum_{n=1}^{\infty} \mathbb{P}(|X_1| > b_n) = \infty$. If one takes $b_n = n^{1/2}(\ln(n))^b$ we obtain the constraint $b > 1/2$ for the almost sure convergence of $n^{-1/2}(\ln(n))^{-b}(X_1 + \cdots + X_n)$ to zero. This is exactly the same constraint as in our dynamical situation.

1.3 A general result for stationary sequences

Before stating the maximal inequality proved in this paper, we shall introduce some definitions and notations.

Definition 1.8. *For any nonnegative random variable X, define the "upper tail" quantile function Q_X by $Q_X(u) = \inf\{t \geq 0 : \mathbb{P}(X > t) \leq u\}$.*

This function is defined on $[0, 1]$, non-increasing, right continuous, and has the same distribution as X. This makes it very convenient to express the tail properties of X using Q_X. For instance, for $0 < \varepsilon < 1$, if the distribution of X has no atom at $Q_X(\varepsilon)$, then

$$\mathbb{E}(X\mathbb{1}_{X>Q_X(\varepsilon)}) = \sup_{\mathbb{P}(A)\leq\varepsilon} \mathbb{E}(X\mathbb{1}_A) = \int_0^\varepsilon Q_X(u)du \,.$$

Definition 1.9. *Let μ be the probability distribution of a random variable X. If Q is an integrable quantile function, let $\widetilde{\mathrm{Mon}}(Q, \mu)$ be the set of functions g which are monotonic on some open interval of $\mathbb{R}$ and null elsewhere and such that $Q_{|g(X)|} \leq Q$. Let $\widetilde{\mathcal{F}}(Q, \mu)$ be the closure in $\mathbb{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^{L} a_\ell f_\ell$, where $\sum_{\ell=1}^{L} |a_\ell| \leq 1$ and f_ℓ belongs to $\widetilde{\mathrm{Mon}}(Q, \mu)$.*

This definition is similar to Definition 1.4, we only use quantile functions instead of tail functions. There is in fact a complete equivalence between these two points of view: if Q is a quantile function and H is its càdlàg inverse, then $\widetilde{\mathrm{Mon}}(Q, \mu) = \mathrm{Mon}(H, \mu)$ and $\widetilde{\mathcal{F}}(Q, \mu) = \mathcal{F}(H, \mu)$.

Let now $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $\theta : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. Let $\mathcal{M}_0$ be a sub-σ-algebra of $\mathcal{A}$ satisfying

$\mathcal{M}_0 \subseteq \theta^{-1}(\mathcal{M}_0)$.

Definition 1.10. *For any integrable random variable X, let us write $X^{(0)} = X - \mathbb{E}(X)$. For any random variable $Y = (Y_1, \cdots, Y_k)$ with values in $\mathbb{R}^k$ and any σ-algebra $\mathcal{F}$, let*

$$\alpha(\mathcal{F}, Y) = \sup_{(x_1,\dots,x_k)\in\mathbb{R}^k} \left\| \mathbb{E}\Big(\prod_{j=1}^{k} (\mathbb{1}_{Y_j \le x_j})^{(0)} \Big| \mathcal{F}\Big)^{(0)} \right\|_1 .$$

For a sequence $\mathbf{Y} = (Y_i)_{i\in\mathbb{Z}}$, where $Y_i = Y_0 \circ \theta^i$ and Y_0 is an $\mathcal{M}_0$-measurable and real-valued random variable, let

$$\alpha_{k,\mathbf{Y}}(n) = \max_{1\le l\le k} \sup_{n\le i_1\le\ldots\le i_l} \alpha(\mathcal{M}_0, (Y_{i_1}, \ldots, Y_{i_l})). \tag{1.9}$$

The following maximal inequality is crucial for the proof of Theorem 1.13 below.

Proposition 1.11. *Let $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$, where $Y_i = Y_0 \circ \theta^i$ and f belongs to $\widetilde{\mathcal{F}}(Q, P_{Y_0})$ (here, P_{Y_0} denotes the distribution of Y_0, and Q is a square integrable quantile function). Define the coefficients $\alpha_{1,\mathbf{Y}}(n)$ and $\alpha_{2,\mathbf{Y}}(n)$ as in (1.9). Let $n \in \mathbb{N}$. Let*

$$R(u) = (\min\{q \in \mathbb{N} : \alpha_{2,\mathbf{Y}}(q) \le u\} \wedge n) Q(u) \text{ and } S(v) = R^{-1}(v) = \inf\{u \in [0,1] : R(u) \le v\} .$$

Let $S_n = \sum_{k=1}^n X_k$. For any $x > 0$, $r \ge 1$, and $s_n > 0$ with $s_n^2 \ge 4n \sum_{i=0}^{n-1} \int_0^{\alpha_{1,\mathbf{Y}}(i)} Q^2(u)du$, one has

$$\mathbb{P}\Big(\sup_{1\le k\le n} |S_k| \ge 5x \Big) \le 4 \exp\left(-\frac{r^2 s_n^2}{8x^2} h\left(\frac{2x^2}{r s_n^2}\right)\right) + n\Big(\frac{6}{x} + \frac{16x}{r s_n^2}\Big) \int_0^{S(x/r)} Q(u)du \,, \tag{1.10}$$

where $h(u) := (1+u)\ln(1+u) - u$.

Remark 1.12. Note that a similar bound for α-mixing sequences in the sense of Rosenblatt (1956) has been proved in Merlevède (2008, Theorem 1). Since $h(u) \ge u \ln(1+u)/2$, under the notation and assumptions of the above theorem, we get that for any $x > 0$ and $r \ge 1$,

$$\mathbb{P}\Big(\sup_{1\le k\le n} |S_k| \ge 5x \Big) \;\le\; 4\left(1 + \frac{2x^2}{r s_n^2}\right)^{-r/8} + n\Big(\frac{6}{x} + \frac{16x}{r s_n^2}\Big) \int_0^{S(x/r)} Q(u)du \,. \tag{1.11}$$

Theorem 1.5 is in fact a corollary of the following theorem, which gives both a precise control of the tail of the partial sums by applying Proposition 1.11, and a strong invariance principle for the partial sums.

Let $\mathcal{I}$ be the σ-algebra of all θ-invariant sets. The map θ is $\mathbb{P}$-ergodic if each element of $\mathcal{I}$ has measure 0 or 1.

Theorem 1.13. *Let Y_i, X_i and S_n be as in Proposition 1.11. Assume that the following* *condition is satisfied:*

$$\sum_{k\geq 1}\int_0^{\alpha_{2,\mathbf{Y}}(k)} Q^2(u)du < \infty\,. \tag{1.12}$$

Then the series $\sigma^2 = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k)$ *converges absolutely to some nonnegative number* σ^2*, and*

$$\sum_{n>0}\frac{1}{n}\mathbb{P}\Big(\sup_{k\in[1,n]}|S_k|\geq A\sqrt{2n\ln(\ln(n))}\Big)<\infty,\ \textit{with}\ A = 20\Big(\sum_{k\geq 0}\int_0^{\alpha_{1,\mathbf{Y}}(k)} Q^2(u)du\Big)^{1/2}. \tag{1.13}$$

Assume moreover that θ *is* $\mathbb{P}$*-ergodic. Then, enlarging* Ω *if necessary, there exists a sequence* $(Z_i)_{i\geq 0}$ *of i.i.d. gaussian random variables with mean zero and variance* σ^2 *such that*

$$\Big|S_n - \sum_{i=1}^n Z_i\Big| = o(\sqrt{n\ln(\ln(n))}),\ \textit{almost surely.} \tag{1.14}$$

Remark 1.14. The strong invariance principle for α-mixing sequences (in the sense of Rosenblatt (1956)) given in Rio (1995) Theorem 2, can be easily deduced from (1.14). Note that the optimality of Rio's result is discussed in Theorem 3 of his paper.

1.4 Dependence coefficients for intermittent maps

Let θ be the shift operator from $\mathbb{R}^\mathbb{Z}$ to $\mathbb{R}^\mathbb{Z}$ defined by $(\theta(x))_i = x_{i+1}$, and let π_i be the projection from $\mathbb{R}^\mathbb{Z}$ to $\mathbb{R}$ defined by $\pi_i(x) = x_i$. Let $\mathbf{Y} = (Y_i)_{i\geq 0}$ be a stationary real-valued Markov chain with transition kernel K and invariant measure ν. By Kolmogorov's extension theorem, there exists a shift-invariant probability $\mathbb{P}$ on $(\mathbb{R}^\mathbb{Z}, (\mathcal{B}(\mathbb{R}))^\mathbb{Z})$, such that $\pi = (\pi_i)_{i\geq 0}$ is distributed as $\mathbf{Y}$. Let $\mathcal{M}_0 = \sigma(\pi_i, i\leq 0)$. We define the coefficient $\alpha_{k,\mathbf{Y}}(n)$ of the chain $(Y_i)_{i\geq 0}$ *via* its extension $(\pi_i)_{i\in\mathbb{Z}}$: $\alpha_{k,\mathbf{Y}}(n) = \alpha_{k,\pi}(n)$.

Note that these coefficients may be written in terms of the kernel K as follows. Let $f^{(0)} = f - \nu(f)$. For any non-negative integers $n_1, n_2, \ldots, n_k$, and any bounded measurable functions $f_1, f_2, \ldots, f_k$, define

$$K^{(0)(n_1,n_2,\ldots,n_k)}(f_1, f_2, \ldots, f_k) = \big(K^{n_1}(f_1K^{n_2}(f_2K^{n_3}(f_3\cdots K^{n_{k-1}}(f_{k-1}K^{n_k}(f_k))\cdots))))\big)^{(0)}.$$

Let BV_1 be space of bounded variation functions f such that $\|df\|\leq 1$, where $\|df\|$ is the variation norm on $\mathbb{R}$ of the measure df. We have

$$\alpha_{k,\mathbf{Y}}(n) = \sup_{1\leq l\leq k}\ \sup_{n_1\geq n, n_2\geq 0,\ldots n_l\geq 0}\ \sup_{f_1,\ldots,f_l\in \mathrm{BV}_1} \nu\big(|K^{(0)(n_1,n_2,\ldots,n_l)}(f_1^{(0)}, f_2^{(0)}, \ldots, f_l^{(0)})|\big)\,. \tag{1.15}$$

Let us now fix a GPM map T of parameter $\gamma\in(0,1)$. Denote by ν its absolutely continuous invariant probability measure, and by K its Perron-Frobenius operator with respect to ν. Let $\mathbf{Y} = (Y_i)_{i\geq 0}$ be a stationary Markov chain with invariant measure ν and transition kernel K.

The following proposition shows that the iterates of K on BV are uniformly bounded.

Proposition 1.15. *There exists $C > 0$, not depending on n, such that for any* BV *function f, $\|dK^n(f)\| \le C\|df\|$.*

The following covariance inequality implies an estimate on $\alpha_{1,\mathbf{Y}}$.

Proposition 1.16. *There exists $B > 0$ such that, for any bounded function φ, any* BV *function f and any $n > 0$*

$$|\nu(\varphi \circ T^n \cdot (f - \nu(f)))| \le \frac{B}{n^{(1-\gamma)/\gamma}} \|df\| \|\varphi\|_\infty \,. \tag{1.16}$$

Putting together the last two propositions and (1.15), we obtain the following:

Proposition 1.17. *For any positive integer k, there exists a constant C such that, for any $n > 0$,*

$$\alpha_{k,\mathbf{Y}}(n) \le \frac{C}{n^{(1-\gamma)/\gamma}} \,.$$

Proof. Let $f \in \mathrm{BV}_1$ and $g \in \mathrm{BV}$ with $\|g\|_\infty \le 1$. Then, applying Proposition 1.15, we obtain for any $n \ge 0$,

$$\|d(f^{(0)} K^n(g))\| \le \|df\| \|g\|_\infty + \|dK^n(g)\| \|f^{(0)}\|_\infty \le 1 + C\|dg\| \,. \tag{1.17}$$

For $f_1, \dots, f_k \in \mathrm{BV}_1$, let $f = f_1^{(0)} K^{n_2}(f_2^{(0)} K^{n_3}(f_3^{(0)} \cdots K^{n_{k-1}}(f_{k-1}^{(0)} K^{n_k}(f_k^{(0)})) \cdots))$. Iterating Inequality (1.17), we obtain, for any $n_2, \dots, n_k \ge 0$, $\|df\| \le 1 + C + C^2 + \cdots + C^{k-1}$. Together with the bound (1.15) for $\alpha_{k,\mathbf{Y}}(n)$, this implies that

$$\alpha_{k,\mathbf{Y}}(n) \le (1 + C + C^2 + \cdots + C^{k-1}) \alpha_{1,\mathbf{Y}}(n) \,.$$

Now the upper bound (1.16) means exactly that $\alpha_{1,\mathbf{Y}}(n) \le B n^{(\gamma-1)/\gamma}$, which concludes the proof of Proposition (1.17). □

Proposition 1.17 improves on the corresponding upper bound given in Dedecker and Prieur (2009). Let us mention that this upper bound is optimal: the lower bound $\alpha_{k,\mathbf{Y}}(n) \ge C' n^{(\gamma-1)/\gamma}$ was given in Dedecker and Prieur (2009) for Liverani-Saussol-Vaienti maps, and is a consequence in this markovian context of the lower bound for $\nu(\varphi \circ T^n \cdot (f - \nu(f)))$ given by Sarig (2002), Corollary 1. Our techniques imply that this lower bound also holds in the general setting of GPM maps.

In the rest of the paper, we prove the previous results. First, in Section 2, we prove the results of Paragraph 1.3, which are essentially of probabilistic nature. In Section 3, we study the transfer operator of a GPM map T, to prove the dynamical results of Paragraph 1.4. Finally, in the last section, we put together all those results (and arguments of Dedecker and Merlevède (2007)) to prove the main theorems of Paragraph 1.2.

In the rest of this paper, C and D are positive constants that may vary from line to line.

2 Proofs of the probabilistic results

2.1 Proof of Proposition 1.11

Assume first that $X_i = \sum_{\ell=1}^L a_\ell f_\ell(Y_i) - \sum_{\ell=1}^L a_\ell \mathbb{E}(f_\ell(Y_i))$, with f_ℓ belonging to $\widetilde{\mathrm{Mon}}(Q, P_{Y_0})$ and $\sum_{\ell=1}^L |a_\ell| \le 1$. Let $M > 0$ and $g_M(x) = (x \wedge M) \vee (-M)$. For any $i \ge 0$, we first define

$$X_i' = \sum_{\ell=1}^L a_\ell \, g_M \circ f_\ell(Y_i) - \sum_{\ell=1}^L a_\ell \mathbb{E}(g_M \circ f_\ell(Y_i)) \quad \text{and} \quad X_i'' = X_i - X_i' \, .$$

Let $S_n' = \sum_{i=1}^n X_i'$ and $S_n'' = \sum_{i=1}^n X_i''$. Let q be a positive integer and for $1 \le i \le [n/q]$, define the random variables $U_i' = S_{iq}' - S_{iq-q}'$ and $U_i'' = S_{iq}'' - S_{iq-q}''$.

Let us first show that

$$\max_{1 \le k \le n} |S_k| \le \max_{1 \le j \le [n/q]} \Big| \sum_{i=1}^j U_i' \Big| + 2qM + \sum_{k=1}^n |X_k''| \, . \tag{2.1}$$

If the maximum of $|S_k|$ is obtained for $k = k_0$, then for $j_0 = [k_0/q]$,

$$\max_{1 \le k \le n} |S_k| \le \Big| \sum_{i=1}^{j_0} U_i' \Big| + \sum_{i=1}^{j_0} |U_i''| + \sum_{k=qj_0+1}^{k_0} |X_k'| + \sum_{k=qj_0+1}^{k_0} |X_k''| \, .$$

Since $|X_k'| \le 2M \sum_{\ell=1}^L |a_\ell| \le 2M$, and $\sum_{i=1}^{j_0} |U_i''| \le \sum_{k=1}^{qj_0} |X_k''|$, this concludes the proof of (2.1).

For all $i \ge 1$, let $\mathcal{F}_i^U = \mathcal{M}_{iq}$, where $\mathcal{M}_k = \theta^{-k}(\mathcal{M}_0)$. We define a sequence $(\tilde U_i)_{i \ge 1}$ by $\tilde U_i = U_i' - \mathbb{E}(U_i' | \mathcal{F}_{i-2}^U)$. The sequences $(\tilde U_{2i-1})_{i \ge 1}$ and $(\tilde U_{2i})_{i \ge 1}$ are sequences of martingale differences with respect respectively to $(\mathcal{F}_{2i-1}^U)$ and $(\mathcal{F}_{2i}^U)$. Substituting the variables $\tilde U_i$ to the initial variables, in the inequality (2.1), we derive the following upper bound

$$\max_{1 \le k \le n} |S_k| \le 2qM + \max_{2 \le 2j \le [n/q]} \Big| \sum_{i=1}^j \tilde U_{2i} \Big| + \max_{1 \le 2j-1 \le [n/q]} \Big| \sum_{i=1}^j \tilde U_{2i-1} \Big| + \sum_{i=1}^{[n/q]} |U_i' - \tilde U_i| + \sum_{k=1}^n |X_k''| \, . \tag{2.2}$$

Since $\sum_{\ell=1}^L |a_\ell| \le 1$, $|U_i'| \le 2qM$ almost surely. Consequently $|\tilde U_i| \le 4qM$ almost surely. Applying Proposition A.1 of the appendix with $y = 2s_n^2$, we derive that

$$\begin{aligned} \mathbb{P}\Big(\max_{2 \le 2j \le [n/q]} \Big| \sum_{i=1}^j \tilde U_{2i} \Big| \ge x \Big) &\le 2 \exp\left(- \frac{s_n^2}{8(qM)^2} h\left(\frac{2xqM}{s_n^2} \right) \right) \\ &\quad + \mathbb{P}\Big(\sum_{i=1}^{[[n/q]/2]} \mathbb{E}(\tilde U_{2i}^2 | \mathcal{F}_{2(i-1)}^U) \ge 2s_n^2 \Big) \, . \end{aligned} \tag{2.3}$$

Since $\mathbb{E}(\tilde U_{2i}^2|\mathcal{F}^U_{2(i-1)}) \le \mathbb{E}((U'_{2i})^2|\mathcal{F}^U_{2(i-1)})$,

$$\mathbb{P}\Big(\sum_{i=1}^{[[n/q]/2]} \mathbb{E}(\tilde U_{2i}^2|\mathcal{F}^U_{2(i-1)}) \ge 2s_n^2\Big) \le \mathbb{P}\Big(\sum_{i=1}^{[[n/q]/2]} \mathbb{E}((U'_{2i})^2|\mathcal{F}^U_{2(i-1)}) \ge 2s_n^2\Big)\,. \tag{2.4}$$

By stationarity

$$\sum_{i=1}^{[[n/q]/2]} \mathbb{E}((U'_{2i})^2) = [[n/q]/2]\mathbb{E}(S'_q)^2 = [[n/q]/2]\sum_{|i|\le q}(q-|i|)\mathbb{E}(X'_0X'_{|i|})\,.$$

Now,

$$\mathbb{E}(X'_0X'_{|i|}) = \sum_{\ell=1}^{L}\sum_{k=1}^{L} a_\ell a_k \operatorname{Cov}\big(g_M\circ f_\ell(Y_0), g_M\circ f_k(Y_{|i|})\big).$$

Applying Theorem 1.1 in Rio (2000) and noticing that $Q_{|g_M\circ f_\ell(Y_{|i|})|}(u) \le Q_{|f_\ell(Y_{|i|})|}(u) \le Q(u)$, we derive that

$$\big|\operatorname{Cov}\big(g_M\circ f_\ell(Y_0), g_M\circ f_k(Y_{|i|})\big)\big| \le 2\int_0^{2\bar\alpha(g_M\circ f_\ell(Y_0), g_M\circ f_k(Y_{|i|}))} Q^2(u)du\,,$$

where

$$\bar\alpha(g_M\circ f_\ell(Y_0), g_M\circ f_k(Y_{|i|})) = \sup_{(s,t)\in\mathbb{R}^2}\big|\operatorname{Cov}(\mathbb{1}_{g_M\circ f_\ell(Y_0)\le s}, \mathbb{1}_{g_M\circ f_k(Y_{|i|})\le t})\big|\,.$$

Since $g_M\circ f_k$ is monotonic on an interval and zero elsewhere, it follows that $\{g_M\circ f_k(x)\le t\}$ is either some interval or the complement of some interval. Hence

$$\bar\alpha(g_M\circ f_\ell(Y_0), g_M\circ f_k(Y_{|i|})) \le 2\bar\alpha(g_M\circ f_\ell(Y_0), Y_{|i|}) \le \alpha_1(|i|)\,.$$

Consequently since $\sum_{\ell=1}^{L}|a_\ell|\le 1$, we get that

$$\mathbb{E}(X'_0X'_{|i|}) \le 2\int_0^{2\alpha_{1,\mathbf{Y}}(|i|)} Q^2(u)du \le 4\int_0^{\alpha_{1,\mathbf{Y}}(|i|)} Q^2(u)du\,, \tag{2.5}$$

so that

$$\sum_{i=1}^{[[n/q]/2]} \mathbb{E}((U'_{2i})^2) \le 4n\sum_{i=0}^{q-1}\int_0^{\alpha_{1,\mathbf{Y}}(i)} Q^2(u)du \le s_n^2\,.$$

This bound and Markov's inequality imply that

$$\mathbb{P}\Big(\sum_{i=1}^{[[n/q]/2]} \mathbb{E}((U'_{2i})^2|\mathcal{F}^U_{2(i-1)}) \ge 2s_n^2\Big) \le \frac{1}{s_n^2}\sum_{i=1}^{[[n/q]/2]} \mathbb{E}|\mathbb{E}((U'_{2i})^2|\mathcal{F}^U_{2(i-1)}) - \mathbb{E}((U'_{2i})^2)|\,. \tag{2.6}$$

Obviously similar computations allow to treat the quantity $\max_{1\le 2j-1\le[n/q]}|\sum_{i=1}^{j}\tilde U_{2i-1}|$.

Hence we get that

$$\mathbb{P}\Big(\max_{2\le 2j\le [n/q]}\Big|\sum_{i=1}^{j}\tilde U_{2i}\Big| + \max_{1\le 2j-1\le [n/q]}\Big|\sum_{i=1}^{j}\tilde U_{2i-1}\Big| \ge 2x\Big) \le 4\exp\left(-\frac{s_n^2}{8(qM)^2}h\left(\frac{2xqM}{s_n^2}\right)\right) + \frac{1}{s_n^2}\sum_{i=1}^{[n/q]}\mathbb{E}|\mathbb{E}((U_i')^2|\mathcal{M}_{(i-2)q}) - \mathbb{E}((U_i')^2)|\,.$$

By stationarity we have

$$\begin{aligned}\sum_{i=1}^{[n/q]}\|\mathbb{E}((U_i')^2|\mathcal{M}_{(i-2)q}) - \mathbb{E}((U_i')^2)\|_1 &\le \frac{n}{q}\|\mathbb{E}((S_q')^2|\mathcal{M}_{-q}) - \mathbb{E}((S_q')^2)\|_1 \\ &\le \frac{n}{q}\sum_{i=q+1}^{2q}\sum_{j=q+1}^{2q}\|\mathbb{E}(X_i'X_j'|\mathcal{M}_0) - \mathbb{E}(X_i'X_j')\|_1\,.\end{aligned} \tag{2.7}$$

Let us now prove that

$$\|\mathbb{E}(X_i'X_j'|\mathcal{M}_0) - \mathbb{E}(X_i'X_j')\|_1 \le 16M^2\alpha_{2,\mathbf{Y}}(q). \tag{2.8}$$

Setting $A := \mathrm{sign}\{\mathbb{E}(X_i'X_j'|\mathcal{M}_0) - \mathbb{E}(X_i'X_j')\}$, we have that

$$\begin{aligned}\|\mathbb{E}(X_i'X_j'|\mathcal{M}_0) - \mathbb{E}(X_i'X_j')\|_1 &= \mathbb{E}\Big\{A\Big(\mathbb{E}(X_i'X_j'|\mathcal{M}_0) - \mathbb{E}(X_i'X_j')\Big)\Big\} = \mathbb{E}\big((A-\mathbb{E}A)X_i'X_j'\big) \\ &= \sum_{\ell=1}^{L}\sum_{k=1}^{L}a_\ell a_k\mathbb{E}\big((A-\mathbb{E}A)(g_M\circ f_\ell(Y_i) - \mathbb{E}g_M\circ f_\ell(Y_i))(g_M\circ f_k(Y_j) - \mathbb{E}g_M\circ f_k(Y_j))\big)\,.\end{aligned}$$

From Proposition 6.1 and Lemma 6.1 in Dedecker and Rio (2008), noticing that $Q_A(u)\le 1$ and $Q_{|g_M\circ f_\ell(Y_i)|}(u)\le M$, we have that

$$\begin{aligned}&|\mathbb{E}\big((A-\mathbb{E}A)(g_M\circ f_\ell(Y_i) - \mathbb{E}g_M\circ f_\ell(Y_i))(g_M\circ f_k(Y_j) - \mathbb{E}g_M\circ f_k(Y_j))\big)| \\ &\qquad\le 8M^2\bar\alpha(A, g_M\circ f_\ell(Y_i), g_M\circ f_k(Y_j))\,,\end{aligned}$$

where for real valued random variables A, B, V,

$$\bar\alpha(A,B,V) = \sup_{(s,t,u)\in\mathbf{R}^3}\big|\mathbb{E}((\mathbb{1}_{A\le s} - \mathbb{P}(A\le s))(\mathbb{1}_{B\le t} - \mathbb{P}(B\le t))(\mathbb{1}_{V\le u} - \mathbb{P}(V\le u)))\big|\,.$$

For all $i,j\ge q$,

$$\bar\alpha(A, g_M\circ f_\ell(Y_i), g_M\circ f_k(Y_j)) \le 4\bar\alpha(A, Y_i, Y_j) \le 2\alpha_{2,\mathbf{Y}}(q)\,.$$

This concludes the proof of (2.8). Together with (2.7), this yields

$$\sum_{i=1}^{[n/q]} \mathbb{E}|\mathbb{E}((U_i')^2|\mathcal{M}_{(i-2)q}) - \mathbb{E}(U_i')^2| \leq 16nqM^2\alpha_{2,\mathbf{Y}}(q)\,. \tag{2.9}$$

It follows that

$$\mathbb{P}\Big(\max_{2\leq 2j\leq [n/q]}\Big|\sum_{i=1}^{j}\tilde{U}_{2i}\Big| + \max_{1\leq 2j-1\leq [n/q]}\Big|\sum_{i=1}^{j}\tilde{U}_{2i-1}\Big| \geq 2x\Big) \\ \leq 4\exp\left(-\frac{s_n^2}{8(qM)^2}h\left(\frac{2xqM}{s_n^2}\right)\right) + \frac{16nqM}{s_n^2}M\alpha_{2,\mathbf{Y}}(q)\,. \tag{2.10}$$

Now by using Markov's inequality, we get that

$$\mathbb{P}\Big(\sum_{i=1}^{[n/q]}|U_i' - \tilde{U}_i| + \sum_{k=1}^{n}|X_k''| \geq x\Big) \leq \frac{1}{x}\Big(\sum_{i=1}^{[n/q]}\|\mathbb{E}(U_i'|\mathcal{M}_{(i-2)q})\|_1 + \sum_{k=1}^{n}\|X_k''\|_1\Big)\,.$$

By stationarity, we have that

$$\sum_{i=1}^{[n/q]}\|\mathbb{E}(U_i'|\mathcal{M}_{(i-2)q})\|_1 \leq \frac{n}{q}\sum_{i=q+1}^{2q}\|\mathbb{E}(X_i'|\mathcal{M}_0)\|_1\,.$$

Setting $A = \mathrm{sign}\{\mathbb{E}(X_i'|\mathcal{M}_0)\}$, we get that

$$\|\mathbb{E}(X_i'|\mathcal{M}_0)\|_1 = \mathbb{E}((A-\mathbb{E}A)X_i') = \sum_{\ell=1}^{L}a_\ell\mathbb{E}\big((A-\mathbb{E}A)(g_M\circ f_\ell(Y_i) - \mathbb{E}g_M\circ f_\ell(Y_i))\big)$$

Now applying again Theorem 1.1 in Rio (2000), and using the fact that $Q_{|g_M\circ f_\ell(Y_i)|}(u) \leq Q(u)$, we derive that

$$|\mathbb{E}\big((A-\mathbb{E}A)(g_M\circ f_\ell(Y_i) - \mathbb{E}g_M\circ f_\ell(Y_i))\big) \leq 2\int_0^{2\bar{\alpha}(A, g_M\circ f_\ell(Y_i))}Q(u)du\,.$$

Since for all $i \geq q$,

$$\bar{\alpha}(A, g_M\circ f_\ell(Y_i)) \leq 2\bar{\alpha}(A, Y_i) \leq \alpha_{1,\mathbf{Y}}(i) \leq \alpha_{2,\mathbf{Y}}(i)\,,$$

we derive that

$$\|\mathbb{E}(X_i'|\mathcal{M}_0)\|_1 \leq 4\int_0^{\alpha_{2,\mathbf{Y}}(i)}Q(u)du\,, \tag{2.11}$$

which implies that

$$\mathbb{P}\Big(\sum_{i=1}^{[n/q]}|U_i'-\tilde{U}_i|+\sum_{k=1}^{n}|X_k''|\geq x\Big)\leq\frac{4n}{x}\int_0^{\alpha_{2,\mathbf{Y}}(q)}Q(u)du+\frac{1}{x}\sum_{k=1}^{n}\mathbb{E}(|X_k''|)\,. \tag{2.12}$$

Then starting from (2.2), if q and M are chosen in such a way that $qM\leq x$, we derive from (2.10) and (2.12) that

$$\begin{aligned}\mathbb{P}\Big(\max_{1\leq k\leq n}|S_k|\geq 5x\Big)&\leq 4\exp\left(-\frac{s_n^2}{8(qM)^2}h\left(\frac{2xqM}{s_n^2}\right)\right)+\frac{16nqM}{s_n^2}M\alpha_{2,\mathbf{Y}}(q)\\&\quad+\frac{4n}{x}\int_0^{\alpha_{2,\mathbf{Y}}(q)}Q(u)du+\frac{1}{x}\sum_{k=1}^{n}\mathbb{E}(|X_k''|)\,.\end{aligned} \tag{2.13}$$

Now choose $v=S(x/r)$, $q=\min\{q\in\mathbb{N}\,:\,\alpha_{2,\mathbf{Y}}(q)\leq v\}\wedge n$ and $M=Q(v)$. Since R is right continuous, we have $R(S(w))\leq w$ for any w, hence

$$qM=R(v)=R(S(x/r))\leq x/r\leq x\,.$$

Note also that, writing $\varphi_M(x)=(|x|-M)_+$,

$$\sum_{k=1}^{n}\mathbb{E}(|X_k''|)\leq 2\sum_{\ell=1}^{L}|a_\ell|\sum_{k=1}^{n}\mathbb{E}(\varphi_M(f_\ell(Y_k)))$$

and that $Q_{\varphi_M(f_\ell(Y_k))}\leq Q_{|f_\ell(Y_k)|}\mathbb{1}_{[0,v]}\leq Q\mathbb{1}_{[0,v]}$. Consequently

$$\sum_{k=1}^{n}\mathbb{E}(|X_k''|)\leq 2\sum_{\ell=1}^{L}|a_\ell|\sum_{k=1}^{n}\int_0^v Q_{|f_\ell(Y_k)|}(u)du\leq 2n\int_0^v Q(u)du\,. \tag{2.14}$$

Assume first $q<n$. The choice of q then implies that $\alpha_{2,\mathbf{Y}}(q)\leq v$ and $M\alpha_{2,\mathbf{Y}}(q)\leq vQ(v)\leq\int_0^v Q(u)du$. Moreover, as $qM\leq x/r$, we have

$$\frac{1}{(qM)^2}h\left(\frac{2xqM}{s_n^2}\right)\geq\frac{r^2}{x^2}h\left(\frac{2x^2}{rs_n^2}\right),$$

since the function $t\mapsto t^{-2}h(t)$ is decreasing. Together with (2.13) and (2.14), this gives the desired inequality (1.10).

If $q=n$, the previous argument breaks down since we may have $\alpha_{2,\mathbf{Y}}(q)>v$. However, a much simpler argument is available. Indeed, bounding simply X_i' by $2M$, we obtain $\max_{1\leq k\leq n}|S_k|\leq 2qM+\sum_{k=1}^n|X_k''|$. Since $2qM\leq 2x$, this gives

$$\mathbb{P}\Big(\max_{1\leq k\leq n}|S_k|\geq 5x\Big)\leq\frac{1}{x}\sum_{k=1}^{n}\mathbb{E}(|X_k''|).$$

With (2.14), this again implies (1.10).

The proposition is proved for any variable $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$ with $f = \sum_{\ell=1}^{L} a_\ell f_\ell$ and $f_\ell \in \widetilde{\mathrm{Mon}}(Q, P_{Y_0})$, $\sum |a_\ell| \leq 1$. Since these functions are dense in $\widetilde{\mathcal{F}}(Q, P_{Y_0})$ by definition, the result follows by applying Fatou's lemma. $\square$

2.2 Proof of Theorem 1.13

Let us first prove the inequality (1.13). We follow the proof of Theorem 6.4 page 89 in Rio (2000), and we use the same notations: $Lx = \ln(x \vee e)$ and $LLx = \ln(\ln(x \vee e) \vee e)$. Let A be as in (1.13). We apply Proposition 1.11 with

$$r = r_n = 8LLn, \quad x = x_n = (A\sqrt{2nLLn})/5 \quad \text{and} \quad s_n = x_n/\sqrt{r_n}\,.$$

We obtain

$$\sum_{n>0} \frac{1}{n} \mathbb{P}\Big(\sup_{1\leq k \leq n} |S_k| \geq A\sqrt{2nLLn} \Big) \leq 4 \sum_{n>0} \frac{1}{n3^{LLn}} + 22 \sum_{n>0} \frac{1}{x_n} \int_0^{S(x_n/r_n)} Q(u)du\,.$$

Clearly the first series on right hand converges. From the end of the proof of Theorem 6.4 in Rio (2000), we see that the second series on the right hand side converges. This completes the proof of (1.13).

Note that the inequality (1.13) implies that

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2nLLn}} \leq 20 \Big(\sum_{k\geq 0} \int_0^{\alpha_{1,\mathbf{Y}}(k)} Q^2(u)du \Big)^{1/2} \text{ almost surely}\,. \tag{2.15}$$

We turn now to the proof of (1.14). Assume that θ is $\mathbb{P}$-ergodic. In 1973, Gordin (see also Esseen and Janson (1985)) proved that if

$$\sum_{k\geq 1} \|\mathbb{E}(X_k|\mathcal{M}_0)\|_1 < \infty \tag{2.16}$$

and

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{E}\Big(\Big| \sum_{k=1}^{n} X_k \Big| \Big) < \infty\,, \tag{2.17}$$

then $X_0 = D_0 + Z_0 - Z_0 \circ \theta$, where $\|Z_0\|_1 < \infty$, $\mathbb{E}(D_0^2) < \infty$, D_0 is $\mathcal{M}_0$-measurable, and $\mathbb{E}(D_0|\mathcal{M}_{-1}) = 0$.

Notice now that by a similar computation than to get (2.11), we have that

$$\|\mathbb{E}(X_k|\mathcal{F}_0)\|_1 \leq 4 \int_0^{\alpha_{1,\mathbf{Y}}(k)} Q(u)du\,. \tag{2.18}$$

Hence (1.12) implies (2.16). Now clearly (2.17) holds as soon as $\sum_{k=0}^{\infty} |\operatorname{Cov}(X_0, X_k)| < \infty$

which holds under (1.12) by applying the upper bound (2.5) with $M = \infty$ (note that this also justifies the convergence of the series σ^2).

Consequently, if we set $D_i = D_0 \circ \theta^i$, and $Z_i = Z_0 \circ \theta^i$, we then obtain under (1.12) that

$$S_n = M_n + Z_1 - Z_{n+1}\,, \tag{2.19}$$

where $M_n = \sum_{j=1}^n D_j$ is a martingale in $\mathbb{L}^2$ and Z_0 is integrable. Now (1.14) follows by the almost sure invariance principle for martingales (see Theorem 3.1 in Berger (1990)) if we can prove that

$$Z_n = o(\sqrt{nLLn})\,, \quad \text{almost surely.} \tag{2.20}$$

According to the lemma page 428 in Volný and Samek (2000), we have either (2.20) or

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{|Z_n|}{\sqrt{nLLn}} = \infty\Big) = 1\,. \tag{2.21}$$

Using the decomposition (2.19), the fact that M_n satisfies the law of the iterated logarithm and that S_n satisfies (2.15), it is clear that (2.21) cannot hold, which then proves (2.20) and ends the proof of (1.14). □

3 Proofs of the dynamical estimates

If f is supported in $[0, 1]$, let $V(f)$ be the variation of the function f, given by

$$\mathrm{V}(f) = \sup_{x_0<\cdots<x_N} \sum_{i=1}^N |f(x_{i+1}) - f(x_i)|\,,$$

where the x_is are real numbers (not necessarily in $[0, 1]$). Note that $\mathrm{V}(.)$ is a norm and that $\mathrm{V}(f \cdot g) \le \mathrm{V}(f)\,\mathrm{V}(g)$.

Let us fix once and for all a GPM map $T : [0, 1] \to [0, 1]$ of parameter $\gamma \in (0, 1)$. Let $v_k : T_{(k)}I_k \to I_k$ be the inverse branches of T. Consider $M = \{m \in \{1, \dots, d-1\} : 0 \in T_{(m)}I_m\}$, and let $z_0 \in (0, y_1)$ be so small that v_m is well defined on $[0, z_0]$ for any $m \in M$, v_0' is decreasing on $(0, z_0]$ (this is possible since $v_0''(x) < 0$ for small x), and $T_{(k)}I_k \cap [0, z_0] = \emptyset$ for $k \notin M$. Note that $M \ne \emptyset$, since T is topologically transitive.

Define a sequence z_n inductively by $z_n = v_0(z_{n-1})$. Let $J_n = (z_{n+1}, z_n]$, so that T^n is bijective from J_n to $(z_1, z_0]$. Following the procedure in Zweimüller (1998), the invariant measure of T may be constructed as follows: we first consider the first return map on $(z_1, 1]$. It is Rychlik and topologically transitive, hence it admits an invariant measure ν_0 on $(z_1, 1]$ whose density h_0 is bounded from above and below in $(z_1, 1]$ and has bounded variation.

Extending ν_0 to the whole interval by the formula

$$\nu(A) = \nu_0(A \cap (z_1, 1]) + \sum_{n \geq 1} \nu_0(T^{-n}(A) \cap \{\phi > n\}) ,$$

where ϕ is the first return time to $(z_1, 1]$, and then renormalizing, we obtain the invariant probability measure of T. Denoting by h the density of ν, the previous formula becomes, for $x \in [0, z_1]$,

$$h(x) = \sum_{n=0}^{\infty} \sum_{m \in M} |(v_m v_0^n)'(x)| h(v_m v_0^n x). \tag{3.1}$$

Our goal in this paragraph and the next is to study the Perron-Frobenius operator K^n acting on the space BV of bounded variation functions. Let $K(x, y)$ be the kernel corresponding to the operator K. It is given by $K(x, v_k x) = h(v_k x)|v_k'(x)|/h(x)$ for $k \in \{0, \dots, d-1\}$, and $K(x, y) = 0$ if y is not of the form $v_k x$. By definition,

$$K^n f(x_0) = \sum_{x_1, \dots, x_n} K(x_0, x_1) K(x_1, x_2) \dots K(x_{n-1}, x_n) f(x_n) .$$

To understand the behavior of K^n, we will break the trajectories $x_0, \dots, x_n$ of the random walk according to their first and last entrance in the reference set $(z_1, 1]$ – the interest of this set is that T is uniformly expanding there. More precisely, let us define operators A_n, B_n, C_n and T_n as follows: they are defined like K^n but we only sum over trajectories $x_0, \dots, x_n$ such that

- For A_n, $x_0, \dots, x_{n-1} \in [0, z_1]$ and $x_n \in (z_1, 1]$.
- For B_n, $x_0 \in (z_1, 1)$ and $x_1, \dots, x_n \in [0, z_1]$.
- For C_n, $x_0, \dots, x_n \in [0, z_1]$.
- For T_n, $x_0 \in (z_1, 1]$ and $x_n \in (z_1, 1]$.

By construction, one has the decomposition

$$K^n f = \sum_{a+k+b=n} A_a T_k B_b f + C_n f . \tag{3.2}$$

One can give formulas for A_n, B_n and C_n, as follows:

$$A_n f(x) = \mathbb{1}_{[0,z_1]}(x) \sum_{m \in M} \frac{|(v_m v_0^{n-1})'(x)| h(v_m v_0^{n-1} x)}{h(x)} f(v_m v_0^{n-1} x) , \tag{3.3}$$

$$B_n f(x) = \mathbb{1}_{(z_1,z_0]}(x) \frac{(v_0^n)'(x) h(v_0^n x)}{h(x)} f(v_0^n x) , \tag{3.4}$$

$$C_n f(x) = \mathbb{1}_{[0,z_1]}(x) \frac{(v_0^n)'(x) h(v_0^n x)}{h(x)} f(v_0^n x) . \tag{3.5}$$

On the other hand, the operator T_n is less explicit, but it can be studied using operator renewal theory.

Proposition 3.1. *The operator T_n can be decomposed as*

$$T_n f = \left(\int_{(z_1,1]} f d\nu \right) \mathbb{1}_{(z_1,1]} + E_n f \ , \tag{3.6}$$

where the operator E_n satisfies $\mathrm{V}(E_n f) \leq \dfrac{C}{n^{(1-\gamma)/\gamma}} \mathrm{V}(f)$.

Proof. Since this follows closely from the arguments in Sarig (2002), Gouëzel (2004b) and Gouëzel (2007), we will only sketch the proof.

Define an operator R_n by $R_n f(x_0) = \mathbb{1}_{(z_1,1]}(x) \sum K(x_0, x_1) \dots K(x_1, x_n) f(x_n)$, where the summation is over all $x_1, \dots, x_{n-1} \in [0, z_1]$ and $x_n \in (z_1, 1]$: this operator is similar to T_n, but it only takes the first returns to $(z_1, 1]$ into account. Breaking a trajectory into its successive excursions outside of $(z_1, 1]$, it follows that the following renewal equation holds: $T_n = \sum_{\ell=1}^{\infty} \sum_{k_1+\dots+k_\ell=n} R_{k_1} \dots R_{k_\ell}$. In other words, $I + \sum T_n z^n = (I - \sum R_k z^k)^{-1}$, at least as formal series.

In the proof of Lemma 3.1 in Gouëzel (2007), it is shown that the operators R_k act continuously on BV, with a norm bounded by $C/k^{1+1/\gamma}$ – the estimates in Gouëzel do not deal with the factor h, but since this function as well as its inverse have bounded variation on $(z_1, 1]$ they do not change anything. Since this is summable, we can define, for $|z| \leq 1$, an operator $R(z) = \sum R_n z^n$ acting on BV. Moreover, Gouëzel (2007) also proves that the essential spectral radius of this operator is < 1 for any $|z| \leq 1$. Thanks to the topological transitivity of T, it follows that $R(1)$ has a simple eigenvalue at 1 (the corresponding eigenfunction is the constant function 1), while $I - R(z)$ is invertible for $z \neq 1$.

This spectral control makes it possible to apply Theorem 1.1 in Gouëzel (2004b), dealing with renewal sequences of operators as above. Its conclusion implies (3.6). $\square$

With (3.2), we finally obtain that

$$K^n f = \sum_{a+k+b=n} A_a(\mathbb{1}_{(z_1,1]}) \cdot \nu(B_b f) + \sum_{a+k+b=n} A_a E_k B_b f + C_n f \ , \tag{3.7}$$

where

$$\mathrm{V}(E_k f) \leq \frac{C}{k^{(1-\gamma)/\gamma}} \mathrm{V}(f). \tag{3.8}$$

3.1 Proof of Proposition 1.15

We shall prove successively that, for $n > 0$,

$$\mathrm{V}(C_n f) \leq C\,\mathrm{V}(f)\,, \tag{3.9}$$

$$\mathrm{V}(A_n f) \leq C\,\mathrm{V}(f)/(n+1)\,, \tag{3.10}$$

$$\mathrm{V}(B_n f) \leq C\,\mathrm{V}(f)/(n+1)^{1/\gamma}\,. \tag{3.11}$$

The proof of Proposition 1.15 follows from the above upper bounds and from the following elementary lemma.

Lemma 3.2. *Let u_n and v_n be two non increasing sequences such that $u_{[n/2]} \leq Cu_n$ and $v_{[n/2]} \leq Cv_n$. Then*

$$\sum_{i+j=n} u_i v_j \leq C u_n \left(\sum_{j=0}^{n} v_i \right) + C v_n \left(\sum_{i=0}^{n} u_i \right) .$$

Proof. If $i \leq n/2$, we use that v_j is bounded by Cv_n. If $j \leq n/2$, we use that u_i is bounded by Cu_n. □

We can now complete the proof, assuming the bounds (3.9), (3.10), and (3.11):

Proof of Proposition 1.15. Let f be such that $\nu(f) = 0$. We will bound $V(K^n f)$ using the decomposition of $K^n f$ given in (3.7). Using (3.10), (3.8) and (3.11), we get

$$\mathrm{V}\left(\sum_{a+k+b=n} A_a E_k B_b f \right) \leq C\,\mathrm{V}(f) \sum_{a+k+b=n} \frac{1}{(a+1)(k+1)^{(1-\gamma)/\gamma}(b+1)^{1/\gamma}} .$$

By lemma 3.2,

$$\sum_{k+b=j} \frac{1}{(k+1)^{(1-\gamma)/\gamma}(b+1)^{1/\gamma}} \leq \frac{C}{(j+1)^{(1-\gamma)/\gamma}}$$

and

$$\sum_{a+j=n} \frac{1}{(a+1)(j+1)^{(1-\gamma)/\gamma}} \leq C\Big(\frac{\ln(n)}{(n+1)^{(1-\gamma)/\gamma}} \vee \frac{1}{n} \Big) .$$

Consequently,

$$\mathrm{V}\left(\sum_{a+k+b=n} A_a E_k B_b f \right) \leq C\,\mathrm{V}(f) \Big(\frac{\ln(n)}{(n+1)^{(1-\gamma)/\gamma}} \vee \frac{1}{n} \Big) . \tag{3.12}$$

It remains to bound up the first term in (3.7), which can be written

$$\sum_{a=0}^{n} A_a(\mathbb{1}_{(z_1,1]}) \cdot \left(\sum_{b=0}^{n-a} \nu(B_b f) \right) .$$

Now, $\sum_{b=0}^{\infty} \nu(B_b f) = \nu(f) = 0$, so that

$$\left|\sum_{b=0}^{n-a} \nu(B_b f)\right| = \left|\sum_{b>n-a} \nu(B_b f)\right| \leq \sum_{b>n-a} \mathrm{V}(B_b f) \leq \sum_{b>n-a} \frac{C\,\mathrm{V}(f)}{(b+1)^{1/\gamma}} \leq \frac{D\,\mathrm{V}(f)}{(n+1-a)^{(1-\gamma)/\gamma}}\,.$$

By (3.10), $\mathrm{V}(A_a \mathbb{1}_{(z_1,1]}) \leq C/(a+1)$. Consequently,

$$\begin{aligned} \mathrm{V}\left(\sum_{a=0}^{n} A_a(\mathbb{1}_{(z_1,1]}) \cdot \left(\sum_{b=0}^{n-a} \nu(B_b f)\right)\right) &\leq C\,\mathrm{V}(f) \sum_{a=0}^{n} \frac{1}{(a+1)(n+1-a)^{(1-\gamma)/\gamma}} \\ &\leq D\,\mathrm{V}(f) \Big(\frac{\ln(n)}{(n+1)^{(1-\gamma)/\gamma}} \vee \frac{1}{n} \Big)\,, \end{aligned} \tag{3.13}$$

the last inequality following from Lemma 3.2.

Starting from (3.7) and using (3.9), (3.12) and (3.13) we obtain that $\mathrm{V}(K^n f) \leq C\,\mathrm{V}(f)$ for any f such that $\nu(f) = 0$. Now let f be any BV function on $[0,1]$, and let $\|df\|$ be the variation norm of the measure df on $[0,1]$. To conclude the proof, it suffices to note that $\|dK^n(f)\| = \|dK^n(f^{(0)})\| \leq \mathrm{V}(K^n(f^{(0)})) \leq C\,\mathrm{V}(f^{(0)}) \leq 3C\|df\|$. □

It remains to prove the upper bounds (3.9), (3.10), and (3.11). We shall use the following facts, proved e.g. in Liverani, Saussol and Vaienti (1999) or Young (1999). We will denote Lebesgue measure by λ.

1. One has $z_n \sim C/n^{1/\gamma}$ for some $C > 0$. Moreover, $\lambda(J_n) = z_n - z_{n+1} \sim C/n^{(1+\gamma)/\gamma}$ for some $C > 0$. One has
$$h(z_n) \sim C z_n^{-\gamma} \sim Dn\,. \tag{3.14}$$

2. There exists a constant $C > 0$ such that, for all $n \geq 0$ and $k \geq 0$, and for all $x, y \in J_k$,
$$\left|1 - \frac{(v_0^n)'(x)}{(v_0^n)'(y)}\right| \leq C|x-y|\,.$$
Integrating the above inequality, we obtain that
$$C^{-1}\frac{\lambda(J_{n+k})}{\lambda(J_k)} \leq (v_0^n)'(x) \leq C\frac{\lambda(J_{n+k})}{\lambda(J_k)}\,. \tag{3.15}$$

3. The function $(v_0^n)'$ is decreasing on $[0, z_1)$.

The following easy lemma follows from the definition of V.

Lemma 3.3. *If f is nonnegative and monotonic on some interval I, then*

$$\mathrm{V}(\mathbb{1}_I f) \leq C \sup_I |f|. \tag{3.16}$$

If f is positive on some interval I, then

$$\mathrm{V}(\mathbb{1}_I/f) \leq C\,\mathrm{V}(\mathbb{1}_I f)/\min_I |f|^2. \tag{3.17}$$

We shall also use the following lemma on the density h.

Lemma 3.4. *There exists a constant C such that, for any $1 \leq i < j$,*

$$\mathrm{V}(\mathbb{1}_{[z_j,z_i]}h) \leq Cj \quad \text{ and } \mathrm{V}(\mathbb{1}_{[z_j,z_i]}/h) \leq Cj/i^2. \tag{3.18}$$

Proof. We start from the formula (3.1) for h, and the inequality $\mathrm{V}(fg) \leq \mathrm{V}(f)\,\mathrm{V}(g)$, to obtain

$$\mathrm{V}(\mathbb{1}_{[z_j,z_i]}h) \leq \sum_{n=0}^{\infty}\sum_{m\in M} \mathrm{V}(\mathbb{1}_{[z_j,z_i]}(v_0^n)') \cdot \mathrm{V}(\mathbb{1}_{[z_j,z_i]}|v_m' \circ v_0^n|) \cdot \mathrm{V}(\mathbb{1}_{[z_j,z_i]}h \circ v_m \circ v_0^n) \tag{3.19}$$

Since the functions v_m' have bounded variation, and the function h has bounded variation on $(z_1, 1]$ (which contains the image of $v_m v_0^n(0, z_1)$), we get $\mathrm{V}(\mathbb{1}_{[z_j,z_i]}h) \leq C\sum_{n=0}^{\infty}\mathrm{V}(\mathbb{1}_{[z_j,z_i]}(v_0^n)')$. Since the function $(v_0^n)'$ is decreasing on $[z_j, z_i]$, we get by using (3.16)

$$\mathrm{V}(\mathbb{1}_{[z_j,z_i]}h) \leq C\sum_{n=0}^{\infty}(v_0^n)'(z_j) \leq C\sum_{n=0}^{\infty}\frac{\lambda(J_{n+j})}{\lambda(J_j)} = C\frac{z_j}{z_j - z_{j+1}} \leq C\frac{j^{-1/\gamma}}{j^{-1/\gamma-1}} = Cj.$$

This proves the first inequality of the proposition.

To prove the second one, we use (3.17). Since $\min_{[z_j,z_i]}|h| \geq Cz_i^{-\gamma} \geq Ci$, the result follows. □

We can now prove the upper bounds (3.9), (3.10), and (3.11)

Since C_n is given by (3.5), the upper bound (3.9) follows from Lemma 3.5 below.

Lemma 3.5. *There exists $C > 0$ such that, for any $n \geq 1$,*

$$\mathrm{V}\left(\mathbb{1}_{[0,z_1]}\frac{(v_0^n)'(x)h(v_0^n x)}{h(x)}\right) \leq C\,. \tag{3.20}$$

Proof. Since $K1 = 1$, we have $h(x) = v_0'(x)h(v_0 x) + \sum_{m\in M}|v_m'(x)|h(v_m x)$ on $[0, z_1]$. By iterating this equality, we obtain for any $n \in \mathbb{N}$,

$$h(x) = (v_0^n)'(x)h(v_0^n x) + \sum_{j=0}^{n-1}\sum_{m\in M}|(v_m v_0^j)'(x)|h(v_m v_0^j x)\,.$$

Consequently,

$$1 - \frac{(v_0^n)'(x)h(v_0^n x)}{h(x)} = \sum_{j=0}^{n-1}\sum_{m\in M}\frac{(v_0^j)'(x)|v_m'(v_0^j x)|h(v_m v_0^j x)}{h(x)}\,. \tag{3.21}$$

Let s be such that $2^s \le n < 2^{s+1}$. To prove (3.20), we will control, for any k,

$$\mathrm{V}\left(\mathbb{1}_{[z_{2^k}, z_{2^{k-1}}]} \frac{(v_0^n)'(x) h(v_0^n x)}{h(x)}\right).$$

Assume first that $k \le s$. On $[z_{2^k}, z_{2^{k-1}}]$, the function $(v_0^n)'$ is decreasing, so that its variation is bounded in terms of its supremum $(v_0^n)'(z_{2^k}) \le C\lambda(J_{2^k+n})/\lambda(J_{2^k})$. The variation of the function $h \circ v_0^n$ on $[z_{2^k}, z_{2^{k-1}}]$ is the variation of h on $[z_{2^k+n}, z_{2^{k-1}+n}]$, hence by Lemma 3.4 it is bounded by $C(2^k + n)$. This lemma also shows that the variation of $1/h$ is bounded by $C/2^k$. Hence,

$$\begin{aligned}\mathrm{V}\left(\mathbb{1}_{[z_{2^k}, z_{2^{k-1}}]} \frac{(v_0^n)'(x) h(v_0^n x)}{h(x)}\right) &\le C \frac{\lambda(J_{2^k+n})}{\lambda(J_{2^k})} \frac{2^k + n}{2^k} \\ &\le C \frac{(2^k + n)^{-(1+\gamma)/\gamma}}{(2^k)^{-(1+\gamma)/\gamma}} \frac{2^k + n}{2^k} \le C \frac{(2^k)^{1/\gamma}}{n^{1/\gamma}}.\end{aligned}$$

Summing on k, we get

$$\mathrm{V}\left(\mathbb{1}_{[z_{2^{s+1}}, z_1]} \frac{(v_0^n)'(x) h(v_0^n x)}{h(x)}\right) \le C \sum_{k=1}^{s} \frac{(2^k)^{1/\gamma}}{n^{1/\gamma}} \le \frac{C 2^{s/\gamma}}{n^{1/\gamma}} \le C, \tag{3.22}$$

since $2^s \le n$.

Let now $k > s$. The previous upper bound gives a suboptimal control, hence we shall use the right hand term in (3.21). For $0 \le j \le n-1$ and $m \in M$, the variation of $v_m' \circ v_0^j \cdot h \circ v_m \circ v_0^j$ is uniformly bounded (since v_m is C^2 and h has bounded variation on $(z_1, 1]$). Moreover, as above, the variation of $(v_0^j)'$ is bounded by $C\lambda(J_{2^k+j})/\lambda(J_{2^k})$, which is uniformly bounded. Finally, the variation of $1/h$ is at most $C/2^k$, by Lemma 3.4. Consequently,

$$\mathrm{V}\left(\mathbb{1}_{[z_{2^k}, z_{2^{k-1}}]} \left(1 - \frac{(v_0^n)'(x) h(v_0^n x)}{h(x)}\right)\right) \le \sum_{j=0}^{n-1} \frac{C}{2^k} = \frac{Cn}{2^k}.$$

Summing on $k > s$,

$$\mathrm{V}\left(\mathbb{1}_{[0, z_{2^{s+1}}]} \left(1 - \frac{(v_0^n)'(x) h(v_0^n x)}{h(x)}\right)\right) \le Cn \sum_{k=s+1}^{\infty} \frac{1}{2^k} \le \frac{Cn}{2^s} \le D. \tag{3.23}$$

Lemma 3.5 follows by combining (3.22) and (3.23). □

Since A_n is given by (3.3), the upper bound (3.10) follows from Lemma 3.6 below.

Lemma 3.6. *There exists a positive constant C such that, for any $n \ge 1$,*

$$\mathrm{V}\left(\mathbb{1}_{[0, z_1]}(x) \sum_{m \in M} \frac{|(v_m v_0^{n-1})'(x)| h(v_m v_0^{n-1} x)}{h(x)}\right) \le \frac{C}{n}. \tag{3.24}$$

Proof. As in the proof of Lemma 3.5, we control the variation of the functions on $[z_{2^k}, z_{2^k-1}]$. On this interval, the variation of $(v_m v_0^{n-1})'$ is at most $C\lambda(J_{2^k+n})/\lambda(J_{2^k})$, the variation of $h(v_m v_0^{n-1})$ is bounded by C and the variation of $1/h$ is bounded by $C/2^k$. Summing on k, we obtain

$$\mathrm{V}\left(\mathbb{1}_{[0,z_1]}(x)\sum_{m\in M}\frac{|(v_m v_0^{n-1})'(x)|h(v_m v_0^{n-1}x)}{h(x)}\right)$$
$$\leq C\sum_{k=1}^{\infty}\frac{\lambda(J_{2^k+n})}{\lambda(J_{2^k})}\frac{1}{2^k}\leq D\sum_{k=1}^{\infty}\frac{2^{k(1+\gamma)/\gamma}}{(n+2^k)^{(1+\gamma)/\gamma}}\frac{1}{2^k}\,.$$

Let s be such that $2^s \leq n < 2^{s+1}$. We split the sum on the sets $k \leq s$ and $k > s$, and we obtain the upper bound

$$C\sum_{k=1}^{s}\frac{2^{k(1+\gamma)/\gamma}}{(n+1)^{(1+\gamma)/\gamma}2^k}+C\sum_{k=s+1}^{\infty}\frac{1}{2^k}\leq\frac{C2^{s/\gamma}}{(n+1)^{(1+\gamma)/\gamma}}+\frac{1}{2^s}\leq\frac{D}{n}\,.\qquad\square$$

It remains to prove (3.11). Recall that B_n is given by (3.4). On $(z_1, z_0]$, the variation of the function $(v_0^n)'$ is bounded by $C\lambda(J_n)/\lambda(J_0) \leq C/n^{(1+\gamma)/\gamma}$, the variation of $1/h$ is bounded by C, and the variation of $h(v_0^n x)$ is bounded by $\mathrm{V}(\mathbb{1}_{(z_{n+1},z_n]}h) \leq Cn$. This implies the upper bound (3.11). The proof of Proposition 1.15 is complete.

3.2 Proof of Proposition 1.16

To prove Proposition 1.16, we keep the same notations as in the previous paragraphs. The proof follows the line of that of Theorem 2.3.6 in Gouëzel (2004c). Let f be a function in BV with $\nu(f) = 0$, we wish to estimate $\nu(|K^n f|)$ thanks to the decomposition (3.7).

For the term $C_n f$, we have

$$\nu(|C_n(f)|) \leq C\|f\|_\infty \nu(K^n \mathbb{1}_{[0,z_{n+1}]}) = C\|f\|_\infty \nu(\mathbb{1}_{[0,z_{n+1}]})\,.$$

Since $\nu(J_k) \leq C/(k+1)^{1/\gamma}$, it follows that

$$\nu(|C_n(f)|) \leq \frac{C\|f\|_\infty}{(n+1)^{(1-\gamma)/\gamma}}\,. \tag{3.25}$$

We now turn to the term $\sum_{a+k+b=n} A_a E_k B_b f$ in (3.7). Let us first remark that, for any bounded function g,

$$\nu(|A_n(g)|) \leq C\|g\|_\infty \nu(K^n \mathbb{1}_{(z_1,1]\cap T^{-1}[0,z_n]}) = C\|g\|_\infty \nu((z_1, 1] \cap T^{-1}[0, z_n]).$$

Since the density of ν is bounded on $(z_1, 1]$, this quantity is $\leq C\|g\|_\infty z_n$. We obtain

$$\nu(|A_n(g)|) \leq \frac{C\|g\|_\infty}{(n+1)^{1/\gamma}} . \tag{3.26}$$

Using successively (3.26), (3.8) and (3.11), we obtain

$$\begin{aligned}
\nu\Big(\Big|\sum_{a+k+b=n} A_a E_k B_b f\Big|\Big) &\leq C \sum_{a+k+b=n} \frac{\|E_k B_b f\|_\infty}{(a+1)^{1/\gamma}} \\
&\leq C \sum_{a+k+b=n} \frac{\mathrm{V}(f)}{(a+1)^{1/\gamma}(k+1)^{(1-\gamma)/\gamma}(b+1)^{1/\gamma}} \\
&\leq \frac{C\,\mathrm{V}(f)}{(n+1)^{(1-\gamma)/\gamma}} .
\end{aligned} \tag{3.27}$$

We finally turn to the term $\sum_{a+k+b=n} A_a(\mathbb{1}_{(z_1,1]}) \cdot \nu(B_b f)$ in (3.7). From (3.1) and (3.26), we obtain

$$\begin{aligned}
\nu\left(\left|\sum_{a=0}^{n} A_a(\mathbb{1}_{(z_1,1]}) \cdot \left(\sum_{b=0}^{n-a} \nu(B_b f)\right)\right|\right) &\leq C\,\mathrm{V}(f) \sum_{a=0}^{n} \frac{1}{(a+1)^{1/\gamma}(n+1-a)^{(1-\gamma)/\gamma}} \\
&\leq \frac{D\,\mathrm{V}(f)}{(n+1)^{(1-\gamma)/\gamma}} .
\end{aligned} \tag{3.28}$$

We have shown that, if $\nu(f) = 0$, all the terms on the right hand side of (3.7) are bounded by $C\,\mathrm{V}(f)/(n+1)^{(1-\gamma)/\gamma}$. Therefore, $\nu(|K^n f|)$ is bounded by the same quantity. Now let f be any BV function on $[0, 1]$, and let $\|df\|$ be the variation norm of the measure df on $[0, 1]$. To conclude the proof, it suffices to note that

$$\nu(|K^n(f^{(0)})|) \leq \frac{C\,\mathrm{V}(f^{(0)})}{(n+1)^{(1-\gamma)/\gamma}} \leq \frac{3C\|df\|}{(n+1)^{(1-\gamma)/\gamma}} . \quad \square$$

4 Proofs of the main results, Theorems 1.5, 1.6 and 1.7

It is well known that $(T^0, T^1, T^2, \ldots, T^{n-1})$ is distributed as $(Y_n, Y_{n-1}, \ldots, Y_1)$ where $(Y_i)_{i\geq 0}$ is a stationary Markov chain with invariant measure ν and transition kernel K (see for instance Lemma XI.3 in Hennion and Hervé (2001)). Let $X_n = f(Y_n) - \nu(f)$ for some function $f : [0, 1] \to \mathbb{R}$. A common argument of the proofs of Theorems 1.5 and 1.6 is the following inequality: for any $\varepsilon > 0$,

$$\nu\Big(\max_{1\leq k\leq n} \Big|\sum_{i=0}^{k-1}(f \circ T^i - \nu(f))\Big| \geq \varepsilon\Big) \leq \nu\Big(2 \max_{1\leq k\leq n} \Big|\sum_{i=1}^{k} X_i\Big| \geq \varepsilon\Big) . \tag{4.1}$$

Indeed since

$$(f-\nu(f), f\circ T-\nu(f), \dots, f\circ T^{n-1}-\nu(f)) \text{ is distributed as } (X_n, X_{n-1}, \dots, X_1),$$

the following equality holds in distribution

$$\max_{1\le k\le n}\sum_{i=0}^{k-1}(f\circ T^i-\nu(f)) = \max_{1\le k\le n}\sum_{i=k}^{n} X_i\,. \tag{4.2}$$

Notice now that for any $k\in[1,n]$,

$$\sum_{i=k}^{n} X_i = \sum_{i=1}^{n} X_i - \sum_{i=1}^{k-1} X_i\,.$$

Consequently

$$\max_{1\le k\le n}\Big|\sum_{i=k}^{n} X_i\Big| \le \max_{1\le k\le n-1}\Big|\sum_{i=1}^{k} X_i\Big| + \Big|\sum_{i=1}^{n} X_i\Big|\,,$$

which together with (4.2) entails (4.1).

4.1 Proof of Theorem 1.5

According to (4.1), Item 1 of Theorem 1.5 holds as soon as

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}\Big(2\max_{1\le k\le n}\Big|\sum_{i=1}^{k} X_i\Big| \ge A\sqrt{n\ln(\ln(n))}\Big) < \infty\,, \tag{4.3}$$

for some positive constant A. Using the extension $(\pi_i)_{i\in\mathbb{Z}}$ of the chain $(Y_i)_{i\ge 0}$ given at the beginning of Section 1.4, (4.3) follows from the inequality (1.13) of Theorem 1.13 by taking

$$A = 40\sqrt{2}\Big(\sum_{k\ge 1}\int_0^{\alpha_{1,\mathbf{Y}}(k)} Q^2(u)du\Big)^{1/2}.$$

By Theorem 1.13, (1.13) holds as soon as $f\in\widetilde{\mathcal{F}}(Q,\nu)$ and (1.12) holds. In the same way, Item 2 of Theorem 1.5 follows from (1.14) of Theorem 1.13 provided that (1.12) holds.

Now, by Proposition 1.17, $\alpha_{2,\mathbf{Y}}(n) = O(n^{(\gamma-1)/\gamma})$. Hence (1.13) holds as soon as, for $p=2$,

$$f\in\widetilde{\mathcal{F}}(Q,\nu), \quad \text{and} \quad \int_0^1 u^{-\gamma(p-1)/(1-\gamma)}Q^p(u)du < \infty\,. \tag{4.4}$$

If H is the càdlàg inverse of Q, then $f\in\mathcal{F}(H,\nu)$ iff $f\in\widetilde{\mathcal{F}}(Q,\nu)$. Moreover (4.4) holds if and only if

$$f\in\mathcal{F}(H,\nu), \quad \text{and} \quad \int_0^{\infty} x^{p-1}(H(x))^{\frac{1-p\gamma}{1-\gamma}}dx < \infty\,. \tag{4.5}$$

Indeed, setting $v = u^{(1-\gamma p)/(1-\gamma)}$, we get that

$$\int_0^1 u^{-\gamma(p-1)/(1-\gamma)} Q^p(u)du = \frac{1-\gamma}{1-\gamma p}\int_0^1 Q^p(v^{(1-\gamma)/(1-\gamma p)})dv\,.$$

Since H is the càdlàg inverse of Q, we get

$$\int_0^1 Q^p(v^{(1-\gamma)/(1-\gamma p)})dv = \int_0^\infty \big(H(t^{1/p})\big)^{\frac{1-p\gamma}{1-\gamma}}dt = p\int_0^\infty x^{p-1}(H(x))^{\frac{1-p\gamma}{1-\gamma}}dx\,,$$

which concludes the proof.

4.2 Proof of Theorem 1.6

By using (4.1), (1.6) will hold if we can prove that for any $\varepsilon > 0$ and any $p \in (1,2)$, one has

$$\sum_{n=1}^\infty \frac{1}{n}\mathbb{P}\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k X_i\Big| \ge n^{1/p}\varepsilon\Big) < \infty\,. \tag{4.6}$$

According to Theorem 4 in Dedecker and Merlevède (2007), we have that

$$\sum_{n=1}^\infty \frac{1}{n}\mathbb{P}\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k X_i\Big| \ge n^{1/p}\varepsilon\Big) \le C\sum_{i=0}^\infty (i+1)^{p-2}\int_0^{\gamma_i} Q^{p-1}_{|X_0|}\circ G_{|X_0|}(u)du\,, \tag{4.7}$$

where $\gamma_i = \|\mathbb{E}(X_i|\mathcal{M}_0)\|_1$ and $G_{|X_0|}$ is the inverse of $L_{|X_0|}(x) = \int_0^x Q_{|X_0|}(u)du$. We will denote by L and G the same functions constructed from Q, the càdlàg inverse of H. Assume first that $X_i = f(Y_i) - \nu(f)$ with $f = \sum_{\ell=1}^L a_\ell f_\ell$, where $f_\ell \in \widetilde{\mathrm{Mon}}(Q,\nu)$ and $\sum_{\ell=1}^L |a_\ell| \le 1$. According to (2.18)

$$\gamma_i \le 4\int_0^{\alpha_{1,\mathbf{Y}}(i)} Q(u)du\,. \tag{4.8}$$

Since $Q_{|X_0|}(u) \le Q_{|f(Y_0)|}(u) + \nu(f)$, we see that $\int_0^x Q_{|X_0|}(u)du \le 2\int_0^x Q_{|f(Y_0)|}(u)du$. Since $f = \sum a_\ell f_\ell$, we get, according to item (c) of Lemma 2.1 in Rio (2000),

$$\int_0^x Q_{|X_0|}(u)du \le 2\sum_{\ell=1}^L \int_0^x Q_{|a_\ell f_\ell(X_0)|}(u)du \le 2\sum_{\ell=1}^L |a_\ell|\int_0^x Q(u)du\,.$$

Since $\sum_{\ell=1}^L |a_\ell| \le 1$, it follows that $G(u/2) \le G_{|X_0|}(u)$, where G is the inverse of $x \mapsto \int_0^x Q(u)du$. In particular, $G_{|X_0|}(u) \ge G(u/4)$. Since $Q_{|X_0|}$ is non-increasing, it follows that

$$\begin{aligned}\int_0^{\gamma_i} Q^{p-1}_{|X_0|}\circ G_{|X_0|}(u)du &\le \int_0^{\gamma_i} Q^{p-1}_{|X_0|}\circ G(u/4)du = 4\int_0^{\gamma_i/4} Q^{p-1}_{|X_0|}\circ G(v)dv\\ &= 4\int_0^{L(\gamma_i/4)} Q^{p-1}_{|X_0|}(w)Q(w)dw \le 4\int_0^{\alpha_{1,\mathbf{Y}}(i)} Q^{p-1}_{|X_0|}(w)Q(w)dw\,,\end{aligned}$$

where the last inequality follows from (4.8). Let $\alpha_1^{-1}(u) = \sum_{i\geq 0} \mathbb{1}_{u<\alpha_{1,\mathbf{Y}}(i)}$. Since $(\alpha_1^{-1}(u))^{p-1} = \sum_{j\geq 0} \big((j+1)^{p-1} - j^{p-1}\big)\mathbb{1}_{u<\alpha_{1,\mathbf{Y}}(j)}$ and $(j+1)^{p-2} \leq C\big((j+1)^{p-1} - j^{p-1}\big)$, we get

$$\sum_{i=0}^{\infty}(i+1)^{p-2}\int_0^{\gamma_i} Q^{p-1}_{|X_0|}\circ G_{|X_0|}(u)du \leq C\int_0^1 (\alpha_1^{-1}(u))^{p-1}Q^{p-1}_{|X_0|}(u)Q(u)du\,. \tag{4.9}$$

Using Hölder's inequality, we derive that

$$\begin{aligned}\int_0^1 (\alpha_1^{-1}(u))^{p-1}Q^{p-1}_{|X_0|}(u)Q(u)du \leq \Big(\int_0^1 (\alpha_1^{-1}(u))^{p-1}Q^{p}(u)du\Big)^{1/p} \\ \times\Big(\int_0^1 (\alpha_1^{-1}(u))^{p-1}Q^{p}_{|X_0|}(u)du\Big)^{(p-1)/p}.\end{aligned} \tag{4.10}$$

Now note that $Q^p_{|X_0|} = Q_{|X_0|^p}$. By convexity and the fact that $\sum_{\ell=1}^L |a_\ell| \leq 1$,

$$Q_{|X_0|^p}(u) \leq Q_{\sum_{\ell=1}^L |a_\ell||f_\ell(Y_0)-\nu(f_\ell)|^p}(u)\,.$$

Using again item (c) of Lemma 2.1 in Rio (2000), we get that

$$\begin{aligned}\int_0^1 (\alpha_1^{-1}(u))^{p-1}Q^{p}_{|X_0|}(u)du &\leq \sum_{\ell=1}^L |a_\ell|\int_0^1 (\alpha_1^{-1}(u))^{p-1}Q_{|f_\ell(Y_0)-\nu(f_\ell)|^p}(u)du \\ &\leq 2^{p+1}\int_0^1 (\alpha_1^{-1}(u))^{p-1}Q^{p}(u)du\,.\end{aligned} \tag{4.11}$$

It follows that

$$\sum_{i=0}^{\infty}(i+1)^{p-2}\int_0^{\gamma_i} Q^{p-1}_{|X_0|}\circ G_{|X_0|}(u)du \leq C\int_0^1 (\alpha_1^{-1}(u))^{p-1}Q^{p}(u)du\,. \tag{4.12}$$

From (4.7), (4.12) and the fact that $\alpha_{1,\mathbf{Y}}(n) = O(n^{(\gamma-1)/\gamma})$ by Proposition 1.17, it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^k X_i\Big| \geq n^{1/p}\varepsilon\Big) \leq C\int_0^1 u^{-\gamma(p-1)/(1-\gamma)}Q^p(u)du\,,$$

and the same inequality holds for any variable $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$ with $f \in \widetilde{\mathcal{F}}(Q,\nu)$ by applying Fatou's lemma. Hence (4.6) holds as soon as (4.4) holds. Since (4.4) is equivalent to (4.5), the result follows.

4.3 Proof of Theorem 1.7

By using (4.1), (1.6) will hold if we can prove that for any $\varepsilon > 0$, any p in $(1,2]$ and any $b > 1/p$, one has

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\Big(\max_{1\le k\le n} \Big| \sum_{i=1}^{k} X_i \Big| \ge n^{1/p}(\ln(n))^b \varepsilon \Big) < \infty \,. \tag{4.13}$$

Let Q be the càdlàg inverse of H. Note that $f \in \mathcal{F}(H,\nu)$ if and only if $f \in \widetilde{\mathcal{F}}(Q,\nu)$, and that H satisfies (1.7) if and only if $Q(u) \le (Cu)^{-(1-p\gamma)/(p(1-\gamma))}$.

We keep the same notations as in the proof of Theorem 1.6. Assume first that $X_i = \sum_{\ell=1}^{L} a_\ell f_\ell(Y_i) - \sum_{\ell=1}^{L} a_\ell \mathbb{E}(f_\ell(Y_i))$, with $f_\ell \in \widetilde{\mathcal{F}}(Q,\nu)$ and $\sum_{\ell=1}^{L} |a_\ell| \le 1$. Define the function $(\gamma/2)^{-1}(u) = \sum_{i\ge 0} \mathbb{1}_{u<\gamma_i/2}$, where $\gamma_i = \|\mathbb{E}(X_i|\mathcal{M}_0)\|_1$. Let $\bar{R}_{|X_0|}(u) = U_{|X_0|}(u) Q_{|X_0|}(u)$, with $U_{|X_0|} = ((\gamma/2)^{-1} \circ G^{-1}_{|X_0|})$. We apply Inequality (3.9) in Dedecker and Merlevède (2007):

$$\begin{aligned}\mathbb{P}\Big(\max_{1\le k\le n} \Big| \sum_{i=1}^{k} X_i \Big| \ge 5x \Big) &\le \frac{14n}{x} \int_0^1 Q_{|X_0|}(u) \mathbb{1}_{x<\bar{R}_{|X_0|}(u)} du \\ &\quad + \frac{4n}{x^2} \int_0^1 \mathbb{1}_{x\ge \bar{R}_{|X_0|}(u)} \bar{R}_{|X_0|}(u) Q_{|X_0|}(u) du \,.\end{aligned}$$

Taking $x_n = \varepsilon n^{1/p}(\ln(n))^b/5$, and summing in n, we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\Big(\max_{1\le k\le n} \Big| \sum_{i=1}^{k} X_i \Big| \ge n^{1/p}(\ln(n))^b \varepsilon \Big) &\le C \int_0^1 \frac{\bar{R}^{p-1}_{|X_0|}(u)}{(\ln(\bar{R}_{|X_0|}(u)) \vee 1)^{bp}} Q_{|X_0|}(u) du \\ &\le D \int_0^1 \frac{U^{p-1}_{|X_0|}(u)}{(\ln(U_{|X_0|}(u)) \vee 1)^{bp}} Q^p_{|X_0|}(u) du \,.\end{aligned}$$

Now, we make the change of variables $u = G_{|X_0|}(y)$, and we use that $G(y/2) \le G_{|X_0|}(y)$. It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\Big(\max_{1\le k\le n} \Big| \sum_{i=1}^{k} X_i \Big| \ge n^{1/p}(\ln(n))^b \varepsilon \Big) \le C \int_0^{\|X_0\|_1} \frac{((\gamma/2)^{-1}(y))^{p-1}}{(\ln((\gamma/2)^{-1})(y) \vee 1)^{bp}} Q^{p-1}_{|X_0|} \circ G(y/2) dy \,.$$

Let $U(u) = ((\gamma/2)^{-1} \circ 2G^{-1})(u)$, and make the change of variables $u = G(y/2)$. We obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\Big(\max_{1\le k\le n} \Big| \sum_{i=1}^{k} X_i \Big| \ge n^{1/p}(\ln(n))^b \varepsilon \Big) \le C \int_0^1 \frac{U^{p-1}(u)}{(\ln(U(u)) \vee 1)^{bp}} Q^{p-1}_{|X_0|}(u) Q(u) du \,.$$

From (4.8) we infer that $U(u) \le C u^{-\gamma/(1-\gamma)}$, so that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\Big(\max_{1\le k\le n} \Big| \sum_{i=1}^{k} X_i \Big| \ge n^{1/p}(\ln(n))^b \varepsilon \Big) \le C \int_0^1 \frac{u^{-\gamma(p-1)/(1-\gamma)}}{|\ln(u)|^{bp} \vee 1} Q^{p-1}_{|X_0|}(u) Q(u) du \,.$$

Applying Hölder's inequality as in (4.10), and next applying item (c) of Lemma 2.1 in Rio (2000) as in (4.11), it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\Big(\max_{1 \le k \le n} \Big| \sum_{i=1}^{k} X_i \Big| \ge n^{1/p} (\ln(n))^b \varepsilon \Big) \le C \int_0^1 \frac{u^{-\gamma(p-1)/(1-\gamma)}}{|\ln(u)|^{bp} \vee 1} Q^p(u) du \,.$$

Since $Q^p(u) \le (Cu)^{-(1-p\gamma)/(1-\gamma)}$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\Big(\max_{1 \le k \le n} \Big| \sum_{i=1}^{k} X_i \Big| \ge n^{1/p} (\ln(n))^b \varepsilon \Big) \le C \int_0^1 \frac{1}{u(|\ln(u)|^{bp} \vee 1)} du \,, \tag{4.14}$$

and the same inequality holds for any variable $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$ with $f \in \widetilde{\mathcal{F}}(Q, \nu)$ by applying Fatou's lemma. Now the right-hand term in (4.14) is finite as soon as $bp > 1$, which concludes the proof.

A Appendix

We recall a maximal exponential inequality for martingales which is a straightforward consequence of Theorem 3.4 in Pinelis (1994).

Proposition A.1. *Let $(d_j, \mathcal{F}_j)_{j \ge 1}$ be a real-valued martingale difference sequence with $|d_j| \le c$ for all j. Let $M_j = \sum_{i=1}^{j} d_i$. Then for all $x, y > 0$,*

$$\mathbb{P}\Big(\sup_{1 \le j \le n} |M_j| \ge x, \sum_{j=1}^{n} \mathbb{E}(|d_j|^2 | \mathcal{F}_{j-1}) \le y \Big) \le 2 \exp\left(-\frac{y}{c^2} \, h\Big(\frac{xc}{y}\Big) \right) ,$$

where $h(u) = (1+u)\ln(1+u) - u$.

Proof. Let $A_i = \{\sum_{j=1}^{i} \mathbb{E}(|d_j|^2 | \mathcal{F}_{j-1}) \le y\}$, and let $\bar{M}_j$ be the martingale $\bar{M}_j = \sum_{i=1}^{j} d_i \mathbb{1}_{A_i}$. Clearly

$$\begin{aligned} \mathbb{P}\Big(\sup_{1 \le j \le n} |M_j| \ge x, \sum_{j=1}^{n} \mathbb{E}(|d_j|^2 | \mathcal{F}_{j-1}) \le y \Big) &= \mathbb{P}\Big(\sup_{1 \le j \le n} |\bar{M}_j| \ge x, \sum_{j=1}^{n} \mathbb{E}(|d_j|^2 | \mathcal{F}_{j-1}) \le y \Big) \\ &\le \mathbb{P}\Big(\sup_{1 \le j \le n} |\bar{M}_j| \ge x \Big) \,. \end{aligned}$$

To conclude, it suffices to apply Theorem 3.4 in Pinelis (1994) to the martingale $\bar{M}_j$. ☐

References

[1] Berger, E. (1990). An almost sure invariance principle for stationary ergodic sequences of Banach space valued random variables, *Probab. Theory Relat. Fields* **84** 161-201.

[2] Dedecker, J. and Merlevède, F. (2007). Convergence rates in the law of large numbers for Banach-valued dependent variables, *Teor. Veroyatnost. i Primenen* **52** 562-587.

[3] Dedecker, J. and Prieur, C. (2009). Some unbounded functions of intermittent maps for which the central limit theorem holds. *ALEA Lat. Am. J. Probab. Math. Stat.* **5** 29-45.

[4] Dedecker, J. and Rio, E. (2008). On mean central limit theorems for stationary sequences. *Ann. Inst. H. Poincaré Probab. Statist.* **44** 693-726.

[5] Esseen, C.-G. and Janson, S. (1985). On moment conditions for normed sums of independent variables and martingale differences. *Stochastic Process. Appl.* **19** 173-182.

[6] Feller, W. (1966). An introduction to probability theory and its applications. Vol. II. John Wiley & Sons, Inc., New York-London-Sydney, 636 pp.

[7] Feller, W. (1968). An extension of the law of the iterated logarithm to variables without variance. *J. Math. Mech.* **18** 343-355.

[8] Ferrero, P., Haydn, N. and Vaienti, S. (2003). Entropy fluctuations for parabolic maps. *Nonlinearity* **16** 1203-1218.

[9] Gordin, M. I. (1973). Abstracts of communication, T.1:A-K, International conference on probability theory, Vilnius.

[10] Gouëzel, S. (2004a). Central limit theorem and stable laws for intermittent maps. *Probab. Theory Relat. Fields* **128** 82-122.

[11] Gouëzel, S. (2004b). Sharp polynomial estimates for the decay of correlations. *Isr. J. Math.* **139** 29-65.

[12] Gouëzel, S. (2004c). Vitesse de décorrélation et théorèmes limites pour les applications non uniformément dilatantes. Thèse 7526 de l'Université Paris Sud.

[13] Gouëzel, S. (2007). A Borel-Cantelli lemma for intermittent interval maps. *Nonlinearity* **20** 1491-1497.

[14] Hennion, H. and Hervé, L. (2001). Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness. Lecture Notes in Mathematics **1766**, Springer.

[15] Heyde, C. C. (1969). A note concerning behaviour of iterated logarithm type. *Proc. Amer. Math. Soc.* **23** 85-90.

[16] Hofbauer, F. and Keller, G. (1982). Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.* **180** 119-140.

[17] Kontoyiannis, I. (1998). Asymptotic recurrence and waiting times for stationary processes. *J. Theoret. Probab* **11** 795-811.

[18] Liverani, C., Saussol, B. and Vaienti S. (1999). A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems* **19** 671-685.

[19] Melbourne, I. and Nicol, M. (2005). Almost sure invariance principle for nonuniformly hyperbolic systems. *Commun. Math. Phys.* **260** 131-146.

[20] Merlevède, F. (2008). On a maximal inequality for strongly mixing random variables in Hilbert spaces. Application to the compact law of the iterated logarithm. *Publ. Inst. Stat. Univ. Paris* **12** 47-60.

[21] Philipp, W. and Stout, W.F. (1975). Almost sure invariance principle for partial sums of weakly dependent random variables. *Mem. of the Amer. Math. Soc.* **161**. Providence, RI: Amer. Math. Soc.

[22] Pinelis, I. (1994). Optimum bounds for the distributions of martingales in Banach spaces. *Ann. Probab.* **22** 1679-1706.

[23] Pomeau, Y. and Manneville, P. (1980), Intermittent transition to turbulence in dissipative dynamical systems. *Commun. Math. Phys.* **74** 189-197.

[24] Rio, E. (1993). Covariance inequalities for strongly mixing processes. *Ann. Inst. H. Poincaré Probab. Statist.* **29** 587-597.

[25] Rio, E. (1995). The functional law of the iterated logarithm for stationary strongly mixing sequences. *Ann. Probab.* **23** 1188-1203.

[26] Rio, E. (2000). *Théorie asymptotique des processus aléatoires faiblement dépendants. Mathématiques et applications de la SMAI.* **31** Springer.

[27] Rosenblatt, M. (1956). A central limit theorem and a strong mixing condition, *Proc. Nat. Acad. Sci. U. S. A.* **42** 43-47.

[28] Sarig, O. (2002). Subexponential decay of correlations. *Inv. Math.* **150** 629-653.

[29] Stout, W. F. (1974). Almost sure convergence. *Academic Press, New-York.*

[30] Volný, D. and Samek, P. (2000). On the invariance principle and the law of iterated logarithm for stationary processes. *Mathematical physics and stochastic analysis* 424–438, World Sci. Publishing, River Edge.

[31] Young, L.-S. (1999). Recurrence times and rates of mixing. *Israel J. Math.* **110** 153-188.

[32] Zweimüller, R. (1998). Ergodic structure and invariant densities of non-markovian interval maps with indifferent fixed points. *Nonlinearity* **11** 1263-1276.